Chapter 1

Understanding the Derivative

1.1 How do we measure velocity?

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How is the average velocity of a moving object connected to the values of its position function?
- How do we interpret the average velocity of an object geometrically with regard to the graph of its position function?
- How is the notion of instantaneous velocity connected to average velocity?

Introduction

Calculus can be viewed broadly as the study of change. A natural and important question to ask about any changing quantity is "how fast is the quantity changing?" It turns out that in order to make the answer to this question precise, substantial mathematics is required.

We begin with a familiar problem: a ball being tossed straight up in the air from an initial height. From this elementary scenario, we will ask questions about how the ball is moving. These questions will lead us to begin investigating ideas that will be central throughout our study of differential calculus and that have wide-ranging consequences. In a great deal of our thinking about calculus, we will be well-served by remembering this first example and asking ourselves how the various (sometimes abstract) ideas we are considering are related to the simple act of tossing a ball straight up in the air.

Preview Activity 1.1. Suppose that the height s of a ball (in feet) at time t (in seconds) is given by the formula $s(t) = 64 - 16(t-1)^2$.

(a) Construct an accurate graph of $y = s(t)$ on the time interval $0 \le t \le 3$. Label at least six distinct points on the graph, including the three points that correspond to when the ball was released, when the ball reaches its highest point, and when the ball lands.

(b) In everyday language, describe the behavior of the ball on the time interval $0 < t < 1$ and on time interval $1 < t < 3$. What occurs at the instant $t = 1$?

(c) Consider the expression

$$AV_{[0.5,1]} = \frac{s(1) - s(0.5)}{1 - 0.5}.$$

Compute the value of $AV_{[0.5,1]}$. What does this value measure geometrically? What does this value measure physically? In particular, what are the units on $AV_{[0.5,1]}$?

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Position and average velocity

Any moving object has a *position* that can be considered a function of *time*. When this motion is along a straight line, the position is given by a single variable, and we usually let this position be denoted by $s(t)$, which reflects the fact that position is a function of time. For example, we might view $s(t)$ as telling the mile marker of a car traveling on a straight highway at time t in hours; similarly, the function s described in Preview Activity 1.1 is a position function, where position is measured vertically relative to the ground.

Not only does such a moving object have a position associated with its motion, but on any time interval, the object has an *average velocity*. Think, for example, about driving from one location to another: the vehicle travels some number of miles over a certain time interval (measured in hours), from which we can compute the vehicle's average velocity. In this situation, average velocity is the number of miles traveled divided by the time elapsed, which of course is given in *miles per hour*. Similarly, the calculation of $AV_{[0.5,1]}$ in Preview Activity 1.1 found the average velocity of the ball on the time interval $[0.5, 1]$, measured in feet per second.

In general, we make the following definition: for an object moving in a straight line whose position at time t is given by the function $s(t)$, the *average velocity of the object on the interval from $t = a$ to $t = b$*, denoted $AV_{[a,b]}$, is given by the formula

$$AV_{[a,b]} = \frac{s(b) - s(a)}{b - a}.$$

Note well: the units on $AV_{[a,b]}$ are "units of s per unit of t," such as "miles per hour" or "feet per second."

Activity 1.1.

The following questions concern the position function given by $s(t) = 64 - 16(t-1)^2$, which is the same function considered in Preview Activity 1.1.

(a) Compute the average velocity of the ball on each of the following time intervals: $[0.4, 0.8]$, $[0.7, 0.8]$, $[0.79, 0.8]$, $[0.799, 0.8]$, $[0.8, 1.2]$, $[0.8, 0.9]$, $[0.8, 0.81]$, $[0.8, 0.801]$. Include units for each value.

(b) On the provided graph in Figure 1.1, sketch the line that passes through the points $A = (0.4, s(0.4))$ and $B = (0.8, s(0.8))$. What is the meaning of the slope of this line? In light of this meaning, what is a geometric way to interpret each of the values computed in the preceding question?

(c) Use a graphing utility to plot the graph of $s(t) = 64 - 16(t-1)^2$ on an interval containing the value $t = 0.8$. Then, zoom in repeatedly on the point $(0.8, s(0.8))$. What do you observe about how the graph appears as you view it more and more closely?

(d) What do you conjecture is the velocity of the ball at the instant $t = 0.8$? Why?

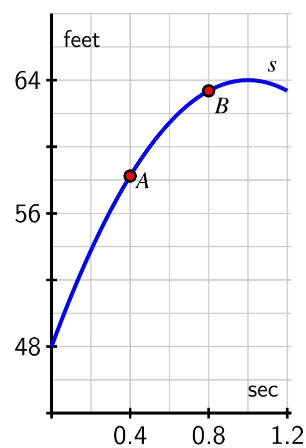

Figure 1.1: A partial plot of $s(t) = 64 - 16(t-1)^2$.

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Instantaneous Velocity

Whether driving a car, riding a bike, or throwing a ball, we have an intuitive sense that any moving object has a velocity at any given moment – a number that measures how fast the object is moving *right now*. For instance, a car's speedometer tells the driver what appears to be the car's velocity at any given instant. In fact, the posted velocity on a speedometer is really an average velocity that is computed over a very small time interval (by computing how many revolutions the tires have undergone to compute distance traveled), since velocity fundamentally comes from considering a change in position divided by a change in time. But if we let the time interval over which average velocity is computed become shorter and shorter, then we can progress from average velocity to *instantaneous* velocity.

Informally, we define the *instantaneous velocity* of a moving object at time $t = a$ to be the value that the average velocity approaches as we take smaller and smaller intervals of time containing $t = a$ to compute the average velocity. We will develop a more formal definition of this momentarily, one that will end up being the foundation of much of our work in first semester calculus. For now, it is fine to think of instantaneous velocity this way: take average velocities on smaller and smaller time intervals, and if those average velocities approach a single number, then that number will be the instantaneous velocity at that point.

Activity 1.2.

Each of the following questions concern $s(t) = 64 - 16(t-1)^2$, the position function from Preview Activity 1.1.

(a) Compute the average velocity of the ball on the time interval $[1.5, 2]$. What is different between this value and the average velocity on the interval $[0, 0.5]$?

(b) Use appropriate computing technology to estimate the instantaneous velocity of the ball at $t = 1.5$. Likewise, estimate the instantaneous velocity of the ball at $t = 2$. Which value is greater?

(c) How is the sign of the instantaneous velocity of the ball related to its behavior at a given point in time? That is, what does positive instantaneous velocity tell you the ball is doing? Negative instantaneous velocity?

(d) Without doing any computations, what do you expect to be the instantaneous velocity of the ball at $t = 1$? Why?

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At this point we have started to see a close connection between average velocity and instantaneous velocity, as well as how each is connected not only to the physical behavior of the moving object but also to the geometric behavior of the graph of the position function. In order to make the link between average and instantaneous velocity more formal, we will introduce the notion of *limit* in Section 1.2. As a preview of that concept, we look at a way to consider the limiting value of average velocity through the introduction of a parameter. Note that if we desire to know the instantaneous velocity at $t = a$ of a moving object with position function s, we are interested in computing average velocities on the interval $[a, b]$ for smaller and smaller intervals. One way to visualize this is to think of the value b as being $b = a + h$, where h is a small number that is allowed to vary. Thus, we observe that the average velocity of the object on the interval $[a, a+h]$ is

$$AV_{[a,a+h]} = \frac{s(a+h) - s(a)}{h},$$

with the denominator being simply h because $(a+h) - a = h$. Initially, it is fine to think of h being a small positive real number; but it is important to note that we allow h to be a small negative number, too, as this enables us to investigate the average velocity of the moving object on intervals prior to $t = a$, as well as following $t = a$. When $h < 0$, $AV_{[a,a+h]}$ measures the average velocity on the interval $[a+h, a]$.

To attempt to find the instantaneous velocity at $t = a$, we investigate what happens as the value of h approaches zero. We consider this further in the following example.

Example 1.1. For a falling ball whose position function is given by $s(t) = 16 - 16t^2$ (where s is measured in feet and t in seconds), find an expression for the average velocity of the ball on a time interval of the form $[0.5, 0.5+h]$ where $-0.5 < h < 0.5$ and $h \neq 0$. Use this expression to compute the average velocity on $[0.5, 0.75]$ and $[0.4, 0.5]$, as well as to make a conjecture about the instantaneous velocity at $t = 0.5$.

Solution. We make the assumptions that $-0.5 < h < 0.5$ and $h \neq 0$ because h cannot be zero (otherwise there is no interval on which to compute average velocity) and because the function only makes sense on the time interval $0 \leq t \leq 1$, as this is the duration of time during which the ball is falling. Observe that we want to compute and simplify

$$AV_{[0.5,0.5+h]} = \frac{s(0.5+h) - s(0.5)}{(0.5+h) - 0.5}.$$

The most unusual part of this computation is finding $s(0.5+h)$. To do so, we follow the rule that defines the function s. In particular, since $s(t) = 16 - 16t^2$, we see that

$$\begin{aligned} s(0.5+h) &= 16 - 16(0.5+h)^2 \\ &= 16 - 16(0.25 + h + h^2) \\ &= 16 - 4 - 16h - 16h^2 \\ &= 12 - 16h - 16h^2. \end{aligned}$$

Now, returning to our computation of the average velocity, we find that

$$\begin{aligned} AV_{[0.5,0.5+h]} &= \frac{s(0.5+h)-s(0.5)}{(0.5+h)-0.5} \\ &= \frac{(12-16h-16h^2)-(16-16(0.5)^2)}{0.5+h-0.5} \\ &= \frac{12-16h-16h^2-12}{h} \\ &= \frac{-16h-16h^2}{h}. \end{aligned}$$

At this point, we note two things: first, the expression for average velocity clearly depends on h, which it must, since as h changes the average velocity will change. Further, we note that since h can never equal zero, we may further simplify the most recent expression. Removing the common factor of h from the numerator and denominator, it follows that

$$AV_{[0.5,0.5+h]} = -16 - 16h.$$

Now, for any small positive or negative value of h, we can compute the average velocity. For instance, to obtain the average velocity on $[0.5, 0.75]$, we let $h = 0.25$, and the average velocity is $-16 - 16(0.25) = -20$ ft/sec. To get the average velocity on $[0.4, 0.5]$, we let $h = -0.1$, which tells us the average velocity is $-16 - 16(-0.1) = -14.4$ ft/sec. Moreover, we can even explore what happens to $AV_{[0.5,0.5+h]}$ as h gets closer and closer to zero. As h approaches zero, $-16h$ will also approach zero, and thus it appears that the instantaneous velocity of the ball at $t = 0.5$ should be -16 ft/sec.

Activity 1.3.

For the function given by $s(t) = 64 - 16(t-1)^2$ from Preview Activity 1.1, find the most simplified expression you can for the average velocity of the ball on the interval $[2, 2+h]$. Use your result to compute the average velocity on $[1.5, 2]$ and to estimate the instantaneous velocity at $t = 2$. Finally, compare your earlier work in Activity 1.1.

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Summary

In this section, we encountered the following important ideas:

- The average velocity on $[a, b]$ can be viewed geometrically as the slope of the line between the points $(a, s(a))$ and $(b, s(b))$ on the graph of $y = s(t)$, as shown in Figure 1.2.

- Given a moving object whose position at time t is given by a function s, the average

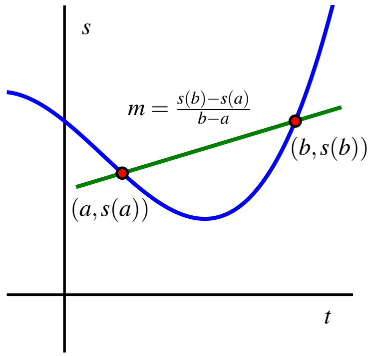

Figure 1.2: The graph of position function s together with the line through $(a, s(a))$ and $(b, s(b))$ whose slope is $m = \frac{s(b)-s(a)}{b-a}$. The line's slope is the average rate of change of s on the interval $[a, b]$.

velocity of the object on the time interval $[a, b]$ is given by $AV_{[a,b]} = \frac{s(b)-s(a)}{b-a}$. Viewing the interval $[a, b]$ as having the form $[a, a+h]$, we equivalently compute average velocity by the formula $AV_{[a,a+h]} = \frac{s(a+h)-s(a)}{h}$.

- The instantaneous velocity of a moving object at a fixed time is estimated by considering average velocities on shorter and shorter time intervals that contain the instant of interest.

Exercises

1. A bungee jumper dives from a tower at time $t = 0$. Her height h (measured in feet) at time t (in seconds) is given by the graph in Figure 1.3.

   In this problem, you may base your answers on estimates from the graph or use the fact that the jumper's height function is given by $s(t) = 100\cos(0.75t) \cdot e^{-0.2t} + 100$.

   (a) What is the change in vertical position of the bungee jumper between $t = 0$ and $t = 15$?

   (b) Estimate the jumper's average velocity on each of the following time intervals: $[0, 15]$, $[0, 2]$, $[1, 6]$, and $[8, 10]$. Include units on your answers.

   (c) On what time interval(s) do you think the bungee jumper achieves her greatest average velocity? Why?

   (d) Estimate the jumper's instantaneous velocity at $t = 5$. Show your work and explain your reasoning, and include units on your answer.

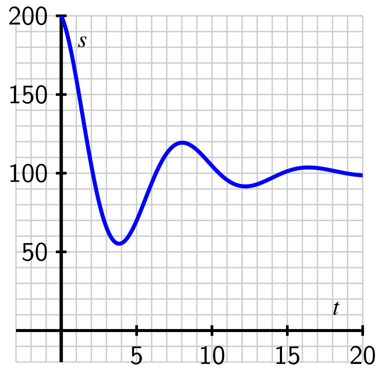

Figure 1.3: A bungee jumper's height function.

(e) Among the average and instantaneous velocities you computed in earlier questions, which are positive and which are negative? What does negative velocity indicate?

2. A diver leaps from a 3 meter springboard. His feet leave the board at time $t = 0$, he reaches his maximum height of 4.5 m at $t = 1.1$ seconds, and enters the water at $t = 2.45$. Once in the water, the diver coasts to the bottom of the pool (depth 3.5 m), touches bottom at $t = 7$, rests for one second, and then pushes off the bottom. From there he coasts to the surface, and takes his first breath at $t = 13$.

(a) Let $s(t)$ denote the function that gives the height of the diver's feet (in meters) above the water at time t. (Note that the "height" of the bottom of the pool is -3.5 meters.) Sketch a carefully labeled graph of $s(t)$ on the provided axes in Figure 1.4. Include scale and units on the vertical axis. Be as detailed as possible.

(b) Based on your graph in (a), what is the average velocity of the diver between $t = 2.45$ and $t = 7$? Is his average velocity the same on every time interval within $[2.45, 7]$?

(c) Let the function $v(t)$ represent the *instantaneous vertical velocity* of the diver at time t (i.e. the speed at which the height function $s(t)$ is changing; note that velocity in the upward direction is positive, while the velocity of a falling object is negative). Based on your understanding of the diver's behavior, as well as your graph of the position function, sketch a carefully labeled graph of $v(t)$ on the axes provided in Figure 1.4. Include scale and units on the vertical axis. Write several sentences that explain how you constructed your graph, discussing when you expect $v(t)$ to be zero, positive, negative, relatively large, and relatively small.

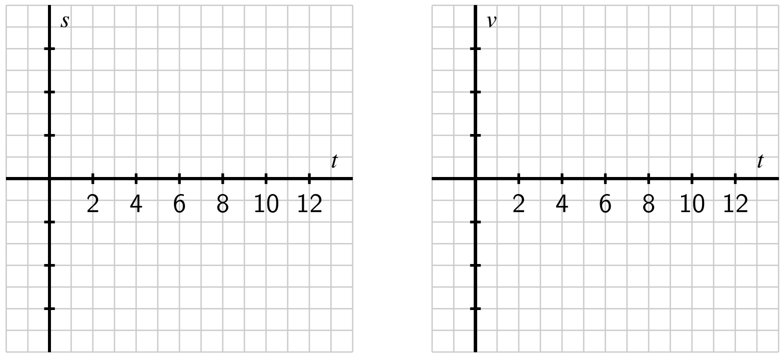

Figure 1.4: Axes for plotting $s(t)$ in part (a) and $v(t)$ in part (c) of the diver problem.

(d) Is there a connection between the two graphs that you can describe? What can you say about the velocity graph when the height function is increasing? decreasing? Make as many observations as you can.

3. According to the U.S. census, the population of the city of Grand Rapids, MI, was 181,843 in 1980; 189,126 in 1990; and 197,800 in 2000.

(a) Between 1980 and 2000, by how many people did the population of Grand Rapids grow?

(b) In an average year between 1980 and 2000, by how many people did the population of Grand Rapids grow?

(c) Just like we can find the average velocity of a moving body by computing change in position over change in time, we can compute the average rate of change of any function f. In particular, the *average rate of change* of a function f over an interval $[a, b]$ is the quotient

$$\frac{f(b) - f(a)}{b - a}.$$

What does the quantity $\frac{f(b)-f(a)}{b-a}$ measure on the graph of $y = f(x)$ over the interval $[a, b]$?

(d) Let $P(t)$ represent the population of Grand Rapids at time t, where t is measured in years from January 1, 1980. What is the average rate of change of P on the interval $t = 0$ to $t = 20$? What are the units on this quantity?

(e) If we assume the population of Grand Rapids is growing at a rate of approximately 4% per decade, we can model the population function with the formula

$$P(t) = 181843(1.04)^{t/10}.$$

Use this formula to compute the average rate of change of the population on the intervals [5, 10], [5, 9], [5, 8], [5, 7], and [5, 6].

(f) How fast do you think the population of Grand Rapids was changing on January 1, 1985? Said differently, at what rate do you think people were being added to the population of Grand Rapids as of January 1, 1985? How many additional people should the city have expected in the following year? Why?

1.2 The notion of limit

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is the mathematical notion of *limit* and what role do limits play in the study of functions?
- What is the meaning of the notation $\lim_{x \to a} f(x) = L$?
- How do we go about determining the value of the limit of a function at a point?
- How do we manipulate average velocity to compute instantaneous velocity??

Introduction

Functions are at the heart of mathematics: a function is a process or rule that associates each individual input to exactly one corresponding output. Students learn in courses prior to calculus that there are many different ways to represent functions, including through formulas, graphs, tables, and even words. For example, the squaring function can be thought of in any of these ways. In words, the squaring function takes any real number x and computes its square. The formulaic and graphical representations go hand in hand, as $y = f(x) = x^2$ is one of the simplest curves to graph. Finally, we can also partially represent this function through a table of values, essentially by listing some of the ordered pairs that lie on the curve, such as $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 4)$.

Functions are especially important in calculus because they often model important phenomena – the location of a moving object at a given time, the rate at which an automobile is consuming gasoline at a certain velocity, the reaction of a patient to the size of a dose of a drug – and calculus can be used to study how these output quantities change in response to changes in the input variable. Moreover, thinking about concepts like average and instantaneous velocity leads us naturally from an initial function to a related, sometimes more complicated function. As one example of this, think about the falling ball whose position function is given by $s(t) = 64 - 16t^2$ and the average velocity of the ball on the interval $[1, x]$. Observe that

$$AV_{[1,x]} = \frac{s(x) - s(1)}{x - 1} = \frac{(64 - 16x^2) - (64 - 16)}{x - 1} = \frac{16 - 16x^2}{x - 1}.$$

Now, two things are essential to note: this average velocity depends on x (indeed, $AV_{[1,x]}$ is a function of x), and our most focused interest in this function occurs near $x = 1$, which is where the function is not defined. Said differently, the function $g(x) = \frac{16-16x^2}{x-1}$ tells us the average velocity of the ball on the interval from $t = 1$ to $t = x$, and if we are interested in the instantaneous velocity of the ball when $t = 1$, we'd like to know what happens to $g(x)$ as x gets closer and closer to 1. At the same time, $g(1)$ is not defined, because it leads to the quotient $0/0$.

This is where the idea of *limits* comes in. By using a limit, we'll be able to allow x to get arbitrarily close, but not equal, to 1 and fully understand the behavior of $g(x)$ near this value. We'll develop key language, notation, and conceptual understanding in what follows, but for now we consider a preliminary activity that uses the graphical interpretation of a function to explore points on a graph where interesting behavior occurs.

Preview Activity 1.2. Suppose that g is the function given by the graph below. Use the graph to answer each of the following questions.

(a) Determine the values $g(-2)$, $g(-1)$, $g(0)$, $g(1)$, and $g(2)$, if defined. If the function value is not defined, explain what feature of the graph tells you this.

(b) For each of the values $a = -1$, $a = 0$, and $a = 2$, complete the following sentence: "As x gets closer and closer (but not equal) to a, $g(x)$ gets as close as we want to ____."

(c) What happens as x gets closer and closer (but not equal) to $a = 1$? Does the function $g(x)$ get as close as we would like to a single value?

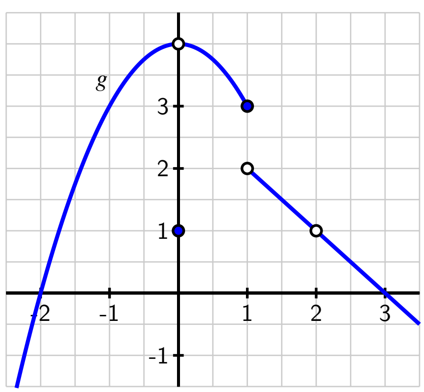

Figure 1.5: Graph of $y = g(x)$ for Preview Activity 1.2.

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The Notion of Limit

Limits can be thought of as a way to study the tendency or trend of a function as the input variable approaches a fixed value, or even as the input variable increases or decreases without bound. We put off the study of the latter idea until further along in the course when we will have some helpful calculus tools for understanding the end behavior of functions. Here, we focus on what it means to say that "a function f has limit L as x approaches a." To begin, we think about a recent example.

In Preview Activity 1.2, you saw that for the given function g, as x gets closer and closer (but not equal) to 0, $g(x)$ gets as close as we want to the value 4. At first, this may feel counterintuitive, because the value of $g(0)$ is 1, not 4. By their very definition, limits regard the behavior of a function ***arbitrarily close to*** a fixed input, but the value of the function ***at*** the fixed input does not matter. More formally[1], we say the following.

Definition 1.1. Given a function f, a fixed input $x = a$, and a real number L, we say that *f has limit L as x approaches a*, and write

$$\lim_{x \to a} f(x) = L$$

provided that we can make $f(x)$ as close to L as we like by taking x sufficiently close (but not equal) to a. If we cannot make $f(x)$ as close to a single value as we would like as x approaches a, then we say that *f does not have a limit as x approaches a.*

For the function g pictured in Figure 1.5, we can make the following observations:

$$\lim_{x \to -1} g(x) = 3, \ \lim_{x \to 0} g(x) = 4, \ \text{and} \ \lim_{x \to 2} g(x) = 1,$$

but g does not have a limit as $x \to 1$. When working graphically, it suffices to ask if the function approaches a single value from each side of the fixed input, while understanding that the function value right at the fixed input is irrelevant. This reasoning explains the values of the first three stated limits. In a situation such as the jump in the graph of g at $x = 1$, the issue is that if we approach $x = 1$ from the left, the function values tend to get as close to 3 as we'd like, but if we approach $x = 1$ from the right, the function values get as close to 2 as we'd like, and there is no single number that all of these function values approach. This is why the limit of g does not exist at $x = 1$.

For any function f, there are typically three ways to answer the question "does f have a limit at $x = a$, and if so, what is the limit?" The first is to reason graphically as we have just done with the example from Preview Activity 1.2. If we have a formula for $f(x)$, there are two additional possibilities: (1) evaluate the function at a sequence of inputs that approach a on either side, typically using some sort of computing technology, and ask if the sequence of outputs seems to approach a single value; (2) use the algebraic form of the function to understand the trend in its output as the input values approach a. The first approach only produces an approximation of the value of the limit, while the latter can

[1]What follows here is not what mathematicians consider the formal definition of a limit. To be completely precise, it is necessary to quantify both what it means to say "as close to L as we like" and "sufficiently close to a." That can be accomplished through what is traditionally called the epsilon-delta definition of limits. The definition presented here is sufficient for the purposes of this text.

often be used to determine the limit exactly. The following example demonstrates both of these approaches, while also using the graphs of the respective functions to help confirm our conclusions.

Example 1.2. For each of the following functions, we'd like to know whether or not the function has a limit at the stated a-values. Use both numerical and algebraic approaches to investigate and, if possible, estimate or determine the value of the limit. Compare the results with a careful graph of the function on an interval containing the points of interest.

(a) $f(x) = \dfrac{4-x^2}{x+2}$; $a = -1$, $a = -2$

(b) $g(x) = \sin\left(\dfrac{\pi}{x}\right)$; $a = 3$, $a = 0$

Solution. We first construct a graph of f along with tables of values near $a = -1$ and $a = -2$.

x	$f(x)$
-0.9	2.9
-0.99	2.99
-0.999	2.999
-0.9999	2.9999
-1.1	3.1
-1.01	3.01
-1.001	3.001
-1.0001	3.0001

x	$f(x)$
-1.9	3.9
-1.99	3.99
-1.999	3.999
-1.9999	3.9999
-2.1	4.1
-2.01	4.01
-2.001	4.001
-2.0001	4.0001

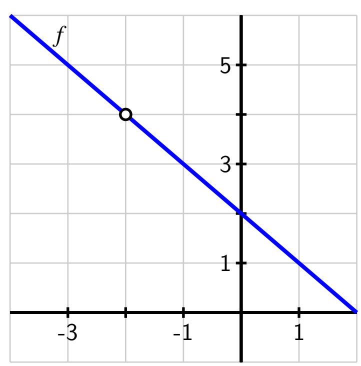

Figure 1.6: Tables and graph for $f(x) = \dfrac{4-x^2}{x+2}$.

From the left table, it appears that we can make f as close as we want to 3 by taking x sufficiently close to -1, which suggests that $\lim_{x\to -1} f(x) = 3$. This is also consistent with the graph of f. To see this a bit more rigorously and from an algebraic point of view, consider the formula for f: $f(x) = \frac{4-x^2}{x+2}$. The numerator and denominator are each polynomial functions, which are among the most well-behaved functions that exist. Formally, such functions are *continuous*[2], which means that the limit of the function at any point is equal

[2]See Section 1.7 for more on the notion of continuity.

to its function value. Here, it follows that as $x \to -1$, $(4 - x^2) \to (4 - (-1)^2) = 3$, and $(x + 2) \to (-1 + 2) = 1$, so as $x \to -1$, the numerator of f tends to 3 and the denominator tends to 1, hence $\lim_{x \to -1} f(x) = \frac{3}{1} = 3$.

The situation is more complicated when $x \to -2$, due in part to the fact that $f(-2)$ is not defined. If we attempt to use a similar algebraic argument regarding the numerator and denominator, we observe that as $x \to -2$, $(4 - x^2) \to (4 - (-2)^2) = 0$, and $(x + 2) \to (-2 + 2) = 0$, so as $x \to -2$, the numerator of f tends to 0 and the denominator tends to 0. We call $0/0$ an ***indeterminate form*** and will revisit several important issues surrounding such quantities later in the course. For now, we simply observe that this tells us there is somehow more work to do. From the table and the graph, it appears that f should have a limit of 4 at $x = -2$. To see algebraically why this is the case, let's work directly with the form of $f(x)$. Observe that

$$\begin{aligned} \lim_{x \to -2} f(x) &= \lim_{x \to -2} \frac{4 - x^2}{x + 2} \\ &= \lim_{x \to -2} \frac{(2 - x)(2 + x)}{x + 2}. \end{aligned}$$

At this point, it is important to observe that since we are taking the limit as $x \to -2$, we are considering x values that are close, but not equal, to -2. Since we never actually allow x to equal -2, the quotient $\frac{2+x}{x+2}$ has value 1 for every possible value of x. Thus, we can simplify the most recent expression above, and now find that

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} 2 - x.$$

Because $2 - x$ is simply a linear function, this limit is now easy to determine, and its value clearly is 4. Thus, from several points of view we've seen that $\lim_{x \to -2} f(x) = 4$.

Next we turn to the function g, and construct two tables and a graph.

x	$g(x)$
2.9	0.84864
2.99	0.86428
2.999	0.86585
2.9999	0.86601
3.1	0.88351
3.01	0.86777
3.001	0.86620
3.0001	0.86604

x	$g(x)$
-0.1	0
-0.01	0
-0.001	0
-0.0001	0
0.1	0
0.01	0
0.001	0
0.0001	0

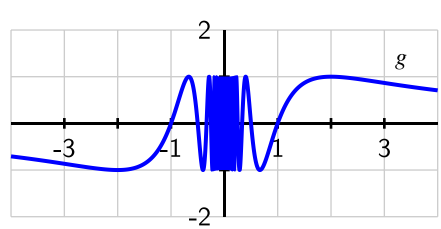

Figure 1.7: Tables and graph for $g(x) = \sin\left(\frac{\pi}{x}\right)$.

First, as $x \to 3$, it appears from the data (and the graph) that the function is approaching approximately 0.866025. To be precise, we have to use the fact that $\frac{\pi}{x} \to \frac{\pi}{3}$, and thus we find that $g(x) = \sin(\frac{\pi}{x}) \to \sin(\frac{\pi}{3})$ as $x \to 3$. The exact value of $\sin(\frac{\pi}{3})$ is $\frac{\sqrt{3}}{2}$, which is approximately 0.8660254038. Thus, we see that

$$\lim_{x\to 3} g(x) = \frac{\sqrt{3}}{2}.$$

As $x \to 0$, we observe that $\frac{\pi}{x}$ does not behave in an elementary way. When x is positive and approaching zero, we are dividing by smaller and smaller positive values, and $\frac{\pi}{x}$ increases without bound. When x is negative and approaching zero, $\frac{\pi}{x}$ decreases without bound. In this sense, as we get close to $x = 0$, the inputs to the sine function are growing rapidly, and this leads to wild oscillations in the graph of g. It is an instructive exercise to plot the function $g(x) = \sin\left(\frac{\pi}{x}\right)$ with a graphing utility and then zoom in on $x = 0$. Doing so shows that the function never settles down to a single value near the origin and suggests that g does not have a limit at $x = 0$.

How do we reconcile this with the righthand table above, which seems to suggest that the limit of g as x approaches 0 may in fact be 0? Here we need to recognize that the data misleads us because of the special nature of the sequence $\{0.1, 0.01, 0.001, \ldots\}$: when we evaluate $g(10^{-k})$, we get $g(10^{-k}) = \sin\left(\frac{\pi}{10^{-k}}\right) = \sin(10^k\pi) = 0$ for each positive integer value of k. But if we take a different sequence of values approaching zero, say $\{0.3, 0.03, 0.003, \ldots\}$, then we find that

$$g(3 \cdot 10^{-k}) = \sin\left(\frac{\pi}{3 \cdot 10^{-k}}\right) = \sin\left(\frac{10^k\pi}{3}\right) = -\frac{\sqrt{3}}{2} \approx -0.866025.$$

That sequence of data would suggest that the value of the limit is $\frac{\sqrt{3}}{2}$. Clearly the function cannot have two different values for the limit, and this shows that g has no limit as $x \to 0$.

An important lesson to take from Example 1.2 is that tables can be misleading when determining the value of a limit. While a table of values is useful for investigating the possible value of a limit, we should also use other tools to confirm the value, if we think the table suggests the limit exists.

Activity 1.4.

Estimate the value of each of the following limits by constructing appropriate tables of values. Then determine the exact value of the limit by using algebra to simplify the function. Finally, plot each function on an appropriate interval to check your result visually.

(a) $\displaystyle\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$

(b) $\lim_{x \to 0} \frac{(2+x)^3 - 8}{x}$

(c) $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$

◁

This concludes a rather lengthy introduction to the notion of limits. It is important to remember that our primary motivation for considering limits of functions comes from our interest in studying the rate of change of a function. To that end, we close this section by revisiting our previous work with average and instantaneous velocity and highlighting the role that limits play.

Instantaneous Velocity

Suppose that we have a moving object whose position at time t is given by a function s. We know that the average velocity of the object on the time interval $[a, b]$ is $AV_{[a,b]} = \frac{s(b)-s(a)}{b-a}$. We define the ***instantaneous velocity*** at a to be the limit of average velocity as b approaches a. Note particularly that as $b \to a$, the length of the time interval gets shorter and shorter (while always including a). In Section 1.3, we will introduce a helpful shorthand notation to represent the instantaneous rate of change. For now, we will write $IV_{t=a}$ for the instantaneous velocity at $t = a$, and thus

$$IV_{t=a} = \lim_{b \to a} AV_{[a,b]} = \lim_{b \to a} \frac{s(b) - s(a)}{b - a}.$$

Equivalently, if we think of the changing value b as being of the form $b = a + h$, where h is some small number, then we may instead write

$$IV_{t=a} = \lim_{h \to 0} AV_{[a,a+h]} = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}.$$

Again, the most important idea here is that to compute instantaneous velocity, we take a limit of average velocities as the time interval shrinks. Two different activities offer the opportunity to investigate these ideas and the role of limits further.

Activity 1.5.

Consider a moving object whose position function is given by $s(t) = t^2$, where s is measured in meters and t is measured in minutes.

(a) Determine the most simplified expression for the average velocity of the object on the interval $[3, 3+h]$, where $h > 0$.

(b) Determine the average velocity of the object on the interval $[3, 3.2]$. Include units on your answer.

(c) Determine the instantaneous velocity of the object when $t = 3$. Include units on your answer.

◁

The closing activity of this section asks you to make some connections among average velocity, instantaneous velocity, and slopes of certain lines.

Activity 1.6.

For the moving object whose position s at time t is given by the graph below, answer each of the following questions. Assume that s is measured in feet and t is measured in seconds.

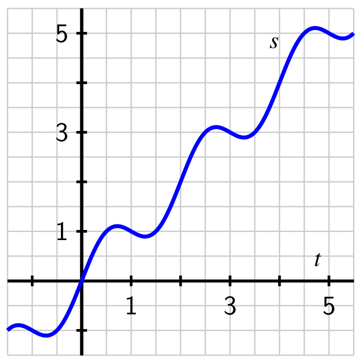

Figure 1.8: Plot of the position function $y = s(t)$ in Activity 1.6.

(a) Use the graph to estimate the average velocity of the object on each of the following intervals: $[0.5, 1]$, $[1.5, 2.5]$, $[0, 5]$. Draw each line whose slope represents the average velocity you seek.

(b) How could you use average velocities or slopes of lines to estimate the instantaneous velocity of the object at a fixed time?

(c) Use the graph to estimate the instantaneous velocity of the object when $t = 2$. Should this instantaneous velocity at $t = 2$ be greater or less than the average velocity on $[1.5, 2.5]$ that you computed in (a)? Why?

◁

Summary

In this section, we encountered the following important ideas:

- Limits enable us to examine trends in function behavior near a specific point. In particular, taking a limit at a given point asks if the function values nearby tend to approach a particular fixed value.

- When we write $\lim_{x \to a} f(x) = L$, we read this as saying "the limit of f as x approaches a is L," and this means that we can make the value of $f(x)$ as close to L as we want by taking x sufficiently close (but not equal) to a.
- If we desire to know $\lim_{x \to a} f(x)$ for a given value of a and a known function f, we can estimate this value from the graph of f or by generating a table of function values that result from a sequence of x-values that are closer and closer to a. If we want the exact value of the limit, we need to work with the function algebraically and see if we can use familiar properties of known, basic functions to understand how different parts of the formula for f change as $x \to a$.
- The instantaneous velocity of a moving object at a fixed time is found by taking the limit of average velocities of the object over shorter and shorter time intervals that all contain the time of interest.

Exercises

1. Consider the function whose formula is $f(x) = \dfrac{16 - x^4}{x^2 - 4}$.

(a) What is the domain of f?

(b) Use a sequence of values of x near $a = 2$ to estimate the value of $\lim_{x \to 2} f(x)$, if you think the limit exists. If you think the limit doesn't exist, explain why.

(c) Evaluate $\lim_{x \to 2} f(x)$ exactly, if the limit exists, or explain how your work shows the limit fails to exist. Here you should use algebra to factor and simplify the numerator and denominator of $f(x)$ as you work to evaluate the limit. Discuss how your findings compare to your results in (b).

(d) True or false: $f(2) = -8$. Why?

(e) True or false: $\frac{16-x^4}{x^2-4} = -4 - x^2$. Why? How is this equality connected to your work above with the function f?

(f) Based on all of your work above, construct an accurate, labeled graph of $y = f(x)$ on the interval $[1, 3]$, and write a sentence that explains what you now know about $\lim_{x \to 2} \dfrac{16 - x^4}{x^2 - 4}$.

2. Let $g(x) = -\dfrac{|x + 3|}{x + 3}$.

(a) What is the domain of g?

(b) Use a sequence of values near $a = -3$ to estimate the value of $\lim_{x \to -3} g(x)$, if you think the limit exists. If you think the limit doesn't exist, explain why.

(c) Evaluate $\lim_{x\to 2} g(x)$ exactly, if the limit exists, or explain how your work shows the limit fails to exist. Here you should use the definition of the absolute value function in the numerator of $g(x)$ as you work to evaluate the limit. Discuss how your findings compare to your results in (b). (**Hint**: $|a| = a$ whenever $a \geq 0$, but $|a| = -a$ whenever $a < 0$.)

(d) True or false: $g(-3) = -1$. Why?

(e) True or false: $-\frac{|x+3|}{x+3} = -1$. Why? How is this equality connected to your work above with the function g?

(f) Based on all of your work above, construct an accurate, labeled graph of $y = g(x)$ on the interval $[-4, -2]$, and write a sentence that explains what you now know about $\lim_{x\to -3} g(x)$.

3. For each of the following prompts, sketch a graph on the provided axes of a function that has the stated properties.

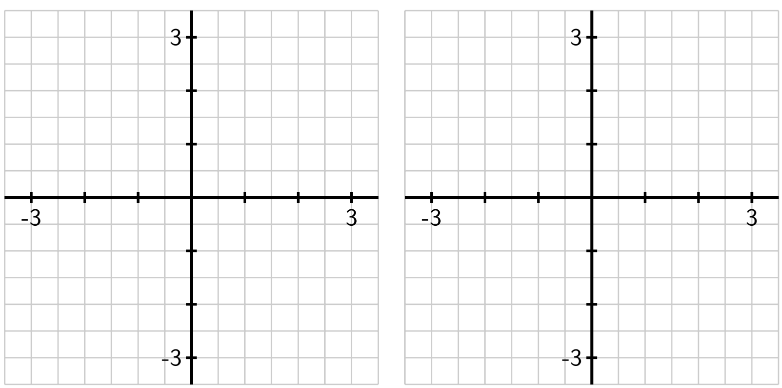

Figure 1.9: Axes for plotting $y = f(x)$ in (a) and $y = g(x)$ in (b).

(a) $y = f(x)$ such that

- $f(-2) = 2$ and $\lim_{x\to -2} f(x) = 1$
- $f(-1) = 3$ and $\lim_{x\to -1} f(x) = 3$
- $f(1)$ is not defined and $\lim_{x\to 1} f(x) = 0$
- $f(2) = 1$ and $\lim_{x\to 2} f(x)$ does not exist.

(b) $y = g(x)$ such that

- $g(-2) = 3$, $g(-1) = -1$, $g(1) = -2$, and $g(2) = 3$
- At $x = -2, -1, 1$ and 2, g has a limit, and its limit equals the value of the function at that point.

- $g(0)$ is not defined and $\lim_{x \to 0} g(x)$ does not exist.

4. A bungee jumper dives from a tower at time $t = 0$. Her height s in feet at time t in seconds is given by $s(t) = 100\cos(0.75t) \cdot e^{-0.2t} + 100$.

(a) Write an expression for the average velocity of the bungee jumper on the interval
$[1, 1 + h]$.

(b) Use computing technology to estimate the value of the limit as $h \to 0$ of the quantity you found in (a).

(c) What is the meaning of the value of the limit in (b)? What are its units?

1.3 The derivative of a function at a point

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How is the average rate of change of a function on a given interval defined, and what does this quantity measure?
- How is the instantaneous rate of change of a function at a particular point defined? How is the instantaneous rate of change linked to average rate of change?
- What is the derivative of a function at a given point? What does this derivative value measure? How do we interpret the derivative value graphically?
- How are limits used formally in the computation of derivatives?

Introduction

An idea that sits at the foundations of calculus is the *instantaneous rate of change* of a function. This rate of change is always considered with respect to change in the input variable, often at a particular fixed input value. This is a generalization of the notion of instantaneous velocity and essentially allows us to consider the question "how do we measure how fast a particular function is changing at a given point?" When the original function represents the position of a moving object, this instantaneous rate of change is precisely velocity, and might be measured in units such as feet per second. But in other contexts, instantaneous rate of change could measure the number of cells added to a bacteria culture per day, the number of additional gallons of gasoline consumed by going one mile per additional mile per hour in a car's velocity, or the number of dollars added to a mortgage payment for each percentage increase in interest rate. Regardless of the presence of a physical or practical interpretation of a function, the instantaneous rate of change may also be interpreted geometrically in connection to the function's graph, and this connection is also foundational to many of the main ideas in calculus.

In what follows, we will introduce terminology and notation that makes it easier to talk about the instantaneous rate of change of a function at a point. In addition, just as instantaneous velocity is defined in terms of average velocity, the more general instantaneous rate of change will be connected to the more general average rate of change. Recall that for a moving object with position function s, its average velocity on the time interval $t = a$ to $t = a + h$ is given by the quotient

$$AV_{[a,a+h]} = \frac{s(a+h) - s(a)}{h}.$$

In a similar way, we make the following definition for an arbitrary function $y = f(x)$.

Definition 1.2. For a function f, the *average rate of change* of f on the interval $[a, a + h]$ is given by the value

$$AV_{[a,a+h]} = \frac{f(a + h) - f(a)}{h}.$$

Equivalently, if we want to consider the average rate of change of f on $[a, b]$, we compute

$$AV_{[a,b]} = \frac{f(b) - f(a)}{b - a}.$$

It is essential to understand how the average rate of change of f on an interval is connected to its graph.

Preview Activity 1.3. Suppose that f is the function given by the graph below and that a and $a + h$ are the input values as labeled on the x-axis. Use the graph in Figure 1.10 to answer the following questions.

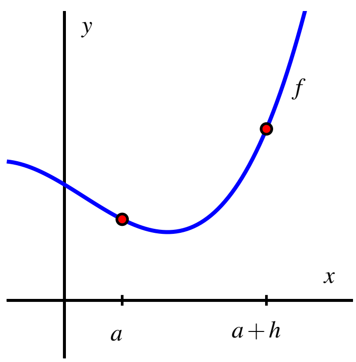

Figure 1.10: Plot of $y = f(x)$ for Preview Activity 1.3.

(a) Locate and label the points $(a, f(a))$ and $(a + h, f(a + h))$ on the graph.

(b) Construct a right triangle whose hypotenuse is the line segment from $(a, f(a))$ to $(a + h, f(a + h))$. What are the lengths of the respective legs of this triangle?

(c) What is the slope of the line that connects the points $(a, f(a))$ and $(a+h, f(a+h))$?

(d) Write a meaningful sentence that explains how the average rate of change of the function on a given interval and the slope of a related line are connected.

⋈

The Derivative of a Function at a Point

Just as we defined instantaneous velocity in terms of average velocity, we now define the instantaneous rate of change of a function at a point in terms of the average rate of change of the function f over related intervals. In addition, we give a special name to "the instantaneous rate of change of f at a," calling this quantity "the *derivative* of f at a," with this value being represented by the shorthand notation $f'(a)$. Specifically, we make the following definition.

Definition 1.3. Let f be a function and $x = a$ a value in the function's domain. We define the ***derivative of*** f ***with respect to*** x ***evaluated at*** $x = a$, denoted $f'(a)$, by the formula

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists.

Aloud, we read the symbol $f'(a)$ as either "f-prime at a" or "the derivative of f evaluated at $x = a$." Much of the next several chapters will be devoted to understanding, computing, applying, and interpreting derivatives. For now, we make the following important notes.

- The derivative of f at the value $x = a$ is defined as the limit of the average rate of change of f on the interval $[a, a+h]$ as $h \to 0$. It is possible for this limit not to exist, so not every function has a derivative at every point.
- We say that a function that has a derivative at $x = a$ is *differentiable* at $x = a$.
- The derivative is a generalization of the instantaneous velocity of a position function: when $y = s(t)$ is a position function of a moving body, $s'(a)$ tells us the instantaneous velocity of the body at time $t = a$.
- Because the units on $\frac{f(a+h)-f(a)}{h}$ are "units of f per unit of x," the derivative has these very same units. For instance, if s measures position in feet and t measures time in seconds, the units on $s'(a)$ are feet per second.
- Because the quantity $\frac{f(a+h)-f(a)}{h}$ represents the slope of the line through $(a, f(a))$ and $(a+h, f(a+h))$, when we compute the derivative we are taking the limit of a collection of slopes of lines, and thus the derivative itself represents the slope of a particularly important line.

While all of the above ideas are important and we will add depth and perspective to them through additional time and study, for now it is most essential to recognize how the derivative of a function at a given value represents the slope of a certain line. Thus, we expand upon the last bullet item above.

As we move from an average rate of change to an instantaneous one, we can think of one point as "sliding towards" another. In particular, provided the function has a derivative at $(a, f(a))$, the point $(a+h, f(a+h))$ will approach $(a, f(a))$ as $h \to 0$. Because this process of taking a limit is a dynamic one, it can be helpful to use computing technology to visualize what the limit is accomplishing. While there are many different options[3], one of the best is a java applet in which the user is able to control the point that is moving. See the examples referenced in the footnote here, or consider building your own, perhaps using the fantastic free program Geogebra[4].

In Figure 1.11, we provide a sequence of figures with several different lines through the points $(a, f(a))$ and $(a+h, f(a+h))$ that are generated by different values of h. These lines (shown in the first three figures in magenta), are often called *secant lines* to the curve $y = f(x)$. A secant line to a curve is simply a line that passes through two points that lie on the curve. For each such line, the slope of the secant line is $m = \frac{f(a+h)-f(a)}{h}$, where the value of h depends on the location of the point we choose. We can see in the diagram how, as $h \to 0$, the secant lines start to approach a single line that passes through the point $(a, f(a))$. In the situation where the limit of the slopes of the secant lines exists, we say that the resulting value is the slope of the *tangent line* to the curve. This tangent line (shown in the right-most figure in green) to the graph of $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ whose slope is $m = f'(a)$.

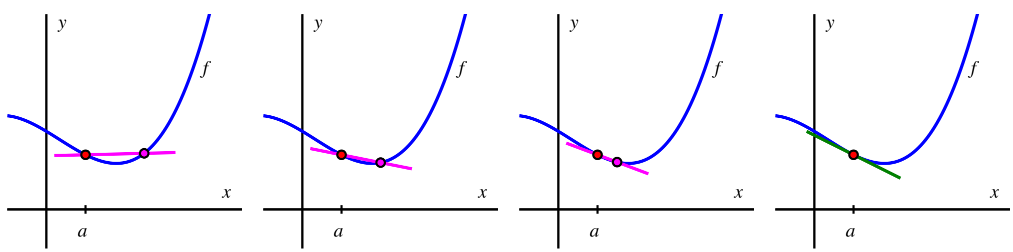

Figure 1.11: A sequence of secant lines approaching the tangent line to f at $(a, f(a))$.

As we will see in subsequent study, the existence of the tangent line at $x = a$ is connected to whether or not the function f looks like a straight line when viewed up close at $(a, f(a))$, which can also be seen in Figure 1.12, where we combine the four graphs in Figure 1.11 into the single one on the left, and then we zoom in on the box centered at $(a, f(a))$, with that view expanded on the right (with two of the secant lines omitted). Note how the tangent line sits relative to the curve $y = f(x)$ at $(a, f(a))$ and how closely it

[3]For a helpful collection of java applets, consider the work of David Austin of Grand Valley State University at http://gvsu.edu/s/5r, and the particularly relevant example at http://gvsu.edu/s/5s. For applets that have been built in Geogebra, a nice example is the work of Marc Renault of Shippensburg University at http://gvsu.edu/s/5p, with the example at http://gvsu.edu/s/5q being especially fitting for our work in this section. There are scores of other examples posted by other authors on the internet.

[4]Available for free download from http://geogebra.org.

resembles the curve near $x = a$.

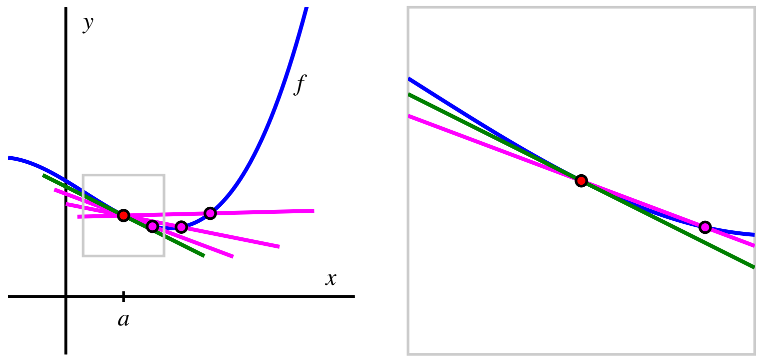

Figure 1.12: A sequence of secant lines approaching the tangent line to f at $(a, f(a))$. At right, we zoom in on the point $(a, f(a))$. The slope of the tangent line (in green) to f at $(a, f(a))$ is given by $f'(a)$.

At this time, it is most important to note that $f'(a)$, the instantaneous rate of change of f with respect to x at $x = a$, also measures the slope of the tangent line to the curve $y = f(x)$ at $(a, f(a))$. The following example demonstrates several key ideas involving the derivative of a function.

Example 1.3. For the function given by $f(x) = x - x^2$, use the limit definition of the derivative to compute $f'(2)$. In addition, discuss the meaning of this value and draw a labeled graph that supports your explanation.

Solution. From the limit definition, we know that

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}.$$

Now we use the rule for f, and observe that $f(2) = 2 - 2^2 = -2$ and $f(2+h) = (2+h) - (2+h)^2$. Substituting these values into the limit definition, we have that

$$f'(2) = \lim_{h \to 0} \frac{(2+h) - (2+h)^2 - (-2)}{h}.$$

Observe that with h in the denominator and our desire to let $h \to 0$, we have to wait to take the limit (that is, we wait to actually let h approach 0). Thus, we do additional

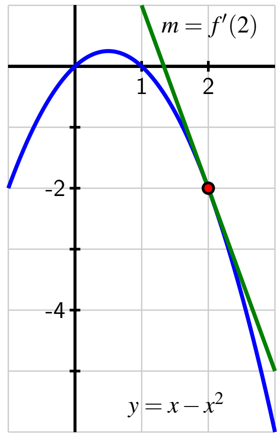

Figure 1.13: The tangent line to $y = x - x^2$ at the point $(2, -2)$.

algebra. Expanding and distributing in the numerator,

$$f'(2) = \lim_{h \to 0} \frac{2 + h - 4 - 4h - h^2 + 2}{h}.$$

Combining like terms, we have

$$f'(2) = \lim_{h \to 0} \frac{-3h - h^2}{h}.$$

Next, we observe that there is a common factor of h in both the numerator and denominator, which allows us to simplify and find that

$$f'(2) = \lim_{h \to 0}(-3 - h).$$

Finally, we are able to take the limit as $h \to 0$, and thus conclude that $f'(2) = -3$.

Now, we know that $f'(2)$ represents the slope of the tangent line to the curve $y = x - x^2$ at the point $(2, -2)$; $f'(2)$ is also the instantaneous rate of change of f at the point $(2, -2)$. Graphing both the function and the line through $(2, -2)$ with slope $m = f'(2) = -3$, we indeed see that by calculating the derivative, we have found the slope of the tangent line at this point, as shown in Figure 1.3.

The following activities will help you explore a variety of key ideas related to derivatives.

Activity 1.7.

Consider the function f whose formula is $f(x) = 3 - 2x$.

(a) What familiar type of function is f? What can you say about the slope of f at every value of x?

(b) Compute the average rate of change of f on the intervals $[1, 4]$, $[3, 7]$, and $[5, 5 + h]$; simplify each result as much as possible. What do you notice about these quantities?

(c) Use the limit definition of the derivative to compute the exact instantaneous rate of change of f with respect to x at the value $a = 1$. That is, compute $f'(1)$ using the limit definition. Show your work. Is your result surprising?

(d) Without doing any additional computations, what are the values of $f'(2)$, $f'(\pi)$, and $f'(-\sqrt{2})$? Why?

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Activity 1.8.

A water balloon is tossed vertically in the air from a window. The balloon's height in feet at time t in seconds after being launched is given by $s(t) = -16t^2 + 16t + 32$. Use this function to respond to each of the following questions.

(a) Sketch an accurate, labeled graph of s on the axes provided in Figure 1.14. You should be able to do this without using computing technology.

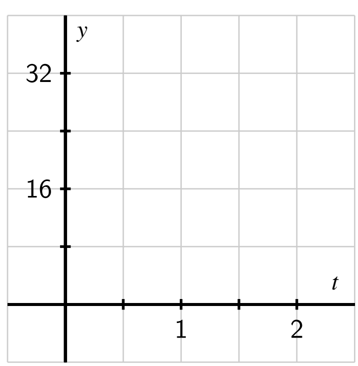

Figure 1.14: Axes for plotting $y = s(t)$ in Activity 1.8.

(b) Compute the average rate of change of s on the time interval $[1, 2]$. Include units on your answer and write one sentence to explain the meaning of the value you found.

(c) Use the limit definition to compute the instantaneous rate of change of s with respect to time, t, at the instant $a = 1$. Show your work using proper notation, include units on your answer, and write one sentence to explain the meaning of the value you found.

(d) On your graph in (a), sketch two lines: one whose slope represents the average rate of change of s on $[1, 2]$, the other whose slope represents the instantaneous rate of change of s at the instant $a = 1$. Label each line clearly.

(e) For what values of a do you expect $s'(a)$ to be positive? Why? Answer the same questions when "positive" is replaced by "negative" and "zero."

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Activity 1.9.

A rapidly growing city in Arizona has its population P at time t, where t is the number of decades after the year 2010, modeled by the formula $P(t) = 25000e^{t/5}$. Use this function to respond to the following questions.

(a) Sketch an accurate graph of P for $t = 0$ to $t = 5$ on the axes provided in Figure 1.15. Label the scale on the axes carefully.

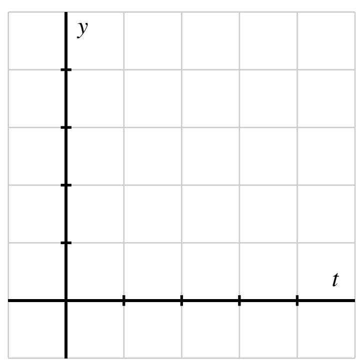

Figure 1.15: Axes for plotting $y = P(t)$ in Activity 1.9.

(b) Compute the average rate of change of P between 2030 and 2050. Include units on your answer and write one sentence to explain the meaning (in everyday language) of the value you found.

(c) Use the limit definition to write an expression for the instantaneous rate of change of P with respect to time, t, at the instant $a = 2$. Explain why this limit is difficult to evaluate exactly.

(d) Estimate the limit in (c) for the instantaneous rate of change of P at the instant $a = 2$ by using several small h values. Once you have determined an accurate estimate of $P'(2)$, include units on your answer, and write one sentence (using everyday language) to explain the meaning of the value you found.

(e) On your graph above, sketch two lines: one whose slope represents the average rate of change of P on $[2, 4]$, the other whose slope represents the instantaneous rate of change of P at the instant $a = 2$.

(f) In a carefully-worded sentence, describe the behavior of $P'(a)$ as a increases in value. What does this reflect about the behavior of the given function P?

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Summary

In this section, we encountered the following important ideas:

- The average rate of change of a function f on the interval $[a, b]$ is $\frac{f(b) - f(a)}{b - a}$. The units on the average rate of change are units of f per unit of x, and the numerical value of the average rate of change represents the slope of the secant line between the points $(a, f(a))$ and $(b, f(b))$ on the graph of $y = f(x)$. If we view the interval as being $[a, a + h]$ instead of $[a, b]$, the meaning is still the same, but the average rate of change is now computed by $\frac{f(a+h) - f(a)}{h}$.
- The instantaneous rate of change with respect to x of a function f at a value $x = a$ is denoted $f'(a)$ (read "the derivative of f evaluated at a" or "f-prime at a") and is defined by the formula
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$
provided the limit exists. Note particularly that the instantaneous rate of change at $x = a$ is the limit of the average rate of change on $[a, a + h]$ as $h \to 0$.
- Provided the derivative $f'(a)$ exists, its value tells us the instantaneous rate of change of f with respect to x at $x = a$, which geometrically is the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. We even say that $f'(a)$ is the ***slope of the curve*** $y = f(x)$ at the point $(a, f(a))$.
- Limits are the link between average rate of change and instantaneous rate of change: they allow us to move from the rate of change over an interval to the rate of change at a single point.

Exercises

1. Consider the graph of $y = f(x)$ provided in Figure 1.16.

(a) On the graph of $y = f(x)$, sketch and label the following quantities:

- the secant line to $y = f(x)$ on the interval $[-3, -1]$ and the secant line to $y = f(x)$ on the interval $[0, 2]$.
- the tangent line to $y = f(x)$ at $x = -3$ and the tangent line to $y = f(x)$ at $x = 0$.

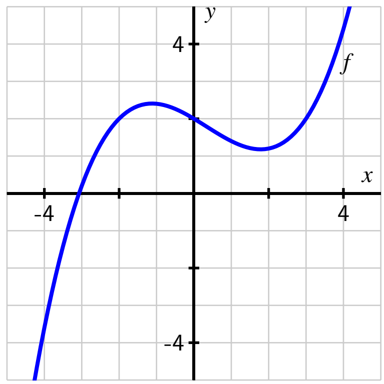

Figure 1.16: Plot of $y = f(x)$.

(b) What is the approximate value of the average rate of change of f on $[-3, -1]$? On $[0, 2]$? How are these values related to your work in (a)?

(c) What is the approximate value of the instantaneous rate of change of f at $x = -3$? At $x = 0$? How are these values related to your work in (a)?

2. For each of the following prompts, sketch a graph on the provided axes in Figure 1.17 of a function that has the stated properties.

(a) $y = f(x)$ such that

- the average rate of change of f on $[-3, 0]$ is -2 and the average rate of change of f on $[1, 3]$ is 0.5, and
- the instantaneous rate of change of f at $x = -1$ is -1 and the instantaneous rate of change of f at $x = 2$ is 1.

(b) $y = g(x)$ such that

- $\frac{g(3)-g(-2)}{5} = 0$ and $\frac{g(1)-g(-1)}{2} = -1$, and
- $g'(2) = 1$ and $g'(-1) = 0$

3. Suppose that the population, P, of China (in billions) can be approximated by the function $P(t) = 1.15(1.014)^t$ where t is the number of years since the start of 1993.

(a) According to the model, what was the total change in the population of China between January 1, 1993 and January 1, 2000? What will be the average rate of change of the population over this time period? Is this average rate of change

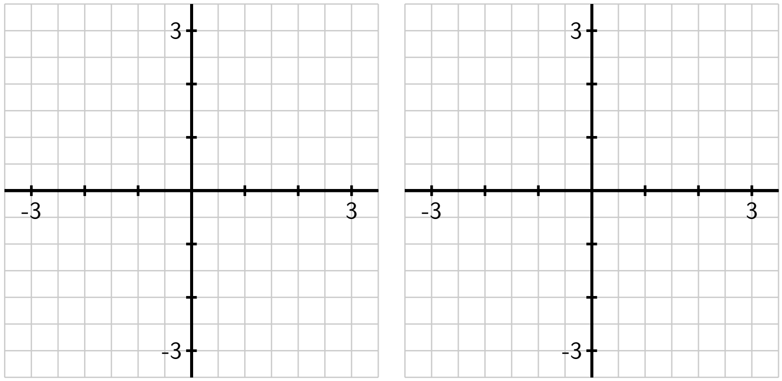

Figure 1.17: Axes for plotting $y = f(x)$ in (a) and $y = g(x)$ in (b).

greater or less than the instantaneous rate of change of the population on January 1, 2000? Explain and justify, being sure to include proper units on all your answers.

(b) According to the model, what is the average rate of change of the population of China in the ten-year period starting on January 1, 2012?

(c) Write an expression involving limits that, if evaluated, would give the exact instantaneous rate of change of the population on today's date. Then estimate the value of this limit (discuss how you chose to do so) and explain the meaning (including units) of the value you have found.

(d) Find an equation for the tangent line to the function $y = P(t)$ at the point where the t-value is given by today's date.

4. The goal of this problem is to compute the value of the derivative at a point for several different functions, where for each one we do so in three different ways, and then to compare the results to see that each produces the same value.

For each of the following functions, use the limit definition of the derivative to compute the value of $f'(a)$ using three different approaches: strive to use the algebraic approach first (to compute the limit exactly), then test your result using numerical evidence (with small values of h), and finally plot the graph of $y = f(x)$ near $(a, f(a))$ along with the appropriate tangent line to estimate the value of $f'(a)$ visually. Compare your findings among all three approaches; if you are unable to complete the algebraic approach, still work numerically and graphically.

(a) $f(x) = x^2 - 3x$, $a = 2$

(b) $f(x) = \frac{1}{x}$, $a = 1$

(c) $f(x) = \sqrt{x}$, $a = 1$

(d) $f(x) = 2 - |x - 1|$, $a = 1$

(e) $f(x) = \sin(x)$, $a = \frac{\pi}{2}$

1.4 The derivative function

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How does the limit definition of the derivative of a function f lead to an entirely new (but related) function f'?
- What is the difference between writing $f'(a)$ and $f'(x)$?
- How is the graph of the derivative function $f'(x)$ connected to the graph of $f(x)$?
- What are some examples of functions f for which f' is not defined at one or more points?

Introduction

Given a function $y = f(x)$, we now know that if we are interested in the instantaneous rate of change of the function at $x = a$, or equivalently the slope of the tangent line to $y = f(x)$ at $x = a$, we can compute the value $f'(a)$. In all of our examples to date, we have arbitrarily identified a particular value of a as our point of interest: $a = 1$, $a = 3$, etc. But it is not hard to imagine that we will often be interested in the derivative value for more than just one a-value, and possibly for many of them. In this section, we explore how we can move from computing simply $f'(1)$ or $f'(3)$ to working more generally with $f'(a)$, and indeed $f'(x)$. Said differently, we will work toward understanding how the so-called process of "taking the derivative" generates a new function that is derived from the original function $y = f(x)$. The following preview activity starts us down this path.

Preview Activity 1.4. Consider the function $f(x) = 4x - x^2$.

(a) Use the limit definition to compute the following derivative values: $f'(0)$, $f'(1)$, $f'(2)$, and $f'(3)$.

(b) Observe that the work to find $f'(a)$ is the same, regardless of the value of a. Based on your work in (a), what do you conjecture is the value of $f'(4)$? How about $f'(5)$? (Note: you should *not* use the limit definition of the derivative to find either value.)

(c) Conjecture a formula for $f'(a)$ that depends only on the value a. That is, in the same way that we have a formula for $f(x)$ (recall $f(x) = 4x - x^2$), see if you can use your work above to guess a formula for $f'(a)$ in terms of a.

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How the derivative is itself a function

In your work in Preview Activity 1.4 with $f(x) = 4x - x^2$, you may have found several patterns. One comes from observing that $f'(0) = 4$, $f'(1) = 2$, $f'(2) = 0$, and $f'(3) = -2$. That sequence of values leads us naturally to conjecture that $f'(4) = -4$ and $f'(5) = -6$. Even more than these individual numbers, if we consider the role of 0, 1, 2, and 3 in the process of computing the value of the derivative through the limit definition, we observe that the particular number has very little effect on our work. To see this more clearly, we compute $f'(a)$, where a represents a number to be named later. Following the now standard process of using the limit definition of the derivative,

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{4(a+h) - (a+h)^2 - (4a - a^2)}{h} \\ &= \lim_{h\to 0} \frac{4a + 4h - a^2 - 2ha - h^2 - 4a + a^2}{h} \\ &= \lim_{h\to 0} \frac{4h - 2ha - h^2}{h} \\ &= \lim_{h\to 0} \frac{h(4 - 2a - h)}{h} \\ &= \lim_{h\to 0} (4 - 2a - h). \end{aligned}$$

Here we observe that neither 4 nor $2a$ depend on the value of h, so as $h \to 0$, $(4-2a-h) \to (4 - 2a)$. Thus, $f'(a) = 4 - 2a$.

This observation is consistent with the specific values we found above: e.g., $f'(3) = 4 - 2(3) = -2$. And indeed, our work with a confirms that while the particular value of a at which we evaluate the derivative affects the value of the derivative, that value has almost no bearing on the process of computing the derivative. We note further that the letter being used is immaterial: whether we call it a, x, or anything else, the derivative at a given value is simply given by "4 minus 2 times the value." We choose to use x for consistency with the original function given by $y = f(x)$, as well as for the purpose of graphing the derivative function, and thus we have found that for the function $f(x) = 4x - x^2$, it follows that $f'(x) = 4 - 2x$.

Because the value of the derivative function is so closely linked to the graphical behavior of the original function, it makes sense to look at both of these functions plotted on the same domain. In Figure 1.18, on the left we show a plot of $f(x) = 4x - x^2$ together with a selection of tangent lines at the points we've considered above. On the right, we show a plot of $f'(x) = 4 - 2x$ with emphasis on the heights of the derivative graph at the same selection of points. Notice the connection between colors in the left and right graph: the green tangent line on the original graph is tied to the green point on the right graph in the following way: *the slope of the tangent line* at a point on the lefthand graph is the

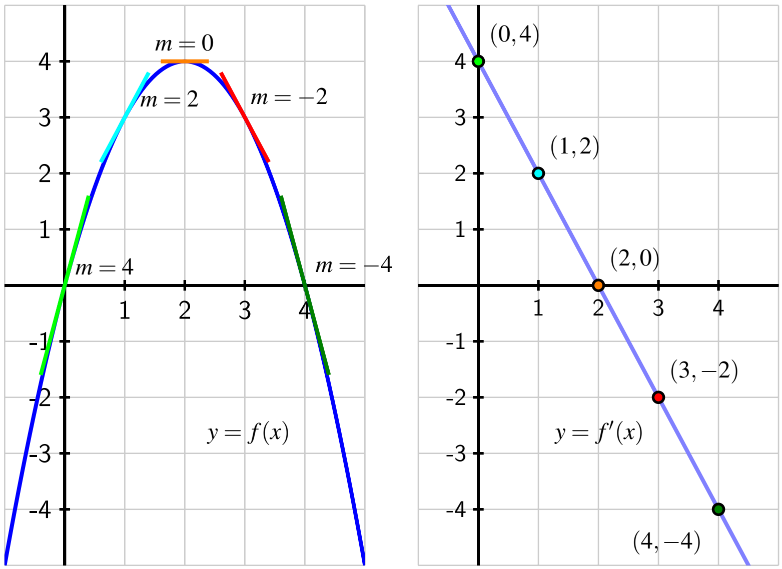


Figure 1.18: The graphs of $f(x) = 4x - x^2$ (at left) and $f'(x) = 4 - 2x$ (at right). Slopes on the graph of f correspond to heights on the graph of f'.

same as the *height* at the corresponding point on the righthand graph. That is, at each respective value of x, the slope of the tangent line to the original function at that x-value is the same as the height of the derivative function at that x-value. Do note, however, that the units on the vertical axes are different: in the left graph, the vertical units are simply the output units of f. On the righthand graph of $y = f'(x)$, the units on the vertical axis are units of f per unit of x.

Of course, this relationship between the graph of a function $y = f(x)$ and its derivative is a dynamic one. An excellent way to explore how the graph of $f(x)$ generates the graph of $f'(x)$ is through a java applet. See, for instance, the applets at http://gvsu.edu/s/5C or http://gvsu.edu/s/5D, via the sites of Austin and Renault[5].

In Section 1.3 when we first defined the derivative, we wrote the definition in terms of a value a to find $f'(a)$. As we have seen above, the letter a is merely a placeholder, and it often makes more sense to use x instead. For the record, here we restate the definition of the derivative.

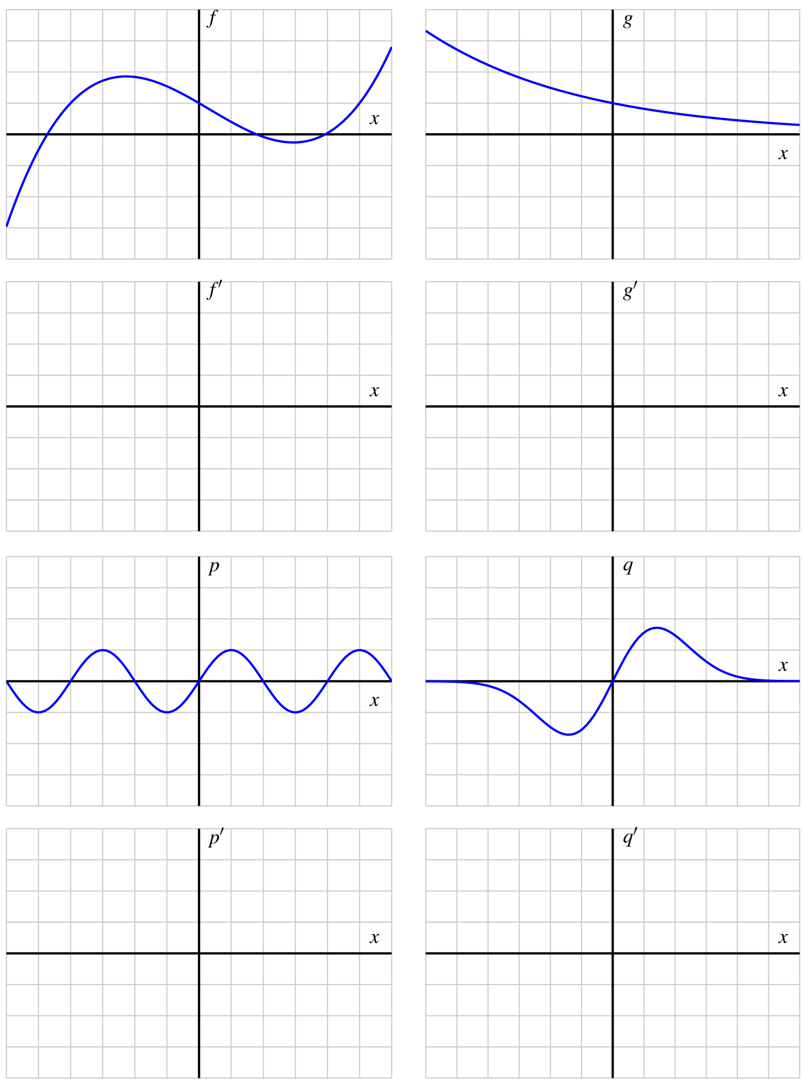
Definition 1.4. Let f be a function and x a value in the function's domain. We define the *derivative of f with respect to x at the value x*, denoted $f'(x)$, by the formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided this limit exists.

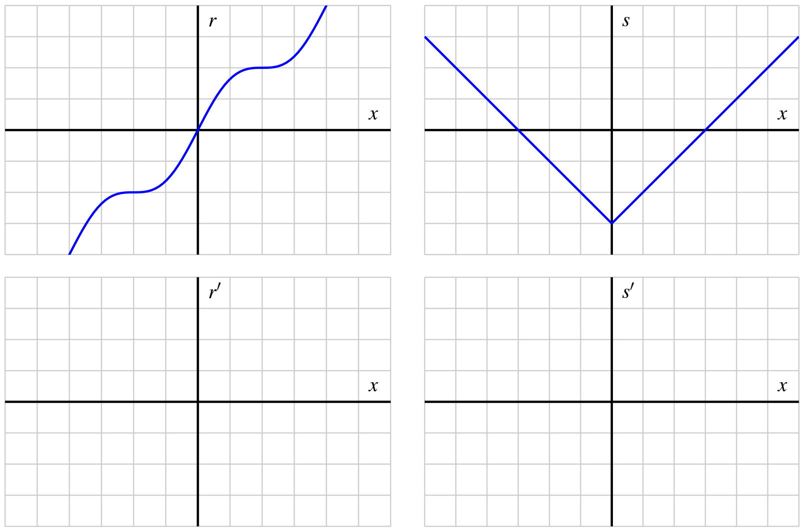
[5] David Austin, http://gvsu.edu/s/5r; Marc Renault, http://gvsu.edu/s/5p.

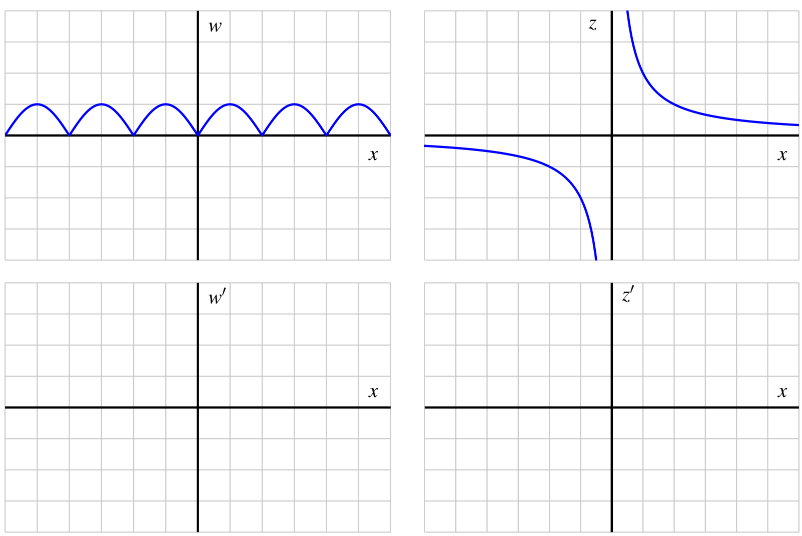
We now may take two different perspectives on thinking about the derivative function: given a graph of $y = f(x)$, how does this graph lead to the graph of the derivative function $y = f'(x)$? and given a formula for $y = f(x)$, how does the limit definition of the derivative generate a formula for $y = f'(x)$? Both of these issues are explored in the following activities.

Activity 1.10.

For each given graph of $y = f(x)$, sketch an approximate graph of its derivative function, $y = f'(x)$, on the axes immediately below. The scale of the grid for the graph of f is 1×1; assume the horizontal scale of the grid for the graph of f' is identical to that for f. If necessary, adjust and label the vertical scale on the axes for f'.







When you are finished with all 8 graphs, write several sentences that describe your overall process for sketching the graph of the derivative function, given the graph the original function. What are the values of the derivative function that you tend to identify first? What do you do thereafter? How do key traits of the graph of the derivative function exemplify properties of the graph of the original function?

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For a dynamic investigation that allows you to experiment with graphing f' when given the graph of f, see http://gvsu.edu/s/8y.[6]

[6]Marc Renault, Calculus Applets Using Geogebra.

Now, recall the opening example of this section: we began with the function $y = f(x) = 4x - x^2$ and used the limit definition of the derivative to show that $f'(a) = 4 - 2a$, or equivalently that $f'(x) = 4 - 2x$. We subsequently graphed the functions f and f' as shown in Figure 1.18. Following Activity 1.10, we now understand that we could have constructed a fairly accurate graph of $f'(x)$ *without* knowing a formula for either f or f'. At the same time, it is ideal to know a formula for the derivative function whenever it is possible to find one.

In the next activity, we further explore the more algebraic approach to finding $f'(x)$: given a formula for $y = f(x)$, the limit definition of the derivative will be used to develop a formula for $f'(x)$.

Activity 1.11.

For each of the listed functions, determine a formula for the derivative function. For the first two, determine the formula for the derivative by thinking about the nature of the given function and its slope at various points; do not use the limit definition. For the latter four, use the limit definition. Pay careful attention to the function names and independent variables. It is important to be comfortable with using letters other than f and x. For example, given a function $p(z)$, we call its derivative $p'(z)$.

(a) $f(x) = 1$

(b) $g(t) = t$

(c) $p(z) = z^2$

(d) $q(s) = s^3$

(e) $F(t) = \frac{1}{t}$

(f) $G(y) = \sqrt{y}$

◁

Summary

In this section, we encountered the following important ideas:

- The limit definition of the derivative, $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, produces a value for each x at which the derivative is defined, and this leads to a new function whose formula is $y = f'(x)$. Hence we talk both about a given function f and its derivative f'. It is especially important to note that taking the derivative is a process that starts with a given function (f) and produces a new, related function (f').

- There is essentially no difference between writing $f'(a)$ (as we did regularly in Section 1.3) and writing $f'(x)$. In either case, the variable is just a placeholder that is used to define the rule for the derivative function.

- Given the graph of a function $y = f(x)$, we can sketch an approximate graph of its derivative $y = f'(x)$ by observing that *heights* on the derivative's graph correspond to *slopes* on the original function's graph.
- In Activity 1.10, we encountered some functions that had sharp corners on their graphs, such as the shifted absolute value function. At such points, the derivative fails to exist, and we say that f is not differentiable there. For now, it suffices to understand this as a consequence of the jump that must occur in the derivative function at a sharp corner on the graph of the original function.

Exercises

1. Let f be a function with the following properties: f is differentiable at every value of x (that is, f has a derivative at every point), $f(-2) = 1$, and $f'(-2) = -2$, $f'(-1) = -1$, $f'(0) = 0$, $f'(1) = 1$, and $f'(2) = 2$.

(a) On the axes provided at left in Figure 1.19, sketch a possible graph of $y = f(x)$. Explain why your graph meets the stated criteria.

(b) On the axes at right in Figure 1.19, sketch a possible graph of $y = f'(x)$. What type of curve does the provided data suggest for the graph of $y = f'(x)$?

(c) Conjecture a formula for the function $y = f(x)$. Use the limit definition of the derivative to determine the corresponding formula for $y = f'(x)$. Discuss both graphical and algebraic evidence for whether or not your conjecture is correct.

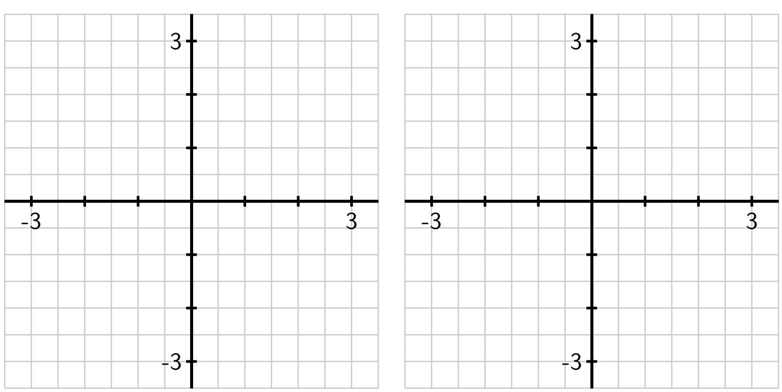

Figure 1.19: Axes for plotting $y = f(x)$ in (a) and $y = f'(x)$ in (b).

2. Consider the function $g(x) = x^2 - x + 3$.

(a) Use the limit definition of the derivative to determine a formula for $g'(x)$.

(b) Use a graphing utility to plot both $y = g(x)$ and your result for $y = g'(x)$; does your formula for $g'(x)$ generate the graph you expected?

(c) Use the limit definition of the derivative to find a formula for $p'(x)$ where $p(x) = 5x^2 - 4x + 12$.

(d) Compare and contrast the formulas for $g'(x)$ and $p'(x)$ you have found. How do the constants 5, 4, 12, and 3 affect the results?

3. Let g be a continuous function (that is, one with no jumps or holes in the graph) and suppose that a graph of $y = g'(x)$ is given by the graph on the right in Figure 1.20.

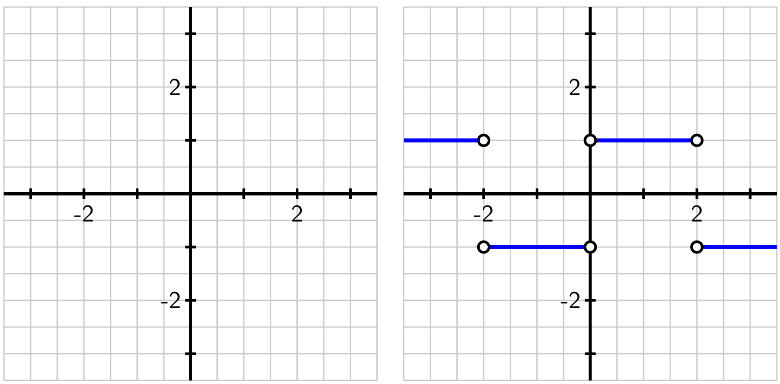

Figure 1.20: Axes for plotting $y = g(x)$ and, at right, the graph of $y = g'(x)$.

(a) Observe that for every value of x that satisfies $0 < x < 2$, the value of $g'(x)$ is constant. What does this tell you about the behavior of the graph of $y = g(x)$ on this interval?

(b) On what intervals other than $0 < x < 2$ do you expect $y = g(x)$ to be a linear function? Why?

(c) At which values of x is $g'(x)$ not defined? What behavior does this lead you to expect to see in the graph of $y = g(x)$?

(d) Suppose that $g(0) = 1$. On the axes provided at left in Figure 1.20, sketch an accurate graph of $y = g(x)$.

4. For each graph that provides an original function $y = f(x)$ in Figure 1.21 (on the following page), your task is to sketch an approximate graph of its derivative function, $y = f'(x)$, on the axes immediately below. View the scale of the grid for the graph of f as being 1×1, and assume the horizontal scale of the grid for the graph of f' is identical to that for f. If you need to adjust the vertical scale on the axes for the graph of f', you should label that accordingly.

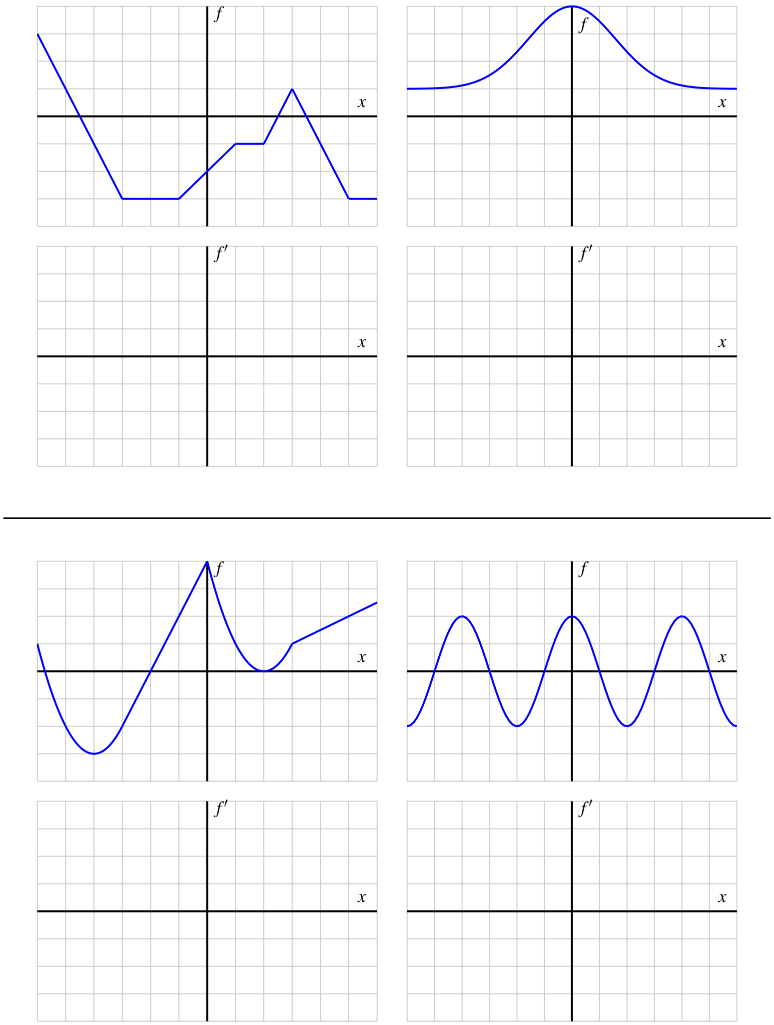

Figure 1.21: Graphs of $y = f(x)$ and grids for plotting the corresponding graph of $y = f'(x)$.

1.5 Interpreting, estimating, and using the derivative

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- In contexts other than the position of a moving object, what does the derivative of a function measure?
- What are the units on the derivative function f', and how are they related to the units of the original function f?
- What is a central difference, and how can one be used to estimate the value of the derivative at a point from given function data?
- Given the value of the derivative of a function at a point, what can we infer about how the value of the function changes nearby?

Introduction

An interesting and powerful feature of mathematics is that it can often be thought of both in abstract terms and in applied ones. For instance, calculus can be developed almost entirely as an abstract collection of ideas that focus on properties of arbitrary functions. At the same time, calculus can also be very directly connected to our experience of physical reality by considering functions that represent meaningful processes. We have already seen that for a position function $y = s(t)$, say for a ball being tossed straight up in the air, the ball's velocity at time t is given by $v(t) = s'(t)$, the derivative of the position function. Further, recall that if $s(t)$ is measured in feet at time t, the units on $v(t) = s'(t)$ are feet per second.

In what follows in this section, we investigate several different functions, each with specific physical meaning, and think about how the units on the independent variable, dependent variable, and the derivative function add to our understanding. To start, we consider the familiar problem of a position function of a moving object.

Preview Activity 1.5. One of the longest stretches of straight (and flat) road in North America can be found on the Great Plains in the state of North Dakota on state highway 46, which lies just south of the interstate highway I-94 and runs through the town of Gackle. A car leaves town (at time $t = 0$) and heads east on highway 46; its position in miles from Gackle at time t in minutes is given by the graph of the function in Figure 1.22. Three important points are labeled on the graph; where the curve looks linear, assume that it is indeed a straight line.

(a) In everyday language, describe the behavior of the car over the provided time

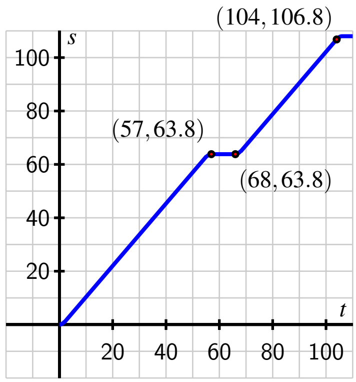

Figure 1.22: The graph of $y = s(t)$, the position of the car along highway 46, which tells its distance in miles from Gackle, ND, at time t in minutes.

interval. In particular, discuss what is happening on the time intervals $[57, 68]$ and $[68, 104]$.

(b) Find the slope of the line between the points $(57, 63.8)$ and $(104, 106.8)$. What are the units on this slope? What does the slope represent?

(c) Find the average rate of change of the car's position on the interval $[68, 104]$. Include units on your answer.

(d) Estimate the instantaneous rate of change of the car's position at the moment $t = 80$. Write a sentence to explain your reasoning and the meaning of this value.

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Units of the derivative function

As we now know, the derivative of the function f at a fixed value x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and this value has several different interpretations. If we set $x = a$, one meaning of $f'(a)$ is the slope of the tangent line at the point $(a, f(a))$.

In alternate notation, we also sometimes equivalently write $\frac{df}{dx}$ or $\frac{dy}{dx}$ instead of $f'(x)$, and these notations helps us to further see the units (and thus the meaning) of the derivative as it is viewed as *the instantaneous rate of change of f with respect to x*. Note that the units on the slope of the secant line, $\frac{f(x+h)-f(x)}{h}$, are "units of f per unit of x." Thus, when we take the limit to get $f'(x)$, we get these same units on the derivative $f'(x)$: units of f per unit of x. Regardless of the function f under consideration (and regardless of the variables being used), it is helpful to remember that the units on the derivative function are "units of output per unit of input," in terms of the input and output of the original function.

For example, say that we have a function $y = P(t)$, where P measures the population of a city (in thousands) at the start of year t (where $t = 0$ corresponds to 2010 AD), and we are told that $P'(2) = 21.37$. What is the meaning of this value? Well, since P is measured in thousands and t is measured in years, we can say that the instantaneous rate of change of the city's population with respect to time at the start of 2012 is 21.37 thousand people per year. We therefore expect that in the coming year, about 21,370 people will be added to the city's population.

Toward more accurate derivative estimates

It is also helpful to recall, as we first experienced in Section 1.3, that when we want to estimate the value of $f'(x)$ at a given x, we can use the *difference quotient* $\frac{f(x+h)-f(x)}{h}$ with a relatively small value of h. In doing so, we should use both positive and negative values of h in order to make sure we account for the behavior of the function on both sides of the point of interest. To that end, we consider the following brief example to demonstrate the notion of a *central difference* and its role in estimating derivatives.

Example 1.4. Suppose that $y = f(x)$ is a function for which three values are known: $f(1) = 2.5$, $f(2) = 3.25$, and $f(3) = 3.625$. Estimate $f'(2)$.

Solution. We know that $f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$. But since we don't have a graph for $y = f(x)$ nor a formula for the function, we can neither sketch a tangent line nor evaluate the limit exactly. We can't even use smaller and smaller values of h to estimate the limit. Instead, we have just two choices: using $h = -1$ or $h = 1$, depending on which point we pair with $(2, 3.25)$.

So, one estimate is

$$f'(2) \approx \frac{f(1) - f(2)}{1 - 2} = \frac{2.5 - 3.25}{-1} = 0.75.$$

The other is

$$f'(2) \approx \frac{f(3) - f(2)}{3 - 2} = \frac{3.625 - 3.25}{1} = 0.375.$$

Since the first approximation looks only backward from the point $(2, 3.25)$ and the second approximation looks only forward from $(2, 3.25)$, it makes sense to average these two values in order to account for behavior on both sides of the point of interest. Doing so, we find that

$$f'(2) \approx \frac{0.75 + 0.375}{2} = 0.5625.$$

The intuitive approach to average the two estimates found in Example 1.4 is in fact the best possible estimate to $f'(2)$ when we have just two function values for f on opposite sides of the point of interest. To see why, we think about the diagram in Figure 1.23, which

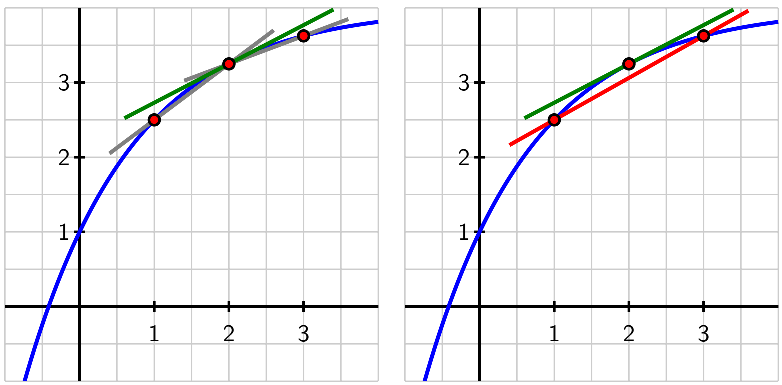

Figure 1.23: At left, the graph of $y = f(x)$ along with the secant line through $(1, 2.5)$ and $(2, 3.25)$, the secant line through $(2, 3.25)$ and $(3, 3.625)$, as well as the tangent line. At right, the same graph along with the secant line through $(1, 2.5)$ and $(3, 3.625)$, plus the tangent line.

shows a possible function $y = f(x)$ that satisfies the data given in Example 1.4. On the left, we see the two secant lines with slopes that come from computing the *backward difference* $\frac{f(1)-f(2)}{1-2} = 0.75$ and from the *forward difference* $\frac{f(3)-f(2)}{3-2} = 0.375$. Note how the first such line's slope over-estimates the slope of the tangent line at $(2, f(2))$, while the second line's slope underestimates $f'(2)$. On the right, however, we see the secant line whose slope is given by the *central difference*

$$\frac{f(3) - f(1)}{3 - 1} = \frac{3.625 - 2.5}{2} = \frac{1.125}{2} = 0.5625.$$

Note that this central difference has the exact same value as the average of the forward difference and backward difference (and it is straightforward to explain why this always holds), and moreover that the central difference yields a very good approximation to the derivative's value, in part because the secant line that uses both a point before and after the point of tangency yields a line that is closer to being parallel to the tangent line.

In general, the central difference approximation to the value of the first derivative is given by

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h},$$

and this quantity measures the slope of the secant line to $y = f(x)$ through the points $(a-h, f(a-h))$ and $(a+h, f(a+h))$. Anytime we have symmetric data surrounding a point at which we desire to estimate the derivative, the central difference is an ideal choice for so doing.

The following activities will further explore the meaning of the derivative in several different contexts while also viewing the derivative from graphical, numerical, and algebraic perspectives.

Activity 1.12.

A potato is placed in an oven, and the potato's temperature F (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time t is measured in minutes.

t	$F(t)$
0	70
15	180.5
30	251
45	296
60	324.5
75	342.8
90	354.5

(a) Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t = 30$. Include units on your answer.

(b) Use a central difference to estimate the instantaneous rate of change of the temperature of the potato at $t = 60$. Include units on your answer.

(c) Without doing any calculation, which do you expect to be greater: $F'(75)$ or $F'(90)$? Why?

(d) Suppose it is given that $F(64) = 330.28$ and $F'(64) = 1.341$. What are the units on these two quantities? What do you expect the temperature of the potato to be when $t = 65$? when $t = 66$? Why?

(e) Write a couple of careful sentences that describe the behavior of the temperature of the potato on the time interval $[0, 90]$, as well as the behavior of the instantaneous rate of change of the temperature of the potato on the same time interval.

◁

Activity 1.13.

A company manufactures rope, and the total cost of producing r feet of rope is $C(r)$ dollars.

(a) What does it mean to say that $C(2000) = 800$?

(b) What are the units of $C'(r)$?

(c) Suppose that $C(2000) = 800$ and $C'(2000) = 0.35$. Estimate $C(2100)$, and justify your estimate by writing at least one sentence that explains your thinking.

(d) Which of the following statements do you think is true, and why?

- $C'(2000) < C'(3000)$
- $C'(2000) = C'(3000)$
- $C'(2000) > C'(3000)$

(e) Suppose someone claims that $C'(5000) = -0.1$. What would the practical meaning of this derivative value tell you about the approximate cost of the next foot of rope? Is this possible? Why or why not?

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Activity 1.14.

Researchers at a major car company have found a function that relates gasoline consumption to speed for a particular model of car. In particular, they have determined that the consumption C, in **liters per kilometer**, at a given speed s, is given by a function $C = f(s)$, where s is the car's speed in **kilometers per hour**.

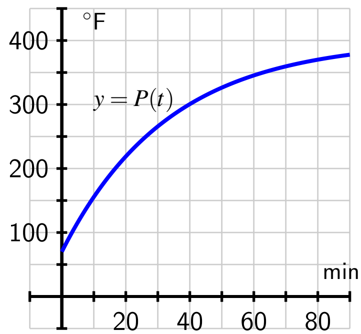
(a) Data provided by the car company tells us that $f(80) = 0.015$, $f(90) = 0.02$, and $f(100) = 0.027$. Use this information to estimate the instantaneous rate of change of fuel consumption with respect to speed at $s = 90$. Be as accurate as possible, use proper notation, and include units on your answer.

(b) By writing a complete sentence, interpret the meaning (in the context of fuel consumption) of "$f(80) = 0.015$."

(c) Write at least one complete sentence that interprets the meaning of the value of $f'(90)$ that you estimated in (a).

◁

In Section 1.4, we learned how use to the graph of a given function f to plot the graph of its derivative, f'. It is important to remember that when we do so, not only does the scale on the vertical axis often have to change to accurately represent f', but the units on that axis also differ. For example, suppose that $P(t) = 400 - 330e^{-0.03t}$ tells us the temperature in degrees Fahrenheit of a potato in an oven at time t in minutes. In Figure 1.24, we sketch the graph of P on the left and the graph of P' on the right.



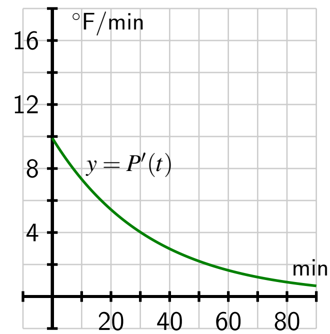


Figure 1.24: Plot of $P(t) = 400 - 330e^{-0.03t}$ at left, and its derivative $P'(t)$ at right.

Note how not only are the vertical scales different in size, but different in units, as the units of P are °F, while those of P' are °F/min. In all cases where we work with functions that have an applied context, it is helpful and instructive to think carefully about units involved and how they further inform the meaning of our computations.

Summary

In this section, we encountered the following important ideas:

- Regardless of the context of a given function $y = f(x)$, the derivative always measures the instantaneous rate of change of the output variable with respect to the input variable.

- The units on the derivative function $y = f'(x)$ are units of f per unit of x. Again, this measures how fast the output of the function f changes when the input of the function changes.

- The central difference approximation to the value of the first derivative is given by

  $$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h},$$

  and this quantity measures the slope of the secant line to $y = f(x)$ through the points $(a - h, f(a - h))$ and $(a + h, f(a + h))$. The central difference generates a good approximation of the derivative's value any time we have symmetric data surrounding a point of interest.

- Knowing the derivative and function values at a single point enables us to estimate other function values nearby. If, for example, we know that $f'(7) = 2$, then we know that at $x = 7$, the function f is increasing at an instantaneous rate of 2 units of output for every one unit of input. Thus, we expect $f(8)$ to be approximately 2 units greater than $f(7)$. The value is approximate because we don't know that the rate of change stays the same as x changes.

Exercises

1. A cup of coffee has its temperature F (in degrees Fahrenheit) at time t given by the function $F(t) = 75 + 110e^{-0.05t}$, where time is measured in minutes.

   (a) Use a central difference with $h = 0.01$ to estimate the value of $F'(10)$.

   (b) What are the units on the value of $F'(10)$ that you computed in (a)? What is the practical meaning of the value of $F'(10)$?

   (c) Which do you expect to be greater: $F'(10)$ or $F'(20)$? Why?

   (d) Write a sentence that describes the behavior of the function $y = F'(t)$ on the time interval $0 \le t \le 30$. How do you think its graph will look? Why?

2. The temperature change T (in Fahrenheit degrees), in a patient, that is generated by a dose q (in milliliters), of a drug, is given by the function $T = f(q)$.

   (a) What does it mean to say $f(50) = 0.75$? Write a complete sentence to explain, using correct units.

   (b) A person's sensitivity, s, to the drug is defined by the function $s(q) = f'(q)$. What are the units of sensitivity?

   (c) Suppose that $f'(50) = -0.02$. Write a complete sentence to explain the meaning of this value. Include in your response the information given in (a).

3. The velocity of a ball that has been tossed vertically in the air is given by $v(t) = 16-32t$, where v is measured in feet per second, and t is measured in seconds. The ball is in the air from $t = 0$ until $t = 2$.

   (a) When is the ball's velocity greatest?

   (b) Determine the value of $v'(1)$. Justify your thinking.

   (c) What are the units on the value of $v'(1)$? What does this value and the corresponding units tell you about the behavior of the ball at time $t = 1$?

   (d) What is the physical meaning of the function $v'(t)$?

4. The value, V, of a particular automobile (in dollars) depends on the number of miles, m, the car has been driven, according to the function $V = h(m)$.

   (a) Suppose that $h(40000) = 15500$ and $h(55000) = 13200$. What is the average rate of change of h on the interval $[40000, 55000]$, and what are the units on this value?

(b) In addition to the information given in (a), say that $h(70000) = 11100$. Determine the best possible estimate of $h'(55000)$ and write one sentence to explain the meaning of your result, including units on your answer.

(c) Which value do you expect to be greater: $h'(30000)$ or $h'(80000)$? Why?

(d) Write a sentence to describe the long-term behavior of the function $V = h(m)$, plus another sentence to describe the long-term behavior of $h'(m)$. Provide your discussion in practical terms regarding the value of the car and the rate at which that value is changing.

1.6 The second derivative

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How does the derivative of a function tell us whether the function is increasing or decreasing at a point or on an interval?
- What can we learn by taking the derivative of the derivative (to achieve the *second* derivative) of a function f?
- What does it mean to say that a function is concave up or concave down? How are these characteristics connected to certain properties of the derivative of the function?
- What are the units of the second derivative? How do they help us understand the rate of change of the rate of change?

Introduction

Given a differentiable function $y = f(x)$, we know that its derivative, $y = f'(x)$, is a related function whose output at a value $x = a$ tells us the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$. That is, heights on the derivative graph tell us the values of slopes on the original function's graph. Therefore, the derivative tells us important information about the function f.

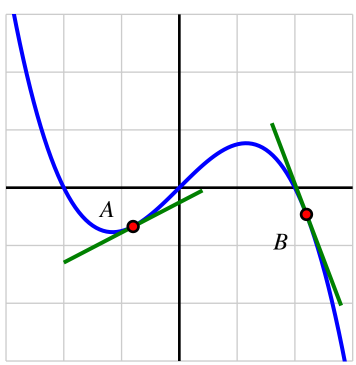


Figure 1.25: Two tangent lines on a graph demonstrate how the slope of the tangent line tells us whether the function is rising or falling, as well as whether it is doing so rapidly or slowly.

At any point where $f'(x)$ is positive, it means that the slope of the tangent line to f is positive, and therefore the function f is increasing (or rising) at that point. Similarly, if $f'(a)$ is negative, we know that the graph of f is decreasing (or falling) at that point.

In the next part of our study, we work to understand not only *whether* the function f is increasing or decreasing at a point or on an interval, but also *how* the function f is increasing or decreasing. Comparing the two tangent lines shown in Figure 1.25, we see that at point A, the value of $f'(x)$ is positive and relatively close to zero, which coincides with the graph rising slowly. By contrast, at point B, the derivative is negative and relatively large in absolute value, which is tied to the fact that f is decreasing rapidly at B. It also makes sense to not only ask whether the value of the derivative function is positive or negative and whether the derivative is large or small, but also to ask "how is the derivative changing?"

We also now know that the derivative, $y = f'(x)$, is itself a function. This means that we can consider taking its derivative – the derivative of the derivative – and therefore ask questions like "what does the derivative of the derivative tell us about how the original function behaves?" As we have done regularly in our work to date, we start with an investigation of a familiar problem in the context of a moving object.

Preview Activity 1.6. The position of a car driving along a straight road at time t in minutes is given by the function $y = s(t)$ that is pictured in Figure 1.26. The car's position function has units measured in thousands of feet. For instance, the point $(2, 4)$ on the graph indicates that after 2 minutes, the car has traveled 4000 feet.

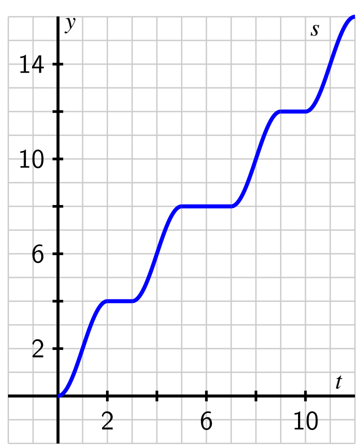


Figure 1.26: The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time t in minutes.

(a) In everyday language, describe the behavior of the car over the provided time interval. In particular, you should carefully discuss what is happening on each of the time intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, $[3, 4]$, and $[4, 5]$, plus provide commentary overall on what the car is doing on the interval $[0, 12]$.

(b) On the lefthand axes provided in Figure 1.27, sketch a careful, accurate graph of $y = s'(t)$.

(c) What is the meaning of the function $y = s'(t)$ in the context of the given problem? What can we say about the car's behavior when $s'(t)$ is positive? when $s'(t)$ is zero? when $s'(t)$ is negative?

(d) Rename the function you graphed in (b) to be called $y = v(t)$. Describe the behavior of v in words, using phrases like "v is increasing on the interval . . ." and "v is constant on the interval . . .."

(e) Sketch a graph of the function $y = v'(t)$ on the righthand axes provide in Figure 1.27. Write at least one sentence to explain how the behavior of $v'(t)$ is connected to the graph of $y = v(t)$.

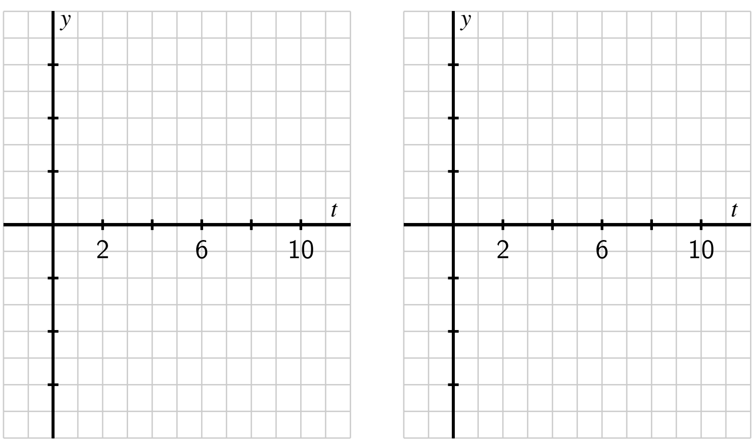


Figure 1.27: Axes for plotting $y = v(t) = s'(t)$ and $y = v'(t)$.

⋈

Increasing, decreasing, or neither

When we look at the graph of a function, there are features that strike us naturally, and common language can be used to name these features. In many different settings so far, we have intuitively used the words *increasing* and *decreasing* to describe a function's graph.

Here we connect these terms more formally to a function's behavior on an interval of input values.

Definition 1.5. Given a function $f(x)$ defined on the interval (a, b), we say that f *is increasing on* (a, b) provided that for all x, y in the interval (a, b), if $x < y$, then $f(x) < f(y)$. Similarly, we say that f *is decreasing on* (a, b) provided that for all x, y in the interval (a, b), if $x < y$, then $f(x) > f(y)$.

Simply put, an increasing function is one that is rising as we move from left to right along the graph, and a decreasing function is one that falls as the value of the input increases. For a function that has a derivative, we can use the sign of the derivative to determine whether or not the function is increasing or decreasing.

> Let f be a function that is differentiable on an interval (a, b). We say that f is increasing on (a, b) if and only if $f'(x) > 0$ for every x such that $a < x < b$; similarly, f is decreasing on (a, b) if and only if $f'(x) < 0$. If $f'(a) = 0$, then we say f is neither increasing nor decreasing at $x = a$.

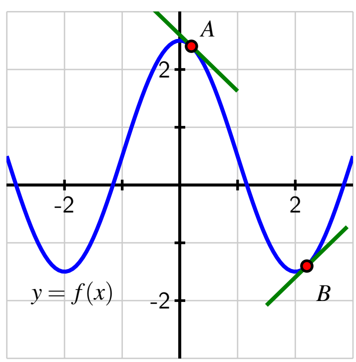


Figure 1.28: A function that is decreasing on the intervals $-3 < x < -2$ and $0 < x < 2$ and increasing on $-2 < x < 0$ and $2 < x < 3$.

For example, the function pictured in Figure 1.28 is increasing on the entire interval $-2 < x < 0$. Note that at both $x = \pm 2$ and $x = 0$, we say that f is neither increasing nor decreasing, because $f'(x) = 0$ at these values.

The Second Derivative

For any function, we are now accustomed to investigating its behavior by thinking about its derivative. Given a function f, its derivative is a new function, one that is given by the rule

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Because f' is itself a function, it is perfectly feasible for us to consider the derivative of the derivative, which is the new function $y = [f'(x)]'$. We call this resulting function *the second derivative* of $y = f(x)$, and denote the second derivative by $y = f''(x)$. Due to the presence of multiple possible derivatives, we will sometimes call f' "the first derivative" of f, rather than simply "the derivative" of f. Formally, the second derivative is defined by the limit definition of the derivative of the first derivative:

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}.$$

We note that all of the established meaning of the derivative function still holds, so when we compute $y = f''(x)$, this new function measures slopes of tangent lines to the curve $y = f'(x)$, as well as the instantaneous rate of change of $y = f'(x)$. In other words, just as the first derivative measures the rate at which the original function changes, the second derivative measures the rate at which the first derivative changes. This means that the second derivative tracks the instantaneous rate of change of the instantaneous rate of change of f. That is, the second derivative will help us to understand how the rate of change of the original function is itself changing.

Concavity

In addition to asking *whether* a function is increasing or decreasing, it is also natural to inquire *how* a function is increasing or decreasing. To begin, there are three basic behaviors that an increasing function can demonstrate on an interval, as pictured in Figure 1.29: the function can increase more and more rapidly, increase at the same rate, or increase in a way that is slowing down. Fundamentally, we are beginning to think about how a particular curve bends, with the natural comparison being made to lines, which don't bend at all. More than this, we want to understand how the bend in a function's graph is tied to behavior characterized by the first derivative of the function.

For the leftmost curve in Figure 1.29, picture a sequence of tangent lines to the curve. As we move from left to right, the slopes of those tangent lines will increase. Therefore, the rate of change of the pictured function is increasing, and this explains why we say this function is *increasing at an increasing rate*. For the rightmost graph in Figure 1.29, observe that as x increases, the function increases but the slope of the tangent line decreases, hence this function is *increasing at a decreasing rate*.

Of course, similar options hold for how a function can decrease. Here we must be extra careful with our language, since decreasing functions involve negative slopes, and negative numbers present an interesting situation in the tension between common language and mathematical language. For example, it can be tempting to say that "-100 is bigger than -2." But we must remember that when we say one number is greater than another, this describes how the numbers lie on a number line: $x < y$ provided that x lies to the left of y. So of course, -100 is less than -2. Informally, it might be helpful to say that

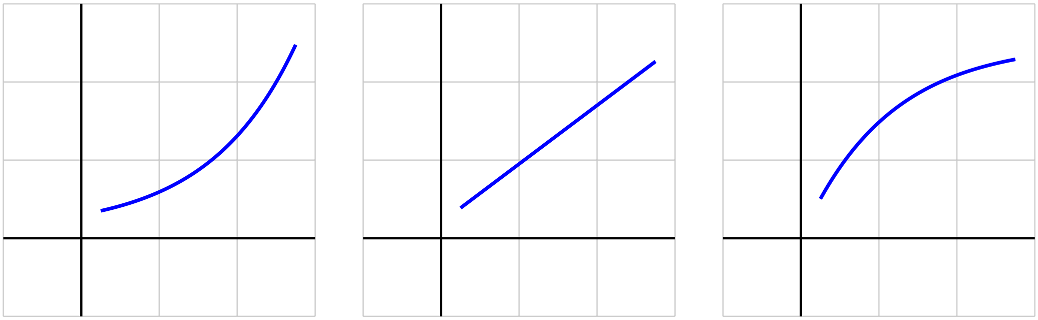


Figure 1.29: Three functions that are all increasing, but doing so at an increasing rate, at a constant rate, and at a decreasing rate, respectively.

"-100 is more negative than -2." This leads us to note particularly that when a function's values are negative, and those values subsequently get more negative, the function must be decreasing.

Now consider the three graphs shown in Figure 1.30. Clearly the middle graph demonstrates the behavior of a function decreasing at a constant rate. If we think about a sequence of tangent lines to the first curve that progress from left to right, we see that the slopes of these lines get less and less negative as we move from left to right. That means that the values of the first derivative, while all negative, are increasing, and thus we say that the leftmost curve is *decreasing at an increasing rate*.

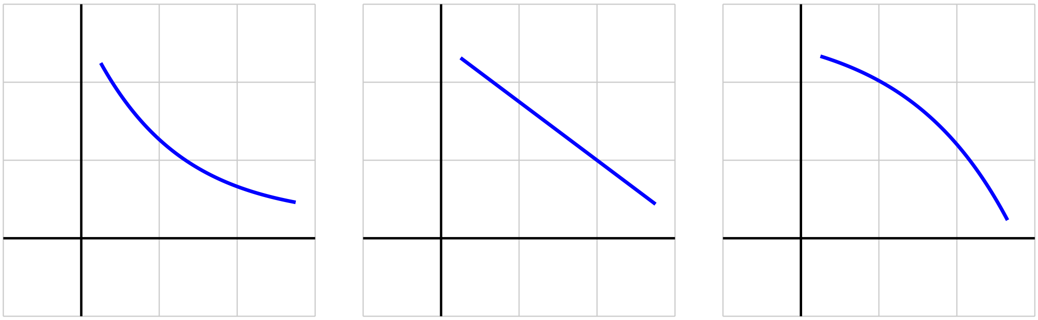


Figure 1.30: From left to right, three functions that are all decreasing, but doing so in different ways.

This leaves only the rightmost curve in Figure 1.30 to consider. For that function, the slope of the tangent line is negative throughout the pictured interval, but as we move from left to right, the slopes get more and more negative. Hence the slope of the curve is decreasing, and we say that the function is *decreasing at a decreasing rate.*

This leads us to introduce the notion of *concavity* which provides simpler language to describe some of these behaviors. Informally, when a curve opens up on a given interval, like the upright parabola $y = x^2$ or the exponential growth function $y = e^x$, we say that the curve is *concave up* on that interval. Likewise, when a curve opens down, such as the parabola $y = -x^2$ or the opposite of the exponential function $y = -e^x$, we say that the function is *concave down.* This behavior is linked to both the first and second derivatives of the function.

In Figure 1.31, we see two functions along with a sequence of tangent lines to each. On the lefthand plot where the function is concave up, observe that the tangent lines to the curve always lie below the curve itself and that, as we move from left to right, the slope of the tangent line is increasing. Said differently, the function f is concave up on the interval shown because its derivative, f', is increasing on that interval. Similarly, on the righthand plot in Figure 1.31, where the function shown is concave down, there we see that the tangent lines alway lie above the curve and that the value of the slope of the tangent line is decreasing as we move from left to right. Hence, what makes f concave down on the interval is the fact that its derivative, f', is decreasing.

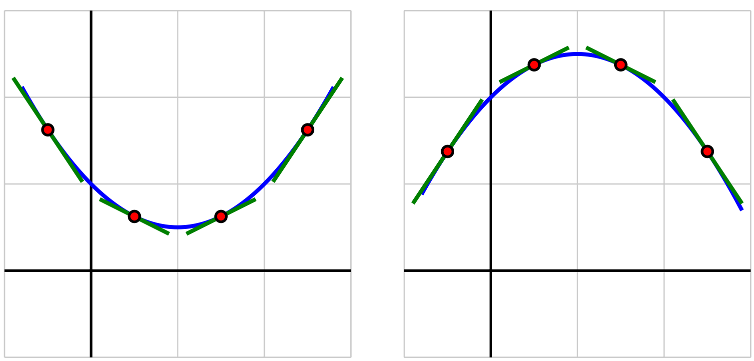


Figure 1.31: At left, a function that is concave up; at right, one that is concave down.

We state these most recent observations formally as the definitions of the terms *concave up* and *concave down.*

Definition 1.6. Let f be a differentiable function on an interval (a, b). Then f is *concave up* on (a, b) if and only if f' is increasing on (a, b); f is *concave down* on (a, b) if and only if f' is decreasing on (a, b).

The following activities lead us to further explore how the first and second derivatives of a function determine the behavior and shape of its graph. We begin by revisiting

Preview Activity 1.6.

Activity 1.15.

The position of a car driving along a straight road at time t in minutes is given by the function $y = s(t)$ that is pictured in Figure 1.32. The car's position function has units measured in thousands of feet. Remember that you worked with this function and sketched graphs of $y = v(t) = s'(t)$ and $y = v'(t)$ in Preview Activity 1.6.

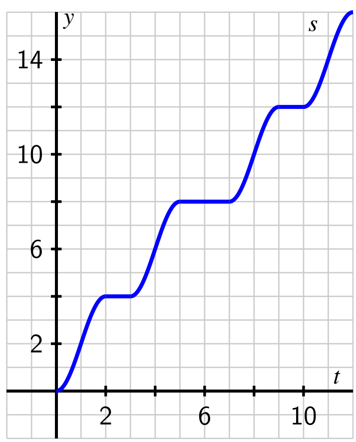


Figure 1.32: The graph of $y = s(t)$, the position of the car (measured in thousands of feet from its starting location) at time t in minutes.

(a) On what intervals is the position function $y = s(t)$ increasing? decreasing? Why?

(b) On which intervals is the velocity function $y = v(t) = s'(t)$ increasing? decreasing? neither? Why?

(c) *Acceleration* is defined to be the instantaneous rate of change of velocity, as the acceleration of an object measures the rate at which the velocity of the object is changing. Say that the car's acceleration function is named $a(t)$. How is $a(t)$ computed from $v(t)$? How is $a(t)$ computed from $s(t)$? Explain.

(d) What can you say about s'' whenever s' is increasing? Why?

(e) Using only the words *increasing*, *decreasing*, *constant*, *concave up*, *concave down*, and *linear*, complete the following sentences. For the position function s with velocity v and acceleration a,

- on an interval where v is positive, s is ______________________.
- on an interval where v is negative, s is ______________________.
- on an interval where v is zero, s is ______________________.

- on an interval where a is positive, v is ____________________.
- on an interval where a is negative, v is ____________________.
- on an interval where a is zero, v is ____________________.
- on an interval where a is positive, s is ____________________.
- on an interval where a is negative, s is ____________________.
- on an interval where a is zero, s is ____________________.

◁

The context of position, velocity, and acceleration is an excellent one in which to understand how a function, its first derivative, and its second derivative are related to one another. In Activity 1.15, we can replace s, v, and a with an arbitrary function f and its derivatives f' and f'', and essentially all the same observations hold. In particular, note that f' is increasing if and only if f is concave up, and similarly f' is increasing if and only if f'' is positive. Likewise, f' is decreasing if and only if f is concave down, and f' is decreasing if and only if f'' is negative.

Activity 1.16.

A potato is placed in an oven, and the potato's temperature F (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time t is measured in minutes. In Activity 1.12, we computed approximations to $F'(30)$ and $F'(60)$ using central differences. Those values and more are provided in the second table below, along with several others computed in the same way.

t	$F(t)$
0	70
15	180.5
30	251
45	296
60	324.5
75	342.8
90	354.5

t	$F'(t)$
0	NA
15	6.03
30	3.85
45	2.45
60	1.56
75	1.00
90	NA

(a) What are the units on the values of $F'(t)$?

(b) Use a central difference to estimate the value of $F''(30)$.

(c) What is the meaning of the value of $F''(30)$ that you have computed in (b) in terms of the potato's temperature? Write several careful sentences that discuss, with appropriate units, the values of $F(30)$, $F'(30)$, and $F''(30)$, and explain the overall behavior of the potato's temperature at this point in time.

(d) Overall, is the potato's temperature increasing at an increasing rate, increasing at a constant rate, or increasing at a decreasing rate? Why?

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Activity 1.17.

This activity builds on our experience and understanding of how to sketch the graph of f' given the graph of f.

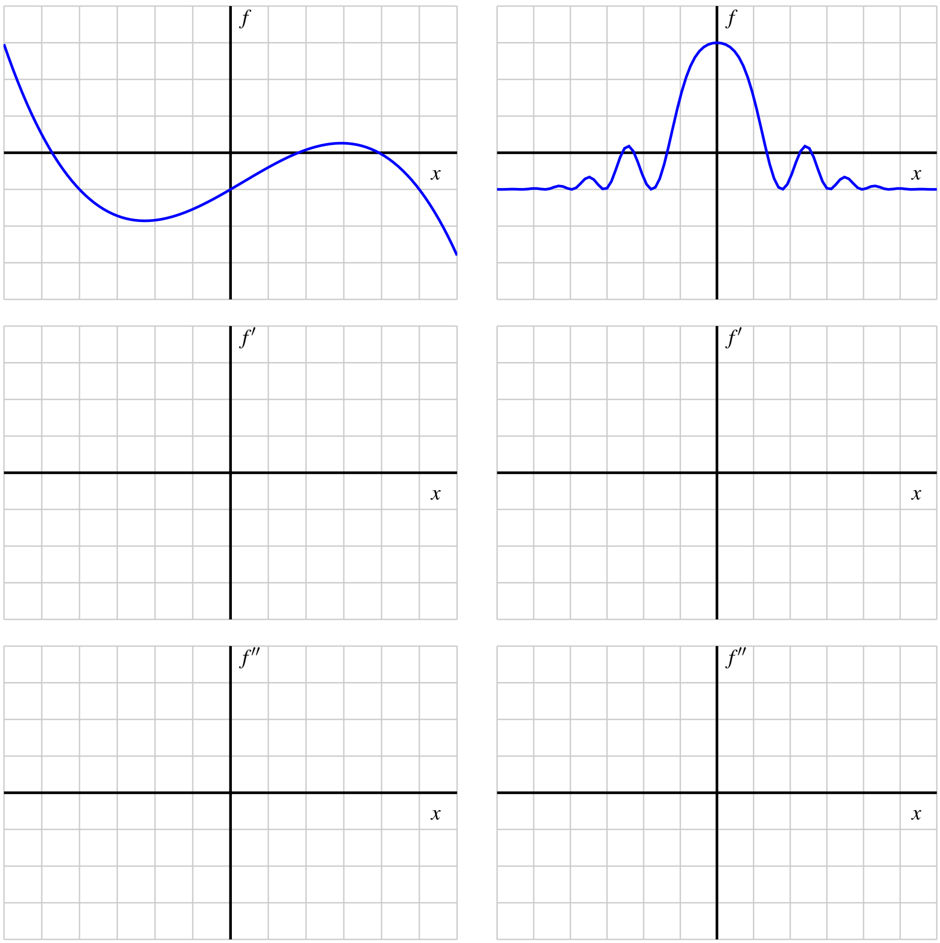

Figure 1.33: Two given functions f, with axes provided for plotting f' and f'' below.

In Figure 1.33, given the respective graphs of two different functions f, sketch the corresponding graph of f' on the first axes below, and then sketch f'' on the second set of axes. In addition, for each, write several careful sentences in the spirit of those in Activity 1.15 that connect the behaviors of f, f', and f''. For instance, write something such as

> f' is ____________________ on the interval _______, which is connected to the fact that f is ____________________ on the same interval _______,

and f'' is ____________________ on the interval as well

but of course with the blanks filled in. Throughout, view the scale of the grid for the graph of f as being 1×1, and assume the horizontal scale of the grid for the graph of f' is identical to that for f. If you need to adjust the vertical scale on the axes for the graph of f' or f'', you should label that accordingly.

◁

Summary

In this section, we encountered the following important ideas:

- A differentiable function f is increasing at a point or on an interval whenever its first derivative is positive, and decreasing whenever its first derivative is negative.
- By taking the derivative of the derivative of a function f, we arrive at the second derivative, f''. The second derivative measures the instantaneous rate of change of the first derivative, and thus the sign of the second derivative tells us whether or not the slope of the tangent line to f is increasing or decreasing.
- A differentiable function is concave up whenever its first derivative is increasing (or equivalently whenever its second derivative is positive), and concave down whenever its first derivative is decreasing (or equivalently whenever its second derivative is negative). Examples of functions that are everywhere concave up are $y = x^2$ and $y = e^x$; examples of functions that are everywhere concave down are $y = -x^2$ and $y = -e^x$.
- The units on the second derivative are "units of output per unit of input per unit of input." They tell us how the value of the derivative function is changing in response to changes in the input. In other words, the second derivative tells us the rate of change of the rate of change of the original function.

Exercises

1. Suppose that $y = f(x)$ is a differentiable function for which the following information is known: $f(2) = -3$, $f'(2) = 1.5$, $f''(2) = -0.25$.

(a) Is f increasing or decreasing at $x = 2$? Is f concave up or concave down at $x = 2$?

(b) Do you expect $f(2.1)$ to be greater than -3, equal to -3, or less than -3? Why?

(c) Do you expect $f'(2.1)$ to be greater than 1.5, equal to 1.5, or less than 1.5? Why?

(d) Sketch a graph of $y = f(x)$ near $(2, f(2))$ and include a graph of the tangent line.

2. For a certain function $y = g(x)$, its derivative is given by the function pictured in Figure 1.34.

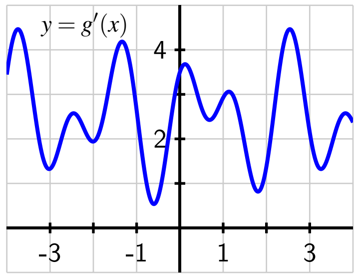

Figure 1.34: The graph of $y = g'(x)$.

(a) What is the approximate slope of the tangent line to $y = g(x)$ at the point $(2, g(2))$?

(b) How many real number solutions can there be to the equation $g(x) = 0$? Justify your conclusion fully and carefully by explaining what you know about how the graph of g must behave based on the given graph of g'.

(c) On the interval $-3 < x < 3$, how many times does the concavity of g change? Why?

(d) Use the provided graph to estimate the value of $g''(2)$.

3. A bungee jumper's height h (in feet) at time t (in seconds) is given in part by the data in the following table:

t	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$h(t)$	200	184.2	159.9	131.9	104.7	81.8	65.5	56.8	55.5	60.4	69.8

t	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
$h(t)$	81.6	93.7	104.4	112.6	117.7	119.4	118.2	114.8	110.0	104.7

(a) Use the given data to estimate $h'(4.5)$, $h'(5)$, and $h'(5.5)$. At which of these times is the bungee jumper rising most rapidly?

(b) Use the given data and your work in (a) to estimate $h''(5)$.

(c) What physical property of the bungee jumper does the value of $h''(5)$ measure? What are its units?

(d) Based on the data, on what approximate time intervals is the function $y = h(t)$ concave down? What is happening to the velocity of the bungee jumper on these time intervals?

4. For each prompt that follows, sketch a possible graph of a function on the interval $-3 < x < 3$ that satisfies the stated properties.

(a) $y = f(x)$ such that f is increasing on $-3 < x < 3$, f is concave up on $-3 < x < 0$, and f is concave down on $0 < x < 3$.

(b) $y = g(x)$ such that g is increasing on $-3 < x < 3$, g is concave down on $-3 < x < 0$, and g is concave up on $0 < x < 3$.

(c) $y = h(x)$ such that h is decreasing on $-3 < x < 3$, h is concave up on $-3 < x < -1$, neither concave up nor concave down on $-1 < x < 1$, and h is concave down on $1 < x < 3$.

(d) $y = p(x)$ such that p is decreasing and concave down on $-3 < x < 0$ and p is increasing and concave down on $0 < x < 3$.

1.7 Limits, Continuity, and Differentiability

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What does it mean graphically to say that f has limit L as $x \to a$? How is this connected to having a left-hand limit at $x = a$ and having a right-hand limit at $x = a$?
- What does it mean to say that a function f is continuous at $x = a$? What role do limits play in determining whether or not a function is continuous at a point?
- What does it mean graphically to say that a function f is differentiable at $x = a$? How is this connected to the function being locally linear?
- How are the characteristics of a function having a limit, being continuous, and being differentiable at a given point related to one another?

Introduction

In Section 1.2, we learned about how the concept of limits can be used to study the trend of a function near a fixed input value. As we study such trends, we are fundamentally interested in knowing how well-behaved the function is at the given point, say $x = a$. In this present section, we aim to expand our perspective and develop language and understanding to quantify how the function acts and how its value changes near a particular point. Beyond thinking about whether or not the function has a limit L at $x = a$, we will also consider the value of the function $f(a)$ and how this value is related to $\lim_{x\to a} f(x)$, as well as whether or not the function has a derivative $f'(a)$ at the point of interest. Throughout, we will build on and formalize ideas that we have encountered in several settings.

We begin to consider these issues through the following preview activity that asks you to consider the graph of a function with a variety of interesting behaviors.

Preview Activity 1.7. A function f defined on $-4 < x < 4$ is given by the graph in Figure 1.35. Use the graph to answer each of the following questions. Note: to the right of $x = 2$, the graph of f is exhibiting infinite oscillatory behavior similar to the function $\sin(\frac{\pi}{x})$ that we encountered in the key example early in Section 1.2.

(a) For each of the values $a = -3, -2, -1, 0, 1, 2, 3$, determine whether or not $\lim\limits_{x\to a} f(x)$ exists. If the function has a limit L at a given point, state the value of the limit using the notation $\lim\limits_{x\to a} f(x) = L$. If the function does not have a limit at a given point, write a sentence to explain why.

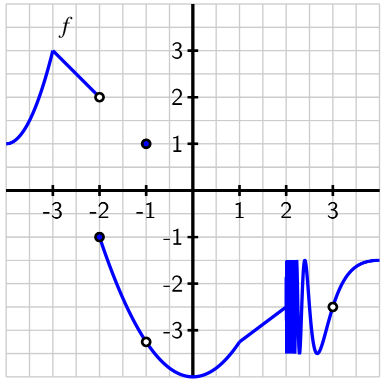

Figure 1.35: The graph of $y = f(x)$.

(b) For each of the values of a from part (a) where f has a limit, determine the value of $f(a)$ at each such point. In addition, for each such a value, does $f(a)$ have the same value as $\lim_{x \to a} f(x)$?

(c) For each of the values $a = -3, -2, -1, 0, 1, 2, 3$, determine whether or not $f'(a)$ exists. In particular, based on the given graph, ask yourself if it is reasonable to say that f has a tangent line at $(a, f(a))$ for each of the given a-values. If so, visually estimate the slope of the tangent line to find the value of $f'(a)$.

⋈

Having a limit at a point

In Section 1.2, we first encountered limits and learned that we say that f has limit L as x approaches a and write $\lim_{x \to a} f(x) = L$ provided that we can make the value of $f(x)$ as close to L as we like by taking x sufficiently close (but not equal to) a. Here, we expand further on this definition and focus in more depth on what it means for a function not to have a limit at a given value.

Essentially there are two behaviors that a function can exhibit at a point where it fails to have a limit. In Figure 1.36, at left we see a function f whose graph shows a jump at $a = 1$. In particular, if we let x approach 1 from the left side, the value of f approaches 2, while if we let x go to 1 from the right, the value of f tends to 3. Because the value of f does not approach a single number as x gets arbitrarily close to 1 from both sides, we know that f does not have a limit at $a = 1$.

Since f does approach a single value on each side of $a = 1$, we can introduce the notion of *left* and *right* (or *one-sided*) limits. We say that f *has limit* L_1 *as* x *approaches* a

from the left and write

$$\lim_{x \to a^-} f(x) = L_1$$

provided that we can make the value of $f(x)$ as close to L_1 as we like by taking x sufficiently close to a while always having $x < a$. In this case, we call L_1 the left-hand limit of f as x approaches a. Similarly, we say L_2 *is the right-hand limit of* f *as* x *approaches* a and write

$$\lim_{x \to a^+} f(x) = L_2$$

provided that we can make the value of $f(x)$ as close to L_2 as we like by taking x sufficiently close to a while always having $x > a$. In the graph of the function f in Figure 1.36, we see that

$$\lim_{x \to 1^-} f(x) = 2 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 3$$

and precisely because the left and right limits are not equal, the overall limit of f as $x \to 1$ fails to exist.

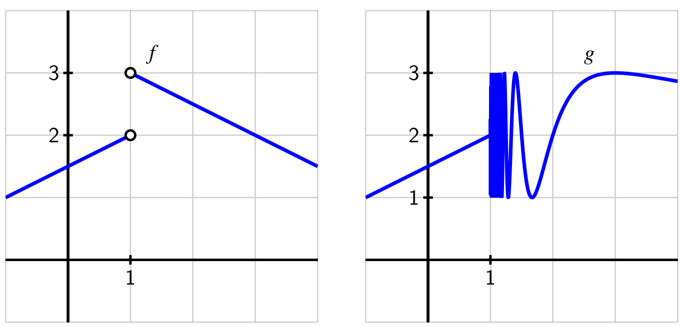

Figure 1.36: Functions f and g that each fail to have a limit at $a = 1$.

For the function g pictured at right in Figure 1.36, the function fails to have a limit at $a = 1$ for a different reason. While the function does not have a jump in its graph at $a = 1$, it is still not the case that g approaches a single value as x approaches 1. In particular, due to the infinitely oscillating behavior of g to the right of $a = 1$, we say that the right-hand limit of g as $x \to 1^+$ does not exist, and thus $\lim_{x \to 1} g(x)$ does not exist.

To summarize, anytime either a left- or right-hand limit fails to exist or the left- and right-hand limits are not equal to each other, the overall limit will not exist. Said differently,

> A function f has limit L as $x \to a$ if and only if
>
> $$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$
>
> That is, a function has a limit at $x = a$ if and only if both the left- and right-hand limits at $x = a$ exist and share the same value.

In Preview Activity 1.7, the function f given in Figure 1.35 only fails to have a limit at two values: at $a = -2$ (where the left- and right-hand limits are 2 and -1, respectively) and at $x = 2$, where $\lim_{x \to 2^+} f(x)$ does not exist). Note well that even at values like $a = -1$ and $a = 0$ where there are holes in the graph, the limit still exists.

Activity 1.18.

Consider a function that is piecewise-defined according to the formula

$$f(x) = \begin{cases} 3(x+2)+2 & \text{for } -3 < x < -2 \\ \frac{2}{3}(x+2)+1 & \text{for } -2 \le x < -1 \\ \frac{2}{3}(x+2)+1 & \text{for } -1 < x < 1 \\ 2 & \text{for } x = 1 \\ 4-x & \text{for } x > 1 \end{cases}$$

Use the given formula to answer the following questions.

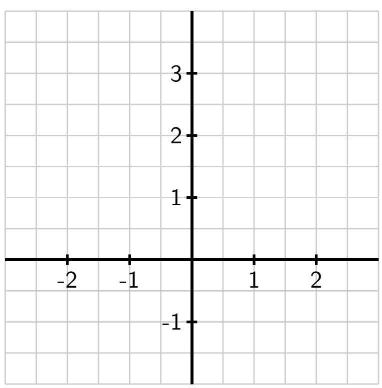

Figure 1.37: Axes for plotting the function $y = f(x)$ in Activity 1.18.

(a) For each of the values $a = -2, -1, 0, 1, 2$, compute $f(a)$.

(b) For each of the values $a = -2, -1, 0, 1, 2$, determine $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.

(c) For each of the values $a = -2, -1, 0, 1, 2$, determine $\lim_{x \to a} f(x)$. If the limit fails to exist, explain why by discussing the left- and right-hand limits at the relevant a-value.

(d) For which values of a is the following statement true?

$$\lim_{x \to a} f(x) \neq f(a)$$

(e) On the axes provided in Figure 1.37, sketch an accurate, labeled graph of $y = f(x)$. Be sure to carefully use open circles (∘) and filled circles (•) to represent key points on the graph, as dictated by the piecewise formula.

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Being continuous at a point

Intuitively, a function is continuous if we can draw it without ever lifting our pencil from the page. Alternatively, we might say that the graph of a continuous function has no jumps or holes in it. We first consider three specific situations in Figure 1.38 where all three functions have a limit at $a = 1$, and then work to make the idea of continuity more precise.

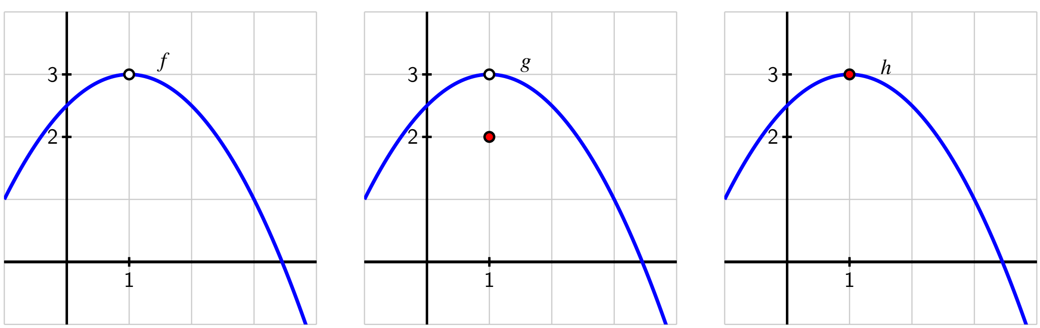

Figure 1.38: Functions f, g, and h that demonstrate subtly different behaviors at $a = 1$.

Note that $f(1)$ is not defined, which leads to the resulting hole in the graph of f at $a = 1$. We will naturally say that f is *not continuous at* $a = 1$. For the next function g in in Figure 1.38, we observe that while $\lim_{x \to 1} g(x) = 3$, the value of $g(1) = 2$, and thus the limit does not equal the function value. Here, too, we will say that g is *not continuous*, even though the function is defined at $a = 1$. Finally, the function h appears to be the most well-behaved of all three, since at $a = 1$ its limit and its function value agree. That is,

$$\lim_{x \to 1} h(x) = 3 = h(1).$$

With no hole or jump in the graph of h at $a = 1$, we desire to say that h is *continuous* there.

More formally, we make the following definition.

Definition 1.7. A function f is *continuous at* $x = a$ provided that

(a) f has a limit as $x \to a$,

(b) f is defined at $x = a$, and

(c) $\lim_{x \to a} f(x) = f(a)$.

Conditions (a) and (b) are technically contained implicitly in (c), but we state them explicitly to emphasize their individual importance. In words, (c) essentially says that a function is continuous at $x = a$ provided that its limit as $x \to a$ exists and equals its function value at $x = a$. If a function is continuous at every point in an interval $[a, b]$, we say the function is "continuous on $[a, b]$." If a function is continuous at every point in its domain, we simply say the function is "continuous." Thus, continuous functions are particularly nice: to evaluate the limit of a continuous function at a point, all we need to do is evaluate the function.

For example, consider $p(x) = x^2 - 2x + 3$. It can be proved that every polynomial is a continuous function at every real number, and thus if we would like to know $\lim_{x \to 2} p(x)$, we simply compute

$$\lim_{x \to 2}(x^2 - 2x + 3) = 2^2 - 2 \cdot 2 + 3 = 3.$$

This route of substituting an input value to evaluate a limit works anytime we know the function being considered is continuous. Besides polynomial functions, all exponential functions and the sine and cosine functions are continuous at every point, as are many other familiar functions and combinations thereof.

Activity 1.19.

This activity builds on your work in Preview Activity 1.7, using the same function f as given by the graph that is repeated in Figure 1.39

(a) At which values of a does $\lim_{x \to a} f(x)$ not exist?

(b) At which values of a is $f(a)$ not defined?

(c) At which values of a does f have a limit, but $\lim_{x \to a} f(x) \neq f(a)$?

(d) State all values of a for which f is not continuous at $x = a$.

(e) Which condition is stronger, and hence implies the other: f has a limit at $x = a$ or f is continuous at $x = a$? Explain, and hence complete the following sentence: "If f ______________________ at $x = a$, then f ______________________ at $x = a$," where you complete the blanks with *has a limit* and *is continuous*, using each phrase once.

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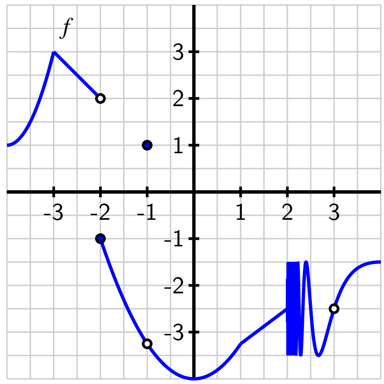

Figure 1.39: The graph of $y = f(x)$ for Activity 1.19.

Being differentiable at a point

We recall that a function f is said to be differentiable at $x = a$ whenever $f'(a)$ exists. Moreover, for $f'(a)$ to exist, we know that the function $y = f(x)$ must have a tangent line at the point $(a, f(a))$, since $f'(a)$ is precisely the slope of this line. In order to even ask if f has a tangent line at $(a, f(a))$, it is necessary that f be continuous at $x = a$: if f fails to have a limit at $x = a$, if $f(a)$ is not defined, or if $f(a)$ does not equal the value of $\lim_{x \to a} f(x)$, then it doesn't even make sense to talk about a tangent line to the curve at this point.

Indeed, it can be proved formally that if a function f is differentiable at $x = a$, then it must be continuous at $x = a$. So, if f is not continuous at $x = a$, then it is automatically the case that f is not differentiable there. For example, in Figure 1.38 from our early discussion of continuity, both f and g fail to be differentiable at $x = 1$ because neither function is continuous at $x = 1$. But can a function fail to be differentiable at a point where the function is continuous?

In Figure 1.40, we revisit the situation where a function has a sharp corner at a point, something we encountered several times in Section 1.4. For the pictured function f, we observe that f is clearly continuous at $a = 1$, since $\lim_{x \to 1} f(x) = 1 = f(1)$.

But the function f in Figure 1.40 is not differentiable at $a = 1$ because $f'(1)$ fails to exist. One way to see this is to observe that $f'(x) = -1$ for every value of x that is less than 1, while $f'(x) = +1$ for every value of x that is greater than 1. That makes it seem that either $+1$ or -1 would be equally good candidates for the value of the derivative at $x = 1$. Alternately, we could use the limit definition of the derivative to attempt to compute $f'(1)$, and discover that the derivative does not exist. A similar problem will be investigated in Activity 1.20. Finally, we can also see visually that the function f in Figure 1.40 does not have a tangent line. When we zoom in on $(1, 1)$ on the graph of f, no matter how closely we examine the function, it will always look like a "V", and never like a single line, which

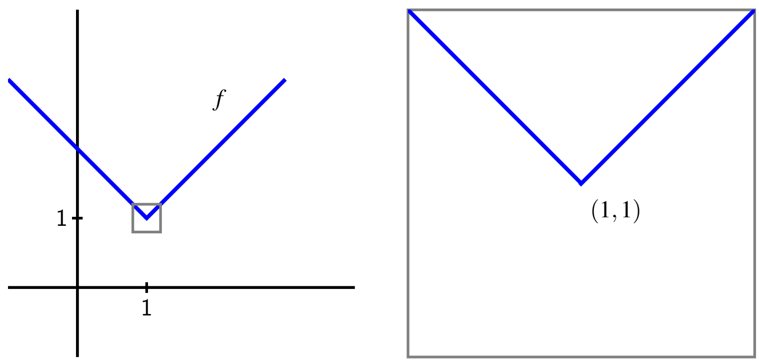

Figure 1.40: A function f that is continuous at $a = 1$ but not differentiable at $a = 1$; at right, we zoom in on the point $(1, 1)$ in a magnified version of the box in the left-hand plot.

tells us there is no possibility for a tangent line there.

To make a more general observation, if a function does have a tangent line at a given point, when we zoom in on the point of tangency, the function and the tangent line should appear essentially indistinguishable[7]. Conversely, if we have a function such that when we zoom in on a point the function looks like a single straight line, then the function should have a tangent line there, and thus be differentiable. Hence, a function that is differentiable at $x = a$ will, up close, look more and more like its tangent line at $(a, f(a))$, and thus we say that a function is differentiable at $x = a$ is *locally linear*.

To summarize the preceding discussion of differentiability and continuity, we make several important observations.

- If f is differentiable at $x = a$, then f is continuous at $x = a$. Equivalently, if f fails to be continuous at $x = a$, then f will not be differentiable at $x = a$.
- A function can be continuous at a point, but not be differentiable there. In particular, a function f is not differentiable at $x = a$ if the graph has a sharp corner (or *cusp*) at the point $(a, f(a))$.
- If f is differentiable at $x = a$, then f is locally linear at $x = a$. That is, when a function is differentiable, it looks linear when viewed up close because it resembles its tangent line there.

[7]See, for instance, http://gvsu.edu/s/6J for an applet (due to David Austin, GVSU) where zooming in shows the increasing similarity between the tangent line and the curve.

Activity 1.20.

In this activity, we explore two different functions and classify the points at which each is not differentiable. Let g be the function given by the rule $g(x) = |x|$, and let f be the function that we have previously explored in Preview Activity 1.7, whose graph is given again in Figure 1.41.

(a) Reasoning visually, explain why g is differentiable at every point x such that $x \neq 0$.

(b) Use the limit definition of the derivative to show that $g'(0) = \lim_{h \to 0} \frac{|h|}{h}$.

(c) Explain why $g'(0)$ fails to exist by using small positive and negative values of h.

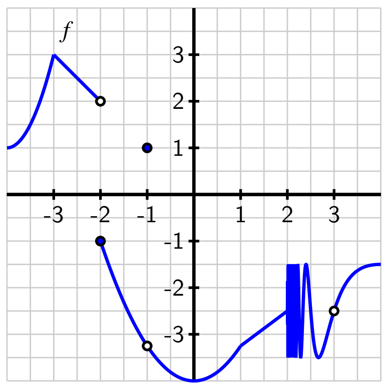

Figure 1.41: The graph of $y = f(x)$ for Activity 1.20.

(d) State all values of a for which f is not differentiable at $x = a$. For each, provide a reason for your conclusion.

(e) True or false: if a function p is differentiable at $x = b$, then $\lim_{x \to b} p(x)$ must exist. Why?

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Summary

In this section, we encountered the following important ideas:

- A function f has limit L as $x \to a$ if and only if f has a left-hand limit at $x = a$, has a right-hand limit at $x = a$, and the left- and right-hand limits are equal. Visually, this means that there can be a hole in the graph at $x = a$, but the function must approach the same single value from either side of $x = a$.

- A function f is continuous at $x = a$ whenever $f(a)$ is defined, f has a limit as $x \to a$, and the value of the limit and the value of the function agree. This guarantees that there is not a hole or jump in the graph of f at $x = a$.
- A function f is differentiable at $x = a$ whenever $f'(a)$ exists, which means that f has a tangent line at $(a, f(a))$ and thus f is locally linear at the value $x = a$. Informally, this means that the function looks like a line when viewed up close at $(a, f(a))$ and that there is not a corner point or cusp at $(a, f(a))$.
- Of the three conditions discussed in this section (having a limit at $x = a$, being continuous at $x = a$, and being differentiable at $x = a$), the strongest condition is being differentiable, and the next strongest is being continuous. In particular, if f is differentiable at $x = a$, then f is also continuous at $x = a$, and if f is continuous at $x = a$, then f has a limit at $x = a$.

Exercises

1. Consider the graph of the function $y = p(x)$ that is provided in Figure 1.42. Assume that each portion of the graph of p is a straight line, as pictured.

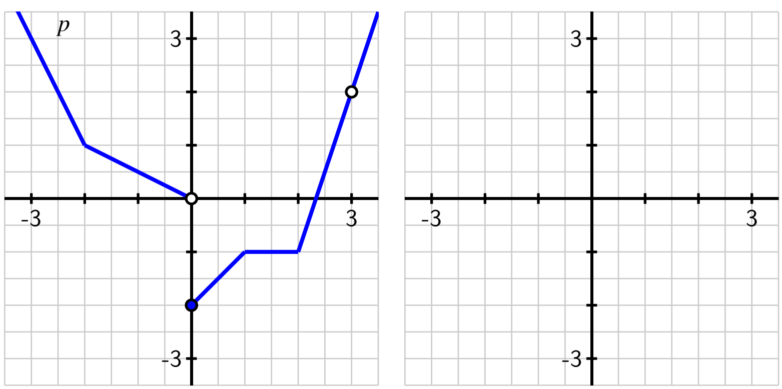


Figure 1.42: At left, the piecewise linear function $y = p(x)$. At right, axes for plotting $y = p'(x)$.

(a) State all values of a for which $\lim_{x\to a} p(x)$ does not exist.

(b) State all values of a for which p is not continuous at a.

(c) State all values of a for which p is not differentiable at $x = a$.

(d) On the axes provided in Figure 1.42, sketch an accurate graph of $y = p'(x)$.

2. For each of the following prompts, give an example of a function that satisfies the stated criteria. A formula or a graph, with reasoning, is sufficient for each. If no such example is possible, explain why.

(a) A function f that is continuous at $a = 2$ but not differentiable at $a = 2$.

(b) A function g that is differentiable at $a = 3$ but does not have a limit at $a = 3$.

(c) A function h that has a limit at $a = -2$, is defined at $a = -2$, but is not continuous at $a = -2$.

(d) A function p that satisfies all of the following:

- $p(-1) = 3$ and $\lim_{x\to -1} p(x) = 2$
- $p(0) = 1$ and $p'(0) = 0$
- $\lim_{x\to 1} p(x) = p(1)$ and $p'(1)$ does not exist

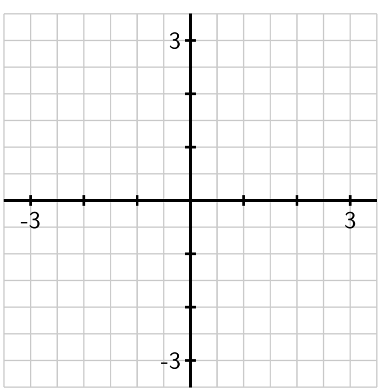
3. Let $h(x)$ be a function whose derivative $y = h'(x)$ is given by the graph on the right in Figure 1.43.

(a) Based on the graph of $y = h'(x)$, what can you say about the behavior of the function $y = h(x)$?

(b) At which values of x is $y = h'(x)$ not defined? What behavior does this lead you to expect to see in the graph of $y = h(x)$?

(c) Is it possible for $y = h(x)$ to have points where h is not continuous? Explain your answer.

(d) On the axes provided at left, sketch at least two distinct graphs that are possible functions $y = h(x)$ that each have a derivative $y = h'(x)$ that matches the provided graph at right. Explain why there are multiple possibilities for $y = h(x)$.



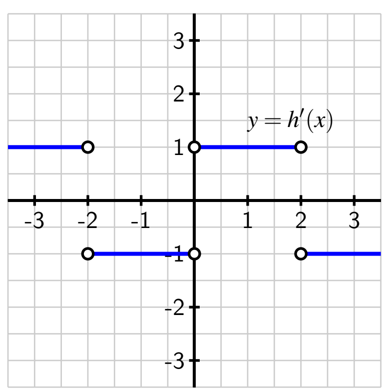


Figure 1.43: Axes for plotting $y = h(x)$ and, at right, the graph of $y = h'(x)$.

4. Consider the function $g(x) = \sqrt{|x|}$.

(a) Use a graph to explain visually why g is not differentiable at $x = 0$.

(b) Use the limit definition of the derivative to show that

$$g'(0) = \lim_{h \to 0} \frac{\sqrt{|h|}}{h}.$$

(c) Investigate the value of $g'(0)$ by estimating the limit in (b) using small positive and negative values of h. For instance, you might compute $\frac{\sqrt{|-0.01|}}{0.01}$. Be sure to use several different values of h (both positive and negative), including ones closer to 0 than 0.01. What do your results tell you about $g'(0)$?

(d) Use your graph in (a) to sketch an approximate graph of $y = g'(x)$.

1.8 The Tangent Line Approximation

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is the formula for the general tangent line approximation to a differentiable function $y = f(x)$ at the point $(a, f(a))$?
- What is the principle of local linearity and what is the local linearization of a differentiable function f at a point $(a, f(a))$?
- How does knowing just the tangent line approximation tell us information about the behavior of the original function itself near the point of approximation? How does knowing the second derivative's value at this point provide us additional knowledge of the original function's behavior?

Introduction

Among all functions, linear functions are simplest. One of the powerful consequences of a function $y = f(x)$ being differentiable at a point $(a, f(a))$ is that, up close, the function $y = f(x)$ is locally linear and looks like its tangent line at that point. In certain circumstances, this allows us to approximate the original function f with a simpler function L that is linear: this can be advantageous when we have limited information about f or when f is computationally or algebraically complicated. We will explore all of these situations in what follows.

It is essential to recall that when f is differentiable at $x = a$, the value of $f'(a)$ provides the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$. By knowing both a point on the line and the slope of the line we are thus able to find the equation of the tangent line. Preview Activity 1.8 will refresh these concepts through a key example and set the stage for further study.

Preview Activity 1.8. Consider the function $y = g(x) = -x^2 + 3x + 2$.

(a) Use the limit definition of the derivative to compute a formula for $y = g'(x)$.

(b) Determine the slope of the tangent line to $y = g(x)$ at the value $x = 2$.

(c) Compute $g(2)$.

(d) Find an equation for the tangent line to $y = g(x)$ at the point $(2, g(2))$. Write your result in point-slope form[8].

[8]Recall that a line with slope m that passes through (x_0, y_0) has equation $y - y_0 = m(x - x_0)$, and this is

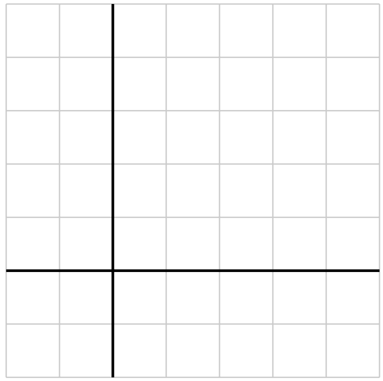


Figure 1.44: Axes for plotting $y = g(x)$ and its tangent line to the point $(2, g(2))$.

(e) On the axes provided in Figure 1.44, sketch an accurate, labeled graph of $y = g(x)$ along with its tangent line at the point $(2, g(2))$.

⋈

The tangent line

Given a function f that is differentiable at $x = a$, we know that we can determine the slope of the tangent line to $y = f(x)$ at $(a, f(a))$ by computing $f'(a)$. The resulting tangent line through $(a, f(a))$ with slope $m = f'(a)$ has its equation in point-slope form given by

$$y - f(a) = f'(a)(x - a),$$

which we can also express as $y = f'(a)(x - a) + f(a)$. Note well: there is a major difference between $f(a)$ and $f(x)$ in this context. The former is a constant that results from using the given fixed value of a, while the latter is the general expression for the rule that defines the function. The same is true for $f'(a)$ and $f'(x)$: we must carefully distinguish between these expressions. Each time we find the tangent line, we need to evaluate the function and its derivative at a fixed a-value.

In Figure 1.45, we see a labeled plot of the graph of a function f and its tangent line at the point $(a, f(a))$. Notice how when we zoom in we see the local linearity of f more clearly highlighted as the function and its tangent line are nearly indistinguishable up close. This can also be seen dynamically in the java applet at http://gvsu.edu/s/6J.

the *point-slope form* of the equation.

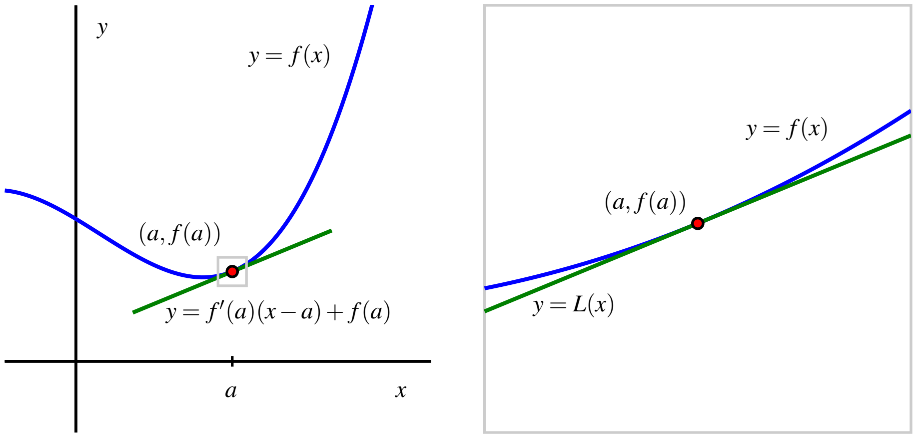


Figure 1.45: A function $y = f(x)$ and its tangent line at the point $(a, f(a))$: at left, from a distance, and at right, up close. At right, we label the tangent line function by $y = L(x)$ and observe that for x near a, $f(x) \approx L(x)$.

The local linearization

A slight change in perspective and notation will enable us to be more precise in discussing how the tangent line to $y = f(x)$ at $(a, f(a))$ approximates f near $x = a$. Taking the equation for the tangent line and solving for y, we observe that the tangent line is given by

$$y = f'(a)(x - a) + f(a)$$

and moreover that this line is itself a function of x. Replacing the variable y with the expression $L(x)$, we call

$$L(x) = f'(a)(x - a) + f(a)$$

the *local linearization of f* at the point $(a, f(a))$. In this notation, it is particularly important to observe that $L(x)$ is nothing more than a new name for the tangent line, and that for x close to a, we have that $f(x) \approx L(x)$.

Say, for example, that we know that a function $y = f(x)$ has its tangent line approximation given by $L(x) = 3 - 2(x - 1)$ at the point $(1, 3)$, but we do not know anything else about the function f. If we are interested in estimating a value of $f(x)$ for x near 1, such as $f(1.2)$, we can use the fact that $f(1.2) \approx L(1.2)$ and hence

$$f(1.2) \approx L(1.2) = 3 - 2(1.2 - 1) = 3 - 2(0.2) = 2.6.$$

Again, much of the new perspective here is only in notation since $y = L(x)$ is simply a new name for the tangent line function. In light of this new notation and our observations above, we note that since $L(x) = f(a) + f'(a)(x - a)$ and $L(x) \approx f(x)$ for x near a, it also follows that we can write

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

The next activity explores some additional important properties of the local linearization $y = L(x)$ to a function f at given a-value.

Activity 1.21.

Suppose it is known that for a given differentiable function $y = g(x)$, its local linearization at the point where $a = -1$ is given by $L(x) = -2 + 3(x + 1)$.

(a) Compute the values of $L(-1)$ and $L'(-1)$.

(b) What must be the values of $g(-1)$ and $g'(-1)$? Why?

(c) Do you expect the value of $g(-1.03)$ to be greater than or less than the value of $g(-1)$? Why?

(d) Use the local linearization to estimate the value of $g(-1.03)$.

(e) Suppose that you also know that $g''(-1) = 2$. What does this tell you about the graph of $y = g(x)$ at $a = -1$?

(f) For x near -1, sketch the graph of the local linearization $y = L(x)$ as well as a possible graph of $y = g(x)$ on the axes provided in Figure 1.46.

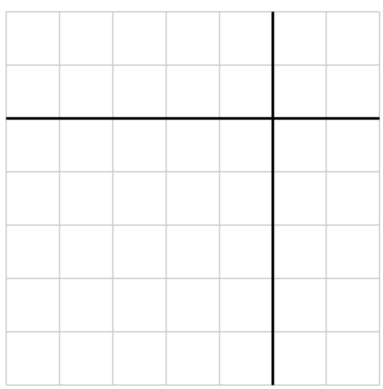


Figure 1.46: Axes for plotting $y = L(x)$ and $y = g(x)$.

◁

As we saw in the example provided by Activity 1.21, the local linearization $y = L(x)$ is a linear function that shares two important values with the function $y = f(x)$ that it is derived from. In particular, observe that since $L(x) = f(a) + f'(a)(x - a)$, it follows that $L(a) = f(a)$. In addition, since L is a linear function, its derivative is its slope. Hence,

$L'(x) = f'(a)$ for every value of x, and specifically $L'(a) = f'(a)$. Therefore, we see that L is a linear function that has both the same value and the same slope as the function f at the point $(a, f(a))$.

In situations where we know the linear approximation $y = L(x)$, we therefore know the original function's value and slope at the point of tangency. What remains unknown, however, is the shape of the function f at the point of tangency. There are essentially four possibilities, as enumerated in Figure 1.47.

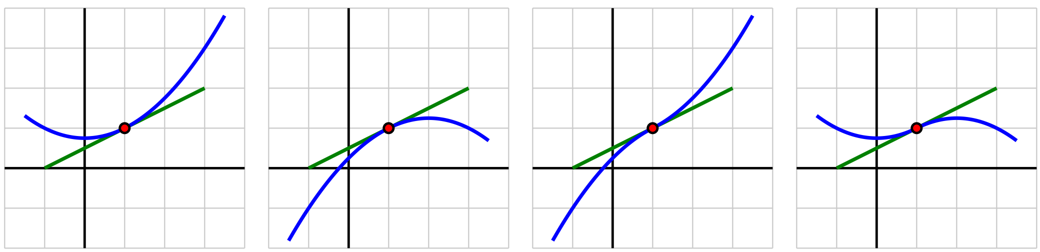


Figure 1.47: Four possible graphs for a nonlinear differentiable function and how it can be situated relative to its tangent line at a point.

These stem from the fact that there are three options for the value of the second derivative: either $f''(a) < 0$, $f''(a) = 0$, or $f''(a) > 0$. If $f''(a) > 0$, then we know the graph of f is concave up, and we see the first possibility on the left, where the tangent line lies entirely below the curve. If $f''(a) < 0$, then we find ourselves in the second situation (from left) where f is concave down and the tangent line lies above the curve. In the situation where $f''(a) = 0$ and f'' changes sign at $x = a$, the concavity of the graph will change, and we will see either the third or fourth option[9]. A fifth option (that is not very interesting) can occur, which is where the function f is linear, and so $f(x) = L(x)$ for all values of x.

The plots in Figure 1.47 highlight yet another important thing that we can learn from the concavity of the graph near the point of tangency: whether the tangent line lies above or below the curve itself. This is key because it tells us whether or not the tangent line approximation's values will be too large or too small in comparison to the true value of f. For instance, in the first situation in the leftmost plot in Figure 1.47 where $f''(a) > 0$, since the tangent line falls below the curve, we know that $L(x) \leq f(x)$ for all values of x near a.

We explore these ideas further in the following activity.

Activity 1.22.

This activity concerns a function $f(x)$ about which the following information is known:

[9]It is possible to have $f''(a) = 0$ and have f'' not change sign at $x = a$, in which case the graph will look like one of the first two options.

- f is a differentiable function defined at every real number x
- $f(2) = -1$
- $y = f'(x)$ has its graph given in Figure 1.48

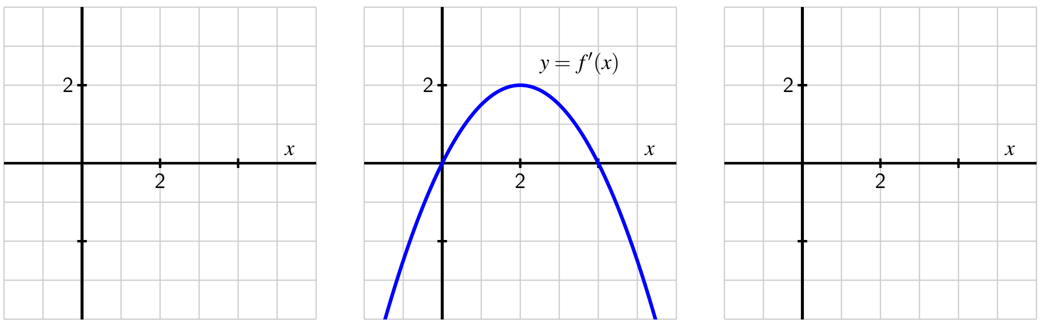


Figure 1.48: At center, a graph of $y = f'(x)$; at left, axes for plotting $y = f(x)$; at right, axes for plotting $y = f''(x)$.

Your task is to determine as much information as possible about f (especially near the value $a = 2$) by responding to the questions below.

(a) Find a formula for the tangent line approximation, $L(x)$, to f at the point $(2, -1)$.

(b) Use the tangent line approximation to estimate the value of $f(2.07)$. Show your work carefully and clearly.

(c) Sketch a graph of $y = f''(x)$ on the righthand grid in Figure 1.48; label it appropriately.

(d) Is the slope of the tangent line to $y = f(x)$ increasing, decreasing, or neither when $x = 2$? Explain.

(e) Sketch a possible graph of $y = f(x)$ near $x = 2$ on the lefthand grid in Figure 1.48. Include a sketch of $y = L(x)$ (found in part (a)). Explain how you know the graph of $y = f(x)$ looks like you have drawn it.

(f) Does your estimate in (b) over- or under-estimate the true value of $f(2.07)$? Why?

◁

The idea that a differentiable function looks linear and can be well-approximated by a linear function is an important one that finds wide application in calculus. For example, by approximating a function with its local linearization, it is possible to develop an effective algorithm to estimate the zeroes of a function. Local linearity also helps us to make further sense of certain challenging limits. For instance, we have seen that a limit such as

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

is indeterminate because both its numerator and denominator tend to 0. While there is no algebra that we can do to simplify $\frac{\sin(x)}{x}$, it is straightforward to show that the linearization of $f(x) = \sin(x)$ at the point $(0, 0)$ is given by $L(x) = x$. Hence, for values of x near 0, $\sin(x) \approx x$. As such, for values of x near 0,

$$\frac{\sin(x)}{x} \approx \frac{x}{x} = 1,$$

which makes plausible the fact that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

These ideas and other applications of local linearity will be explored later on in our work.

Summary

In this section, we encountered the following important ideas:

- The tangent line to a differentiable function $y = f(x)$ at the point $(a, f(a))$ is given in point-slope form by the equation

$$y - f(a) = f'(a)(x - a).$$

- The principle of local linearity tells us that if we zoom in on a point where a function $y = f(x)$ is differentiable, the function should become indistinguishable from its tangent line. That is, a differentiable function looks linear when viewed up close. We rename the tangent line to be the function $y = L(x)$ where $L(x) = f(a) + f'(a)(x - a)$ and note that $f(x) \approx L(x)$ for all x near $x = a$.

- If we know the tangent line approximation $L(x) = f(a) + f'(a)(x - a)$, then because $L(a) = f(a)$ and $L'(a) = f'(a)$, we also know both the value and the derivative of the function $y = f(x)$ at the point where $x = a$. In other words, the linear approximation tells us the height and slope of the original function. If, in addition, we know the value of $f''(a)$, we then know whether the tangent line lies above or below the graph of $y = f(x)$ depending on the concavity of f.

Exercises

1. A certain function $y = p(x)$ has its local linearization at $a = 3$ given by $L(x) = -2x + 5$.

(a) What are the values of $p(3)$ and $p'(3)$? Why?

(b) Estimate the value of $p(2.79)$.

(c) Suppose that $p''(3) = 0$ and you know that $p''(x) < 0$ for $x < 3$. Is your estimate in (b) too large or too small?

(d) Suppose that $p''(x) > 0$ for $x > 3$. Use this fact and the additional information above to sketch an accurate graph of $y = p(x)$ near $x = 3$. Include a sketch of $y = L(x)$ in your work.

2. A potato is placed in an oven, and the potato's temperature F (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time t is measured in minutes.

t	$F(t)$
0	70
15	180.5
30	251
45	296
60	324.5
75	342.8
90	354.5

(a) Use a central difference to estimate $F'(60)$. Use this estimate as needed in subsequent questions.

(b) Find the local linearization $y = L(t)$ to the function $y = F(t)$ at the point where $a = 60$.

(c) Determine an estimate for $F(63)$ by employing the local linearization.

(d) Do you think your estimate in (c) is too large or too small? Why?

3. An object moving along a straight line path has a differentiable position function $y = s(t)$; $s(t)$ measures the object's position relative to the origin at time t. It is known that at time $t = 9$ seconds, the object's position is $s(9) = 4$ feet (i.e., 4 feet to the right of the origin). Furthermore, the object's instantaneous velocity at $t = 9$ is -1.2 feet per second, and its acceleration at the same instant is 0.08 feet per second per second.

(a) Use local linearity to estimate the position of the object at $t = 9.34$.

(b) Is your estimate likely too large or too small? Why?

(c) In everyday language, describe the behavior of the moving object at $t = 9$. Is it moving toward the origin or away from it? Is its velocity increasing or decreasing?

4. For a certain function f, its derivative is known to be $f'(x) = (x-1)e^{-x^2}$. Note that you do not know a formula for $y = f(x)$.

(a) At what x-value(s) is $f'(x) = 0$? Justify your answer algebraically, but include a graph of f' to support your conclusion.

(b) Reasoning graphically, for what intervals of x-values is $f''(x) > 0$? What does this tell you about the behavior of the original function f? Explain.

(c) Assuming that $f(2) = -3$, estimate the value of $f(1.88)$ by finding and using the tangent line approximation to f at $x = 2$. Is your estimate larger or smaller than the true value of $f(1.88)$? Justify your answer.