Chapter 2

Computing Derivatives

2.1 Elementary derivative rules

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What are alternate notations for the derivative?
- How can we sometimes use the algebraic structure of a function $f(x)$ to easily compute a formula for $f'(x)$?
- What is the derivative of a power function of the form $f(x) = x^n$? What is the derivative of an exponential function of form $f(x) = a^x$?
- If we know the derivative of $y = f(x)$, how is the derivative of $y = kf(x)$ computed, where k is a constant?
- If we know the derivatives of $y = f(x)$ and $y = g(x)$, how is the derivative of $y = f(x) + g(x)$ computed?

Introduction

In Chapter 1, we developed the concept of the derivative of a function. We now know that the derivative f' of a function f measures the instantaneous rate of change of f with respect to x as well as the slope of the tangent line to $y = f(x)$ at any given value of x. To date, we have focused primarily on interpreting the derivative graphically or, in the context of functions in a physical setting, as a meaningful rate of change. To actually calculate the value of the derivative at a specific point, we have typically relied on the limit definition of the derivative.

In this present chapter, we will investigate how the limit definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

leads to interesting patterns and rules that enable us to quickly find a formula for $f'(x)$ based on the formula for $f(x)$ *without* using the limit definition directly. For example, we already know that if $f(x) = x$, then it follows that $f'(x) = 1$. While we could use the limit definition of the derivative to confirm this, we know it to be true because $f(x)$ is a linear function with slope 1 at every value of x. One of our goals is to be able to take standard functions, say ones such as $g(x) = 4x^7 - \sin(x) + 3e^x$, and, based on the algebraic form of the function, be able to apply shortcuts to almost immediately determine the formula for $g'(x)$.

Preview Activity 2.1. Functions of the form $f(x) = x^n$, where $n = 1, 2, 3, \ldots$, are often called *power functions*. The first two questions below revisit work we did earlier in Chapter 1, and the following questions extend those ideas to higher powers of x.

(a) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^2$.

(b) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^3$.

(c) Use the limit definition of the derivative to find $f'(x)$ for $f(x) = x^4$. (Hint: $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. Apply this rule to $(x+h)^4$ within the limit definition.)

(d) Based on your work in (a), (b), and (c), what do you conjecture is the derivative of $f(x) = x^5$? Of $f(x) = x^{13}$?

(e) Conjecture a formula for the derivative of $f(x) = x^n$ that holds for any positive integer n. That is, given $f(x) = x^n$ where n is a positive integer, what do you think is the formula for $f'(x)$?

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Some Key Notation

In addition to our usual f' notation for the derivative, there are other ways to symbolically denote the derivative of a function, as well as the instruction to take the derivative. We know that if we have a function, say $f(x) = x^2$, that we can denote its derivative by $f'(x)$, and we write $f'(x) = 2x$. Equivalently, if we are thinking more about the relationship between y and x, we sometimes denote the derivative of y with respect to x with the symbol

$$\frac{dy}{dx}$$

which we read "dee-y dee-x." This notation comes from the fact that the derivative is related to the slope of a line, and slope is measured by $\frac{\Delta y}{\Delta x}$. Note that while we read $\frac{\Delta y}{\Delta x}$ as "change in y over change in x," for the derivative symbol $\frac{dy}{dx}$, we view this is a single symbol, not a quotient of two quantities[1]. For example, if $y = x^2$, we'll write that the derivative is $\frac{dy}{dx} = 2x$.

Furthermore, we use a variant of $\frac{dy}{dx}$ notation to convey the instruction to take the derivative of a certain quantity with respect to a given variable. In particular, if we write

$$\frac{d}{dx}[\square]$$

this means "take the derivative of the quantity in $\square$ with respect to x." To continue our example above with the squaring function, here we may write $\frac{d}{dx}[x^2] = 2x$.

It is important to note that the independent variable can be different from x. If we have $f(z) = z^2$, we then write $f'(z) = 2z$. Similarly, if $y = t^2$, we can say $\frac{dy}{dt} = 2t$. And changing the variable and derivative notation once more, it is also true that $\frac{d}{dq}[q^2] = 2q$. This notation may also be applied to second derivatives: $f''(z) = \frac{d}{dz}\left[\frac{df}{dz}\right] = \frac{d^2 f}{dz^2}$.

In what follows, we'll be working to widely expand our repertoire of functions for which we can quickly compute the corresponding derivative formula

Constant, Power, and Exponential Functions

So far, we know the derivative formula for two important classes of functions: constant functions and power functions. For the first kind, observe that if $f(x) = c$ is a constant function, then its graph is a horizontal line with slope zero at every point. Thus, $\frac{d}{dx}[c] = 0$. We summarize this with the following rule.

Constant Functions: For any real number c, if $f(x) = c$, then $f'(x) = 0$.

Thus, if $f(x) = 7$, then $f'(x) = 0$. Similarly, $\frac{d}{dx}[\sqrt{3}] = 0$.

For power functions, from your work in Preview Activity 2.1, you have conjectured that for any positive integer n, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$. Not only can this rule be formally proved to hold for any positive integer n, but also for any nonzero real number (positive or negative).

Power Functions: For any nonzero real number, if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

This rule for power functions allows us to find derivatives such as the following: if $g(z) = z^{-3}$, then $g'(z) = -3z^{-4}$. Similarly, if $h(t) = t^{7/5}$, then $\frac{dh}{dt} = \frac{7}{5}t^{2/5}$; likewise, $\frac{d}{dq}[q^{\pi}] = \pi q^{\pi-1}$.

[1]That is, we do *not* say "dee-y over dee-x."

As we next turn to thinking about derivatives of combinations of basic functions, it will be instructive to have one more type of basic function whose derivative formula we know. For now, we simply state this rule without explanation or justification; we will explore why this rule is true in one of the exercises at the end of this section, plus we will encounter graphical reasoning for why the rule is plausible in Preview Activity 2.2.

Exponential Functions: For any positive real number a, if $f(x) = a^x$, then $f'(x) = a^x \ln(a)$.

For instance, this rule tells us that if $f(x) = 2^x$, then $f'(x) = 2^x \ln(2)$. Similarly, for $p(t) = 10^t$, $p'(t) = 10^t \ln(10)$. It is especially important to note that when $a = e$, where e is the base of the natural logarithm function, we have that

$$\frac{d}{dx}[e^x] = e^x \ln(e) = e^x$$

since $\ln(e) = 1$. This is an extremely important property of the function e^x: its derivative function is itself!

Finally, note carefully the distinction between power functions and exponential functions: in power functions, the variable is in the base, as in x^2, while in exponential functions, the variable is in the power, as in 2^x. As we can see from the rules, this makes a big difference in the form of the derivative.

The following activity will check your understanding of the derivatives of the three basic types of functions noted above.

Activity 2.1.

Use the three rules above to determine the derivative of each of the following functions. For each, state your answer using full and proper notation, labeling the derivative with its name. For example, if you are given a function $h(z)$, you should write "$h'(z) =$" or "$\frac{dh}{dz} =$" as part of your response.

(a) $f(t) = \pi$

(b) $g(z) = 7^z$

(c) $h(w) = w^{3/4}$

(d) $p(x) = 3^{1/2}$

(e) $r(t) = (\sqrt{2})^t$

(f) $\frac{d}{dq}[q^{-1}]$

(g) $m(t) = \frac{1}{t^3}$

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Constant Multiples and Sums of Functions

Of course, most of the functions we encounter in mathematics are more complicated than being simply constant, a power of a variable, or a base raised to a variable power. In this section and several following, we will learn how to quickly compute the derivative of a function constructed as an algebraic combination of basic functions. For instance, we'd like to be able to understand how to take the derivative of a polynomial function such as $p(t) = 3t^5 - 7t^4 + t^2 - 9$, which is a function made up of constant multiples and sums of powers of t. To that end, we develop two new rules: the Constant Multiple Rule and the Sum Rule.

Say we have a function $y = f(x)$ whose derivative formula is known. How is the derivative of $y = kf(x)$ related to the derivative of the original function? Recall that when we multiply a function by a constant k, we vertically stretch the graph by a factor of $|k|$ (and reflect the graph across $y = 0$ if $k < 0$). This vertical stretch affects the slope of the graph, making the slope of the function $y = kf(x)$ be k times as steep as the slope of $y = f(x)$. In terms of the derivative, this is essentially saying that when we multiply a function by a factor of k, we change the value of its derivative by a factor of k as well. Thus[2], the Constant Multiple Rule holds:

> **The Constant Multiple Rule**: For any real number k, if $f(x)$ is a differentiable function with derivative $f'(x)$, then $\frac{d}{dx}[kf(x)] = kf'(x)$.

In words, this rule says that "the derivative of a constant times a function is the constant times the derivative of the function." For example, if $g(t) = 3 \cdot 5^t$, we have $g'(t) = 3 \cdot 5^t \ln(5)$. Similarly, $\frac{d}{dz}[5z^{-2}] = 5(-2z^{-3})$.

Next we examine what happens when we take a sum of two functions. If we have $y = f(x)$ and $y = g(x)$, we can compute a new function $y = (f + g)(x)$ by adding the outputs of the two functions: $(f + g)(x) = f(x) + g(x)$. Not only does this result in the value of the new function being the sum of the values of the two known functions, but also the slope of the new function is the sum of the slopes of the known functions. Therefore[3], we arrive at the following Sum Rule for derivatives:

> **The Sum Rule**: If $f(x)$ and $g(x)$ are differentiable functions with derivatives $f'(x)$ and $g'(x)$ respectively, then $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$.

In words, the Sum Rule tells us that "the derivative of a sum is the sum of the derivatives." It also tells us that any time we take a sum of two differentiable functions, the result must also be differentiable. Furthermore, because we can view the difference function $y = (f - g)(x) = f(x) - g(x)$ as $y = f(x) + (-1 \cdot g(x))$, the Sum Rule and Constant Multiple

[2]The Constant Multiple Rule can be formally proved as a consequence of properties of limits, using the limit definition of the derivative.

[3]Like the Constant Multiple Rule, the Sum Rule can be formally proved as a consequence of properties of limits, using the limit definition of the derivative.

Rules together tell us that $\frac{d}{dx}[f(x) + (-1 \cdot g(x))] = f'(x) - g'(x)$, or that "the derivative of a difference is the difference of the derivatives." Hence we can now compute derivatives of sums and differences of elementary functions. For instance, $\frac{d}{dw}(2^w + w^2) = 2^w \ln(2) + 2w$, and if $h(q) = 3q^6 - 4q^{-3}$, then $h'(q) = 3(6q^5) - 4(-3q^{-4}) = 18q^5 + 12q^{-4}$.

Activity 2.2.

Use only the rules for constant, power, and exponential functions, together with the Constant Multiple and Sum Rules, to compute the derivative of each function below with respect to the given independent variable. Note well that we do not yet know any rules for how to differentiate the product or quotient of functions. This means that you may have to do some algebra first on the functions below before you can actually use existing rules to compute the desired derivative formula. In each case, label the derivative you calculate with its name using proper notation such as $f'(x)$, $h'(z)$, dr/dt, etc.

(a) $f(x) = x^{5/3} - x^4 + 2^x$

(b) $g(x) = 14e^x + 3x^5 - x$

(c) $h(z) = \sqrt{z} + \frac{1}{z^4} + 5^z$

(d) $r(t) = \sqrt{53}\,t^7 - \pi e^t + e^4$

(e) $s(y) = (y^2 + 1)(y^2 - 1)$

(f) $q(x) = \dfrac{x^3 - x + 2}{x}$

(g) $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$

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In the same way that we have shortcut rules to help us find derivatives, we introduce some language that is simpler and shorter. Often, rather than say "take the derivative of f," we'll instead say simply "differentiate f." This phrasing is tied to the notion of having a derivative to begin with: if the derivative exists at a point, we say "f is differentiable," which is tied to the fact that f can be differentiated.

As we work more and more with the algebraic structure of functions, it is important to strive to develop a big picture view of what we are doing. Here, we can note several general observations based on the rules we have so far. One is that the derivative of any polynomial function will be another polynomial function, and that the degree of the derivative is one less than the degree of the original function. For instance, if $p(t) = 7t^5 - 4t^3 + 8t$, p is a degree 5 polynomial, and its derivative, $p'(t) = 35t^4 - 12t^2 + 8$, is a degree 4 polynomial. Additionally, the derivative of any exponential function is another exponential function: for example, if $g(z) = 7 \cdot 2^z$, then $g'(z) = 7 \cdot 2^z \ln(2)$, which is also exponential.

Furthermore, while our current emphasis is on learning shortcut rules for finding derivatives without directly using the limit definition, we should be certain not to lose sight of the fact that all of the meaning of the derivative still holds that we developed in Chapter 1. That is, anytime we compute a derivative, that derivative measures the instantaneous rate of change of the original function, as well as the slope of the tangent line at any selected point on the curve. The following activity asks you to combine the just-developed derivative rules with some key perspectives that we studied in Chapter 1.

Activity 2.3.

Each of the following questions asks you to use derivatives to answer key questions about functions. Be sure to think carefully about each question and to use proper notation in your responses.

(a) Find the slope of the tangent line to $h(z) = \sqrt{z} + \frac{1}{z}$ at the point where $z = 4$.

(b) A population of cells is growing in such a way that its total number in millions is given by the function $P(t) = 2(1.37)^t + 32$, where t is measured in days.

i. Determine the instantaneous rate at which the population is growing on day 4, and include units on your answer.

ii. Is the population growing at an increasing rate or growing at a decreasing rate on day 4? Explain.

(c) Find an equation for the tangent line to the curve $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$ at the point where $a = -1$.

(d) What is the difference between being asked to find the *slope* of the tangent line (asked in (a)) and the *equation* of the tangent line (asked in (c))?

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Summary

In this section, we encountered the following important ideas:

- Given a differentiable function $y = f(x)$, we can express the derivative of f in several different notations: $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, and $\frac{d}{dx}[f(x)]$.

- The limit definition of the derivative leads to patterns among certain families of functions that enable us to compute derivative formulas without resorting directly to the limit definition. For example, if f is a power function of the form $f(x) = x^n$, then $f'(x) = nx^{n-1}$ for any real number n other than 0. This is called the Rule for Power Functions.

- We have stated a rule for derivatives of exponential functions in the same spirit as the rule for power functions: for any positive real number a, if $f(x) = a^x$, then $f'(x) = a^x \ln(a)$.

- If we are given a constant multiple of a function whose derivative we know, or a sum of functions whose derivatives we know, the Constant Multiple and Sum Rules make it straightforward to compute the derivative of the overall function. More formally, if $f(x)$ and $g(x)$ are differentiable with derivatives $f'(x)$ and $g'(x)$ and a and b are constants, then

$$\frac{d}{dx}\left[af(x) + bg(x)\right] = af'(x) + bg'(x).$$

Exercises

1. Let f and g be differentiable functions for which the following information is known: $f(2) = 5$, $g(2) = -3$, $f'(2) = -1/2$, $g'(2) = 2$.

(a) Let h be the new function defined by the rule $h(x) = 3f(x) - 4g(x)$. Determine $h(2)$ and $h'(2)$.

(b) Find an equation for the tangent line to $y = h(x)$ at the point $(2, h(2))$.

(c) Let p be the function defined by the rule $p(x) = -2f(x) + \frac{1}{2}g(x)$. Is p increasing, decreasing, or neither at $a = 2$? Why?

(d) Estimate the value of $p(2.03)$ by using the local linearization of p at the point $(2, p(2))$.

2. Let functions p and q be the piecewise linear functions given by their respective graphs in Figure 2.1. Use the graphs to answer the following questions.

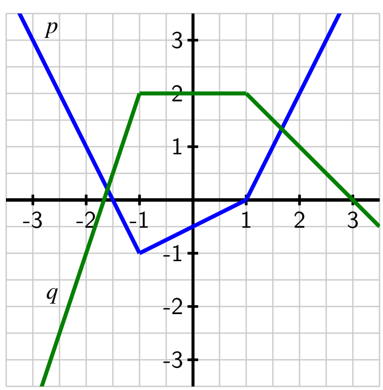

Figure 2.1: The graphs of p (in blue) and q (in green).

(a) At what values of x is p not differentiable? At what values of x is q not differentiable? Why?

(b) Let $r(x) = p(x) + 2q(x)$. At what values of x is r not differentiable? Why?

(c) Determine $r'(-2)$ and $r'(0)$.

(d) Find an equation for the tangent line to $y = r(x)$ at the point $(2, r(2))$.

3. Consider the functions $r(t) = t^t$ and $s(t) = \arccos(t)$, for which you are given the facts that $r'(t) = t^t(\ln(t) + 1)$ and $s'(t) = -\frac{1}{\sqrt{1-t^2}}$. Do not be concerned with where these derivative formulas come from. We restrict our interest in both functions to the domain $0 < t < 1$.

(a) Let $w(t) = 3t^t - 2\arccos(t)$. Determine $w'(t)$.

(b) Find an equation for the tangent line to $y = w(t)$ at the point $(\frac{1}{2}, w(\frac{1}{2}))$.

(c) Let $v(t) = t^t + \arccos(t)$. Is v increasing or decreasing at the instant $t = \frac{1}{2}$? Why?

4. Let $f(x) = a^x$. The goal of this problem is to explore how the value of a affects the derivative of $f(x)$, without assuming we know the rule for $\frac{d}{dx}[a^x]$ that we have stated and used in earlier work in this section.

(a) Use the limit definition of the derivative to show that

$$f'(x) = \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}.$$

(b) Explain why it is also true that

$$f'(x) = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}.$$

(c) Use computing technology and small values of h to estimate the value of

$$L = \lim_{h \to 0} \frac{a^h - 1}{h}$$

when $a = 2$. Do likewise when $a = 3$.

(d) Note that it would be ideal if the value of the limit L was 1, for then f would be a particularly special function: its derivative would be simply a^x, which would mean that its derivative is itself. By experimenting with different values of a between 2 and 3, try to find a value for a for which

$$L = \lim_{h \to 0} \frac{a^h - 1}{h} = 1.$$

(e) Compute $\ln(2)$ and $\ln(3)$. What does your work in (b) and (c) suggest is true about $\frac{d}{dx}[2^x]$ and $\frac{d}{dx}[3^x]$?

(f) How do your investigations in (d) lead to a particularly important fact about the function $f(x) = e^x$?

2.2 The sine and cosine functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a graphical justification for why $\frac{d}{dx}[a^x] = a^x \ln(a)$?
- What do the graphs of $y = \sin(x)$ and $y = \cos(x)$ suggest as formulas for their respective derivatives?
- Once we know the derivatives of $\sin(x)$ and $\cos(x)$, how do previous derivative rules work when these functions are involved?

Introduction

Throughout Chapter 2, we will be working to develop shortcut derivative rules that will help us to bypass the limit definition of the derivative in order to quickly determine the formula for $f'(x)$ when we are given a formula for $f(x)$. In Section 2.1, we learned the rule for power functions, that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$, and justified this in part due to results from different n-values when applying the limit definition of the derivative. We also stated the rule for exponential functions, that if a is a positive real number and $f(x) = a^x$, then $f'(x) = a^x \ln(a)$. Later in this present section, we are going to work to conjecture formulas for the sine and cosine functions, primarily through a graphical argument. To help set the stage for doing so, the following preview activity asks you to think about exponential functions and why it is reasonable to think that the derivative of an exponential function is a constant times the exponential function itself.

Preview Activity 2.2. Consider the function $g(x) = 2^x$, which is graphed in Figure 2.2.

(a) At each of $x = -2, -1, 0, 1, 2$, use a straightedge to sketch an accurate tangent line to $y = g(x)$.

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point in (a).

(c) Use the limit definition of the derivative to estimate $g'(0)$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = g(x)$ at $x = 0$ in (b).

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of $y = g'(x)$ on the axes adjacent to the graph of $y = g(x)$.

(e) Write at least one sentence that explains why it is reasonable to think that $g'(x) = cg(x)$, where c is a constant. In addition, calculate $\ln(2)$, and then discuss how this value, combined with your work above, reasonably suggests that $g'(x) = 2^x \ln(2)$.

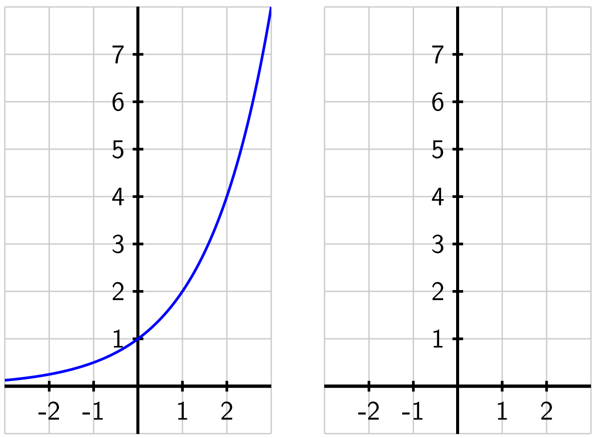

Figure 2.2: At left, the graph of $y = g(x) = 2^x$. At right, axes for plotting $y = g'(x)$.

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The sine and cosine functions

The sine and cosine functions are among the most important functions in all of mathematics. Sometimes called the *circular* functions due to their genesis in the unit circle, these periodic functions play a key role in modeling repeating phenomena such as the location of a point on a bicycle tire, the behavior of an oscillating mass attached to a spring, tidal elevations, and more. Like polynomial and exponential functions, the sine and cosine functions are considered basic functions, ones that are often used in the building of more complicated functions. As such, we would like to know formulas for $\frac{d}{dx}[\sin(x)]$ and $\frac{d}{dx}[\cos(x)]$, and the next two activities lead us to that end.

Activity 2.4.

Consider the function $f(x) = \sin(x)$, which is graphed in Figure 2.3 below. Note carefully that the grid in the diagram does not have boxes that are 1×1, but rather approximately 1.57×1, as the horizontal scale of the grid is $\pi/2$ units per box.

(a) At each of $x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, use a straightedge to sketch an accurate tangent line to $y = f(x)$.

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point. Pay careful attention to the scale of the grid.

(c) Use the limit definition of the derivative to estimate $f'(0)$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = f(x)$ at $x = 0$ in (b). Using periodicity, what does this result suggest about $f'(2\pi)$? about $f'(-2\pi)$?

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of $y = f'(x)$ on the axes adjacent to the graph of $y = f(x)$.

(e) What familiar function do you think is the derivative of $f(x) = \sin(x)$?

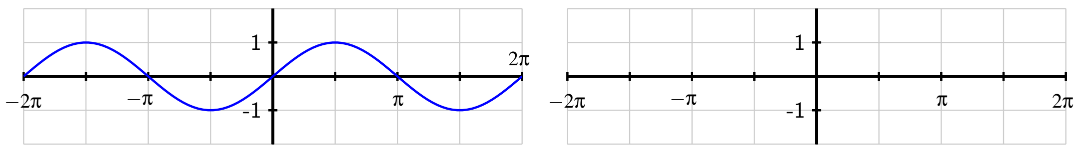

Figure 2.3: At left, the graph of $y = f(x) = \sin(x)$.

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Activity 2.5.

Consider the function $g(x) = \cos(x)$, which is graphed in Figure 2.4 below. Note carefully that the grid in the diagram does not have boxes that are 1×1, but rather approximately 1.57×1, as the horizontal scale of the grid is $\pi/2$ units per box.

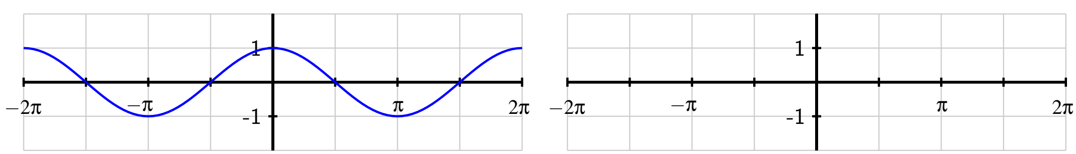

Figure 2.4: At left, the graph of $y = g(x) = \cos(x)$.

(a) At each of $x = -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, use a straightedge to sketch an accurate tangent line to $y = g(x)$.

(b) Use the provided grid to estimate the slope of the tangent line you drew at each point. Again, note the scale of the axes and grid.

(c) Use the limit definition of the derivative to estimate $g'(\frac{\pi}{2})$ by using small values of h, and compare the result to your visual estimate for the slope of the tangent line to $y = g(x)$ at $x = \frac{\pi}{2}$ in (b). Using periodicity, what does this result suggest about $g'(-\frac{3\pi}{2})$? can symmetry on the graph help you estimate other slopes easily?

(d) Based on your work in (a), (b), and (c), sketch an accurate graph of $y = g'(x)$ on the axes adjacent to the graph of $y = g(x)$.

(e) What familiar function do you think is the derivative of $g(x) = \cos(x)$?

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The results of the two preceding activities suggest that the sine and cosine functions not only have the beautiful interrelationships that are learned in a course in trigonometry – connections such as the identities $\sin^2(x) + \cos^2(x) = 1$ and $\cos(x - \frac{\pi}{2}) = \sin(x)$ – but that they are even further linked through calculus, as the derivative of each involves the other. The following rules summarize the results of the activities[4].

> **Sine and Cosine Functions**: For all real numbers x,
>
> $$\frac{d}{dx}[\sin(x)] = \cos(x) \text{ and } \frac{d}{dx}[\cos(x)] = -\sin(x)$$

We have now added two additional functions to our library of basic functions whose derivatives we know: power functions, exponential functions, and the sine and cosine functions. The constant multiple and sum rules still hold, of course, and all of the inherent meaning of the derivative persists, regardless of the functions that are used to constitute a given choice of $f(x)$. The following activity puts our new knowledge of the derivatives of $\sin(x)$ and $\cos(x)$ to work.

Activity 2.6.

Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

(a) Determine the derivative of $h(t) = 3\cos(t) - 4\sin(t)$.

(b) Find the exact slope of the tangent line to $y = f(x) = 2x + \frac{\sin(x)}{2}$ at the point where $x = \frac{\pi}{6}$.

(c) Find the equation of the tangent line to $y = g(x) = x^2 + 2\cos(x)$ at the point where $x = \frac{\pi}{2}$.

(d) Determine the derivative of $p(z) = z^4 + 4^z + 4\cos(z) - \sin(\frac{\pi}{2})$.

(e) The function $P(t) = 24 + 8\sin(t)$ represents a population of a particular kind of animal that lives on a small island, where P is measured in hundreds and t is measured in decades since January 1, 2010. What is the instantaneous rate of change of P on January 1, 2030? What are the units of this quantity? Write a sentence in everyday language that explains how the population is behaving at this point in time.

[4]These two rules may be formally proved using the limit definition of the derivative and the expansion identities for $\sin(x + h)$ and $\cos(x + h)$.

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Summary

In this section, we encountered the following important ideas:

- If we consider the graph of an exponential function $f(x) = a^x$ (where $a > 1$), the graph of $f'(x)$ behaves similarly, appearing exponential and as a possibly scaled version of the original function a^x. For $f(x) = 2^x$, careful analysis of the graph and its slopes suggests that $\frac{d}{dx}[2^x] = 2^x \ln(2)$, which is a special case of the rule we stated in Section 2.1.
- By carefully analyzing the graphs of $y = \sin(x)$ and $y = \cos(x)$, plus using the limit definition of the derivative at select points, we found that $\frac{d}{dx}[\sin(x)] = \cos(x)$ and $\frac{d}{dx}[\cos(x)] = -\sin(x)$.
- We note that all previously encountered derivative rules still hold, but now may also be applied to functions involving the sine and cosine, plus all of the established meaning of the derivative applies to these trigonometric functions as well.

Exercises

1. Suppose that $V(t) = 24 \cdot 1.07^t + 6\sin(t)$ represents the value of a person's investment portfolio in thousands of dollars in year t, where $t = 0$ corresponds to January 1, 2010.

(a) At what instantaneous rate is the portfolio's value changing on January 1, 2012? Include units on your answer.

(b) Determine the value of $V''(2)$. What are the units on this quantity and what does it tell you about how the portfolio's value is changing?

(c) On the interval $0 \le t \le 20$, graph the function $V(t) = 24 \cdot 1.07^t + 6\sin(t)$ and describe its behavior in the context of the problem. Then, compare the graphs of the functions $A(t) = 24 \cdot 1.07^t$ and $V(t) = 24 \cdot 1.07^t + 6\sin(t)$, as well as the graphs of their derivatives $A'(t)$ and $V'(t)$. What is the impact of the term $6\sin(t)$ on the behavior of the function $V(t)$?

2. Let $f(x) = 3\cos(x) - 2\sin(x) + 6$.

(a) Determine the exact slope of the tangent line to $y = f(x)$ at the point where $a = \frac{\pi}{4}$.

(b) Determine the tangent line approximation to $y = f(x)$ at the point where $a = \pi$.

(c) At the point where $a = \frac{\pi}{2}$, is f increasing, decreasing, or neither?

(d) At the point where $a = \frac{3\pi}{2}$, does the tangent line to $y = f(x)$ lie above the curve, below the curve, or neither? How can you answer this question without even graphing the function or the tangent line?

3. In this exercise, we explore how the limit definition of the derivative more formally shows that $\frac{d}{dx}[\sin(x)] = \cos(x)$. Letting $f(x) = \sin(x)$, note that the limit definition of the derivative tells us that

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

(a) Recall the trigonometric identity for the sine of a sum of angles α and β: $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. Use this identity and some algebra to show that

$$f'(x) = \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h}.$$

(b) Next, note that as h changes, x remains constant. Explain why it therefore makes sense to say that

$$f'(x) = \sin(x) \cdot \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h}.$$

(c) Finally, use small values of h to estimate the values of the two limits in (c):

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{\sin(h)}{h}.$$

(d) What do your results in (c) thus tell you about $f'(x)$?

(e) By emulating the steps taken above, use the limit definition of the derivative to argue convincingly that $\frac{d}{dx}[\cos(x)] = -\sin(x)$.

2.3 The product and quotient rules

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How does the algebraic structure of a function direct us in computing its derivative using shortcut rules?
- How do we compute the derivative of a product of two basic functions in terms of the derivatives of the basic functions?
- How do we compute the derivative of a quotient of two basic functions in terms of the derivatives of the basic functions?
- How do the product and quotient rules combine with the sum and constant multiple rules to expand the library of functions we can quickly differentiate?

Introduction

So far, the basic functions we know how to differentiate include power functions (x^n), exponential functions (a^x), and the two fundamental trigonometric functions ($\sin(x)$ and $\cos(x)$). With the sum rule and constant multiple rules, we can also compute the derivative of combined functions such as

$$f(x) = 7x^{11} - 4 \cdot 9^x + \pi \sin(x) - \sqrt{3}\cos(x),$$

because the function f is fundamentally a sum of basic functions. Indeed, we can now quickly say that $f'(x) = 77x^{10} - 4 \cdot 9^x \ln(9) + \pi \cos(x) + \sqrt{3}\sin(x)$.

But we can of course combine basic functions in ways other than multiplying them by constants and taking sums and differences. For example, we could consider the function that results from a product of two basic functions, such as

$$p(z) = z^3 \cos(z),$$

or another that is generated by the quotient of two basic functions, one like

$$q(t) = \frac{\sin(t)}{2^t}.$$

While the derivative of a sum is the sum of the derivatives, it turns out that the rules for computing derivatives of products and quotients are more complicated. In what follows we explore why this is the case, what the product and quotient rules actually say, and work to expand our repertoire of functions we can easily differentiate. To start, Preview Activity 2.3

asks you to investigate the derivative of a product and quotient of two polynomials.

Preview Activity 2.3. Let f and g be the functions defined by $f(t) = 2t^2$ and $g(t) = t^3 + 4t$.

(a) Determine $f'(t)$ and $g'(t)$.

(b) Let $p(t) = 2t^2(t^3 + 4t)$ and observe that $p(t) = f(t) \cdot g(t)$. Rewrite the formula for p by distributing the $2t^2$ term. Then, compute $p'(t)$ using the sum and constant multiple rules.

(c) True or false: $p'(t) = f'(t) \cdot g'(t)$.

(d) Let $q(t) = \dfrac{t^3 + 4t}{2t^2}$ and observe that $q(t) = \dfrac{g(t)}{f(t)}$. Rewrite the formula for q by dividing each term in the numerator by the denominator and simplify to write q as a sum of constant multiples of powers of t. Then, compute $q'(t)$ using the sum and constant multiple rules.

(e) True or false: $q'(t) = \dfrac{g'(t)}{f'(t)}$.

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The product rule

As parts (b) and (d) of Preview Activity 2.3 show, it is not true in general that the derivative of a product of two functions is the product of the derivatives of those functions. Indeed, the rule for differentiating a function of the form $p(x) = f(x) \cdot g(x)$ in terms of the derivatives of f and g is more complicated than simply taking the product of the derivatives of f and g. To see further why this is the case, as well as to begin to understand how the product rule actually works, we consider an example involving meaningful functions.

Say that an investor is regularly purchasing stock in a particular company. Let $N(t)$ be a function that represents the number of shares owned on day t, where $t = 0$ represents the first day on which shares were purchased. Further, let $S(t)$ be a function that gives the value of one share of the stock on day t; note that the units on $S(t)$ are dollars per share. Moreover, to compute the total value on day t of the stock held by the investor, we use the function $V(t) = N(t) \cdot S(t)$. By taking the product

$$V(t) = N(t) \text{ shares} \cdot S(t) \text{ dollars per share},$$

we have the total value in dollars of the shares held. Observe that over time, both the number of shares and the value of a given share will vary. The derivative $N'(t)$ measures the rate at which the number of shares held is changing, while $S'(t)$ measures the rate at which the value per share is changing. The big question we'd like to answer is: how do these respective rates of change affect the rate of change of the total value function?

To help better understand the relationship among changes in N, S, and V, let's consider some specific data. Suppose that on day 100, the investor owns 520 shares of stock and the stock's current value is \$27.50 per share. This tells us that $N(100) = 520$ and $S(100) = 27.50$. In addition, say that on day 100, the investor purchases an additional 12 shares (so the number of shares held is rising at a rate of 12 shares per day), and that on that same day the price of the stock is rising at a rate of 0.75 dollars per share per day. Viewed in calculus notation, this tells us that $N'(100) = 12$ (shares per day) and $S'(100) = 0.75$ (dollars per share per day). At what rate is the value of the investor's total holdings changing on day 100?

Observe that the increase in total value comes from two sources: the growing number of shares, and the rising value of each share. If only the number of shares is rising (and the value of each share is constant), the rate at which which total value would rise is found by computing the product of the current value of the shares with the rate at which the number of shares is changing. That is, the rate at which total value would change is given by

$$S(100) \cdot N'(100) = 27.50 \, \frac{\text{dollars}}{\text{share}} \cdot 12 \, \frac{\text{shares}}{\text{day}} = 330 \, \frac{\text{dollars}}{\text{day}}.$$

Note particularly how the units make sense and explain that we are finding the rate at which the total value V is changing, measured in dollars per day. If instead the number of shares is constant, but the value of each share is rising, then the rate at which the total value would rise is found similarly by taking the product of the number of shares with the rate of change of share value. In particular, the rate total value is rising is

$$N(100) \cdot S'(100) = 520 \, \text{shares} \cdot 0.75 \, \frac{\text{dollars per share}}{\text{day}} = 390 \, \frac{\text{dollars}}{\text{day}}.$$

Of course, when both the number of shares is changing and the value of each share is changing, we have to include both of these sources, and hence the rate at which the total value is rising is

$$V'(100) = S(100) \cdot N'(100) + N(100) \cdot S'(100) = 330 + 390 = 720 \, \frac{\text{dollars}}{\text{day}}.$$

This tells us that we expect the total value of the investor's holdings to rise by about \$720 on the 100th day.[5]

[5]While this example highlights why the product rule is true, there are some subtle issues to recognize. For one, if the stock's value really does rise exactly \$0.75 on day 100, and the number of shares really rises by 12 on day 100, then we'd expect that $V(101) = N(101) \cdot S(101) = 532 \cdot 28.25 = 15029$. If, as noted above, we expect the total value to rise by \$720, then with $V(100) = N(100) \cdot S(100) = 520 \cdot 27.50 = 14300$, then it seems like we should find that $V(101) = V(100) + 720 = 15020$. Why do the two results differ by 9? One way to understand why this difference occurs is to recognize that $N'(100) = 12$ represents an ***instantaneous*** rate of

Next, we expand our perspective from the specific example above to the more general and abstract setting of a product p of two differentiable functions, f and g. If we have $P(x) = f(x) \cdot g(x)$, our work above suggests that $P'(x) = f(x)g'(x) + g(x)f'(x)$. Indeed, a formal proof using the limit definition of the derivative can be given to show that the following rule, called the *product rule*, holds in general.

Product Rule: If f and g are differentiable functions, then their product $P(x) = f(x) \cdot g(x)$ is also a differentiable function, and

$$P'(x) = f(x)g'(x) + g(x)f'(x).$$

In light of the earlier example involving shares of stock, the product rule also makes sense intuitively: the rate of change of P should take into account both how fast f and g are changing, as well as how large f and g are at the point of interest. Furthermore, we note in words what the product rule says: if P is the product of two functions f (the first function) and g (the second), then "the derivative of P is the first times the derivative of the second, plus the second times the derivative of the first." It is often a helpful mental exercise to say this phrasing aloud when executing the product rule.

For example, if $P(z) = z^3 \cdot \cos(z)$, we can now use the product rule to differentiate P. The first function is z^3 and the second function is $\cos(z)$. By the product rule, P' will be given by the first, z^3, times the derivative of the second, $-\sin(z)$, plus the second, $\cos(z)$, times the derivative of the first, $3z^2$. That is,

$$P'(z) = z^3(-\sin(z)) + \cos(z)3z^2 = -z^3\sin(z) + 3z^2\cos(z).$$

The following activity further explores the use of the product rule.

Activity 2.7.

Use the product rule to answer each of the questions below. Throughout, be sure to carefully label any derivative you find by name. It is not necessary to algebraically simplify any of the derivatives you compute.

(a) Let $m(w) = 3w^{17}4^w$. Find $m'(w)$.

(b) Let $h(t) = (\sin(t) + \cos(t))t^4$. Find $h'(t)$.

(c) Determine the slope of the tangent line to the curve $y = f(x)$ at the point where $a = 1$ if f is given by the rule $f(x) = e^x \sin(x)$.

(d) Find the tangent line approximation $L(x)$ to the function $y = g(x)$ at the point where $a = -1$ if g is given by the rule $g(x) = (x^2 + x)2^x$.

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change, while our (informal) discussion has also thought of this number as the total change in the number of shares over the course of a single day. The formal proof of the product rule reconciles this issue by taking the limit as the change in the input tends to zero.

The quotient rule

Because quotients and products are closely linked, we can use the product rule to understand how to take the derivative of a quotient. In particular, let $Q(x)$ be defined by $Q(x) = f(x)/g(x)$, where f and g are both differentiable functions. We desire a formula for Q' in terms of f, g, f', and g'. It turns out that Q is differentiable everywhere that $g(x) \neq 0$. Moreover, taking the formula $Q = f/g$ and multiplying both sides by g, we can observe that

$$f(x) = Q(x) \cdot g(x).$$

Thus, we can use the product rule to differentiate f. Doing so,

$$f'(x) = Q(x)g'(x) + g(x)Q'(x).$$

Since we want to know a formula for Q', we work to solve this most recent equation for $Q'(x)$, finding first that

$$Q'(x)g(x) = f'(x) - Q(x)g'(x).$$

Dividing both sides by $g(x)$, we have

$$Q'(x) = \frac{f'(x) - Q(x)g'(x)}{g(x)}.$$

Finally, we also recall that $Q(x) = \frac{f(x)}{g(x)}$. Using this expression in the preceding equation and simplifying, we have

$$\begin{aligned} Q'(x) &= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \\ &= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} \cdot \frac{g(x)}{g(x)} \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}. \end{aligned}$$

This shows the fundamental argument for why the ***quotient rule*** holds.

Quotient Rule: If f and g are differentiable functions, then their quotient $Q(x) = \frac{f(x)}{g(x)}$ is also a differentiable function for all x where $g(x) \neq 0$, and

$$Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

Like the product rule, it can be helpful to think of the quotient rule verbally. If a function Q is the quotient of a top function f and a bottom function g, then Q' is given by “the bottom times the derivative of the top, minus the top times the derivative of the bottom, all over the bottom squared." For example, if $Q(t) = \sin(t)/2^t$, then we can identify the top function as $\sin(t)$ and the bottom function as 2^t. By the quotient rule, we then have that Q' will be given by the bottom, 2^t, times the derivative of the top, $\cos(t)$, minus the top, $\sin(t)$, times the derivative of the bottom, $2^t \ln(2)$, all over the bottom squared, $(2^t)^2$. That is,

$$Q'(t) = \frac{2^t \cos(t) - \sin(t) 2^t \ln(2)}{(2^t)^2}.$$

In this particular example, it is possible to simplify $Q'(t)$ by removing a factor of 2^t from both the numerator and denominator, hence finding that

$$Q'(t) = \frac{\cos(t) - \sin(t)\ln(2)}{2^t}.$$

In general, we must be careful in doing any such simplification, as we don't want to correctly execute the quotient rule but then find an incorrect overall derivative due to an algebra error. As such, we will often place more emphasis on correctly using derivative rules than we will on simplifying the result that follows. The next activity further explores the use of the quotient rule.

Activity 2.8.

Use the quotient rule to answer each of the questions below. Throughout, be sure to carefully label any derivative you find by name. That is, if you're given a formula for $f(x)$, clearly label the formula you find for $f'(x)$. It is not necessary to algebraically simplify any of the derivatives you compute.

(a) Let $r(z) = \dfrac{3^z}{z^4 + 1}$. Find $r'(z)$.

(b) Let $v(t) = \dfrac{\sin(t)}{\cos(t) + t^2}$. Find $v'(t)$.

(c) Determine the slope of the tangent line to the curve $R(x) = \dfrac{x^2 - 2x - 8}{x^2 - 9}$ at the point where $x = 0$.

(d) When a camera flashes, the intensity I of light seen by the eye is given by the function

$$I(t) = \frac{100t}{e^t},$$

where I is measured in candles and t is measured in milliseconds. Compute $I'(0.5)$, $I'(2)$, and $I'(5)$; include appropriate units on each value; and discuss the meaning of each.

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Combining rules

One of the challenges to learning to apply various derivative shortcut rules correctly and effectively is recognizing the fundamental structure of a function. For instance, consider the function given by

$$f(x) = x\sin(x) + \frac{x^2}{\cos(x)+2}.$$

How do we decide which rules to apply? Our first task is to recognize the overall structure of the given function. Observe that the function f is fundamentally a sum of two slightly less complicated functions, so we can apply the sum rule[6] and get

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[x\sin(x) + \frac{x^2}{\cos(x)+2}\right] \\ &= \frac{d}{dx}\left[x\sin(x)\right] + \frac{d}{dx}\left[\frac{x^2}{\cos(x)+2}\right] \end{aligned}$$

Now, the left-hand term above is a product, so the product rule is needed there, while the right-hand term is a quotient, so the quotient rule is required. Applying these rules respectively, we find that

$$\begin{aligned} f'(x) &= (x\cos(x) + \sin(x)) + \frac{(\cos(x)+2)2x - x^2(-\sin(x))}{(\cos(x)+2)^2} \\ &= x\cos(x) + \sin(x) + \frac{2x\cos(x) + 4x^2 + x^2\sin(x)}{(\cos(x)+2)^2}. \end{aligned}$$

We next consider how the situation changes with the function defined by

$$s(y) = \frac{y \cdot 7^y}{y^2+1}.$$

Overall, s is a quotient of two simpler function, so the quotient rule will be needed. Here, we execute the quotient rule and use the notation $\frac{d}{dy}$ to defer the computation of the derivative of the numerator and derivative of the denominator. Thus,

$$s'(y) = \frac{(y^2+1)\cdot \frac{d}{dy}[y\cdot 7^y] - y\cdot 7^y \cdot \frac{d}{dy}\left[y^2+1\right]}{(y^2+1)^2}.$$

Now, there remain two derivatives to calculate. The first one, $\frac{d}{dy}[y\cdot 7^y]$ calls for use of the product rule, while the second, $\frac{d}{dy}\left[y^2+1\right]$ takes only an elementary application of the

[6] When taking a derivative that involves the use of multiple derivative rules, it is often helpful to use the notation $\frac{d}{dx}[\ \ ]$ to wait to apply subsequent rules. This is demonstrated in each of the two examples presented here.

sum rule. Applying these rules, we now have

$$s'(y) = \frac{(y^2+1)[y \cdot 7^y \ln(7) + 7^y \cdot 1] - y \cdot 7^y[2y]}{(y^2+1)^2}.$$

While some minor simplification is possible, we are content to leave $s'(y)$ in its current form, having found the desired derivative of s. In summary, to compute the derivative of s, we applied the quotient rule. In so doing, when it was time to compute the derivative of the top function, we used the product rule; at the point where we found the derivative of the bottom function, we used the sum rule.

In general, one of the main keys to success in applying derivative rules is to recognize the structure of the function, followed by the careful and diligent application of relevant derivative rules. The best way to get good at this process is by doing a large number of exercises, and the next activity provides some practice and exploration to that end.

Activity 2.9.

Use relevant derivative rules to answer each of the questions below. Throughout, be sure to use proper notation and carefully label any derivative you find by name.

(a) Let $f(r) = (5r^3 + \sin(r))(4^r - 2\cos(r))$. Find $f'(r)$.

(b) Let $p(t) = \dfrac{\cos(t)}{t^6 \cdot 6^t}$. Find $p'(t)$.

(c) Let $g(z) = 3z^7 e^z - 2z^2 \sin(z) + \dfrac{z}{z^2+1}$. Find $g'(z)$.

(d) A moving particle has its position in feet at time t in seconds given by the function $s(t) = \dfrac{3\cos(t) - \sin(t)}{e^t}$. Find the particle's instantaneous velocity at the moment $t = 1$.

(e) Suppose that $f(x)$ and $g(x)$ are differentiable functions and it is known that $f(3) = -2$, $f'(3) = 7$, $g(3) = 4$, and $g'(3) = -1$. If $p(x) = f(x) \cdot g(x)$ and $q(x) = \dfrac{f(x)}{g(x)}$, calculate $p'(3)$ and $q'(3)$.

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As the algebraic complexity of the functions we are able to differentiate continues to increase, it is important to remember that all of the derivative's meaning continues to hold. Regardless of the structure of the function f, the value of $f'(a)$ tells us the instantaneous rate of change of f with respect to x at the moment $x = a$, as well as the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$.

Summary

In this section, we encountered the following important ideas:

- If a function is a sum, product, or quotient of simpler functions, then we can use the sum, product, or quotient rules to differentiate the overall function in terms of the simpler functions and their derivatives.
- The product rule tells us that if P is a product of differentiable functions f and g according to the rule $P(x) = f(x)g(x)$, then

$$P'(x) = f(x)g'(x) + g(x)f'(x).$$

- The quotient rule tells us that if Q is a quotient of differentiable functions f and g according to the rule $Q(x) = \frac{f(x)}{g(x)}$, then

$$Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

- The product and quotient rules now complement the constant multiple and sum rules and enable us to compute the derivative of any function that consists of sums, constant multiples, products, and quotients of basic functions we already know how to differentiate. For instance, if F has the form

$$F(x) = \frac{2a(x) - 5b(x)}{c(x) \cdot d(x)},$$

then F is fundamentally a quotient, and the numerator is a sum of constant multiples and the denominator is a product. Hence the derivative of F can be found by applying the quotient rule and then using the sum and constant multiple rules to differentiate the numerator and the product rule to differentiate the denominator.

Exercises

1. Let f and g be differentiable functions for which the following information is known: $f(2) = 5$, $g(2) = -3$, $f'(2) = -1/2$, $g'(2) = 2$.

   (a) Let h be the new function defined by the rule $h(x) = g(x) \cdot f(x)$. Determine $h(2)$ and $h'(2)$.

   (b) Find an equation for the tangent line to $y = h(x)$ at the point $(2, h(2))$ (where h is the function defined in (a)).

   (c) Let r be the function defined by the rule $r(x) = \frac{g(x)}{f(x)}$. Is r increasing, decreasing, or neither at $a = 2$? Why?

(d) Estimate the value of $r(2.06)$ (where r is the function defined in (c)) by using the local linearization of r at the point $(2, r(2))$.

2. Consider the functions $r(t) = t^t$ and $s(t) = \arccos(t)$, for which you are given the facts that $r'(t) = t^t(\ln(t) + 1)$ and $s'(t) = -\frac{1}{\sqrt{1-t^2}}$. Do not be concerned with where these derivative formulas come from. We restrict our interest in both functions to the domain $0 < t < 1$.

(a) Let $w(t) = t^t \arccos(t)$. Determine $w'(t)$.

(b) Find an equation for the tangent line to $y = w(t)$ at the point $(\frac{1}{2}, w(\frac{1}{2}))$.

(c) Let $v(t) = \frac{t^t}{\arccos(t)}$. Is v increasing or decreasing at the instant $t = \frac{1}{2}$? Why?

3. Let functions p and q be the piecewise linear functions given by their respective graphs in Figure 2.5. Use the graphs to answer the following questions.

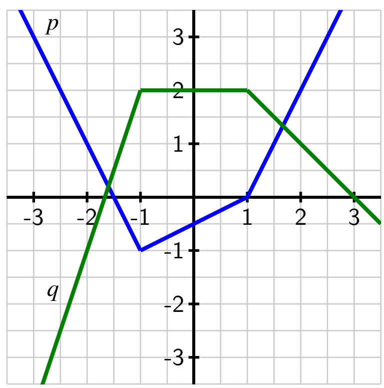

Figure 2.5: The graphs of p (in blue) and q (in green).

(a) Let $r(x) = p(x) \cdot q(x)$. Determine $r'(-2)$ and $r'(0)$.

(b) Are there values of x for which $r'(x)$ does not exist? If so, which values, and why?

(c) Find an equation for the tangent line to $y = r(x)$ at the point $(2, r(2))$.

(d) Let $z(x) = \frac{q(x)}{p(x)}$. Determine $z'(0)$ and $z'(2)$.

(e) Are there values of x for which $z'(x)$ does not exist? If so, which values, and why?

4. A farmer with large land holdings has historically grown a wide variety of crops. With the price of ethanol fuel rising, he decides that it would be prudent to devote more and more of his acreage to producing corn. As he grows more and more corn, he learns efficiencies that increase his yield per acre. In the present year, he used 7000 acres of his land to grow corn, and that land had an average yield of 170 bushels per acre. At the current time, he plans to increase his number of acres devoted to growing corn at a rate of 600 acres/year, and he expects that right now his average yield is increasing at a rate of 8 bushels per acre per year. Use this information to answer the following questions.

(a) Say that the present year is $t = 0$, that $A(t)$ denotes the number of acres the farmer devotes to growing corn in year t, $Y(t)$ represents the average yield in year t (measured in bushels per acre), and $C(t)$ is the total number of bushels of corn the farmer produces. What is the formula for $C(t)$ in terms of $A(t)$ and $Y(t)$? Why?

(b) What is the value of $C(0)$? What does it measure?

(c) Write an expression for $C'(t)$ in terms of $A(t)$, $A'(t)$, $Y(t)$, and $Y'(t)$. Explain your thinking.

(d) What is the value of $C'(0)$? What does it measure?

(e) Based on the given information and your work above, estimate the value of $C(1)$.

5. Let $f(v)$ be the gas consumption (in liters/km) of a car going at velocity v (in km/hour). In other words, $f(v)$ tells you how many liters of gas the car uses to go one kilometer if it is traveling at v kilometers per hour. In addition, suppose that $f(80) = 0.05$ and $f'(80) = 0.0004$.

(a) Let $g(v)$ be the distance the same car goes on one liter of gas at velocity v. What is the relationship between $f(v)$ and $g(v)$? Hence find $g(80)$ and $g'(80)$.

(b) Let $h(v)$ be the gas consumption in liters per hour of a car going at velocity v. In other words, $h(v)$ tells you how many liters of gas the car uses in one hour if it is going at velocity v. What is the algebraic relationship between $h(v)$ and $f(v)$? Hence find $h(80)$ and $h'(80)$.

(c) How would you explain the practical meaning of these function and derivative values to a driver who knows no calculus? Include units on each of the function and derivative values you discuss in your response.

2.4 Derivatives of other trigonometric functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What are the derivatives of the tangent, cotangent, secant, and cosecant functions?
- How do the derivatives of $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$ combine with other derivative rules we have developed to expand the library of functions we can quickly differentiate?

Introduction

One of the powerful themes in trigonometry is that the entire subject emanates from a very simple idea: locating a point on the unit circle.

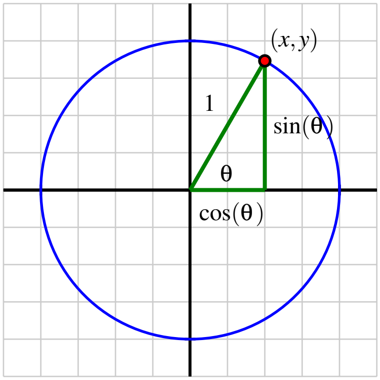

Figure 2.6: The unit circle and the definition of the sine and cosine functions.

Because each angle θ corresponds to one and only one point (x, y) on the unit circle, the x- and y-coordinates of this point are each functions of θ. Indeed, this is the very definition of $\cos(\theta)$ and $\sin(\theta)$: $\cos(\theta)$ is the x-coordinate of the point on the unit circle corresponding to the angle θ, and $\sin(\theta)$ is the y-coordinate. From this simple definition, all of trigonometry is founded. For instance, the fundamental trigonometric identity,

$$\sin^2(\theta) + \cos^2(\theta) = 1,$$

is a restatement of the Pythagorean Theorem, applied to the right triangle shown in Figure 2.6.

We recall as well that there are four other trigonometric functions, each defined in terms of the sine and/or cosine functions. These six trigonometric functions together offer us a wide range of flexibility in problems involving right triangles. The tangent function is defined by $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, while the cotangent function is its reciprocal: $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$. The secant function is the reciprocal of the cosine function, $\sec(\theta) = \frac{1}{\cos(\theta)}$, and the cosecant function is the reciprocal of the sine function, $\csc(\theta) = \frac{1}{\sin(\theta)}$.

Because we know the derivatives of the sine and cosine function, and the other four trigonometric functions are defined in terms of these familiar functions, we can now develop shortcut differentiation rules for the tangent, cotangent, secant, and cosecant functions. In this section's preview activity, we work through the steps to find the derivative of $y = \tan(x)$.

Preview Activity 2.4. Consider the function $f(x) = \tan(x)$, and remember that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

(a) What is the domain of f?

(b) Use the quotient rule to show that one expression for $f'(x)$ is

$$f'(x) = \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)}.$$

(c) What is the Fundamental Trigonometric Identity? How can this identity be used to find a simpler form for $f'(x)$?

(d) Recall that $\sec(x) = \frac{1}{\cos(x)}$. How can we express $f'(x)$ in terms of the secant function?

(e) For what values of x is $f'(x)$ defined? How does this set compare to the domain of f?

⋈

Derivatives of the cotangent, secant, and cosecant functions

In Preview Activity 2.4, we found that the derivative of the tangent function can be expressed in several ways, but most simply in terms of the secant function. Next, we develop the derivative of the cotangent function.

Let $g(x) = \cot(x)$. To find $g'(x)$, we observe that $g(x) = \frac{\cos(x)}{\sin(x)}$ and apply the quotient rule. Hence

$$\begin{aligned} g'(x) &= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\ &= -\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)} \end{aligned}$$

By the Fundamental Trigonometric Identity, we see that $g'(x) = -\frac{1}{\sin^2(x)}$; recalling that $\csc(x) = \frac{1}{\sin(x)}$, it follows that we can most simply express g' by the rule

$$g'(x) = -\csc^2(x).$$

Note that neither g nor g' is defined when $\sin(x) = 0$, which occurs at every integer multiple of π. Hence we have the following rule.

Cotangent Function: For all real numbers x such that $x \neq k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$,

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x).$$

Observe that the shortcut rule for the cotangent function is very similar to the rule we discovered in Preview Activity 2.4 for the tangent function.

Tangent Function: For all real numbers x such that $x \neq \frac{(2k+1)\pi}{2}$, where $k = \pm 1, \pm 2, \ldots$,

$$\frac{d}{dx}[\tan(x)] = \sec^2(x).$$

In the next two activities, we develop the rules for differentiating the secant and cosecant functions.

Activity 2.10.

Let $h(x) = \sec(x)$ and recall that $\sec(x) = \frac{1}{\cos(x)}$.

(a) What is the domain of h?

(b) Use the quotient rule to develop a formula for $h'(x)$ that is expressed completely in terms of $\sin(x)$ and $\cos(x)$.

(c) How can you use other relationships among trigonometric functions to write $h'(x)$ only in terms of $\tan(x)$ and $\sec(x)$?

(d) What is the domain of h'? How does this compare to the domain of h?

◁

Activity 2.11.

Let $p(x) = \csc(x)$ and recall that $\csc(x) = \frac{1}{\sin(x)}$.

(a) What is the domain of p?

(b) Use the quotient rule to develop a formula for $p'(x)$ that is expressed completely in terms of $\sin(x)$ and $\cos(x)$.

(c) How can you use other relationships among trigonometric functions to write $p'(x)$ only in terms of $\cot(x)$ and $\csc(x)$?

(d) What is the domain of p'? How does this compare to the domain of p?

◁

The quotient rule has thus enabled us to determine the derivatives of the tangent, cotangent, secant, and cosecant functions, expanding our overall library of basic functions we can differentiate. Moreover, we observe that just as the derivative of any polynomial function is a polynomial, and the derivative of any exponential function is another exponential function, so it is that the derivative of any basic trigonometric function is another function that consists of basic trigonometric functions. This makes sense because all trigonometric functions are periodic, and hence their derivatives will be periodic, too.

As has been and will continue to be the case throughout our work in Chapter 2, the derivative retains all of its fundamental meaning as an instantaneous rate of change and as the slope of the tangent line to the function under consideration. Our present work primarily expands the list of functions for which we can quickly determine a formula for the derivative. Moreover, with the addition of $\tan(x)$, $\cot(x)$, $\sec(x)$, and $\csc(x)$ to our library of basic functions, there are many more functions we can differentiate through the sum, constant multiple, product, and quotient rules.

Activity 2.12.

Answer each of the following questions. Where a derivative is requested, be sure to label the derivative function with its name using proper notation.

(a) Let $f(x) = 5\sec(x) - 2\csc(x)$. Find the slope of the tangent line to f at the point where $x = \frac{\pi}{3}$.

(b) Let $p(z) = z^2 \sec(z) - z\cot(z)$. Find the instantaneous rate of change of p at the point where $z = \frac{\pi}{4}$.

(c) Let $h(t) = \dfrac{\tan(t)}{t^2+1} - 2e^t\cos(t)$. Find $h'(t)$.

(d) Let $g(r) = \dfrac{r\sec(r)}{5^r}$. Find $g'(r)$.

(e) When a mass hangs from a spring and is set in motion, the object's position oscillates in a way that the size of the oscillations decrease. This is usually called a *damped oscillation*. Suppose that for a particular object, its displacement from equilibrium (where the object sits at rest) is modeled by the function

$$s(t) = \frac{15\sin(t)}{e^t}.$$

Assume that s is measured in inches and t in seconds. Sketch a graph of this function for $t \geq 0$ to see how it represents the situation described. Then compute ds/dt, state the units on this function, and explain what it tells you about the object's motion. Finally, compute and interpret $s'(2)$.

◁

Summary

In this section, we encountered the following important ideas:

- The derivatives of the other four trigonometric functions are

$$\frac{d}{dx}[\tan(x)] = \sec^2(x), \quad \frac{d}{dx}[\cot(x)] = -\csc^2(x),$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x), \text{ and } \frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x).$$

  Each derivative exists and is defined on the same domain as the original function. For example, both the tangent function and its derivative are defined for all real numbers x such that $x \neq \frac{k\pi}{2}$, where $k = \pm1, \pm2, \ldots$.

- The above four rules for the derivatives of the tangent, cotangent, secant, and cosecant can be used along with the rules for power functions, exponential functions, and the sine and cosine, as well as the sum, constant multiple, product, and quotient rules, to quickly differentiate a wide range of different functions.

Exercises

1. An object moving vertically has its height at time t (measured in feet, with time in seconds) given by the function $h(t) = 3 + \frac{2\cos(t)}{1.2^t}$.

(a) What is the object's instantaneous velocity when $t = 2$?

(b) What is the object's acceleration at the instant $t = 2$?

(c) Describe in everyday language the behavior of the object at the instant $t = 2$.

2. Let $f(x) = \sin(x)\cot(x)$.

(a) Use the product rule to find $f'(x)$.

(b) True or false: for all real numbers x, $f(x) = \cos(x)$.

(c) Explain why the function that you found in (a) is almost the opposite of the sine function, but not quite. (Hint: convert all of the trigonometric functions in (a) to sines and cosines, and work to simplify. Think carefully about the domain of f and the domain of f'.)

3. Let $p(z)$ be given by the rule

$$p(z) = \frac{z\tan(z)}{z^2\sec(z) + 1} + 3e^z + 1.$$

(a) Determine $p'(z)$.

(b) Find an equation for the tangent line to p at the point where $z = 0$.

(c) At $z = 0$, is p increasing, decreasing, or neither? Why?

2.5 The chain rule

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a composite function and how do we recognize its structure algebraically?
- Given a composite function $C(x) = f(g(x))$ that is built from differentiable functions f and g, how do we compute $C'(x)$ in terms of f, g, f', and g'? What is the statement of the Chain Rule?

Introduction

In addition to learning how to differentiate a variety of basic functions, we have also been developing our ability to understand how to use rules to differentiate certain algebraic combinations of them. For example, we not only know how to take the derivative of $f(x) = \sin(x)$ and $g(x) = x^2$, but now we can quickly find the derivative of each of the following combinations of f and g:

$$s(x) = 3x^2 - 5\sin(x),$$

$$p(x) = x^2 \sin(x), \text{ and}$$

$$q(x) = \frac{\sin(x)}{x^2}.$$

Finding s' uses the sum and constant multiple rules, determining p' requires the product rule, and q' can be attained with the quotient rule. Again, we note the importance of recognizing the algebraic structure of a given function in order to find its derivative: $s(x) = 3g(x) - 5f(x)$, $p(x) = g(x) \cdot f(x)$, and $q(x) = \frac{f(x)}{g(x)}$.

There is one more natural way to algebraically combine basic functions, and that is by *composing* them. For instance, let's consider the function

$$C(x) = \sin(x^2),$$

and observe that any input x passes through a *chain* of functions. In particular, in the process that defines the function $C(x)$, x is first squared, and then the sine of the result is taken. Using an arrow diagram,

$$x \longrightarrow x^2 \longrightarrow \sin(x^2).$$

In terms of the elementary functions f and g, we observe that x is first input in the function g, and then the result is used as the input in f. Said differently, we can write

$$C(x) = f(g(x)) = \sin(x^2)$$

and say that C is the ***composition*** of f and g. We will refer to g, the function that is first applied to x, as the ***inner*** function, while f, the function that is applied to the result, is the ***outer*** function.

The main question that we answer in the present section is: given a composite function $C(x) = f(g(x))$ that is built from differentiable functions f and g, how do we compute $C'(x)$ in terms of f, g, f', and g'? In the same way that the rate of change of a product of two functions, $p(x) = f(x) \cdot g(x)$, depends on the behavior of both f and g, it makes sense intuitively that the rate of change of a composite function $C(x) = f(g(x))$ will also depend on some combination of f and g and their derivatives. The rule that describes how to compute C' in terms of f and g and their derivatives will be called the ***chain rule***.

But before we can learn what the chain rule says and why it works, we first need to be comfortable decomposing composite functions so that we can correctly identify the inner and outer functions, as we did in the example above with $C(x) = \sin(x^2)$.

Preview Activity 2.5. For each function given below, identify its fundamental algebraic structure. In particular, is the given function a sum, product, quotient, or composition of basic functions? If the function is a composition of basic functions, state a formula for the inner function g and the outer function f so that the overall composite function can be written in the form $f(g(x))$. If the function is a sum, product, or quotient of basic functions, use the appropriate rule to determine its derivative.

(a) $h(x) = \tan(2^x)$

(b) $p(x) = 2^x \tan(x)$

(c) $r(x) = (\tan(x))^2$

(d) $m(x) = e^{\tan(x)}$

(e) $w(x) = \sqrt{x} + \tan(x)$

(f) $z(x) = \sqrt{\tan(x)}$

⋈

The chain rule

One of the challenges of differentiating a composite function is that it often cannot be written in an alternate algebraic form. For instance, the function $C(x) = \sin(x^2)$ cannot be expanded or otherwise rewritten, so it presents no alternate approaches to taking the derivative. But other composite functions can be expanded or simplified, and these present a way to begin to explore how the chain rule might have to work. To that end, we consider two examples of composite functions that present alternate means of finding the derivative.

Example 2.1. Let $f(x) = -4x + 7$ and $g(x) = 3x - 5$. Determine a formula for $C(x) = f(g(x))$ and compute $C'(x)$. How is C' related to f and g and their derivatives?

Solution. By the rules given for f and g,

$$\begin{aligned} C(x) &= f(g(x)) \\ &= f(3x-5) \\ &= -4(3x-5)+7 \\ &= -12x+20+7 \\ &= -12x+27. \end{aligned}$$

Thus, $C'(x) = -12$. Noting that $f'(x) = -4$ and $g'(x) = 3$, we observe that C' appears to be the product of f' and g'.

From one perspective, Example 2.1 may be too elementary. Linear functions are the simplest of all functions, and perhaps composing linear functions (which yields another linear function) does not exemplify the true complexity that is involved with differentiating a composition of more complicated functions. At the same time, we should remember the perspective that any differentiable function is *locally* linear, so any function with a derivative behaves like a line when viewed up close. From this point of view, the fact that the derivatives of f and g are multiplied to find the derivative of their composition turns out to be a key insight.

We now consider a second example involving a nonlinear function to gain further understanding of how differentiating a composite function involves the basic functions that combine to form it.

Example 2.2. Let $C(x) = \sin(2x)$. Use the double angle identity to rewrite C as a product of basic functions, and use the product rule to find C'. Rewrite C' in the simplest form possible.

Solution. By the double angle identity for the sine function,

$$C(x) = \sin(2x) = 2\sin(x)\cos(x).$$

Applying the product rule and simplifying,

$$C'(x) = 2\sin(x)(-\sin(x)) + \cos(x)(2\cos(x)) = 2(\cos^2(x) - \sin^2(x)).$$

Next, we recall that one of the double angle identities for the cosine function tells us that

$$\cos(2x) = \cos^2(x) - \sin^2(x).$$

Substituting this result in our expression for $C'(x)$, we now have that

$$C'(x) = 2\cos(2x).$$

So from Example 2.2, we see that if $C(x) = \sin(2x)$, then $C'(x) = 2\cos(2x)$. Letting $g(x) = 2x$ and $f(x) = \sin(x)$, we observe that $C(x) = f(g(x))$. Moreover, with $g'(x) = 2$ and $f'(x) = \cos(x)$, it follows that we can view the structure of $C'(x)$ as

$$C'(x) = 2\cos(2x) = g'(x)f'(g(x)).$$

In this example, we see that for the composite function $C(x) = f(g(x))$, the derivative C' is (as in the example involving linear functions) constituted by multiplying the derivatives of f and g, but with the special condition that f' is evaluated at $g(x)$, rather than at x.

It makes sense intuitively that these two quantities are involved in understanding the rate of change of a composite function: if we are considering $C(x) = f(g(x))$ and asking how fast C is changing at a given x value as x changes, it clearly matters how fast g is changing at x, as well as how fast f is changing at the value of $g(x)$. It turns out that this structure holds not only for the functions in Examples 2.1 and 2.2, but indeed for all differentiable functions[7] as is stated in the Chain Rule.

Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function C defined by $C(x) = f(g(x))$ is differentiable at x and

$$C'(x) = f'(g(x))g'(x).$$

As with the product and quotient rules, it is often helpful to think verbally about what the chain rule says: "If C is a composite function defined by an outer function f and an inner function g, then C' is given by the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function."

At least initially in working particular examples requiring the chain rule, it can also be helpful to clearly identify the inner function g and outer function f, compute their

[7] Like other differentiation rules, the Chain Rule can be proved formally using the limit definition of the derivative.

derivatives individually, and then put all of the pieces together to generate the derivative of the overall composite function. To see what we mean by this, consider the function

$$r(x) = (\tan(x))^2.$$

The function r is composite, with inner function $g(x) = \tan(x)$ and outer function $f(x) = x^2$. Organizing the key information involving f, g, and their derivatives, we have

$$\begin{array}{ll} f(x) = x^2 & g(x) = \tan(x) \\ f'(x) = 2x & g'(x) = \sec^2(x) \\ f'(g(x)) = 2\tan(x) & \end{array}$$

Applying the chain rule, which tells us that $r'(x) = f'(g(x))g'(x)$, we find that for $r(x) = (\tan(x))^2$, its derivative is

$$r'(x) = 2\tan(x)\sec^2(x).$$

As a side note, we remark that another way to write $r(x)$ is $r(x) = \tan^2(x)$. Observe that in this format, the composite nature of the function is more implicit, but this is common notation for powers of trigonometric functions: $\cos^4(x)$, $\sin^5(x)$, and $\sec^2(x)$ are all composite functions, with the outer function a power function and the inner function a trigonometric one.

The chain rule now substantially expands the library of functions we can differentiate, as the following activity demonstrates.

Activity 2.13.

For each function given below, identify an inner function g and outer function f to write the function in the form $f(g(x))$. Then, determine $f'(x)$, $g'(x)$, and $f'(g(x))$, and finally apply the chain rule to determine the derivative of the given function.

(a) $h(x) = \cos(x^4)$

(b) $p(x) = \sqrt{\tan(x)}$

(c) $s(x) = 2^{\sin(x)}$

(d) $z(x) = \cot^5(x)$

(e) $m(x) = (\sec(x) + e^x)^9$

◁

Using multiple rules simultaneously

The chain rule now joins the sum, constant multiple, product, and quotient rules in our collection of the different techniques for finding the derivative of a function through understanding its algebraic structure and the basic functions that constitute it. It takes substantial practice to get comfortable with navigating multiple rules in a single problem; using proper notation and taking a few extra steps can be particularly helpful as well. We demonstrate with an example and then provide further opportunity for practice in the following activity.

Example 2.3. Find a formula for the derivative of $h(t) = 3^{t^2+2t}\sec^4(t)$.

Solution. We first observe that the most basic structure of h is that it is the product of two functions: $h(t) = a(t)\cdot b(t)$ where $a(t) = 3^{t^2+2t}$ and $b(t) = \sec^4(t)$. Therefore, we see that we will need to use the product rule to differentiate h. When it comes time to differentiate a and b in their roles in the product rule, we observe that since each is a composite function, the chain rule will be needed. We therefore begin by working separately to compute $a'(t)$ and $b'(t)$.

Writing $a(t) = f(g(t)) = 3^{t^2+2t}$, and finding the derivatives of f and g, we have

$$\begin{aligned} f(t) &= 3^t & \qquad g(t) &= t^2 + 2t \\ f'(t) &= 3^t \ln(3) & \qquad g'(t) &= 2t + 2 \\ f'(g(t)) &= 3^{t^2+2t}\ln(3) & & \end{aligned}$$

Thus, by the chain rule, it follows that $a'(t) = f'(g(t))g'(t) = 3^{t^2+2t}\ln(3)(2t+2)$.

Turning next to b, we write $b(t) = r(s(t)) = \sec^4(t)$ and find the derivatives of r and g. Doing so,

$$\begin{aligned} r(t) &= t^4 & \qquad s(t) &= \sec(t) \\ r'(t) &= 4t^3 & \qquad s'(t) &= \sec(t)\tan(t) \\ r'(s(t)) &= 4\sec^3(t) & & \end{aligned}$$

By the chain rule, we now know that $b'(t) = r'(s(t))s'(t) = 4\sec^3(t)\sec(t)\tan(t) = 4\sec^4(t)\tan(t)$.

Now we are finally ready to compute the derivative of the overall function h. Recalling that $h(t) = 3^{t^2+2t}\sec^4(t)$, by the product rule we have

$$h'(t) = 3^{t^2+2t}\frac{d}{dt}[\sec^4(t)] + \sec^4(t)\frac{d}{dt}[3^{t^2+2t}].$$

From our work above with a and b, we know the derivatives of 3^{t^2+2t} and $\sec^4(t)$, and therefore

$$h'(t) = 3^{t^2+2t}4\sec^4(t)\tan(t) + \sec^4(t)3^{t^2+2t}\ln(3)(2t+2).$$

Activity 2.14.

For each of the following functions, find the function's derivative. State the rule(s) you use, label relevant derivatives appropriately, and be sure to clearly identify your overall answer.

(a) $p(r) = 4\sqrt{r^6 + 2e^r}$

(b) $m(v) = \sin(v^2)\cos(v^3)$

(c) $h(y) = \dfrac{\cos(10y)}{e^{4y} + 1}$

(d) $s(z) = 2^{z^2 \sec(z)}$

(e) $c(x) = \sin(e^{x^2})$

◁

The chain rule now adds substantially to our ability to do different familiar problems that involve derivatives. Whether finding the equation of the tangent line to a curve, the instantaneous velocity of a moving particle, or the instantaneous rate of change of a certain quantity, if the function under consideration involves a composition of other functions, the chain rule is indispensable.

Activity 2.15.

Use known derivative rules, including the chain rule, as needed to answer each of the following questions.

(a) Find an equation for the tangent line to the curve $y = \sqrt{e^x + 3}$ at the point where $x = 0$.

(b) If $s(t) = \dfrac{1}{(t^2 + 1)^3}$ represents the position function of a particle moving horizontally along an axis at time t (where s is measured in inches and t in seconds), find the particle's instantaneous velocity at $t = 1$. Is the particle moving to the left or right at that instant?

(c) At sea level, air pressure is 30 inches of mercury. At an altitude of h feet above sea level, the air pressure, P, in inches of mercury, is given by the function

$$P = 30e^{-0.0000323h}.$$

Compute dP/dh and explain what this derivative function tells you about air pressure, including a discussion of the units on dP/dh. In addition, determine how fast the air pressure is changing for a pilot of a small plane passing through an altitude of 1000 feet.

(d) Suppose that $f(x)$ and $g(x)$ are differentiable functions and that the following information about them is known:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	2	-5	-3	4
2	-3	4	-1	2

If $C(x)$ is a function given by the formula $f(g(x))$, determine $C'(2)$. In addition, if $D(x)$ is the function $f(f(x))$, find $D'(-1)$.

◁

The composite version of basic function rules

As we gain more experience with differentiating complicated functions, we will become more comfortable in the process of simply writing down the derivative without taking multiple steps. We demonstrate part of this perspective here by showing how we can find a composite rule that corresponds to two of our basic functions. For instance, we know that $\frac{d}{dx}[\sin(x)] = \cos(x)$. If we instead want to know

$$\frac{d}{dx}[\sin(u(x))],$$

where u is a differentiable function of x, then this requires the chain rule with the sine function as the outer function. Applying the chain rule,

$$\frac{d}{dx}[\sin(u(x))] = \cos(u(x)) \cdot u'(x).$$

Similarly, since $\frac{d}{dx}[a^x] = a^x \ln(a)$, it follows by the chain rule that

$$\frac{d}{dx}[a^{u(x)}] = a^{u(x)} \ln(a) \cdot u'(x).$$

In the process of getting comfortable with derivative rules, an excellent exercise is to write down a list of all basic functions whose derivatives are known, list those derivatives, and then write the corresponding chain rule for the composite version with the inner function being an unknown function $u(x)$ and the outer function being the known basic function. These versions of the chain rule are particularly simple when the inner function is linear, since the derivative of a linear function is a constant. For instance,

$$\frac{d}{dx}\left[(5x+7)^{10}\right] = 10(5x+7)^9 \cdot 5,$$

$$\frac{d}{dx}\left[\tan(17x)\right] = 17\sec^2(17x), \text{ and}$$

$$\frac{d}{dx}\left[e^{-3x}\right] = -3e^{-3x}.$$

Summary

In this section, we encountered the following important ideas:

- A composite function is one where the input variable x first passes through one function, and then the resulting output passes through another. For example, the function $h(x) = 2^{\sin(x)}$ is composite since $x \longrightarrow \sin(x) \longrightarrow 2^{\sin(x)}$.
- Given a composite function $C(x) = f(g(x))$ that is built from differentiable functions f and g, the chain rule tells us that we compute $C'(x)$ in terms of f, g, f', and g' according to the formula
$$C'(x) = f'(g(x))g'(x).$$

Exercises

1. Consider the basic functions $f(x) = x^3$ and $g(x) = \sin(x)$.

(a) Let $h(x) = f(g(x))$. Find the exact instantaneous rate of change of h at the point where $x = \frac{\pi}{4}$.

(b) Which function is changing most rapidly at $x = 0.25$: $h(x) = f(g(x))$ or $r(x) = g(f(x))$? Why?

(c) Let $h(x) = f(g(x))$ and $r(x) = g(f(x))$. Which of these functions has a derivative that is periodic? Why?

2. Let $u(x)$ be a differentiable function. For each of the following functions, determine the derivative. Each response will involve u and/or u'.

(a) $p(x) = e^{u(x)}$

(b) $q(x) = u(e^x)$

(c) $r(x) = \cot(u(x))$

(d) $s(x) = u(\cot(x))$

(e) $a(x) = u(x^4)$

(f) $b(x) = u^4(x)$

3. Let functions p and q be the piecewise linear functions given by their respective graphs in Figure 2.7. Use the graphs to answer the following questions.

(a) Let $C(x) = p(q(x))$. Determine $C'(0)$ and $C'(3)$.

(b) Find a value of x for which $C'(x)$ does not exist. Explain your thinking.

(c) Let $Y(x) = q(q(x))$ and $Z(x) = q(p(x))$. Determine $Y'(-2)$ and $Z'(0)$.

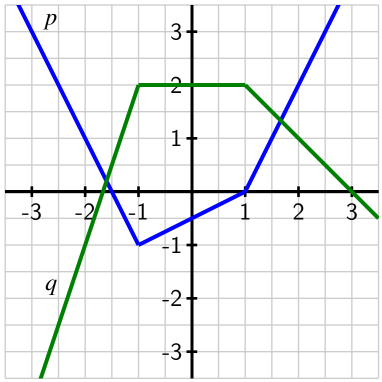

Figure 2.7: The graphs of p (in blue) and q (in green).

4. If a spherical tank of radius 4 feet has h feet of water present in the tank, then the volume of water in the tank is given by the formula

$$V = \frac{\pi}{3} h^2 (12 - h).$$

(a) At what instantaneous rate is the volume of water in the tank changing with respect to the *height* of the water at the instant $h = 1$? What are the units on this quantity?

(b) Now suppose that the height of water in the tank is being regulated by an inflow and outflow (e.g., a faucet and a drain) so that the height of the water at time t is given by the rule $h(t) = \sin(\pi t) + 1$, where t is measured in hours (and h is still measured in feet). At what rate is the height of the water changing with respect to time at the instant $t = 2$?

(c) Continuing under the assumptions in (b), at what instantaneous rate is the volume of water in the tank changing with respect to *time* at the instant $t = 2$?

(d) What are the main differences between the rates found in (a) and (c)? Include a discussion of the relevant units.

2.6 Derivatives of Inverse Functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is the derivative of the natural logarithm function?
- What are the derivatives of the inverse trigonometric functions $\arcsin(x)$ and $\arctan(x)$?
- If g is the inverse of a differentiable function f, how is g' computed in terms of f, f', and g?

Introduction

Much of mathematics centers on the notion of function. Indeed, throughout our study of calculus, we are investigating the behavior of functions, often doing so with particular emphasis on how fast the output of the function changes in response to changes in the input. Because each function represents a process, a natural question to ask is whether or not the particular process can be reversed. That is, if we know the output that results from the function, can we determine the input that led to it? Connected to this question, we now also ask: if we know how fast a particular process is changing, can we determine how fast the inverse process is changing?

As we have noted, one of the most important functions in all of mathematics is the natural exponential function $f(x) = e^x$. Because the natural logarithm, $g(x) = \ln(x)$, is the inverse of the natural exponential function, the natural logarithm is similarly important. One of our goals in this section is to learn how to differentiate the logarithm function, and thus expand our library of basic functions with known derivative formulas. First, we investigate a more familiar setting to refresh some of the basic concepts surrounding functions and their inverses.

Preview Activity 2.6. The equation $y = \frac{5}{9}(x - 32)$ relates a temperature given in x degrees Fahrenheit to the corresponding temperature y measured in degrees Celcius.

(a) Solve the equation $y = \frac{5}{9}(x-32)$ for x to write x (Fahrenheit temperature) in terms of y (Celcius temperature).

(b) Let $C(x) = \frac{5}{9}(x - 32)$ be the function that takes a Fahrenheit temperature as input and produces the Celcius temperature as output. In addition, let $F(y)$ be the function that converts a temperature given in y degrees Celcius to the temperature $F(y)$ measured in degrees Fahrenheit. Use your work in (a) to write a formula for $F(y)$.

(c) Next consider the new function defined by $p(x) = F(C(x))$. Use the formulas for F and C to determine an expression for $p(x)$ and simplify this expression as much as possible. What do you observe?

(d) Now, let $r(y) = C(F(y))$. Use the formulas for F and C to determine an expression for $r(y)$ and simplify this expression as much as possible. What do you observe?

(e) What is the value of $C'(x)$? of $F'(y)$? How do these values appear to be related?

⋈

Basic facts about inverse functions

A function $f : A \to B$ is a rule that associates each element in the set A to one and only one element in the set B. We call A the ***domain*** of f and B the ***codomain*** of f. If there exists a function $g : B \to A$ such that $g(f(a)) = a$ for every possible choice of a in the set A and $f(g(b)) = b$ for every b in the set B, then we say that g is the ***inverse*** of f. We often use the notation f^{-1} (read "f-inverse") to denote the inverse of f. Perhaps the most essential thing to observe about the inverse function is that it undoes the work of f. Indeed, if $y = f(x)$, then

$$f^{-1}(y) = f^{-1}(f(x)) = x,$$

and this leads us to another key observation: writing $y = f(x)$ and $x = f^{-1}(y)$ say the exact same thing. The only difference between the two equations is one of perspective – one is solved for x, while the other is solved for y.

Here we briefly remind ourselves of some key facts about inverse functions. For a function $f : A \to B$,

- f has an inverse if and only if f is one-to-one[8] and onto[9];
- provided f^{-1} exists, the domain of f^{-1} is the codomain of f, and the codomain of f^{-1} is the domain of f;
- $f^{-1}(f(x)) = x$ for every x in the domain of f and $f(f^{-1}(y)) = y$ for every y in the codomain of f;
- $y = f(x)$ if and only if $x = f^{-1}(y)$.

The last stated fact reveals a special relationship between the graphs of f and f^{-1}. In particular, if we consider $y = f(x)$ and a point (x, y) that lies on the graph of f, then it is also true that $x = f^{-1}(y)$, which means that the point (y, x) lies on the graph of f^{-1}.

[8] A function f is *one-to-one* provided that no two distinct inputs lead to the same output.

[9] A function f is *onto* provided that every possible element of the codomain can be realized as an output of the function for some choice of input from the domain.

This shows us that the graphs of f and f^{-1} are the reflections of one another across the line $y = x$, since reflecting across $y = x$ is precisely the geometric action that swaps the coordinates in an ordered pair. In Figure 2.8, we see this exemplified for the function $y = f(x) = 2^x$ and its inverse, with the points $(-1, \frac{1}{2})$ and $(\frac{1}{2}, -1)$ highlighting the reflection of the curves across $y = x$.

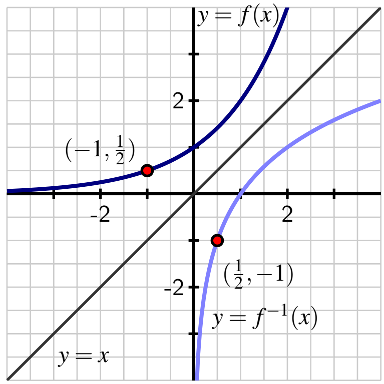

Figure 2.8: A graph of a function $y = f(x)$ along with its inverse, $y = f^{-1}(x)$.

To close our review of important facts about inverses, we recall that the natural exponential function $y = f(x) = e^x$ has an inverse function, and its inverse is the natural logarithm, $x = f^{-1}(y) = \ln(y)$. Indeed, writing $y = e^x$ is interchangeable with $x = \ln(y)$, plus $\ln(e^x) = x$ for every real number x and $e^{\ln(y)} = y$ for every positive real number y.

The derivative of the natural logarithm function

In what follows, we determine a formula for the derivative of $g(x) = \ln(x)$. To do so, we take advantage of the fact that we know the derivative of the natural exponential function, which is the inverse of g. In particular, we know that writing $g(x) = \ln(x)$ is equivalent to writing $e^{g(x)} = x$. Now we differentiate both sides of this most recent equation. In particular, we observe that

$$\frac{d}{dx}\left[e^{g(x)}\right] = \frac{d}{dx}[x].$$

The righthand side is simply 1; applying the chain rule to the left side, we find that

$$e^{g(x)}g'(x) = 1.$$

Since our goal is to determine $g'(x)$, we solve for $g'(x)$, so

$$g'(x) = \frac{1}{e^{g(x)}}.$$

Finally, we recall that since $g(x) = \ln(x)$, $e^{g(x)} = e^{\ln(x)} = x$, and thus

$$g'(x) = \frac{1}{x}.$$

Natural Logarithm: For all positive real numbers x, $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$.

This rule for the natural logarithm function now joins our list of other basic derivative rules that we have already established. There are two particularly interesting things to note about the fact that $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$. One is that this rule is restricted to only apply to positive values of x, as these are the only values for which the original function is defined. The other is that for the first time in our work, differentiating a basic function of a particular type has led to a function of a very different nature: the derivative of the natural logarithm is not another logarithm, nor even an exponential function, but rather a rational one.

Derivatives of logarithms may now be computed in concert with all of the rules known to date. For instance, if $f(t) = \ln(t^2 + 1)$, then by the chain rule, $f'(t) = \frac{1}{t^2+1} \cdot 2t$.

Activity 2.16.

For each function given below, find its derivative.

(a) $h(x) = x^2 \ln(x)$

(b) $p(t) = \dfrac{\ln(t)}{e^t + 1}$

(c) $s(y) = \ln(\cos(y) + 2)$

(d) $z(x) = \tan(\ln(x))$

(e) $m(z) = \ln(\ln(z))$

◁

In addition to the important rule we have derived for the derivative of the natural log functions, there are additional interesting connections to note between the graphs of $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$.

In Figure 2.9, we are reminded that since the natural exponential function has the property that its derivative is itself, the slope of the tangent to $y = e^x$ is equal to the height of the curve at that point. For instance, at the point $A = (\ln(0.5), 0.5)$, the slope of the tangent line is $m_A = 0.5$, and at $B = (\ln(5), 5)$, the tangent line's slope is $m_B = 5$. At the corresponding points A' and B' on the graph of the natural logarithm function (which come from reflecting across the line $y = x$), we know that the slope of the tangent line is the reciprocal of the x-coordinate of the point (since $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$). Thus, with $A' = (0.5, \ln(0.5))$, we have $m_{A'} = \frac{1}{0.5} = 2$, and at $B' = (5, \ln(5))$, $m_{B'} = \frac{1}{5}$.

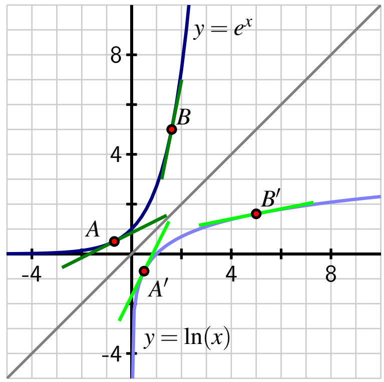

Figure 2.9: A graph of the function $y = e^x$ along with its inverse, $y = \ln(x)$, where both functions are viewed using the input variable x.

In particular, we observe that $m_{A'} = \frac{1}{m_A}$ and $m_{B'} = \frac{1}{m_B}$. This is not a coincidence, but in fact holds for any curve $y = f(x)$ and its inverse, provided the inverse exists. One rationale for why this is the case is due to the reflection across $y = x$: in so doing, we essentially change the roles of x and y, thus reversing the rise and run, which leads to the slope of the inverse function at the reflected point being the reciprocal of the slope of the original function. At the close of this section, we will also look at how the chain rule provides us with an algebraic formulation of this general phenomenon.

Inverse trigonometric functions and their derivatives

Trigonometric functions are periodic, so they fail to be one-to-one, and thus do not have inverses. However, if we restrict the domain of each trigonometric function, we can force the function to be one-to-one. For instance, consider the sine function on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Because no output of the sine function is repeated on this interval, the function is one-to-one and thus has an inverse. In particular, if we view $f(x) = \sin(x)$ as having domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and codomain $[-1, 1]$, then there exists an inverse function f^{-1} such that

$$f^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}].$$

We call f^{-1} the *arcsine* (or inverse sine) function and write $f^{-1}(y) = \arcsin(y)$. It is especially important to remember that writing

$$y = \sin(x) \quad \text{and} \quad x = \arcsin(y)$$

say the exact same thing. We often read "the arcsine of y" as "the angle whose sine is

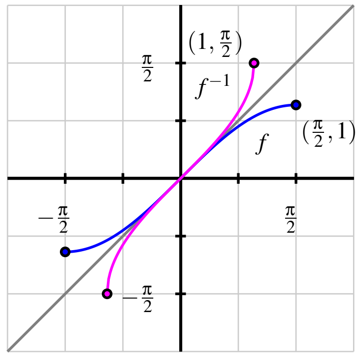

Figure 2.10: A graph of $f(x) = \sin(x)$ (in blue), restricted to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$, along with its inverse, $f^{-1}(x) = \arcsin(x)$ (in magenta).

y." For example, we say that $\frac{\pi}{6}$ is the angle whose sine is $\frac{1}{2}$, which can be written more concisely as $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$, which is equivalent to writing $\sin(\frac{\pi}{6}) = \frac{1}{2}$.

Next, we determine the derivative of the arcsine function. Letting $h(x) = \arcsin(x)$, our goal is to find $h'(x)$. Since $h(x)$ is the angle whose sine is x, it is equivalent to write

$$\sin(h(x)) = x.$$

Differentiating both sides of the previous equation, we have

$$\frac{d}{dx}[\sin(h(x))] = \frac{d}{dx}[x],$$

and by the fact that the righthand side is simply 1 and by the chain rule applied to the left side,

$$\cos(h(x))h'(x) = 1.$$

Solving for $h'(x)$, it follows that

$$h'(x) = \frac{1}{\cos(h(x))}.$$

Finally, we recall that $h(x) = \arcsin(x)$, so the denominator of $h'(x)$ is the function $\cos(\arcsin(x))$, or in other words, "the cosine of the angle whose sine is x." A bit of right triangle trigonometry allows us to simplify this expression considerably.

Let's say that $\theta = \arcsin(x)$, so that θ is the angle whose sine is x. From this, it follows that we can picture θ as an angle in a right triangle with hypotenuse 1 and a vertical leg of length x, as shown in Figure 2.11. The horizontal leg must be $\sqrt{1-x^2}$, by the Pythagorean Theorem. Now, note particularly that $\theta = \arcsin(x)$ since $\sin(\theta) = x$, and

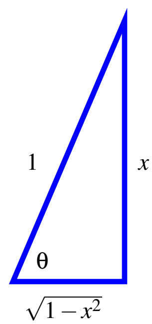

Figure 2.11: The right triangle that corresponds to the angle $\theta = \arcsin(x)$.

recall that we want to know a different expression for $\cos(\arcsin(x))$. From the figure, $\cos(\arcsin(x)) = \cos(\theta) = \sqrt{1-x^2}$.

Thus, returning to our earlier work where we established that if $h(x) = \arcsin(x)$, then $h'(x) = \frac{1}{\cos(\arcsin(x))}$, we have now shown that

$$h'(x) = \frac{1}{\sqrt{1-x^2}}.$$

Inverse sine: For all real numbers x such that $-1 < x < 1$, $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$.

Activity 2.17.

The following prompts in this activity will lead you to develop the derivative of the inverse tangent function.

(a) Let $r(x) = \arctan(x)$. Use the relationship between the arctangent and tangent functions to rewrite this equation using only the tangent function.

(b) Differentiate both sides of the equation you found in (a). Solve the resulting equation for $r'(x)$, writing $r'(x)$ as simply as possible in terms of a trigonometric function evaluated at $r(x)$.

(c) Recall that $r(x) = \arctan(x)$. Update your expression for $r'(x)$ so that it only involves trigonometric functions and the independent variable x.

(d) Introduce a right triangle with angle θ so that $\theta = \arctan(x)$. What are the three sides of the triangle?

(e) In terms of only x and 1, what is the value of $\cos(\arctan(x))$?

(f) Use the results of your work above to find an expression involving only 1 and x for $r'(x)$.

◁

While derivatives for other inverse trigonometric functions can be established similarly, we primarily limit ourselves to the arcsine and arctangent functions. With these rules added to our library of derivatives of basic functions, we can differentiate even more functions using derivative shortcuts. In Activity 2.18, we see each of these rules at work.

Activity 2.18.

Determine the derivative of each of the following functions.

(a) $f(x) = x^3 \arctan(x) + e^x \ln(x)$

(b) $p(t) = 2^{t \arcsin(t)}$

(c) $h(z) = (\arcsin(5z) + \arctan(4 - z))^{27}$

(d) $s(y) = \cot(\arctan(y))$

(e) $m(v) = \ln(\sin^2(v) + 1)$

(f) $g(w) = \arctan\left(\frac{\ln(w)}{1 + w^2}\right)$

◁

The link between the derivative of a function and the derivative of its inverse

In Figure 2.9, we saw an interesting relationship between the slopes of tangent lines to the natural exponential and natural logarithm functions at points that corresponded to reflection across the line $y = x$. In particular, we observed that for a point such as $(\ln(2), 2)$ on the graph of $f(x) = e^x$, the slope of the tangent line at this point is $f'(\ln(2)) = 2$, while at the corresponding point $(2, \ln(2))$ on the graph of $f^{-1}(x) = \ln(x)$, the slope of the tangent line at this point is $(f^{-1})'(2) = \frac{1}{2}$, which is the reciprocal of $f'(\ln(2))$.

That the two corresponding tangent lines having slopes that are reciprocals of one another is not a coincidence. If we consider the general setting of a differentiable function f with differentiable inverse g such that $y = f(x)$ if and only if $x = g(y)$, then we know that $f(g(x)) = x$ for every x in the domain of f^{-1}. Differentiating both sides of this equation with respect to x, we have

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[x],$$

and by the chain rule,

$$f'(g(x))g'(x) = 1.$$

Solving for $g'(x)$, we have $g'(x) = \frac{1}{f'(g(x))}$. Here we see that the slope of the tangent line to the inverse function g at the point $(x, g(x))$ is precisely the reciprocal of the slope of the tangent line to the original function f at the point $(g(x), f(g(x))) = (g(x), x)$.

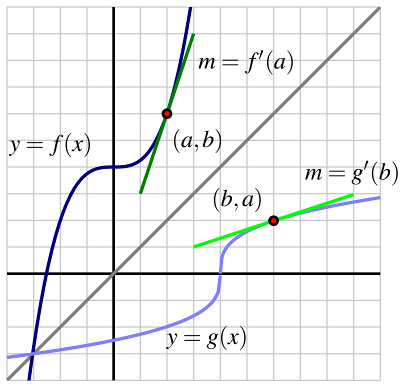

Figure 2.12: A graph of function $y = f(x)$ along with its inverse, $y = g(x) = f^{-1}(x)$. Observe that the slopes of the two tangent lines are reciprocals of one another.

To see this more clearly, consider the graph of the function $y = f(x)$ shown in Figure 2.12, along with its inverse $y = g(x)$. Given a point (a, b) that lies on the graph of f, we know that (b, a) lies on the graph of g; said differently, $f(a) = b$ and $g(b) = a$. Now, applying the rule that $g'(x) = 1/f'(g(x))$ to the value $x = b$, we have

$$g'(b) = \frac{1}{f'(g(b))} = \frac{1}{f'(a)},$$

which is precisely what we see in the figure: the slope of the tangent line to g at (b, a) is the reciprocal of the slope of the tangent line to f at (a, b), since these two lines are reflections of one another across the line $y = x$.

Derivative of an inverse function: Suppose that f is a differentiable function with inverse g and that (a, b) is a point that lies on the graph of f at which $f'(a) \neq 0$. Then

$$g'(b) = \frac{1}{f'(a)}.$$

More generally, for any x in the domain of g', we have $g'(x) = 1/f'(g(x))$.

The rules we derived for $\ln(x)$, $\arcsin(x)$, and $\arctan(x)$ are all just specific examples of this general property of the derivative of an inverse function. For example, with

$g(x) = \ln(x)$ and $f(x) = e^x$, it follows that

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}.$$

Summary

In this section, we encountered the following important ideas:

- For all positive real numbers x, $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$.
- For all real numbers x such that $-1 < x < 1$, $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$. In addition, for all real numbers x, $\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$.
- If g is the inverse of a differentiable function f, then for any point x in the domain of g', $g'(x) = \frac{1}{f'(g(x))}$.

Exercises

1. Determine the derivative of each of the following functions. Use proper notation and clearly identify the derivative rules you use.

(a) $f(x) = \ln(2\arctan(x) + 3\arcsin(x) + 5)$

(b) $r(z) = \arctan(\ln(\arcsin(z)))$

(c) $q(t) = \arctan^2(3t)\arcsin^4(7t)$

(d) $g(v) = \ln\left(\frac{\arctan(v)}{\arcsin(v) + v^2}\right)$

2. Consider the graph of $y = f(x)$ provided in Figure 2.13 and use it to answer the following questions.

(a) Use the provided graph to estimate the value of $f'(1)$.

(b) Sketch an approximate graph of $y = f^{-1}(x)$. Label at least three distinct points on the graph that correspond to three points on the graph of f.

(c) Based on your work in (a), what is the value of $(f^{-1})'(-1)$? Why?

3. Let $f(x) = \frac{1}{4}x^3 + 4$.

(a) Sketch a graph of $y = f(x)$ and explain why f is an invertible function.

(b) Let g be the inverse of f and determine a formula for g.

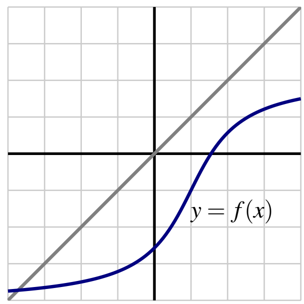

Figure 2.13: A function $y = f(x)$ for use in Exercise 2.

(c) Compute $f'(x)$, $g'(x)$, $f'(2)$, and $g'(6)$. What is the special relationship between $f'(2)$ and $g'(6)$? Why?

4. Let $h(x) = x + \sin(x)$.

(a) Sketch a graph of $y = h(x)$ and explain why h must be invertible.

(b) Explain why it does not appear to be algebraically possible to determine a formula for h^{-1}.

(c) Observe that the point $(\frac{\pi}{2}, \frac{\pi}{2} + 1)$ lies on the graph of $y = h(x)$. Determine the value of $(h^{-1})'(\frac{\pi}{2} + 1)$.

2.7 Derivatives of Functions Given Implicitly

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What does it mean to say that a curve is an implicit function of x, rather than an explicit function of x?
- How does implicit differentiation enable us to find a formula for $\frac{dy}{dx}$ when y is an implicit function of x?
- In the context of an implicit curve, how can we use $\frac{dy}{dx}$ to answer important questions about the tangent line to the curve?

Introduction

In all of our studies with derivatives to date, we have worked in a setting where we can express a formula for the function of interest explicitly in terms of x. But there are many interesting curves that are determined by an equation involving x and y for which it is impossible to solve for y in terms of x. Perhaps the simplest and most natural of all such

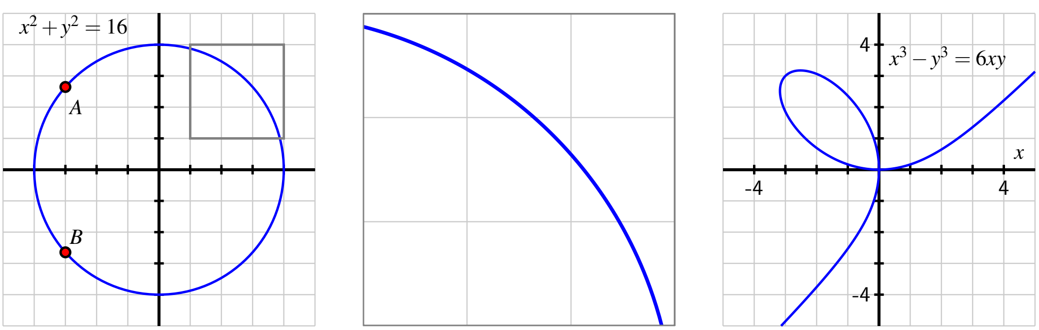

Figure 2.14: At left, the circle given by $x^2 + y^2 = 16$. In the middle, the portion of the circle $x^2 + y^2 = 16$ that has been highlighted in the box at left. And at right, the lemniscate given by $x^3 - y^3 = 6xy$.

curves are circles. Because of the circle's symmetry, for each x value strictly between the endpoints of the horizontal diameter, there are two corresponding y-values. For instance, in Figure 2.14, we have labeled $A = (-3, \sqrt{7})$ and $B = (-3, -\sqrt{7})$, and these points demonstrate that the circle fails the vertical line test. Hence, it is impossible to represent the circle through a single function of the form $y = f(x)$. At the same time, portions of the circle can be represented explicitly as a function of x, such as the highlighted arc that is magnified in the center of Figure 2.14. Moreover, it is evident that the circle is locally linear, so we ought to be able to find a tangent line to the curve at every point; thus, it makes sense to wonder if we can compute $\frac{dy}{dx}$ at any point on the circle, even though we cannot write y explicitly as a function of x. Finally, we note that the righthand curve in Figure 2.14 is called a *lemniscate* and is just one of many fascinating possibilities for implicitly given curves.

In working with implicit functions, we will often be interested in finding an equation for $\frac{dy}{dx}$ that tells us the slope of the tangent line to the curve at a point (x, y). To do so, it will be necessary for us to work with y while thinking of y as a function of x, but without being able to write an explicit formula for y in terms of x. The following preview activity reminds us of some ways we can compute derivatives of functions in settings where the function's formula is not known. For instance, recall the earlier example $\frac{d}{dx}[e^{u(x)}] = e^{u(x)}u'(x)$.

Preview Activity 2.7. Let f be a differentiable function of x (whose formula is not known) and recall that $\frac{d}{dx}[f(x)]$ and $f'(x)$ are interchangeable notations. Determine each of the following derivatives of combinations of explicit functions of x, the unknown function f, and an arbitrary constant c.

(a) $\dfrac{d}{dx}\left[x^2 + f(x)\right]$

(b) $\dfrac{d}{dx}\left[x^2 f(x)\right]$

(c) $\dfrac{d}{dx}\left[c + x + f(x)^2\right]$

(d) $\dfrac{d}{dx}\left[f(x^2)\right]$

(e) $\dfrac{d}{dx}\left[xf(x) + f(cx) + cf(x)\right]$

⋈

Implicit Differentiation

Because a circle is perhaps the simplest of all curves that cannot be represented explicitly as a single function of x, we begin our exploration of implicit differentiation with the example of the circle given by $x^2 + y^2 = 16$. It is visually apparent that this curve is locally linear, so it makes sense for us to want to find the slope of the tangent line to the curve at any point, and moreover to think that the curve is differentiable. The big question is: how do we find a formula for $\frac{dy}{dx}$, the slope of the tangent line to the circle at a given point on the circle? By viewing y as an ***implicit***[10] function of x, we essentially think of y as some function whose formula $f(x)$ is unknown, but which we can differentiate. Just as y represents an unknown formula, so too its derivative with respect to x, $\frac{dy}{dx}$, will be (at least temporarily) unknown.

Consider the equation $x^2 + y^2 = 16$ and view y as an unknown differentiable function of x. Differentiating both sides of the equation with respect to x, we have

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}\left[16\right].$$

On the right, the derivative of the constant 16 is 0, and on the left we can apply the sum rule, so it follows that

$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = 0.$$

Next, it is essential that we recognize the different roles being played by x and y. Since x is the independent variable, it is the variable with respect to which we are differentiating, and thus $\frac{d}{dx}\left[x^2\right] = 2x$. But y is the dependent variable and y is an implicit function of x. Thus, when we want to compute $\frac{d}{dx}[y^2]$ it is identical to the situation in Preview Activity 2.7 where we computed $\frac{d}{dx}[f(x)^2]$. In both situations, we have an unknown function being squared, and we seek the derivative of the result. This requires the chain rule, by which we find that $\frac{d}{dx}[y^2] = 2y^1\frac{dy}{dx}$. Therefore, continuing our work in differentiating both sides of $x^2 + y^2 = 16$, we now have that

$$2x + 2y\frac{dy}{dx} = 0.$$

Since our goal is to find an expression for $\frac{dy}{dx}$, we solve this most recent equation for $\frac{dy}{dx}$. Subtracting $2x$ from both sides and dividing by $2y$,

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

There are several important things to observe about the result that $\frac{dy}{dx} = -\frac{x}{y}$. First, this expression for the derivative involves both x and y. It makes sense that this should be the case, since for each value of x between -4 and 4, there are two corresponding points on the circle, and the slope of the tangent line is different at each of these points. Second, this formula is entirely consistent with our understanding of circles. If we consider the radius from the origin to the point (a, b), the slope of this line segment is $m_r = \frac{b}{a}$. The tangent line to the circle at (a, b) will be perpendicular to the radius, and thus have slope $m_t = -\frac{a}{b}$, as shown in Figure 2.15. Finally, the slope of the tangent line is zero at $(0, 4)$ and

[10]Essentially the idea of an implicit function is that it can be broken into pieces where each piece can be viewed as an explicit function of x, and the combination of those pieces constitutes the full implicit function. For the circle, we could choose to take the top half as one explicit function of x, and the bottom half as another.

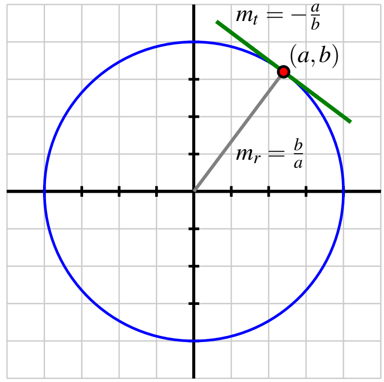

Figure 2.15: The circle given by $x^2 + y^2 = 16$ with point (a, b) on the circle and the tangent line at that point, with labeled slopes of the radial line, m_r, and tangent line, m_t.

$(0, -4)$, and is undefined at $(-4, 0)$ and $(4, 0)$; all of these values are consistent with the formula $\frac{dy}{dx} = -\frac{x}{y}$.

We consider the following more complicated example to investigate and demonstrate some additional algebraic issues that arise in problems involving implicit differentiation.

Example 2.4. For the curve given implicitly by $x^3 + y^2 - 2xy = 2$, shown in Figure 2.16, find the slope of the tangent line at $(-1, 1)$.

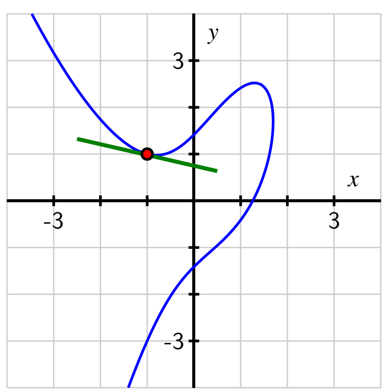

Figure 2.16: The curve $x^3 + y^2 - 2xy = 2$.

Solution. We begin by differentiating the curve's equation implicitly. Taking the derivative of each side with respect to x,

$$\frac{d}{dx}\left[x^3 + y^2 - 2xy\right] = \frac{d}{dx}\left[2\right],$$

by the sum rule and the fact that the derivative of a constant is zero, we have

$$\frac{d}{dx}[x^3] + \frac{d}{dx}[y^2] - \frac{d}{dx}[2xy] = 0.$$

For the three derivatives we now must execute, the first uses the simple power rule, the second requires the chain rule (since y is an implicit function of x), and the third necessitates the product rule (again since y is a function of x). Applying these rules, we now find that

$$3x^2 + 2y\frac{dy}{dx} - [2x\frac{dy}{dx} + 2y] = 0.$$

Remembering that our goal is to find an expression for $\frac{dy}{dx}$ so that we can determine the slope of a particular tangent line, we want to solve the preceding equation for $\frac{dy}{dx}$. To do so, we get all of the terms involving $\frac{dy}{dx}$ on one side of the equation and then factor. Expanding and then subtracting $3x^2 - 2y$ from both sides, it follows that

$$2y\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 3x^2.$$

Factoring the left side to isolate $\frac{dy}{dx}$, we have

$$\frac{dy}{dx}(2y - 2x) = 2y - 3x^2.$$

Finally, we divide both sides by $(2y - 2x)$ and conclude that

$$\frac{dy}{dx} = \frac{2y - 3x^2}{2y - 2x}.$$

Here again, the expression for $\frac{dy}{dx}$ depends on both x and y. To find the slope of the tangent line at $(-1, 1)$, we substitute this point in the formula for $\frac{dy}{dx}$, using the notation

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{2(1) - 3(-1)^2}{2(1) - 2(-1)} = -\frac{1}{4}.$$

This value matches our visual estimate of the slope of the tangent line shown in Figure 2.16.

Example 2.4 shows that it is possible when differentiating implicitly to have multiple terms involving $\frac{dy}{dx}$. Regardless of the particular curve involved, our approach will be similar each time. After differentiating, we expand so that each side of the equation is a sum of terms, some of which involve $\frac{dy}{dx}$. Next, addition and subtraction are used to get all terms involving $\frac{dy}{dx}$ on one side of the equation, with all remaining terms on the other. Finally, we factor to get a single instance of $\frac{dy}{dx}$, and then divide to solve for $\frac{dy}{dx}$.

Note, too, that since $\frac{dy}{dx}$ is often a function of both x and y, we use the notation

$$\left.\frac{dy}{dx}\right|_{(a,b)}$$

to denote the evaluation of $\frac{dy}{dx}$ at the point (a, b). This is analogous to writing $f'(a)$ when f' depends on a single variable.

Finally, there is a big difference between writing $\frac{d}{dx}$ and $\frac{dy}{dx}$. For example,

$$\frac{d}{dx}[x^2 + y^2]$$

gives an instruction to take the derivative with respect to x of the quantity $x^2 + y^2$, presumably where y is a function of x. On the other hand,

$$\frac{dy}{dx}(x^2 + y^2)$$

means the product of the derivative of y with respect to x with the quantity $x^2 + y^2$. Understanding this notational subtlety is essential.

The following activities present opportunities to explore several different problems involving implicit differentiation.

Activity 2.19.

Consider the curve defined by the equation $x = y^5 - 5y^3 + 4y$, whose graph is pictured in Figure 2.17.

(a) Explain why it is not possible to express y as an explicit function of x.

(b) Use implicit differentiation to find a formula for dy/dx.

(c) Use your result from part (b) to find an equation of the line tangent to the graph of $x = y^5 - 5y^3 + 4y$ at the point $(0, 1)$.

(d) Use your result from part (b) to determine all of the points at which the graph of $x = y^5 - 5y^3 + 4y$ has a vertical tangent line.

◁

Two natural questions to ask about any curve involve where the tangent line can be vertical or horizontal. To be horizontal, the slope of the tangent line must be zero, while to be vertical, the slope must be undefined. It is typically the case when differentiating

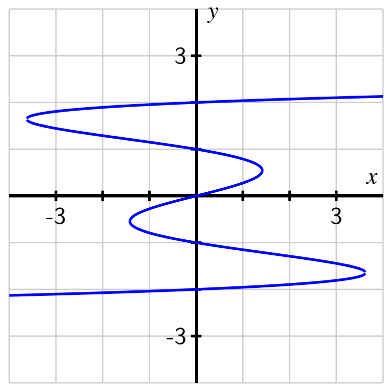

Figure 2.17: The curve $x = y^5 - 5y^3 + 4y$.

implicitly that the formula for $\frac{dy}{dx}$ is expressed as a quotient of functions of x and y, say

$$\frac{dy}{dx} = \frac{p(x, y)}{q(x, y)}.$$

Thus, we observe that the tangent line will be horizontal precisely when the numerator is zero and the denominator is nonzero, making the slope of the tangent line zero. Similarly, the tangent line will be vertical whenever $q(x, y) = 0$ and $p(x, y) \neq 0$, making the slope undefined. If both x and y are involved in an equation such as $p(x, y) = 0$, we try to solve for one of them in terms of the other, and then use the resulting condition in the original equation that defines the curve to find an equation in a single variable that we can solve to determine the point(s) that lie on the curve at which the condition holds. It is not always possible to execute the desired algebra due to the possibly complicated combinations of functions that often arise.

Activity 2.20.

Consider the curve defined by the equation $y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$, whose graph is pictured in Figure 2.18. Through implicit differentiation, it can be shown that

$$\frac{dy}{dx} = \frac{(x-1)(x-2) + x(x-2) + x(x-1)}{(y^2-1)(y-2) + 2y^2(y-2) + y(y^2-1)}.$$

Use this fact to answer each of the following questions.

(a) Determine all points (x, y) at which the tangent line to the curve is horizontal. (Use technology appropriately to find the needed zeros of the relevant polynomial function.)

(b) Determine all points (x, y) at which the tangent line is vertical. (Use technology appropriately to find the needed zeros of the relevant polynomial function.)

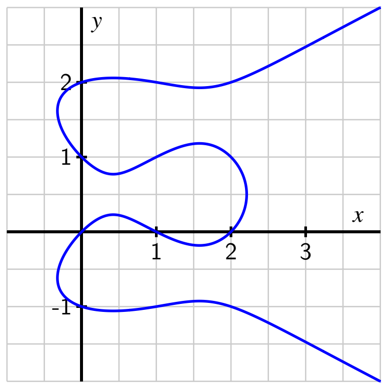

Figure 2.18: The curve $y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$.

(c) Find the equation of the tangent line to the curve at one of the points where $x = 1$.

◁

The closing activity in this section offers more opportunities to practice implicit differentiation.

Activity 2.21.

For each of the following curves, use implicit differentiation to find dy/dx and determine the equation of the tangent line at the given point.

(a) $x^3 - y^3 = 6xy, \quad (-3, 3)$

(b) $\sin(y) + y = x^3 + x, \quad (0, 0)$

(c) $3xe^{-xy} = y^2, \quad (0.619061, 1)$

◁

Summary

In this section, we encountered the following important ideas:

- When we have an equation involving x and y where y cannot be solved for explicitly in terms of x, but where portions of the curve can be thought of as being generated by explicit functions of x, we say that y is an implicit function of x. A good example of such a curve is the unit circle.
- In the process of implicit differentiation, we take the equation that generates an implicitly given curve and differentiate both sides with respect to x while treating y as a function of x. In so doing, the chain rule leads $\frac{dy}{dx}$ to arise, and then we may subsequently solve for $\frac{dy}{dx}$ using algebra.

- While $\frac{dy}{dx}$ may now involve both the variables x and y, $\frac{dy}{dx}$ still measures the slope of the tangent line to the curve, and thus this derivative may be used to decide when the tangent line is horizontal ($\frac{dy}{dx} = 0$) or vertical ($\frac{dy}{dx}$ is undefined), or to find the equation of the tangent line at a particular point on the curve.

Exercises

1. Consider the curve given by the equation $2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$. Find all points at which the tangent line to the curve is horizontal or vertical. Be sure to use a graphing utility to plot this implicit curve and to visually check the results of algebraic reasoning that you use to determine where the tangent lines are horizontal and vertical.

2. For the curve given by the equation $\sin(x + y) + \cos(x - y) = 1$, find the equation of the tangent line to the curve at the point $(\frac{\pi}{2}, \frac{\pi}{2})$.

3. Implicit differentiation enables us a different perspective from which to see why the rule $\frac{d}{dx}[a^x] = a^x \ln(a)$ holds, if we assume that $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$. This exercise leads you through the key steps to do so.

   (a) Let $y = a^x$. Rewrite this equation using the natural logarithm function to write x in terms of y (and the constant a).

   (b) Differentiate both sides of the equation you found in (a) with respect to x, keeping in mind that y is implicitly a function of x.

   (c) Solve the equation you found in (b) for $\frac{dy}{dx}$, and then use the definition of y to write $\frac{dy}{dx}$ solely in terms of x. What have you found?

2.8 Using Derivatives to Evaluate Limits

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can derivatives be used to help us evaluate indeterminate limits of the form $\frac{0}{0}$?
- What does it mean to say that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to a} f(x) = \infty$?
- How can derivatives assist us in evaluating indeterminate limits of the form $\frac{\infty}{\infty}$?

Introduction

Because differential calculus is based on the definition of the derivative, and the definition of the derivative involves a limit, there is a sense in which all of calculus rests on limits. In addition, the limit involved in the limit definition of the derivative is one that always generates an indeterminate form of $\frac{0}{0}$. If f is a differentiable function for which $f'(x)$ exists, then when we consider

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

it follows that not only does $h \to 0$ in the denominator, but also $(f(x+h) - f(x)) \to 0$ in the numerator, since f is continuous. Thus, the fundamental form of the limit involved in the definition of $f'(x)$ is $\frac{0}{0}$. Remember, saying a limit has an indeterminate form only means that we don't yet know its value and have more work to do: indeed, limits of the form $\frac{0}{0}$ can take on any value, as is evidenced by evaluating $f'(x)$ for varying values of x for a function such as $f'(x) = x^2$.

Of course, we have learned many different techniques for evaluating the limits that result from the derivative definition, and including a large number of shortcut rules that enable us to evaluate these limits quickly and easily. In this section, we turn the situation upside-down: rather than using limits to evaluate derivatives, we explore how to use derivatives to evaluate certain limits. This topic will combine several different ideas, including limits, derivative shortcuts, local linearity, and the tangent line approximation.

Preview Activity 2.8. Let h be the function given by $h(x) = \dfrac{x^5 + x - 2}{x^2 - 1}$.

(a) What is the domain of h?

(b) Explain why $\displaystyle\lim_{x\to 1} \frac{x^5 + x - 2}{x^2 - 1}$ results in an indeterminate form.

(c) Next we will investigate the behavior of both the numerator and denominator of h near the point where $x = 1$. Let $f(x) = x^5 + x - 2$ and $g(x) = x^2 - 1$. Find the local linearizations of f and g at $a = 1$, and call these functions $L_f(x)$ and $L_g(x)$, respectively.

(d) Explain why $h(x) \approx \dfrac{L_f(x)}{L_g(x)}$ for x near $a = 1$.

(e) Using your work from (c) and (d), evaluate

$$\lim_{x \to 1} \frac{L_f(x)}{L_g(x)}.$$

What do you think your result tells us about $\lim_{x \to 1} h(x)$?

(f) Investigate the function $h(x)$ graphically and numerically near $x = 1$. What do you think is the value of $\lim_{x \to 1} h(x)$?

⋈

Using derivatives to evaluate indeterminate limits of the form $\frac{0}{0}$.

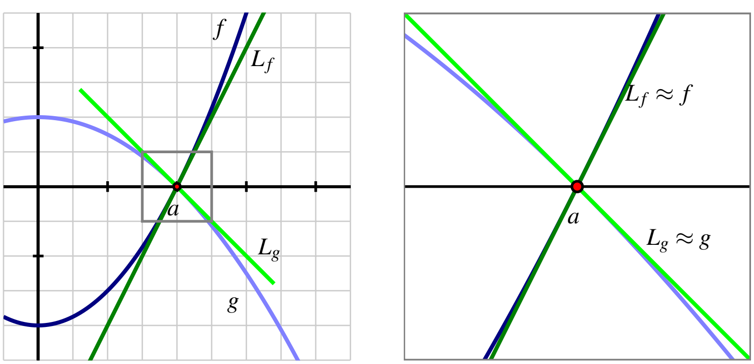

Figure 2.19: At left, the graphs of f and g near the value a, along with their tangent line approximations L_f and L_g at $x = a$. At right, zooming in on the point a and the four graphs.

The fundamental idea of Preview Activity 2.8 – that we can evaluate an indeterminate limit of the form $\frac{0}{0}$ by replacing each of the numerator and denominator with their local linearizations at the point of interest – can be generalized in a way that enables us to easily evaluate a wide range of limits. We begin by assuming that we have a function

$h(x)$ that can be written in the form $h(x) = \frac{f(x)}{g(x)}$ where f and g are both differentiable at $x = a$ and for which $f(a) = g(a) = 0$. We are interested in finding a way to evaluate the indeterminate limit given by $\lim_{x \to a} h(x)$. In Figure 2.19, we see a visual representation of the situation involving such functions f and g. In particular, we see that both f and g have an x-intercept at the point where $x = a$. In addition, since each function is differentiable, each is locally linear, and we can find their respective tangent line approximations L_f and L_g at $x = a$, which are also shown in the figure. Since we are interested in the limit of $\frac{f(x)}{g(x)}$ as $x \to a$, the individual behaviors of $f(x)$ and $g(x)$ as $x \to a$ are key to understand. Here, we take advantage of the fact that each function and its tangent line approximation become indistinguishable as $x \to a$.

First, let's reall that $L_f(x) = f'(a)(x - a) + f(a)$ and $L_g(x) = g'(a)(x - a) + g(a)$. The critical observation we make is that when taking the limit, because x is getting arbitrarily close to a, we can replace f with L_f and replace g with L_g, and thus we observe that

$$\begin{aligned} \lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{L_f(x)}{L_g(x)} \\ &= \lim_{x \to a} \frac{f'(a)(x-a) + f(a)}{g'(a)(x-a) + g(a)}. \end{aligned}$$

Next, we remember a key fundamental assumption: that both $f(a) = 0$ and $g(a) = 0$, as this is precisely what makes the original limit indeterminate. Substituting these values for $f(a)$ and $g(a)$ in the limit above, we now have

$$\begin{aligned} \lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} \\ &= \lim_{x \to a} \frac{f'(a)}{g'(a)}, \end{aligned}$$

where the latter equality holds since x is approaching (but not equal to) a, so $\frac{x-a}{x-a} = 1$. Finally, we note that $\frac{f'(a)}{g'(a)}$ is constant with respect to x, and thus

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

We have, of course, implicitly made the assumption that $g'(a) \neq 0$, which is essential to the overall limit having the value $\frac{f'(a)}{g'(a)}$. We summarize our work above with the statement of L'Hopital's Rule, which is the formal name of the result we have shown.

L'Hopital's Rule: Let f and g be differentiable at $x = a$, and suppose that $f(a) = g(a) = 0$ and that $g'(a) \neq 0$. Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.

In practice, we typically work with a slightly more general version of L'Hopital's Rule, which states that (under the identical assumptions as the boxed rule above and the extra assumption that g' is continuous at $x = a$)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the righthand limit exists. This form reflects the fundamental benefit of L'Hopital's Rule: if $\frac{f(x)}{g(x)}$ produces an indeterminate limit of form $\frac{0}{0}$ as $x \to a$, it is equivalent to consider the limit of the quotient of the two functions' derivatives, $\frac{f'(x)}{g'(x)}$. For example, if we consider the limit from Preview Activity 2.8,

$$\lim_{x\to 1}\frac{x^5 + x - 2}{x^2 - 1},$$

by L'Hopital's Rule we have that

$$\lim_{x\to 1}\frac{x^5 + x - 2}{x^2 - 1} = \lim_{x\to 1}\frac{5x^4 + 1}{2x} = \frac{6}{2} = 3.$$

By being able to replace the numerator and denominator with their respective derivatives, we often move from an indeterminate limit to one whose value we can easily determine.

Activity 2.22.

Evaluate each of the following limits. If you use L'Hopital's Rule, indicate where it was used, and be certain its hypotheses are met before you apply it.

(a) $\lim_{x\to 0}\frac{\ln(1+x)}{x}$

(b) $\lim_{x\to \pi}\frac{\cos(x)}{x}$

(c) $\lim_{x\to 1}\frac{2\ln(x)}{1 - e^{x-1}}$

(d) $\lim_{x\to 0}\frac{\sin(x) - x}{\cos(2x) - 1}$

◁

While L'Hopital's Rule can be applied in an entirely algebraic way, it is important to remember that the genesis of the rule is graphical: the main idea is that the slopes of the tangent lines to f and g at $x = a$ determine the value of the limit of $\frac{f(x)}{g(x)}$ as $x \to a$. We see this in Figure 2.20, which is a modified version of Figure 2.19, where we can see from the grid that $f'(a) = 2$ and $g'(a) = -1$, hence by L'Hopital's Rule,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \frac{2}{-1} = -2.$$

Indeed, what we observe is that it's not the fact that f and g both approach zero that

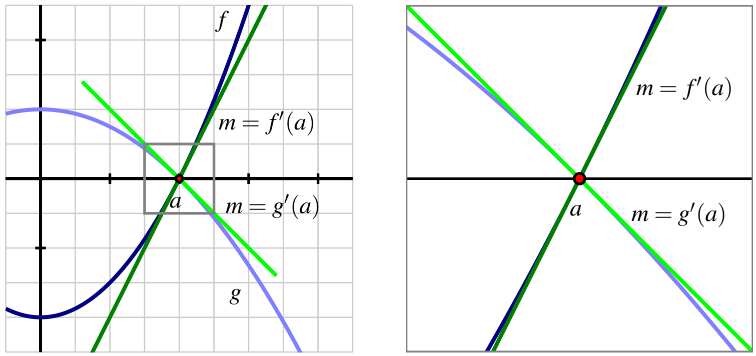

Figure 2.20: Two functions f and g that satisfy L'Hopital's Rule.

matters most, but rather the *rate* at which each approaches zero that determines the value of the limit. This is a good way to remember what L'Hopital's Rule says: if $f(a) = g(a) = 0$, the the limit of $\frac{f(x)}{g(x)}$ as $x \to a$ is given by the ratio of the slopes of f and g at $x = a$.

Activity 2.23.

In this activity, we reason graphically from the following figure to evaluate limits of ratios of functions about which some information is known.

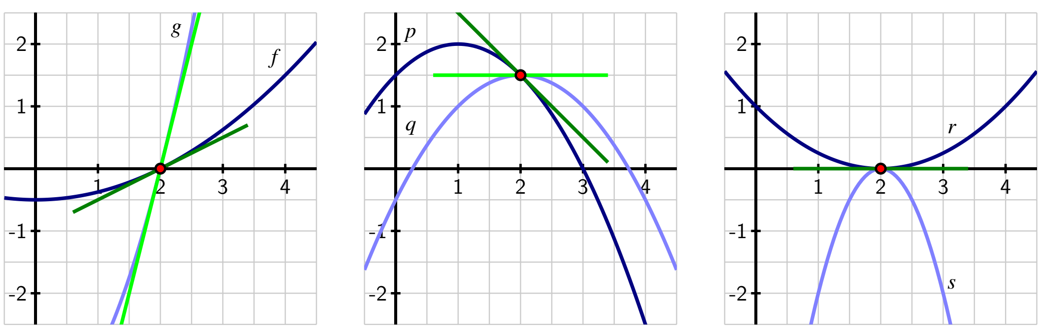

Figure 2.21: Three graphs referenced in the questions of Activity 2.23.

(a) Use the left-hand graph to determine the values of $f(2)$, $f'(2)$, $g(2)$, and $g'(2)$. Then, evaluate

$$\lim_{x \to 2} \frac{f(x)}{g(x)}.$$

(b) Use the middle graph to find $p(2)$, $p'(2)$, $q(2)$, and $q'(2)$. Then, determine the value of

$$\lim_{x\to 2} \frac{p(x)}{q(x)}.$$

(c) Use the right-hand graph to compute $r(2)$, $r'(2)$, $s(2)$, $s'(2)$. Explain why you cannot determine the exact value of

$$\lim_{x\to 2} \frac{r(x)}{s(x)}$$

without further information being provided, but that you can determine the sign of $\lim_{x\to 2} \frac{r(x)}{s(x)}$. In addition, state what the sign of the limit will be, with justification.

◁

Limits involving ∞

The concept of infinity, denoted ∞, arises naturally in calculus, like it does in much of mathematics. It is important to note from the outset that ∞ is a concept, but not a number itself. Indeed, the notion of ∞ naturally invokes the idea of limits. Consider, for example, the function $f(x) = \frac{1}{x}$, whose graph is pictured in Figure 2.22. We note that $x = 0$ is not

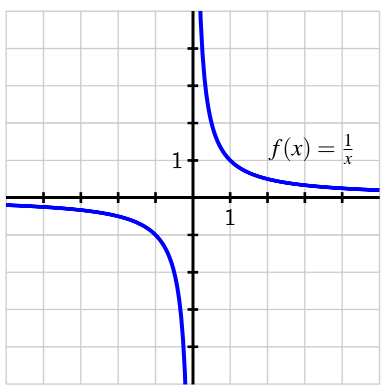

Figure 2.22: The graph of $f(x) = \frac{1}{x}$.

in the domain of f, so we may naturally wonder what happens as $x \to 0$. As $x \to 0^+$, we observe that $f(x)$ *increases without bound*. That is, we can make the value of $f(x)$ as large as we like by taking x closer and closer (but not equal) to 0, while keeping $x > 0$. This is a good way to think about what infinity represents: a quantity is tending to infinity if there is no single number that the quantity is always less than.

Recall that when we write $\lim_{x\to a} f(x) = L$, this means that can make $f(x)$ as close to L

as we'd like by taking x sufficiently close (but not equal) to a. We thus expand this notation and language to include the possibility that either L or a can be ∞. For instance, for $f(x) = \frac{1}{x}$, we now write

$$\lim_{x\to 0^+} \frac{1}{x} = \infty,$$

by which we mean that we can make $\frac{1}{x}$ as large as we like by taking x sufficiently close (but not equal) to 0. In a similar way, we naturally write

$$\lim_{x\to\infty} \frac{1}{x} = 0,$$

since we can make $\frac{1}{x}$ as close to 0 as we'd like by taking x sufficiently large (i.e., by letting x increase without bound).

In general, we understand the notation $\lim\limits_{x\to a} f(x) = \infty$ to mean that we can make $f(x)$ as large as we'd like by taking x sufficiently close (but not equal) to a, and the notation $\lim\limits_{x\to\infty} f(x) = L$ to mean that we can make $f(x)$ as close to L as we'd like by taking x sufficiently large. This notation applies to left- and right-hand limits, plus we can also use limits involving $-\infty$. For example, returning to Figure 2.22 and $f(x) = \frac{1}{x}$, we can say that

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Finally, we write

$$\lim_{x\to\infty} f(x) = \infty$$

when we can make the value of $f(x)$ as large as we'd like by taking x sufficiently large. For example,

$$\lim_{x\to\infty} x^2 = \infty.$$

Note particularly that limits involving infinity identify *vertical* and *horizontal asymptotes* of a function. If $\lim_{x\to a} f(x) = \infty$, then $x = a$ is a vertical asymptote of f, while if $\lim_{x\to\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of f. Similar statements can be made using $-\infty$, as well as with left- and right-hand limits as $x \to a^-$ or $x \to a^+$.

In precalculus classes, it is common to study the *end behavior* of certain families of functions, by which we mean the behavior of a function as $x \to \infty$ and as $x \to -\infty$. Here we briefly examine a library of some familiar functions and note the values of several limits involving ∞.

For the natural exponential function e^x, we note that $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$, while for the related exponential decay function e^{-x}, observe that these limits are reversed, with $\lim_{x\to\infty} e^{-x} = 0$ and $\lim_{x\to-\infty} e^{-x} = \infty$. Turning to the natural logarithm function, we have $\lim_{x\to 0^+} \ln(x) = -\infty$ and $\lim_{x\to\infty} \ln(x) = \infty$. While both e^x and $\ln(x)$ grow without bound as $x \to \infty$, the exponential function does so much more quickly than the logarithm function does. We'll soon use limits to quantify what we mean by "quickly."

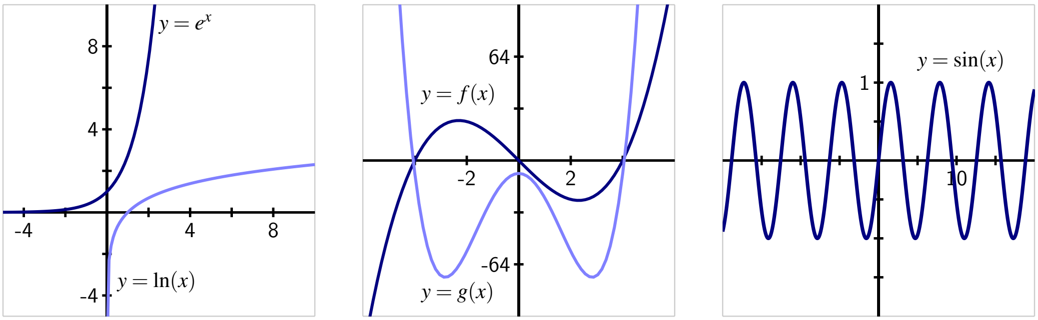

Figure 2.23: Graphs of some familiar functions whose end behavior as $x \to \pm\infty$ is known. In the middle graph, $f(x) = x^3 - 16x$ and $g(x) = x^4 - 16x^2 - 8$.

For polynomial functions of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$, the end behavior depends on the sign of a_n and whether the highest power n is even or odd. If n is even and a_n is positive, then $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = \infty$, as in the plot of g in Figure 2.23. If instead a_n is negative, then $\lim_{x\to\infty} p(x) = -\infty$ and $\lim_{x\to-\infty} p(x) = -\infty$. In the situation where n is odd, then either $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = -\infty$ (which occurs when a_n is positive, as in the graph of f in Figure 2.23), or $\lim_{x\to\infty} p(x) = -\infty$ and $\lim_{x\to-\infty} p(x) = \infty$ (when a_n is negative).

A function can fail to have a limit as $x \to \infty$. For example, consider the plot of the sine function at right in Figure 2.23. Because the function continues oscillating between -1 and 1 as $x \to \infty$, we say that $\lim_{x\to\infty} \sin(x)$ does not exist.

Finally, it is straightforward to analyze the behavior of any rational function as $x \to \infty$. Consider, for example, the function

$$q(x) = \frac{3x^2 - 4x + 5}{7x^2 + 9x - 10}.$$

Note that both $(3x^2 - 4x + 5) \to \infty$ as $x \to \infty$ and $(7x^2 + 9x - 10) \to \infty$ as $x \to \infty$. Here we say that $\lim_{x\to\infty} q(x)$ has indeterminate form $\frac{\infty}{\infty}$, much like we did when we encountered limits of the form $\frac{0}{0}$. We can determine the value of this limit through a standard algebraic approach. Multiplying the numerator and denominator each by $\frac{1}{x^2}$, we find that

$$\begin{aligned} \lim_{x\to\infty} q(x) &= \lim_{x\to\infty} \frac{3x^2 - 4x + 5}{7x^2 + 9x - 10} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{3 - 4\frac{1}{x} + 5\frac{1}{x^2}}{7 + 9\frac{1}{x} - 10\frac{1}{x^2}} = \frac{3}{7} \end{aligned}$$

since $\frac{1}{x^2} \to 0$ and $\frac{1}{x} \to 0$ as $x \to \infty$. This shows that the rational function q has a horizontal asymptote at $y = \frac{3}{7}$. A similar approach can be used to determine the limit of any rational function as $x \to \infty$.

But how should we handle a limit such as

$$\lim_{x\to\infty} \frac{x^2}{e^x}?$$

Here, both $x^2 \to \infty$ and $e^x \to \infty$, but there is not an obvious algebraic approach that enables us to find the limit's value. Fortunately, it turns out that L'Hopital's Rule extends to cases involving infinity.

L'Hopital's Rule (∞): If f and g are differentiable and both approach zero or both approach $\pm\infty$ as $x \to a$ (where a is allowed to be ∞) , then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

(To be technically correct, we need to the additional hypothesis that $g'(x) \neq 0$ on an open interval that contains a or in every neighborhood of infinity if a is ∞; this is almost always met in practice.)

To evaluate $\lim_{x\to\infty} \frac{x^2}{e^x}$, we observe that we can apply L'Hopital's Rule, since both $x^2 \to \infty$ and $e^x \to \infty$. Doing so, it follows that

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

This updated limit is still indeterminate and of the form $\frac{\infty}{\infty}$, but it is simpler since $2x$ has replaced x^2. Hence, we can apply L'Hopital's Rule again, by which we find that

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x}.$$

Now, since 2 is constant and $e^x \to \infty$ as $x \to \infty$, it follows that $\frac{2}{e^x} \to 0$ as $x \to \infty$, which shows that

$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0.$$

Activity 2.24.

Evaluate each of the following limits. If you use L'Hopital's Rule, indicate where it was used, and be certain its hypotheses are met before you apply it.

(a) $\lim_{x\to\infty} \frac{x}{\ln(x)}$

(b) $\displaystyle\lim_{x\to\infty} \frac{e^x + x}{2e^x + x^2}$

(c) $\displaystyle\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}$

(d) $\displaystyle\lim_{x\to \frac{\pi}{2}^-} \frac{\tan(x)}{x - \frac{\pi}{2}}$

(e) $\displaystyle\lim_{x\to\infty} xe^{-x}$

◁

When we are considering the limit of a quotient of two functions $\frac{f(x)}{g(x)}$ that results in an indeterminate form of $\frac{\infty}{\infty}$, in essence we are asking which function is growing faster without bound. We say that the function g ***dominates*** the function f as $x \to \infty$ provided that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,$$

whereas f dominates g provided that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$. Finally, if the value of $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is finite and nonzero, we say that f and g ***grow at the same rate***. For example, from earlier work we know that $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$, so e^x dominates x^2, while $\lim_{x\to\infty} \frac{3x^2-4x+5}{7x^2+9x-10} = \frac{3}{7}$, so $f(x) = 3x^2 - 4x + 5$ and $g(x) = 7x^2 + 9x - 10$ grow at the same rate.

Summary

In this section, we encountered the following important ideas:

- Derivatives be used to help us evaluate indeterminate limits of the form $\frac{0}{0}$ through L'Hopital's Rule, which is developed by replacing the functions in the numerator and denominator with their tangent line approximations. In particular, if $f(a) = g(a) = 0$ and f and g are differentiable at a, L'Hopital's Rule tells us that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

- When we write $x \to \infty$, this means that x is increasing without bound. We thus use ∞ along with limit notation to write $\lim_{x\to\infty} f(x) = L$, which means we can make $f(x)$ as close to L as we like by choosing x to be sufficiently large, and similarly $\lim_{x\to a} f(x) = \infty$, which means we can make $f(x)$ as large as we like by choosing x sufficiently close to a.

- A version of L'Hopital's Rule also allows us to use derivatives to assist us in evaluating indeterminate limits of the form $\frac{\infty}{\infty}$. In particular, If f and g are differentiable and both approach zero or both approach $\pm\infty$ as $x \to a$ (where a is allowed to be ∞), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Exercises

1. Let f and g be differentiable functions about which the following information is known: $f(3) = g(3) = 0$, $f'(3) = g'(3) = 0$, $f''(3) = -2$, and $g''(3) = 1$. Let a new function h be given by the rule $h(x) = \frac{f(x)}{g(x)}$. On the same set of axes, sketch possible graphs of f and g near $x = 3$, and use the provided information to determine the value of

$$\lim_{x\to 3} h(x).$$

Provide explanation to support your conclusion.

2. Find all vertical and horizontal asymptotes of the function

$$R(x) = \frac{3(x-a)(x-b)}{5(x-a)(x-c)},$$

where a, b, and c are distinct, arbitrary constants. In addition, state all values of x for which R is not continuous. Sketch a possible graph of R, clearly labeling the values of a, b, and c.

3. Consider the function $g(x) = x^{2x}$, which is defined for all $x > 0$. Observe that $\lim_{x\to 0^+} g(x)$ is indeterminate due to its form of 0^0. (Think about how we know that $0^k = 0$ for all $k > 0$, while $b^0 = 1$ for all $b \neq 0$, but that neither rule can apply to 0^0.)

(a) Let $h(x) = \ln(g(x))$. Explain why $h(x) = 2x\ln(x)$.

(b) Next, explain why it is equivalent to write $h(x) = \frac{2\ln(x)}{\frac{1}{x}}$.

(c) Use L'Hopital's Rule and your work in (b) to compute $\lim_{x\to 0^+} h(x)$.

(d) Based on the value of $\lim_{x\to 0^+} h(x)$, determine $\lim_{x\to 0^+} g(x)$.

4. Recall we say that function g dominates function f provided that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

(a) Which function dominates the other: $\ln(x)$ or $\sqrt{x}$?

(b) Which function dominates the other: $\ln(x)$ or $\sqrt[n]{x}$? (n can be any positive integer)

(c) Explain why e^x will dominate any polynomial function.

(d) Explain why x^n will dominate $\ln(x)$ for any positive integer n.

(e) Give any example of two nonlinear functions such that neither dominates the other.