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Optimal Packings of up to 6 Equal Circles on a Triangular Flat Torus

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1 Introduction

We find all the locally maximally dense packings of 1 to 6 equal circles on the quotient of the Euclidean plane by a lattice generated by two unit vectors with a 60 degree angle between them (a standard triangular torus). Packings on the square flat torus (when the lattice vectors are unit and perpendicular) are considered in [15, Section 2.2.5], where dense arrangements for 1 to 19 equal circles are presented and the arrangements for 1 to 4 circles are proven to be globally maximally dense using a technique that partitions a fundamental domain into different regions using a lower bound on the globally best radius. (Articles [11], [17] and [14] also explore packings on square or rectangular tori.)

In this article, we use techniques that are fundamentally different from those above, involve discerning the properties of the packing graphs associated to locally maximally dense arrangements of equal circles and enumerating all the 2-cell embeddings of certain graphs onto a torus. These techniques lead to proofs of the global maximality of the arrangements of 1 to 6 circles on the triangular flat torus and can be applied to packings of n circles (equal or not) on other flat tori. The same techniques are used in [4] in the study of 1-5 equal circles on the square flat torus.

There is an outstanding case of a conjecture of L. Fejes Tóth (in [9]) about the triangular close packing of circles in the plane (where one circle is surrounded by 6 other) being **strongly solid**. That is, if we remove any circle from this packing, called $\mathcal{C}_0$, the remaining packing remains **solid**. A packing, $\mathcal{P}$ is called **solid** if no finite subset of the circles can be rearranged such that the rearranged circles together with the rest of circles form a packing not congruent to $\mathcal{P}$. This is an open case of Fejes Tóth's original conjecture; see the work in [2] for the other cases. (A related concept is uniform stability, where the maximum distance each rearranged circle is allowed to move is bounded. The uniform stability of $\mathcal{C}_0$ was proved in [3] and [1].)

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The study of equal circle packings on the standard triangular torus is related to this conjecture. In the development below it will be proven that if $n = a^2 + ab + b^2$ (a and b integral), then there exist a packing of n circles on the standard triangular torus that lift to the triangular close packing on the plane. Numbers of this form are called **triangular packing number**. The following conjecture implies the above conjecture of L. Fejes Tóth.

Conjecture 1.1. *Let $n > 1$ be a triangular packing number so that $n - 1$ is not a triangular packing number and $\mathcal{C}$ be the packing of n circles on the standard triangular torus that lift to the triangular close packing in the plane. The unique globally maximally dense packing of $n - 1$ circles on the standard triangular torus is congruent to $\mathcal{C}$ with one circle removed.*

This article proves this conjecture in the case of $n = 3$ and $n = 7$.

2 Equal Circle Packings On Flat Tori

Consider the lattice, Λ, generated by two linearly independent basis vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{E}^2$. The quotient of the plane by this lattice is called a **flat torus**. A **fundamental domain** of a flat torus is the set of points in the Euclidean plane, $\{t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \mid 0 \leq t_1, t_2 < 1\}$. Notice that a lattice has many bases and therefore has many fundamental domains. When the basis for the lattice consists of two unit vectors with a 60 degree angle between them, the quotient torus is called the **standard triangular torus**. The **standard basis** for the standard triangular torus is one where $\mathbf{v}_1 = \langle 1, 0\rangle$ and $\mathbf{v}_2 = \langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$. For a point, p, a **lift** of it is another point q in the Euclidean plane that is equivalent to p.

For a given flat torus, an arrangement of equal circles forms a **packing** on the torus if the interiors of the circles are disjoint. The density of a packing is the the ratio of the area covered by the circles to the area of the flat torus. For a fixed number of circles, we can ask for the maximum possible packing density on the torus, which obviously occurs when the common diameter is maximized. Some packings are local maximums of the density function. More precisely, we define two packings with the same number of circles to be **ϵ-close** if there is a one-to-one correspondence between the circles, so that corresponding circles have centers the are all within ϵ of each other. We define a packing $\mathcal{P}$ to be **locally maximally dense** if there exists an $\epsilon > 0$ so that all ϵ-close packings of equal circles have a packing density no larger than that of $\mathcal{P}$. A packing $\mathcal{Q}$ is **globally maximally dense** if it is at least as dense as all other locally maximally dense packing. Rather than searching directly for the globally maximally dense packings, our techniques allow us to determine all the locally maximally dense arrangements of a fixed number of circles on a given flat torus. This allows us to easily determine the globally maximally dense packing(s) and the corresponding diameter that globally maximizes the density function.

The main tool that allows us to form a list of all the locally maximally dense packings on a flat torus is the graph of a packing. More formally, given a packing $\mathcal{P}$ on a flat torus, the **graph associated to $\mathcal{P}$**, denoted $G_{\mathcal{P}}$, has vertices and edges defined as follows. The center of each circle in the packing is associated to a vertex (with a location in the flat torus) of $G_{\mathcal{P}}$ and two vertices of $G_{\mathcal{P}}$ are connected with an edge if and only if the corresponding circles are tangent to each other. (For more details see [6, p. 48].) Notice that each packing of equal circles on a flat torus is associated to an embedding of a graph on a flat torus where all the edges are equal in length. The next section will illuminate some of the properties of packing graphs associated to locally maximally dense packings on a flat torus.

For n circles on a flat torus we can upper bound the diameter (or edge length of the associated packing graph) by using the L. Fejes Tóth-Thue Theorem ([8, pp.160-167], [19]) that states that the most dense packing of equal circles in the Euclidean plane is the triangular close packing where each circle is tangent to six others. The triangular close packing has density $\frac{\pi}{\sqrt{12}}$ and as all the lifts of all n circles in a packing on a flat torus form a packing of the Euclidean plane, the packing density on the torus cannot exceed this bound. In the case of the standard triangular torus this proves the following.

Proposition 2.1 (Diameter Upper Bound). *The common diameter of a packing of n equal circles on a standard triangular torus may not exceed $\frac{1}{\sqrt{n}}$.*

Notice that the centers of the circles in the triangular close packing form a triangular lattice (generated the standard basis $\{\mathbf{v}_1, \mathbf{v}_2\}$). A natural question to ask is when is there a sublattice of the triangular lattice that could be rotated and scaled to the triangular lattice.

Proposition 2.2. *If $n = a^2 + ab + b^2$, with a and b integers (both not zero), then there exist a packing of n circles on the standard triangular torus that lifts to the triangular close packing on the plane.*

Proof. Identify the plane with $\mathbb{C}$, then the standard basis of the triangular lattice becomes $\zeta_1 = 1$ and $\zeta_2 = e^{\frac{\pi}{3}i}$. Consider the integer linear combination of the basis vectors, $\eta_1 = a\zeta_1 + b\zeta_2$ and observe that $\eta_2 = \zeta_2\eta_1 = -b\zeta_1 + (a+b)\zeta_2$, so $\{\eta_1, \eta_2\}$ form a basis for a sublattice of the triangular lattice with equal length vectors with a 60 degree angle between them containing $\eta_1\overline{\eta_1} = a^2 + ab + b^2$ vertices of the triangular lattice. Rotating and scaling this sublattice into the standard lattice and considering the triangular close packing on the triangular lattice yield the desired packing. $\square$

The set of numbers $TP = \left\{a^2 + ab + b^2 \mid a, b \in \mathbb{Z}\right\} \setminus \{0\} = \{1, 3, 4, 7, 9, 12, 13, \ldots\}$ are called **triangular packing numbers**. This is Sloan's sequence A003136, and n has this form if in the prime factorization of n all primes of the form $3a + 2$ occur to even powers only (there is no restriction of primes congruent to 0 or $1 \mod 3$). This proposition together with the uniqueness of the triangular close packing as globally the most dense in the plane imply the following.

Proposition 2.3. *Let n be a triangular packing number, then there exists a unique globally maximally dense arrangement of n circles on the standard triangular torus that lifts to the triangular close packing in the plane. The common diameter of the circles is $\frac{1}{\sqrt{n}}$.*

As progress toward Conjecture 1.1 we note the following.

Theorem 2.1. *Let $n > 1$ be a triangular packing number so that $n - 1$ is not a triangular packing number and $\mathcal{C}$ be the packing of n circles on the standard triangular torus that lift to the triangular close packing in the plane. The packing obtained by removing one circle from $\mathcal{C}$ is locally maximally dense.*

Proof. In [3] it is proven that the triangular close packing with on circle removed is uniformly stable. Their argument hinges on the fact that a regular hexagon in the packing graph (the hole where one circle was removed) surrounded by a layer of equilateral triangles has the strict volume expanding property. For triangular packing numbers greater than or equal to 9, the boundary effects of the torus do not effect this hexagon and one layer of triangular and volume expanding property of the polygons and the fixed area of the torus imply the packing is locally maximally dense. For triangular packing numbers less than or equal to 7 the arguments below prove the result by finding all the locally maximally dense packings of 1 to 6 circles on the standard triangular torus. $\square$

Notice that if you drop the condition that $n - 1$ not be a triangular packing number, Theorem 2.1 fails in the case that $n = 4$.

3 Results From Tensegrity Frameworks

One of the most basic questions that we ask about a circle packing is: What is the minimum number of edges that the graph associated to a circle packing must contain in order for it to be locally maximally dense? Connelly has answered this question in [6] and [7] and the answer comes from the study of tensegrity frameworks and determining when such a framework is rigid. For the convenience of the reader, several definitions and theorems are repeated here to aid the reader in understanding this article. The presentation here is the specialization of a much broader theory to the cases needed for the study of circle packings on a flat torus, for more details about the broader theory see [6] and the references it contains.

The **strut tensegrity framework** associated to a circle packing is the graph associated to it where each edge in the graph is designated as a strut. A **strut** is an edge that is not allowed to decrease in length as its endpoint vertices move in a flat torus. This is appropriate because as we move the vertices of a circle packing graph, we want the common length of the edges to either increase or remain unchanged, in order to increase or maintain the density. If the location of the vertices, in a fundamental domain of a flat torus, of a strut tensegrity framework with n vertices are denoted $p_1, p_2, \ldots, p_n$ then a **flex** of a framework is a collection of n continuous functions $\{p_i(t) \mid i = 1..n\}$ from the interval $[0, 1]$ to the flat torus, where

1. $p_i(0) = p_i$ for all i, and

2. for each pair (i,j) where $\overline{p_i p_j}$ is a strut (i.e. edge) in the framework the distance between $p_i(t)$ and $p_j(t)$ does not decrease for all t in $[0,1]$.

A strut tensegrity framework is considered **rigid** if the only flex of the framework is a trivial flex. In the context of a flat torus, the only **trivial flex** of a framework is the one where, for a fixed vector $\mathbf{v}$, $p_i(t) = p_i + t\mathbf{v}$ for all i and all t in $[0,1]$. That is, all the vertices are translated in the same direction by the same amount.

In the context of strut frameworks, there is another notion of rigidity that is easier check and is equivalent to the definition of rigidity given above. A **infinitesimal flex** of a framework is a collection of n vectors $\{\mathbf{p}'_i \mid i = 1..n\}$ where for each pair (i,j) where $\overline{p_i p_j}$ is a strut

$$(p_i - p_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \geq 0 \tag{1}$$

A strut tensegrity framework is considered **infinitesimally rigid** if the only infinitesimal flex of the framework is a trivial infinitesimal flex. In the context of a flat torus, the only **trivial infinitesimal flex** of a framework is the one where, for a fixed vector $\mathbf{v}$, $\mathbf{p}'_i = \mathbf{v}$ for all i.

We have the following theorem due to Connelly (see [6, p. 49]) that relates these two notions of rigidity

Theorem 3.1. *If a tensegrity framework consists only of struts then the framework is rigid if and only if it is infinitesimally rigid.*

If you compare the definition of a locally maximally dense packing of circles and the definition of a rigid strut framework then it is easy to observe the following.

Observation 3.1. *If the packing graph associated to a circle packing $\mathcal{P}$ is (infinitesimally) rigid as a strut framework then the circle packing $\mathcal{P}$ is locally maximally dense.*

We are now in a position to state the main result of Connelly's from [6, p.56] (specialized to the context of flat tori) that begins to answer the question posed at the start of this section. This is a kind of converse of Observation 3.1.

Theorem 3.2. *Let $\mathcal{P}$ be a packing graph that is locally maximally dense on a flat torus, then the strut framework associated to the packing graph $G_{\mathcal{P}}$ is (infinitesimally) rigid or $G_{\mathcal{P}}$ contains a subgraph $G_{\mathcal{Q}}$ (corresponding to sub-packing $\mathcal{Q}$) whose associated strut framework is (infinitesimally) rigid and $\mathcal{P}$ contains circles that are free to move.*

There exist locally maximally dense packings of equal circles which contain circles that are free to move (i.e. they are not held fixed by their neighbors), but the common diameter of all the circles cannot increase. For example, this occurs in the globally maximally dense arrangement of 7 circles packed into a hard-boundary square where one circle is is free to move. (This result is attributed to Schaer [18] in 1965 by Goldberg [10].) If we remove any circles that are free to move, called **free circles** (also known as **floaters** or **rattlers**), from such an arrangement, then we obtain a locally maximally dense packing for fewer circles in the flat torus. Our search will proceed sequentially from 1 to 6 circles and will find all of the locally maximally dense packings of these number of circles, and we will be able to tell, for any locally maximally dense arrangement, if there is room for another equal circle that could be a free circle in a locally maximally dense arrangement of a larger number of circles packed onto a flat torus. This will allow us to consider only connected graphs in our exhaustive search of all possible packing graphs and locally maximally dense packings without free circles in what follows.

Finally, we observe that we can lower bound the number of edges (and their arrangement) incident to a vertex in the packing graph associated to a locally maximally dense packing with no free circles.

Proposition 3.1. *Let $\mathcal{P}$ be a locally maximally dense packing of circles with no free circles, then no circle in $\mathcal{P}$ has its points of tangency contained in a closed semi-circle. In particular, every circle is tangent to at least three circles.*

Proof. Suppose that $\mathcal{P}$ is a locally maximally dense packing of circles with no free circles, and it contains a circle, C (with center p_1), whose tangencies are restricted to a closed semi-circle. Consider the line l that passes through the center of C so that all points of tangency on C are on l or are on one side of l and let $\mathbf{v}$ be the vector perpendicular to l pointing away from the side of l that contains the points of tangency. Let $\mathbf{p}_1' = \mathbf{v}$ and $\mathbf{p}_i' = \mathbf{0}$ for all other circles in the packing. This assignment of vectors to the vertices of $G_{\mathcal{P}}$ is a non-trivial infinitesimal flex of the strut framework associated to $G_{\mathcal{P}}$ and therefore $G_{\mathcal{P}}$ is not rigid. This contradicts Theorem 3.2. □

4 Packings Of 1 To 4 Circles

Using Inequalities 1, Connelly proves in [7] a lower bound on the number of edges that a packing graph must contain in order for the associated packing to be a locally maximally dense. Specializing Equation 2.1 in [7, p.59] to the flat torus we have the following.

Proposition 4.1. *Let $\mathcal{P}$ be a locally maximally dense packing of n circles on a flat torus with no free circles then the packing graph associated to $\mathcal{P}$ contains at least $2n - 1$ edges.*

This lower bound is powerful and allows us to determine all the locally maximally dense packings of 1 to 4 circles on the standard triangular torus. The only locally maximally dense packing of one circle with diameter 1 is obviously the one obtained from Proposition 2.2 because 1 is a triangular packing number. It is also the only packing where a circle is tangent to itself. See Figure 1.

Proposition 4.2. *The only locally maximally dense packing of two circles on the triangular torus (up to translated variants) is the packing pictured in Figure 2. The common diameter is $\frac{1}{\sqrt{3}}$.*

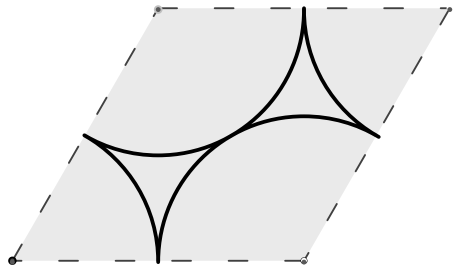


Figure 1: The only locally maximally dense packing of 1 circles on the standard triangular torus and therefore the globally maximally dense packing.

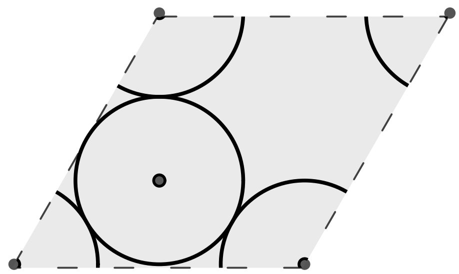


Figure 2: The only locally maximally dense packing of 2 circles on the standard triangular torus and therefore the globally maximally dense packing.

Proof. Consider the fundamental domain of the standard triangular torus with the standard basis. By translations we may fix the first circle at $(0,0)$ and by Proposition 4.1 we must place the second circle in a location where it is tangent to the first circle in at least 3 ways. Using the symmetries of the lattice, we may assume that this circle is located in the triangle with vertices $(0,0)$, $(1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Inside this fundamental domain, if we place this circle anywhere except at $(\frac{1}{2}, \frac{\sqrt{3}}{6})$, then a maximum of two tangencies are formed. Therefore we must must place the second circle at $(\frac{1}{2}, \frac{\sqrt{3}}{6})$. Using Inequality (1) on the three edges of the packing graph it and Observation 3.1 is easy to show that this packing is locally maximally dense. □

Now let us turn our attention to packings of 3 circles on the triangular torus.

Proposition 4.3. *The only locally maximally dense packing of three circles on the triangular torus (up to translated variants) is the packing pictured in Figure 3. The common diameter is* $\frac{1}{\sqrt{3}}$.

Proof. Observe that in the locally maximally dense packing for two circles a third circle can be located at $(1, \frac{\sqrt{3}}{3})$. The resulting packing must be globally maximally dense. Now we demonstrate that this packing is the only locally maximally dense packing by showing that any other packing with the minimum number of tangencies must be congruent to this one.

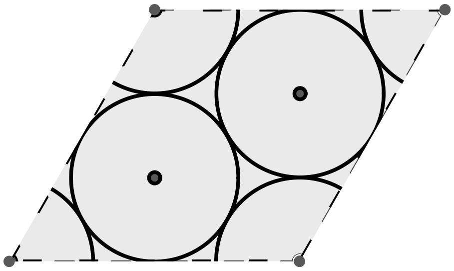


Figure 3: The only locally maximally dense packing of 3 circles on the standard triangular torus and therefore the globally maximally dense packing.

By Proposition 4.1 there must be at least 5 tangencies. There must be at least one pair of circles tangent to each other in at least two different ways (doubly-tangent) because there are only 3 circles and hence $\binom{3}{2} = 3$ possible single tangencies. Call the doubly-tangent circles, C_1 and C_2. By translations, we may assume that circle, C_1 is centered at $p_1 = (0, 0)$ and using the symmetries of the triangular torus we may assume that the second circle, C_2, is tangent to C_1 and the lift of C_1 centered at $(1, 0)$ or the lifts of C_1 at $(1, 0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Either choice of tangency arrangement leads to a similar argument, so we assume that the second circle, C_2 is centered at $p_2 = (\frac{1}{2}, \sqrt{d^2 - \left(\frac{1}{2}\right)^2})$ and d is the common diameter of the circles. Notice that this implies $\frac{1}{2} \leq d \leq \frac{1}{\sqrt{3}}$.

A **disk of exclusion for circle C** is a disk of *radius* d centered at C. By the definition of circle packing, another circle cannot have its center located interior to another circle's disk of exclusion. Let E_1 and E_2 be the disks of exclusion for the circles with centers p_1 and p_2. As we have two tangencies in this packing (the double tangency) we must place one more circles in the fundamental domain to create at least 3 additional tangencies. There are exactly two locations for this circle denoted p_3 and $\overline{p_3}$ that accomplish this. See figure 4.

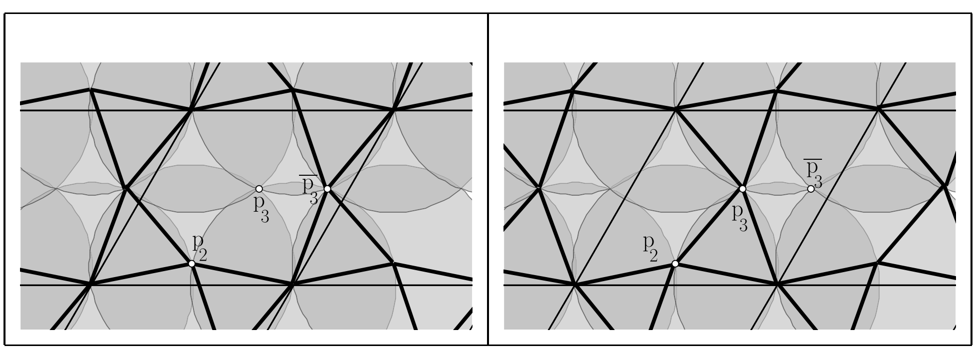


Figure 4: Three circles on the standard triangular torus with two circles of exclusion and the only possible locations of the center of circle C_3.

However, notice that when $\frac{1}{2} \leq d < \frac{1}{\sqrt{3}}$, both of packings are in a family of packings where the common diameter can increase and there each cannot be a locally maximally dense packing. Only when the globally maximal diameter is reached (where the packing graph is a union of triangles) does the packing graph become infinitesimally rigid. This demonstrates that the packing of Figure 3 is the unique locally maximally dense packing. □

Notice that if there is a pair of circles that are tangent to each other in three different ways in a triangular torus then the above arguments implies that the diameter of the circles must be $\frac{1}{\sqrt{3}}$ and Proposition 2.1 about the upper bound on the diameter of a circle packing implies that $n \leq 3$ in order for this diameter to be possible. Therefore we have the following.

Observation 4.1. *If an equal circle packing on the triangular torus contains a pair of circles that are tangent to each other in three different ways, then the packing contains 1,2 or 3 circles.*

Finding all the locally maximally dense packings of four circle easily follows from the fact that 4 is a triangular packing number and the upper bound on the diameter.

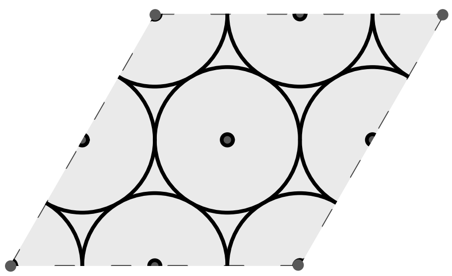


Figure 5: The only locally maximally dense packing of 4 circles on the standard triangular torus and therefore the globally maximally dense packing.

Proposition 4.4. *The only locally maximally dense packing of four circles on the standard triangular torus (up to translated variants) is the packing pictured in Figure 5. The common diameter is* $\frac{1}{\sqrt{4}}$.

Proof. As 4 is a triangular packing number, we know by Proposition 2.2 that the packing in Figure 5 is the unique globally maximally dense arrangement of 4 circles on the standard triangular torus. Suppose that was another locally maximally dense packing with common diameter $d < \frac{1}{\sqrt{4}}$. This restriction on the diameter prevents two circles from being tangent twice on the standard triangular torus. (Notice that if two circles are tangent twice in a packing then as the minimum distance between a circle and its lift is one, the common diameter is at least $\frac{1}{2}$.) Hence in this case there at most $\binom{4}{2} = 6$ tangencies which is less then the minimum 7 needed for a locally maximally dense packing of 4 circles. □

The observation in the second half of this proof allows us to further restrict the packing graph when we have 5 or more circles in a packing on a triangular torus.

Proposition 4.5. *If an equal circle packing on the standard triangular torus contains a pair of circles that are tangent to each other in two different ways, then the packing contains 4 or fewer circles.*

Proof. Suppose circles C_1 and C_2 are tangent to each other in two different ways. In the case of the standard triangular torus, the minimal distance from a circle to a lift of it is one, so if C_2 is tangent to C_1 and a lift of C_1, the triangle inequality implies that the common diameter of the circles is $\frac{1}{2}$ or larger. However, the upper bound on the diameter given in Proposition 2.1 is greater than $\frac{1}{2}$ only if $1 \leq n \leq 4$, so the equal circle packing contains at most 4 circles. $\square$

5 Results From Topological Graph Theory

The results from tensegrity frameworks and a study of the upper bound of the diameter of a packing on a square torus allows us to state many properties of the packing graph associated to a locally maximally dense packing. Collecting these yields the following.

Observation 5.1. *Given a locally maximally dense packing, $\mathcal{P}$, of $n > 4$ equal circles (without any free circles) on a torus, the packing graph $G_{\mathcal{P}}$ satisfies the following conditions.*

1. *It contains at least $2n - 1$ (Proposition 4.1) and at most $3n$ edges (triangular close packing).*
2. *It contains no loops as the only packing of one circle contains a self-tangent circle.*
3. *It contains no multi-edges as every pair of vertices is connected at most once (Observations 4.5 and 4.1).*
4. *Every vertex is connected to at least three others (Proposition 3.1) and at most six others.*

So there are two natural question to ask:

1. For a fixed n, how many abstract graphs satisfy the properties in Observation 5.1?
2. How many ways can those abstract graphs from Question 1 be embedded on a flat torus?

The first question will be discussed in the next section and the second question has been extensively studied by topological graph theorists.

Following the development in [16], let G be a connected graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. The set of neighbors of $i \in V(G)$ is given by $N(i) = \{j \in V(G) \mid ij \in E(G)\}$. A **rotation at i** is a cyclic permutation $\rho_i : N(i) \to N(i)$, and a set $\rho = \{\rho_i\}_{i \in V(G)}$ is called a **rotation scheme on G**. Each rotation scheme on G determines a 2-cell embedding of G onto a surface of genus g and there is a one-to-one correspondence between oriented and labeled embeddings of G and rotation schemes for G given by the following theorem, see [20, Sec. 6.6].

Theorem 5.1. *Each rotation scheme on G determines a 2-cell embedding $M(G)$ of G on a closed oriented 2-manifold M, such that there is an orientation on M which induced a cyclic ordering of the edges $\{i,k\}$ at i in which the immediate successor to $\{i,k\}$ is $\{i,\rho_i(k)\}$, $i=1,\ldots,n$. Conversely, given an oriented labeled 2-cell embedding of G on a closed oriented 2-manifold M there is a corresponding rotation scheme on G determining that embedding.*

This leads naturally to an algorithm that takes a rotation scheme and determines the labeled oriented embedding on a surface. Given a rotation scheme on G, ρ and a list of the oriented edges of G (each edge of G corresponds to two oriented edges), we can determine the faces of the embedding. For each oriented edge ij the next edge in the face is $j\rho_j(i)$, repeating this we can trace the faces in the embedding of G determined by ρ, marking the oriented edges as they are traced. The number of unoriented edges and the number of faces and the Euler formula determine the genus of the surface determined by ρ.

The number of rotation schemes for a given graph is $\prod_{i=1}^{n}(|N(i)|-1)!$, where $|N(i)|$ is the number of vertices adjacent to vertex i. Searching all of these rotation schemes for torus (genus 1) embeddings determines all of the possible oriented labeled 2-cell torus embeddings of a graph G, but many of these are essentially the same. Some are merely relabelings of others, some are identical except for the choice of orientation or both. As circle packing graphs do not come with natural labels or orientations, we are going to regard two rotation schemes, ρ and σ, on G as equivalent if there exists an automorphism α in the automorphism group of G such that $\alpha\rho_i\alpha^{-1}=\sigma_{\alpha(i)}$ or $\alpha\rho_i^{-1}\alpha^{-1}=\sigma_{\alpha(i)}$ for all i.

This gives us an easily programable method for determining all the possible unlabeled, unoriented 2-cell torus embeddings of a given abstract graph onto a topological torus. Once we have a complete answer to question 1 above, this gives us a way to determine a list of *all* potential packing graphs (i.e. 2-cell embeddings of a graph onto a torus) and among these embedded graphs the packing graph associated to *all* locally maximally dense packing must appear so long as it is a 2-cell embedding as the following lemma guarantees.

Lemma 5.1. *If a packing of circles on a flat torus is locally maximally dense without any free circles then the associated packing graph is a 2-cell embedding.*

Proof. Suppose that we have a locally maximally dense packing of circles, $\mathcal{P}$ whose associated packing graph, $G_{\mathcal{P}}$ is not 2-cell embedded on a flat torus. Let R be the region that is not a 2-cell and let B be one component of the polygonal boundary path of edges of R. If B is homotopically trivial (i.e. if $G_{\mathcal{P}}$ is planar) then let C be the smallest circle containing B. As B is a polygonal chain of straight edges, there must be a vertex on C. Clearly all of the tangencies to the circle corresponding to this vertex are contained in a closed semi-circle, contradicting Proposition 3.1.

Now suppose that B is not homotopically trivial. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be a basis for the lattice of the flat torus and translate $G_{\mathcal{P}}$ so that a vertex, v, of B has (1) coordinates of the form $t_1\mathbf{v}_1+0\mathbf{v}_2$ or $0\mathbf{v}_1+t_1\mathbf{v}_2$ for some $t_1\in[0,1)$, and (2) there exists a lift of v that is contained in closure (in $\mathbb{E}^2$) of the fundamental domain so that the line segment, l, connecting v and $\overline{v}$ is in the same homotopy class as B. Locally, on one side of B in the fundamental domain is part of the region R, if l is not contained in R, or all of the edges of B are on l, then the tangencies to the circle corresponding to vertex v are contained in a closed semi-circle, contradicting Proposition 3.1. In the remaining cases, one of the vertices that is furthest from l leads to a circle with tangencies that contradict Proposition 3.1. $\square$

6 Packings of 5 Circles

Packing 5 circles onto a flat torus requires at least 9 and at most 10 tangencies by Observation 5.1. This means that, abstractly, that the packing graph of any locally maximally dense packing is isomorphic to the complete graph on 5 vertices, denoted K_5 or to K_5 with an edge removed.Using the ideas outlined in Section 5, there are 1800 labeled, oriented 2-cell embeddings which reduce to 6 unoriented, unlabeled 2-cell embeddings of K_5 on a flat torus (pictured in Figure 6) and there are 206 labeled, oriented 2-cell embeddings which reduce to 14 unoriented, unlabeled 2-cell embeddings of K_5 minus an edge on a flat torus. Thirteen of these embeddings contain a triangle and the 14$^{\text{th}}$ is Embedding 4 in Figure 6 with an edge removed, see Figure 8. The software by Kocay ([12]) implementing the torus embedding algorithm given in [13] was invaluable in drawing these 20 embeddings on the torus and calculating the automorphism groups of abstract graphs.

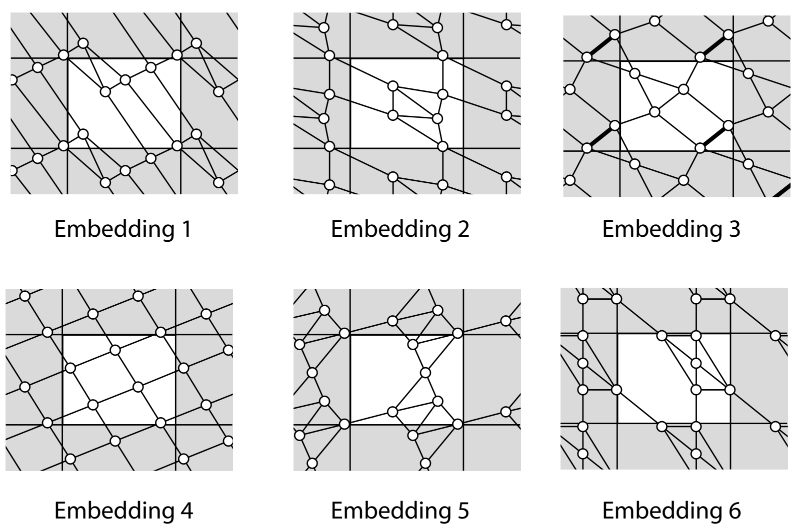


Figure 6: The six inequivalent embeddings of K_5 on a flat torus.

Lemma 6.1. *If the packing graph associated with a locally maximally dense packing of five equal circles on the standard triangular torus contains a triangle, then the packing is congruent to the locally maximally dense*

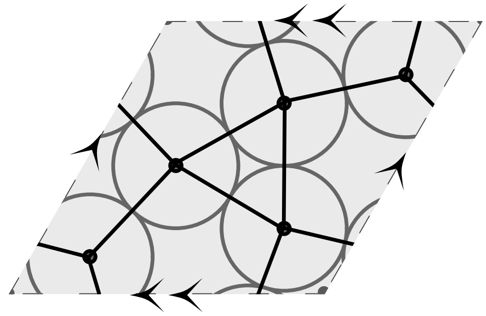


Figure 7: The only locally maximally dense packing of 5 circles on the standard triangular torus and therefore the globally maximally dense packing.

arrangement shown in Figure 7.

Proof. Suppose the packing graph associated to a locally maximally dense packing of five equal circles on the standard triangular torus contain a triangle. Using the symmetries of torus, we may assume that one vertex of the triangle is at the origin in the standard fundamental domain and that line connecting the vertex at the origin to the midpoint of the opposite side makes an angle α with the vector $\langle 1, 0\rangle$ and that $0 \le \alpha \le \frac{\pi}{6}$. Let the locations of the vertices of the triangle be $p_1 = (0,0)$, p_2 and p_3 and the common edge length in the packing graph be d, the diameter of the circles in the packing. A **disk of exclusion for circle C** is a disk of *radius* d centered at C. By the definition of circle packing, another circle cannot have its center located interior to another circle's disk of exclusion. Let E_1, E_2 and E_3 be the disks of exclusion for the circles with centers p_1, p_2 and p_3. As we have three tangencies, by Proposition 4.1, we must place two more circles in the fundamental domain to create at least six additional tangencies. If we assume that the two new circles are tangent to each other, we need to place the new circles in locations where they are tangent to two or three of the circles with centers p_1, p_2 and p_3. That is, we must place at least one circle at a triple intersection and the other at a double intersection point of the boundaries of E_1, E_2 and E_3.

Table 1 illustrates representative arrangements of the disks of exclusion E_0, E_1, and E_2, on the standard triangular torus for various values of d and α. We found all arrangements with a triple intersection point and checked whether it was possible to insert one circle at a triple intersection point and another circle at a second triple intersection point or double intersection point. Here is a brief discussion of the cases from Table 1.

<u>Cases 1, 3, 6, and 7:</u>

There are no triple intersections points of the boundaries of the disks of exclusion.

<u>Case 5:</u>

The union of the interiors of the three disks of exclusion cover the entire fundamental domain.

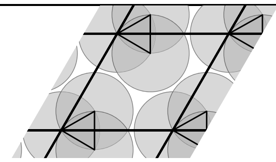
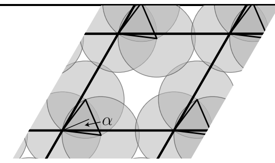
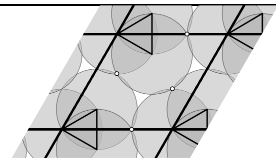
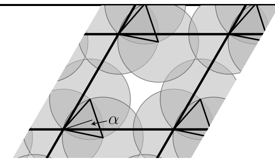
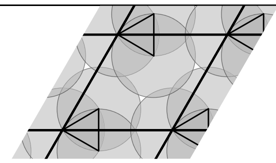
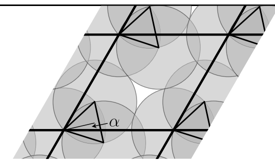
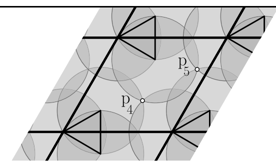
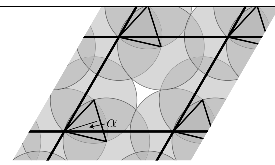
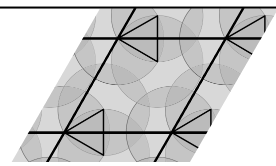
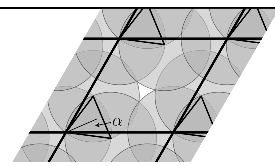
	$\alpha = 0$	$0 < \alpha < \frac{\pi}{6}$
$0 \le d < \frac{1}{1+\sqrt{3}}$	 1	 6
$d = \frac{1}{1+\sqrt{3}}$	 2	 7
$\frac{1}{1+\sqrt{3}} < d < \frac{\sqrt{11}}{4} - \frac{\sqrt{3}}{4}$	 3	 8
$d = \frac{\sqrt{11}}{4} - \frac{\sqrt{3}}{4}$	 4	 9
$\frac{\sqrt{11}}{4} - \frac{\sqrt{3}}{4} < d \le \frac{1}{\sqrt{5}}$	 5	 10

Table 1: Representative arrangements of the disks of exclusion E_0, E_1, and E_2, on the standard triangular torus for various values of d and α.

Cases 8, 9, and 10:

In these cases, for a fixed value of d there is exactly one value of α for which a triple intersection point occurs. At each fixed d value, all other values of α lead to arrangements containing double, but not triple, intersection points. When triple intersection points are formed the diameter of the region inside the fundamental domain but outside of the disks of exclusion is less than d and there is no way to add two additional circles.

Case 2:

There are three triple intersection points and six double intersection points and this leads to two possible packings (up to symmetry), however both of these packings are not locally maximally dense.

Case 4:

The only points in the fundamental domain exterior to union of the interior of the disks of exclusion are $p_4 = (\frac{\sqrt{3}\sqrt{11}}{12} + \frac{1}{4}, \frac{\sqrt{3}}{6})$, and $p_5 = (\frac{\sqrt{3}\sqrt{11}}{12} + \frac{3}{4}, \frac{\sqrt{3}}{3})$. When circles C_4 and C_5 are centered at points p_4 and p_5, the five circles are arranged in the locally maximally dense packing shown in Figure 7.

By exhaustive search of the feasible values of α and d this demonstrate that the only locally maximally dense packing of 5 circles on the standard triangular torus that contain a triangle is the arrangement indicated in case 4 of Table 1. $\square$

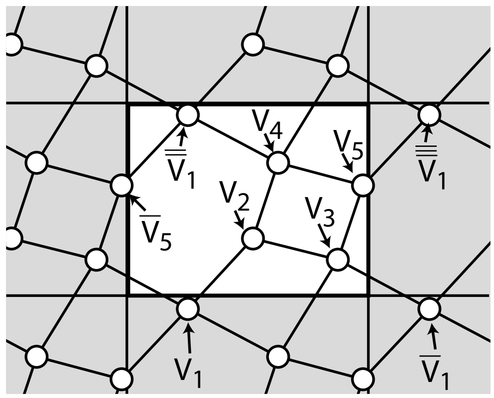


Figure 8: The only embeddings of K_5 minus an edge without a triangle.

Proposition 6.1. *There exists a unique locally maximally dense equal circle packing of 5 circles on the standard triangular torus with packing graph Embedding 3 in Figure 6 with the bold face edge removed. See also Figure 7. The common diameter is* $\frac{\sqrt{11}-\sqrt{3}}{4}$.

Proof. Notice that of the 20 possible embedding that could be the packing graph of a locally maximally dense packing, 18 contain a triangle and by Lemma 6.1, the only one that leads to a locally maximally dense packing is Embedding 3 in Figure 6 with the bold edge removed. This is the embedding shown in Figure 7. The only other possibility is Embedding 4 of Figure 6 or this embedding with an edge removed. If these are equilateral embeddings with edge length d, then the distance between v_1 and $\overline{v}_1$ in Figure 8 (or the corresponding vertices in Embedding 4) is $3d$. As these vertices must differ by a lattice vector, the d is upper bounded by $\frac{1}{\sqrt{5}}$, then $3d < \sqrt{3}$ and they must be distance one apart. The same argument implies that the pairs $\{v_1, \overline{\overline{v}}_1\}$, $\{\overline{v}_1, \overline{\overline{\overline{v}}}_1\}$ and $\{\overline{\overline{v}}_1, \overline{\overline{\overline{v}}}_1\}$ must also be a unit distance apart. The pairs $\{v_1, \overline{\overline{\overline{v}}}_1\}$ and $\{\overline{v}_1, \overline{\overline{v}}_1\}$ can be at most $(2+\sqrt{3})d$ apart, so they must also be a unit distance apart. This implies that there is a vertex and three lifts and all 6 pair of these are a unit length apart, which is impossible on the standard triangular lattice and therefore neither of these can be a packing graphs on the standard triangular torus. $\square$

7 All Locally Maximally Dense Packings Of 6 Circles

Packing 6 circles onto a flat torus requires at least 11 tangencies by the first property of Observation 5.1; the second property implies that there is at most 15 tangencies. This means that abstractly that the packing graph of any locally maximally dense is isomorphic to the complete graph on 6 vertices, denoted K_6 or to K_6 with between one and four edges removed. There are 13 possible abstract (ignoring the embedding) graphs that respect the properties listed in Observation 5.1. Using the ideas outlined in Section 5 Table 2 indicates the number of toroidal embeddings of each of the 13 abstract graphs and the number

Number of Edges Removed from K_6	Labeled and Oriented Embeddings	Unlabeled and Unoriented Embeddings	Number Remain After Prop. 7.1	Number Remain After Props. 7.1 to 7.4
0	1800	4	2	1
1	1584	24	5	1
2 (Type 1)	1488	66	6	0
2 (Type 2)	1056	47	12	4
3 (Type 1)	1016	140	13	2
3 (Type 2)	684	88	19	2
3 (Type 3)	1224	22	2	1
3 (Type 4)	524	17	8	3
4 (Type 1)	450	122	22	5
4 (Type 2)	332	52	15	5
4 (Type 3)	696	33	5	2
4 (Type 4)	408	20	2	1
4 (Type 5)	308	25	11	7
Total	11570	660	122	34

Table 2: Data for the toroidal embeddings of K_6 with between zero and four edges removed.

7.1 Eliminating Potential Packing Graphs

From these 660 potential packing graphs (i.e. 2-cell embeddings of an abstract graph), we can eliminate many of them using the following proposition.

Proposition 7.1. *If an equal circle packing on a flat torus is locally maximally dense with no free circles, then each vertex in its packing graph is not surrounded by any one of the following: (1) two triangles and a polygon,*

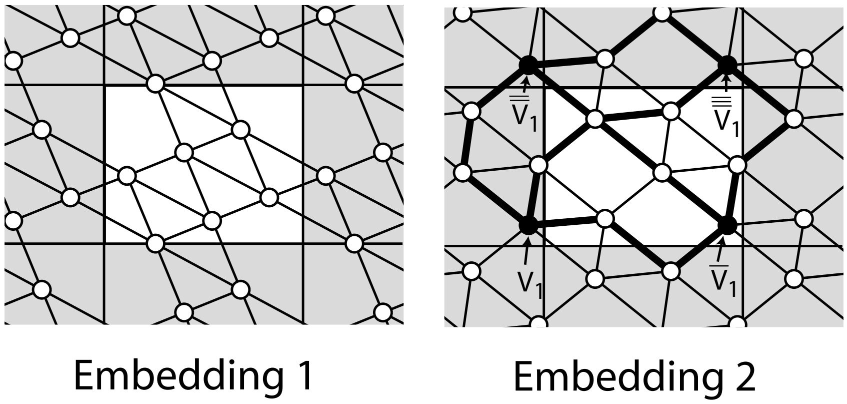


Figure 9: The two inequivalent embeddings of K_6 on a flat torus.

(2) three triangles and a polygon, (3) five triangles, (4) four triangles and a quadrilateral, (5) six polygons with at least one non-triangle, (6) a triangle, a quadrilateral and a polygon, (7) two triangles and two quadrilaterals, (8) three quadrilaterals, or (9) seven (or more) polygons.

Proof. This follows by observing that an equal circle packing graph is equilateral and this forbids certain face patterns around a vertex. For example a vertex can't be surrounded by three equilateral triangles and a polygon because the tangencies at that vertex would be contained in a closed semi-circle violating Proposition 3.1. Five equilateral triangles cannot surround a vertex because the angles around any vertex must be 2π. If a vertex is surrounded by three rhombi then three new edges are forced because the angles in a rhombus in a equal circle packing graph are between 60 and 120 degrees. The other cases follow similarly. □

Applying this proposition to the 660 potential packing graphs eliminates all but 122 of them as is indicated in the 4$^{\text{th}}$ column of Table 2. Here are some useful simple geometric propositions that help eliminate possible packing graphs

Proposition 7.2. *Let G be a potential packing graph. If G contains a chain of edges $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$, where A, B, C, and D are vertices (or lifts of vertices) such that $\overline{AB}$ is parallel to $\overline{CD}$, A and D are on the same side of $\overleftrightarrow{BC}$ and there is no edge connecting A and D, then G is not the packing graph of an equal circle packing on a flat torus.*

Proof. If G was a packing graph of an equal circle packing then $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$ would all be the same length, d. The other conditions imply that the distance from A to D is also d and therefore they should be connected with an edge, a contradiction to the hypotheses of the proposition. □

Proposition 7.3. *Let G be the packing graph of an equal circle packing on a flat torus in which there exists a cycle $v_1v_2v_3v_4v_5v_6$ where v_i are different vertices (or lifts of vertices). If $\overline{v_1v_2} \parallel \overline{v_4v_5}$, then either $\overline{v_2v_3} \parallel \overline{v_6v_1}$*

and $\overline{v_3v_4} \parallel \overline{v_5v_6}$*, or* $\overline{v_2v_3} \parallel \overline{v_5v_6}$ *and* $\overline{v_3v_4} \parallel \overline{v_6v_1}$*.*

Proof. Since $\overline{v_1v_2} \parallel \overline{v_4v_5}$ and the length of $\overline{v_1v_2}$ is the same as $\overline{v_4v_5}$, then $v_1v_2v_4v_5$ is a parallelogram. This implies that $\overline{v_2v_4} \parallel \overline{v_5v_1}$ and and the length of $\overline{v_2v_4}$ is the same as $\overline{v_5v_1}$. Hence $\triangle v_2v_3v_4 \cong \triangle v_5v_6v_1$ by the Side-Side-Side triangle congruence theorem and the conclusions follows. □

By combining Proposition 7.3 with Proposition 7.2, we were able to eliminate 43 potential packing graphs. For example, consider Figure 10. Since [3265] is a rhombus, then [32] ∥ [56]. In the cycle [564231], since [32] ∥ [56] by Proposition 7.3, we have that edge [13] is parallel to either [64] or [42]. In either case we can use Proposition 7.2, to show that Figure 10 is not a packing graph of any packing.

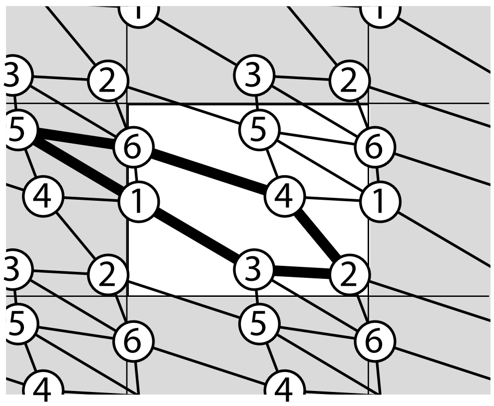


Figure 10: A 6 cycle in bold face that helps eliminate this potential packing graph.

Proposition 7.4. *If in a potential packing graph there are two edges (with a vertex, v, between them) that lay on a straight line and the vertex v is (1) surrounded by 3 polygons contained in one of the half-planes determined by this straight line so that at least one of the 3 polygons is not a triangle or (2) surrounded by 4 or more polygons contained in one of the half-planes or (3) v is surrounded by a triangle and rhombus contained in one half-plane, then the graph is not the packing graph associated to an equal circle packing on a flat torus.*

Proof. We sketch the proof of case 1, the other follow similarly. Since at least one of the 3 polygons surrounding v on the half-plane determined by the line is not a triangle and the sum of the three angles in the polygons at v is π, the angle at v in one of the non-triangles must $\frac{\pi}{3}$ or less. This either forces a new edge in the potential packing graph or is impossible in a packing graph. □

Using Proposition 7.4, we were able to eliminate 27 other embeddings. For example consider Figure 11. Since [6231] and [6352] are rhombi, then [13] ∥ [35]. This implies that [135] is a straight line. Notice that there are two triangles and one rhombus on one side of this straight line and the angle in the rhombus at vertex 3 is 60 degree, which forces edge [12] which is not an edge in this potential packing graph.

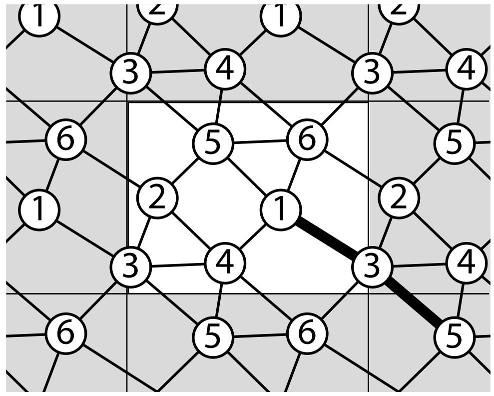


Figure 11: A pair of edges (in boldface) contained in a straight line.

Proposition 7.5. *Let G be the packing graph of a n equal circles on the standard triangular torus. If there is a chain of l edges between two lifts of the same vertex and $l < \sqrt{3n}$, then the two lifts are a unit distance apart.*

Proof. Let $\overline{v}$ and $\overline{\overline{v}}$ be lifts of the same vertex and d the common diameter of the circles. Notice that the furthest apart $\overline{v}$ and $\overline{\overline{v}}$ can be is $d \cdot l$ (when the chain of edges is on a straight line). By Proposition 2.1 $d \leq \frac{1}{\sqrt{n}}$ and the hypothesis on l, $d \cdot l \leq \frac{l}{\sqrt{n}} < \sqrt{3}$. As $\overline{v}$ and $\overline{\overline{v}}$ are lifts of the same vertex they differ by an element of the standard triangular lattice. The lengths of these vectors begin $1, \sqrt{3}, \ldots$, so $\overline{v}$ and $\overline{\overline{v}}$ must be a unit distance apart. □

This is a useful proposition and eliminates Embedding 2 in Figure 9, because the vertex v and its lifts $\overline{v}$, $\overline{\overline{v}}$ and $\overline{\overline{\overline{v}}}$ are a set of four vertices each pair of which is connected by a chain of edges of length 4 or less (in boldface in the Figure 9). This implies that they are all a unit distance apart and such an arrangement is impossible on the triangular torus. This same argument (or an argument where a vertex is found to have 8 unit distance lifts) eliminates 17 other potential packing graphs.

After applying the above propositions we are left with 34 potential packing graphs. Many of these (29) can be eliminated as being the packing graph of any packing on the standard triangular torus using *Ad Hoc* methods. Of the remaining five, 4 are one parameter families of embeddings and do not lead to locally maximally dense arrangements. This proves the following.

Proposition 7.6. *There exists a unique locally maximally dense equal circle packing of 6 circles on the square torus. See Figure 12. The common diameter of the circles is $\frac{1}{\sqrt{7}}$.*

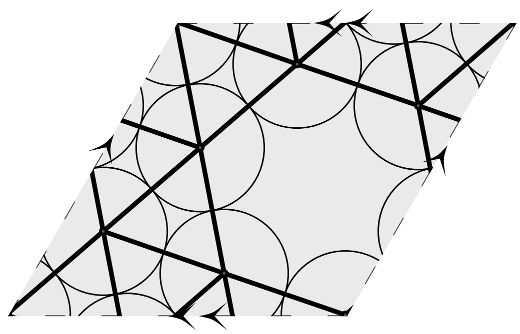


Figure 12: The only locally maximally dense packing of 6 circles on the standard triangular torus and therefore the globally maximally dense packing.

References

[1] Imre Bárány and N. P. Dolbilin, *A stability property of the densest circle packing*, Monatsh. Math. **106** (1988), no. 2, 107–114. MR MR968329 (89j:52011)

[2] A. Bezdek, *Solid packing of circles in the hyperbolic plane*, Studia Sci. Math. Hungar. **14** (1979), no. 1-3, 203–207 (1982). MR MR645529 (84c:52017)

[3] A. Bezdek, K. Bezdek, and R. Connelly, *Finite and uniform stability of sphere packings*, Discrete Comput. Geom. **20** (1998), no. 1, 111–130. MR MR1626699 (99c:52027)

[4] Anna Castelaz, William Dickinson, Daniel Guillot, and Sandi Xhumari, *Optimal packings of up to 5 equal circles on a square flat torus*, Submitted, August 2008.

[5] Robert Connelly, *Rigidity and energy*, Invent. Math. **66** (1982), no. 1, 11–33. MR MR652643 (83m:52012)

[6] ———, *Juxtapositions rigides de cercles et de sphères. I. Juxtapositions finies*, Structural Topology (1988), no. 14, 43–60, Dual French-English text. MR MR939609 (89i:52022)

[7] ———, *Juxtapositions rigides de cercles et de sphères. II. Juxtapositions infinies de mouvement fini*, Structural Topology (1990), no. 16, 57–76, Dual French-English text. MR MR1102004 (92i:52023)

[8] L. Fejes Tòth, *Regular figures*, International Series of Monographs on Pure and Applied Mathematics, vol. 48, Macmillan, New York, 1964.

[9] L. Fejes Tóth, *Solid circle-packings and circle-coverings*, Studia Sci. Math. Hungar. **3** (1968), 401–409. MR MR0238192 (38 #6468)

[10] Micheal Goldberg, *The packing of equal circles in a square*, Math. Mag. **43** (1970), no. 1, 24–30.

[11] Aladár Heppes, *Densest circle packing on the flat torus*, Period. Math. Hungar. **39** (1999), no. 1-3, 129–134, Discrete geometry and rigidity (Budapest, 1999). MR MR1783819 (2001i:52034)

[12] William Kocay, *Groups & graphs – software for graphs, digraphs, and their automorphism groups*, Match **58** (2007), no. 2, 431–443, Presented at the conference "Computers in Scientific Discovery" in 2006, availible from http://www.combinatorialmath.ca/G&G/.

[13] William Kocay, Daniel Neilson, and Ryan Szypowski, *Drawing graphs on the torus*, Ars Combin. **59** (2001), 259–277. MR MR1832459 (2002c:05057)

[14] Boris D. Lubachevsky, Ron L. Graham, and Frank H. Stillinger, *Patterns and structures in disk packings*, Period. Math. Hungar. **34** (1997), no. 1-2, 123–142, 3rd Geometry Festival: an International Conference on Packings, Coverings and Tilings (Budapest, 1996). MR MR1608310 (2000d:52010)

[15] J. B. M. Melissen, *Packing and covering with circles = pakken en overdekken met cirkels : (met een samenvatting in het nederlands) / door johannes bernardus marinus melissen*, Ph.D. thesis, Universiteit Utrecht, Utrecht, 1997.

[16] Bruce P. Mull, Robert G. Rieper, and Arthur T. White, *Enumerating* 2*-cell imbeddings of connected graphs*, Proc. Amer. Math. Soc. **103** (1988), no. 1, 321–330. MR MR938690 (89e:05081)

[17] Andrew Przeworski, *Packing disks on a torus*, Discrete Comput. Geom. **35** (2006), no. 1, 159–174. MR MR2183495 (2007b:52030)

[18] J. Schaer and A. Meir, *On a geometric extremum problem*, Canad. Math. Bull. **8** (1965), 21–27. MR MR0175029 (30 #5215)

[19] Axel Thue, *Selected mathematical papers*, Universitetsforlaget, Oslo, 1977, With an introduction by Carl Ludwig Siegel and a biography by Viggo Brun, Edited by Trygve Nagell, Atle Selberg, Sigmund Selberg, and Knut Thalberg. MR MR0460050 (57 #46)

[20] Arthur T. White, *Graphs of groups on surfaces*, North-Holland Mathematics Studies, vol. 188, North-Holland Publishing Co., Amsterdam, 2001, Interactions and models. MR MR1852593 (2002k:05001)