

Chapter 8

Sequences and Series

8.1 Sequences

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a sequence?
- What does it mean for a sequence to converge?
- What does it mean for a sequence to diverge?

Introduction

We encounter sequences every day. Your monthly rent payments, the annual interest you earn on investments, a list of your car's miles per gallon every time you fill up; all are examples of sequences. Other sequences with which you may be familiar include the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, \dots$$

in which each entry is the sum of the two preceding entries and the triangular numbers

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \dots$$

which are numbers that correspond to the number of vertices seen in the triangles in Figure 8.1. Sequences of integers are of such interest to mathematicians and others that

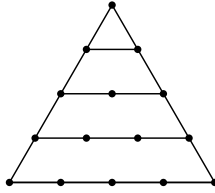


Figure 8.1: Triangular numbers

they have a journal¹ devoted to them and an on-line encyclopedia² that catalogs a huge number of integer sequences and their connections. Sequences are also used in digital recordings and digital images.

To this point, most of our studies in calculus have dealt with continuous information (e.g., continuous functions). The major difference we will see now is that sequences model *discrete* instead of continuous information. We will study ways to represent and work with discrete information in this chapter as we investigate *sequences* and *series*, and ultimately see key connections between the discrete and continuous.

Preview Activity 8.1. Suppose you receive \$5000 through an inheritance. You decide to invest this money into a fund that pays 8% annually, compounded monthly. That means that each month your investment earns $\frac{0.08}{12} \cdot P$ additional dollars, where P is your principal balance at the start of the month. So in the first month your investment earns

$$5000 \left(\frac{0.08}{12} \right)$$

or \$33.33. If you reinvest this money, you will then have \$5033.33 in your account at the end of the first month. From this point on, assume that you reinvest all of the interest you earn.

- How much interest will you earn in the second month? How much money will you have in your account at the end of the second month?
- Complete Table 8.1 to determine the interest earned and total amount of money in this investment each month for one year.
- As we will see later, the amount of money P_n in the account after month n is given by

$$P_n = 5000 \left(1 + \frac{0.08}{12} \right)^n.$$

Use this formula to check your calculations in Table 8.1. Then find the amount of money in the account after 5 years.

¹The Journal of Integer Sequences at <http://www.cs.uwaterloo.ca/journals/JIS/>

²The On-Line Encyclopedia of Integer Sequences at <http://oeis.org/>

Month	Interest earned	Total amount of money in the account
0	\$0	\$5000.00
1	\$33.33	\$5033.33
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		

Table 8.1: Interest

(d) How many years will it be before the account has doubled in value to \$10000?

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Sequences

As our discussion in the introduction and Preview Activity 8.1 illustrate, many discrete phenomena can be represented as lists of numbers (like the amount of money in an account over a period of months). We call these any such list a *sequence*. In other words, a sequence is nothing more than list of terms in some order. To be able to refer to a sequence in a general sense, we often list the entries of the sequence with subscripts,

$$s_1, s_2, \dots, s_n \dots,$$

where the subscript denotes the position of the entry in the sequence. More formally,

Definition 8.1. A sequence is a list of terms s_1, s_2, s_3, \dots in a specified order.

As an alternative to Definition 8.1, we can also consider a sequence to be a function f

whose domain is the set of positive integers. In this context, the sequence s_1, s_2, s_3, \dots would correspond to the function f satisfying $f(n) = s_n$ for each positive integer n . This alternative view will be useful in many situations.

We will often write the sequence

$$s_1, s_2, s_3, \dots$$

using the shorthand notation $\{s_n\}$. The value s_n (alternatively $s(n)$) is called the n th *term* in the sequence. If the terms are all 0 after some fixed value of n , we say the sequence is finite. Otherwise the sequence is infinite. We will work with both finite and infinite sequences, but focus more on the infinite sequences. With infinite sequences, we are often interested in their end behavior and the idea of *convergent* sequences.

Activity 8.1.

- (a) Let s_n be the n th term in the sequence $1, 2, 3, \dots$

Find a formula for s_n and use appropriate technological tools to draw a graph of entries in this sequence by plotting points of the form (n, s_n) for some values of n . Most graphing calculators can plot sequences; directions follow for the TI-84.

- In the MODE menu, highlight SEQ in the FUNC line and press ENTER.
- In the Y= menu, you will now see lines to enter sequences. Enter a value for $n\text{Min}$ (where the sequence starts), a function for $u(n)$ (the n th term in the sequence), and the value of $u_{n\text{Min}}$.
- Set your window coordinates (this involves choosing limits for n as well as the window coordinates XMin, XMax, YMin, and YMax).
- The GRAPH key will draw a plot of your sequence.

Using your knowledge of limits of continuous functions as $x \rightarrow \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \rightarrow \infty$. Explain your reasoning.

- (b) Let s_n be the n th term in the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots$. Find a formula for s_n . Draw a graph of some points in this sequence. Using your knowledge of limits of continuous functions as $x \rightarrow \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \rightarrow \infty$. Explain your reasoning.
- (c) Let s_n be the n th term in the sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots$. Find a formula for s_n . Using your knowledge of limits of continuous functions as $x \rightarrow \infty$, decide if this sequence $\{s_n\}$ has a limit as $n \rightarrow \infty$. Explain your reasoning.

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Next we formalize the ideas from Activity 8.1.

Activity 8.2.

- (a) Recall our earlier work with limits involving infinity in Section 2.8. State clearly what it means for a continuous function f to have a limit L as $x \rightarrow \infty$.
- (b) Given that an infinite sequence of real numbers is a function from the integers to the real numbers, apply the idea from part (a) to explain what you think it means for a sequence $\{s_n\}$ to have a limit as $n \rightarrow \infty$.
- (c) Based on your response to (b), decide if the sequence $\{\frac{1+n}{2+n}\}$ has a limit as $n \rightarrow \infty$. If so, what is the limit? If not, why not?

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In Activities 8.1 and 8.2 we investigated the notion of a sequence $\{s_n\}$ having a limit as n goes to infinity. If a sequence $\{s_n\}$ has a limit as n goes to infinity, we say that the sequence *converges* or is a *convergent sequence*. If the limit of a convergent sequence is the number L , we use the same notation as we did for continuous functions and write

$$\lim_{n \rightarrow \infty} s_n = L.$$

If a sequence $\{s_n\}$ does not converge then we say that the sequence $\{s_n\}$ *diverges*. Convergence of sequences is a major idea in this section and we describe it more formally as follows.

A sequence $\{s_n\}$ of real numbers converges to a number L if we can make all values of s_k for $k \geq n$ as close to L as we want by choosing n to be sufficiently large.

Remember, the idea of sequence having a limit as $n \rightarrow \infty$ is the same as the idea of a continuous function having a limit as $x \rightarrow \infty$. The only new wrinkle here is that our sequences are discrete instead of continuous.

We conclude this section with a few more examples in the following activity.

Activity 8.3.

Use graphical and/or algebraic methods to determine whether each of the following sequences converges or diverges.

- (a) $\{\frac{1+2n}{3n-2}\}$
- (b) $\{\frac{5+3^n}{10+2^n}\}$
- (c) $\{\frac{10^n}{n!}\}$ (where ! is the *factorial* symbol and $n! = n(n-1)(n-2) \cdots (2)(1)$ for any positive integer n (as convention we define $0!$ to be 1)).

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Summary

In this section, we encountered the following important ideas:

- A sequence is a list of objects in a specified order. We will typically work with sequences of real numbers and can also think of a sequence as a function from the positive integers to the set of real numbers.
- A sequence $\{s_n\}$ of real numbers converges to a number L if we can make every value of s_k for $k \geq n$ as close as we want to L by choosing n sufficiently large.
- A sequence diverges if it does not converge.

Exercises

1. Finding limits of convergent sequences can be a challenge. However, there is a useful tool we can adapt from our study of limits of continuous functions at infinity to use to find limits of sequences. We illustrate in this exercise with the example of the sequence

$$\frac{\ln(n)}{n}.$$

- (a) Calculate the first 10 terms of this sequence. Based on these calculations, do you think the sequence converges or diverges? Why?
- (b) For this sequence, there is a corresponding continuous function f defined by

$$f(x) = \frac{\ln(x)}{x}.$$

Draw the graph of $f(x)$ on the interval $[0, 10]$ and then plot the entries of the sequence on the graph. What conclusion do you think we can draw about the sequence $\left\{\frac{\ln(n)}{n}\right\}$ if $\lim_{x \rightarrow \infty} f(x) = L$? Explain.

- (c) Note that $f(x)$ has the indeterminate form $\frac{\infty}{\infty}$ as x goes to infinity. What idea from differential calculus can we use to calculate $\lim_{x \rightarrow \infty} f(x)$? Use this method to find $\lim_{x \rightarrow \infty} f(x)$. What, then, is $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n}$?
2. We return to the example begun in Preview Activity 8.1 to see how to derive the formula for the amount of money in an account at a given time. We do this in a general setting. Suppose you invest P dollars (called the *principal*) in an account paying $r\%$ interest compounded monthly. In the first month you will receive $\frac{r}{12}$ (here r is in decimal form; e.g., if we have 8% interest, we write $\frac{0.08}{12}$) of the principal P in interest, so you earn

$$P \left(\frac{r}{12} \right)$$

dollars in interest. Assume that you reinvest all interest. Then at the end of the first month your account will contain the original principal P plus the interest, or a total of

$$P_1 = P + P \left(\frac{r}{12} \right) = P \left(1 + \frac{r}{12} \right)$$

dollars.

- (a) Given that your principal is now P_1 dollars, how much interest will you earn in the second month? If P_2 is the total amount of money in your account at the end of the second month, explain why

$$P_2 = P_1 \left(1 + \frac{r}{12} \right) = P \left(1 + \frac{r}{12} \right)^2.$$

- (b) Find a formula for P_3 , the total amount of money in the account at the end of the third month in terms of the original investment P .
- (c) There is a pattern to these calculations. Let P_n the total amount of money in the account at the end of the third month in terms of the original investment P . Find a formula for P_n .

3. Sequences have many applications in mathematics and the sciences. In a recent paper³ the authors write

The incretin hormone glucagon-like peptide-1 (GLP-1) is capable of ameliorating glucose-dependent insulin secretion in subjects with diabetes. However, its very short half-life (1.5-5 min) in plasma represents a major limitation for its use in the clinical setting.

The half-life of GLP-1 is the time it takes for half of the hormone to decay in its medium. For this exercise, assume the half-life of GLP-1 is 5 minutes. So if A is the amount of GLP-1 in plasma at some time t , then only $\frac{A}{2}$ of the hormone will be present after $t + 5$ minutes. Suppose $A_0 = 100$ grams of the hormone are initially present in plasma.

- (a) Let A_1 be the amount of GLP-1 present after 5 minutes. Find the value of A_1 .
- (b) Let A_2 be the amount of GLP-1 present after 10 minutes. Find the value of A_2 .
- (c) Let A_3 be the amount of GLP-1 present after 15 minutes. Find the value of A_3 .
- (d) Let A_4 be the amount of GLP-1 present after 20 minutes. Find the value of A_4 .
- (e) Let A_n be the amount of GLP-1 present after $5n$ minutes. Find a formula for A_n .
- (f) Does the sequence $\{A_n\}$ converge or diverge? If the sequence converges, find its limit and explain why this value makes sense in the context of this problem.

³Hui H, Farilla L, Merkel P, Perfetti R. The short half-life of glucagon-like peptide-1 in plasma does not reflect its long-lasting beneficial effects, *Eur J Endocrinol* 2002 Jun;146(6):863-9.

- (g) Determine the number of minutes it takes until the amount of GLP-1 in plasma is 1 gram.
4. Continuous data is the basis for analog information, like music stored on old cassette tapes or vinyl records. A digital signal like on a CD or MP3 file is obtained by sampling an analog signal at some regular time interval and storing that information. For example, the sampling rate of a compact disk is 44,100 samples per second. So a digital recording is only an approximation of the actual analog information. Digital information can be manipulated in many useful ways that allow for, among other things, noisy signals to be cleaned up and large collections of information to be compressed and stored in much smaller space. While we won't investigate these techniques in this chapter, this exercise is intended to give an idea of the importance of discrete (digital) techniques.

Let f be the continuous function defined by $f(x) = \sin(4x)$ on the interval $[0, 10]$. A graph of f is shown in Figure 8.2. We approximate f by *sampling*, that is by

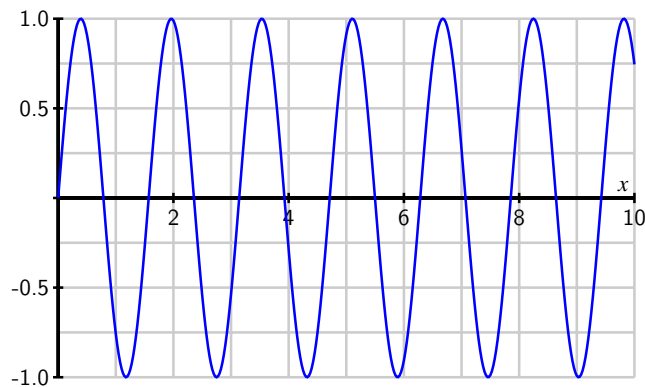


Figure 8.2: The graph of $f(x) = \sin(4x)$ on the interval $[0, 10]$

partitioning the interval $[0, 10]$ into uniform subintervals and recording the values of f at the endpoints.

- (a) Ineffective sampling can lead to several problems in reproducing the original signal. As an example, partition the interval $[0, 10]$ into 8 equal length subintervals and create a list of points (the *sample*) using the endpoints of each subinterval. Plot your sample on graph of f in Figure Figure 8.2. What can you say about the period of your sample as compared to the period of the original function?
- (b) The sampling rate is the number of samples of a signal taken per second. As part (a) illustrates, sampling at too small a rate can cause serious problems with reproducing the original signal (this problem of inefficient sampling leading to an inaccurate approximation is called *aliasing*). There is an elegant theorem

called the Nyquist-Shannon Sampling Theorem that says that human perception is limited, which allows that replacement of a continuous signal with a digital one without any perceived loss of information. This theorem also provides the *lowest* rate at which a signal can be sampled (called the Nyquist rate) without such a loss of information. The theorem states that we should sample at double the maximum desired frequency so that every cycle of the original signal will be sampled at at least two points.

Recall that the frequency of a sinusoidal function is the reciprocal of the period. Identify the frequency of the function f and determine the number of partitions of the interval $[0, 10]$ that give us the Nyquist rate.

- (c) Humans cannot typically pick up signals above 20 kHz. Explain why, then, that information on a compact disk is sampled at 44,100 Hz.
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8.2 Geometric Series

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a geometric series?
- What is a partial sum of a geometric series? What is a simplified form of the n th partial sum of a geometric series?
- Under what conditions does a geometric series converge? What is the sum of a convergent geometric series?

Introduction

Many important sequences are generated through the process of addition. In Preview Activity 8.2, we see a particular example of a special type of sequence that is connected to a sum.

Preview Activity 8.2. Warfarin is an anticoagulant that prevents blood clotting; often it is prescribed to stroke victims in order to help ensure blood flow. The level of warfarin has to reach a certain concentration in the blood in order to be effective.

Suppose warfarin is taken by a particular patient in a 5 mg dose each day. The drug is absorbed by the body and some is excreted from the system between doses. Assume that at the end of a 24 hour period, 8% of the drug remains in the body. Let $Q(n)$ be the amount (in mg) of warfarin in the body before the $(n + 1)$ st dose of the drug is administered.

(a) Explain why $Q(1) = 5 \times 0.08$ mg.

(b) Explain why $Q(2) = (5 + Q(1)) \times 0.08$ mg. Then show that

$$Q(2) = (5 \times 0.08)(1 + 0.08) \text{ mg.}$$

(c) Explain why $Q(3) = (5 + Q(2)) \times 0.08$ mg. Then show that

$$Q(3) = (5 \times 0.08)(1 + 0.08 + 0.08^2) \text{ mg.}$$

(d) Explain why $Q(4) = (5 + Q(3)) \times 0.08$ mg. Then show that

$$Q(4) = (5 \times 0.08)(1 + 0.08 + 0.08^2 + 0.08^3) \text{ mg.}$$

(e) There is a pattern that you should see emerging. Use this pattern to find a formula for $Q(n)$, where n is an arbitrary positive integer.

- (f) Complete Table 8.2 with values of $Q(n)$ for the provided n -values (reporting $Q(n)$ to 10 decimal places). What appears to be happening to the sequence $Q(n)$ as n increases?

$Q(1)$	0.40
$Q(2)$	
$Q(3)$	
$Q(4)$	
$Q(5)$	
$Q(6)$	
$Q(7)$	
$Q(8)$	
$Q(9)$	
$Q(10)$	

Table 8.2: Values of $Q(n)$ for selected values of n

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Geometric Sums

In Preview Activity 8.2 we encountered the sum

$$(5 \times 0.08)(1 + 0.08 + 0.08^2 + 0.08^3 + \cdots + 0.08^{n-1}).$$

In order to evaluate the long-term level of Warfarin in the patient's system, we will want to fully understand the sum in this expression. This sum has the form

$$a + ar + ar^2 + \cdots + ar^{n-1} \tag{8.1}$$

where $a = 5 \times 0.08$ and $r = 0.08$. Such a sum is called a *geometric sum* with ratio r . We will analyze this sum in more detail in the next activity.

Activity 8.4.

Let a and r be real numbers (with $r \neq 1$) and let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

In this activity we will find a shortcut formula for S_n that does not involve a sum of n terms.

- (a) Multiply S_n by r . What does the resulting sum look like?
 (b) Subtract rS_n from S_n and explain why

$$S_n - rS_n = a - ar^n. \quad (8.2)$$

- (c) Solve equation (8.2) for S_n to find a simple formula for S_n that does not involve adding n terms.

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We can summarize the result of Activity 8.4 in the following way.

A geometric sum S_n is a sum of the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}, \quad (8.3)$$

where a and r are real numbers such that $r \neq 1$. The geometric sum S_n can be written more simply as

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r}. \quad (8.4)$$

We now apply equation (8.4) to the example involving warfarin from Preview Activity 8.2. Recall that

$$Q(n) = (5 \times 0.08)(1 + 0.08 + 0.08^2 + 0.08^3 + \cdots + 0.08^{n-1}) \text{ mg},$$

so $Q(n)$ is a geometric sum with $a = 5 \times 0.08 = 0.4$ and $r = 0.08$. Thus,

$$Q(n) = 0.4 \left(\frac{1 - 0.08^n}{1 - 0.08} \right) = \frac{1}{2.3} (1 - 0.08^n).$$

Notice that as n goes to infinity, the value of 0.08^n goes to 0. So,

$$\lim_{n \rightarrow \infty} Q(n) = \lim_{n \rightarrow \infty} \frac{1}{2.3} (1 - 0.08^n) = \frac{1}{2.3} \approx 0.435.$$

Therefore, the long-term level of Warfarin in the blood under these conditions is $\frac{1}{2.3}$, which is approximately 0.435 mg.

To determine the long-term effect of Warfarin, we considered a geometric sum of n terms, and then considered what happened as n was allowed to grow without bound. In this sense, we were actually interested in an infinite geometric sum (the result of letting n go to infinity in the finite sum). We call such an infinite geometric sum a *geometric series*.

Definition 8.2. A geometric series is an infinite sum of the form

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n. \quad (8.5)$$

The value of r in the geometric series (8.5) is called the *common ratio* of the series because the ratio of the $(n + 1)$ st term ar^n to the n th term ar^{n-1} is always r .

Geometric series are very common in mathematics and arise naturally in many different situations. As a familiar example, suppose we want to write the number with repeating decimal expansion

$$N = 0.1212\overline{12}$$

as a rational number. Observe that

$$\begin{aligned} N &= 0.12 + 0.0012 + 0.000012 + \cdots \\ &= \left(\frac{12}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right) + \left(\frac{12}{100}\right)\left(\frac{1}{100}\right)^2 + \cdots, \end{aligned}$$

which is an infinite geometric series with $a = \frac{12}{100}$ and $r = \frac{1}{100}$. In the same way that we were able to find a shortcut formula for the value of a (finite) geometric sum, we would like to develop a formula for the value of a (infinite) geometric series. We explore this idea in the following activity.

Activity 8.5.

Let $r \neq 1$ and a be real numbers and let

$$S = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

be an infinite geometric series. For each positive integer n , let

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Recall that

$$S_n = a \frac{1 - r^n}{1 - r}.$$

- What should we allow n to approach in order to have S_n approach S ?
- What is the value of $\lim_{n \rightarrow \infty} r^n$ for
 - $|r| > 1$?
 - $|r| < 1$?

Explain.

- If $|r| < 1$, use the formula for S_n and your observations in (a) and (b) to explain

why S is finite and find a resulting formula for S .

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From our work in Activity 8.5, we can now find the value of the geometric series $N = (\frac{12}{100}) + (\frac{12}{100})(\frac{1}{100}) + (\frac{12}{100})(\frac{1}{100})^2 + \dots$. In particular, using $a = \frac{12}{100}$ and $r = \frac{1}{100}$, we see that

$$N = \frac{12}{100} \left(\frac{1}{1 - \frac{1}{100}} \right) = \frac{12}{100} \left(\frac{100}{99} \right) = \frac{4}{33}.$$

It is important to notice that a geometric sum is simply the sum of a finite number of terms of a geometric series. In other words, the geometric sum S_n for the geometric series

$$\sum_{k=0}^{\infty} ar^k$$

is

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k.$$

We also call this sum S_n the *n*th *partial sum* of the geometric series. We summarize our recent work with geometric series as follows.

- A geometric series is an infinite sum of the form

$$a + ar + ar^2 + \dots = \sum_{n=0}^{\infty} ar^n, \quad (8.6)$$

where a and r are real numbers such that $r \neq 0$.

- The *n*th partial sum S_n of the geometric series is

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}.$$

- If $|r| < 1$, then using the fact that $S_n = a \frac{1-r^n}{1-r}$, it follows that the sum S of the geometric series (8.6) is

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} a \frac{1-r^n}{1-r} = \frac{a}{1-r}$$

Activity 8.6.

The formulas we have derived for the geometric series and its partial sum so far have assumed we begin indexing our sums at $n = 0$. If instead we have a sum that does not

begin at $n = 0$, we can factor out common terms and use our established formulas. This process is illustrated in the examples in this activity.

(a) Consider the sum

$$\sum_{k=1}^{\infty} (2) \left(\frac{1}{3}\right)^k = (2) \left(\frac{1}{3}\right) + (2) \left(\frac{1}{3}\right)^2 + (2) \left(\frac{1}{3}\right)^3 + \cdots .$$

Remove the common factor of $(2)\left(\frac{1}{3}\right)$ from each term and hence find the sum of the series.

(b) Next let a and r be real numbers with $-1 < r < 1$. Consider the sum

$$\sum_{k=3}^{\infty} ar^k = ar^3 + ar^4 + ar^5 + \cdots .$$

Remove the common factor of ar^3 from each term and find the sum of the series.

(c) Finally, we consider the most general case. Let a and r be real numbers with $-1 < r < 1$, let n be a positive integer, and consider the sum

$$\sum_{k=n}^{\infty} ar^k = ar^n + ar^{n+1} + ar^{n+2} + \cdots .$$

Remove the common factor of ar^n from each term to find the sum of the series.

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Summary

In this section, we encountered the following important ideas:

- A geometric series is an infinite sum of the form

$$\sum_{k=0}^{\infty} ar^k$$

where a and r are real numbers and $r \neq 0$.

- For the geometric series $\sum_{k=0}^{\infty} ar^k$, its n th partial sum is

$$S_n = \sum_{k=0}^{n-1} ar^k.$$

An alternate formula for the n th partial sum is

$$S_n = a \frac{1 - r^n}{1 - r}.$$

Whenever $|r| < 1$, the infinite geometric series $\sum_{k=0}^{\infty} ar^k$ has the finite sum $\frac{a}{1-r}$.

Exercises

1. There is an old question that is often used to introduce the power of geometric growth. Here is one version. Suppose you are hired for a one month (30 days, working every day) job and are given two options to be paid.

Option 1. You can be paid \$500 per day or

Option 2. You can be paid 1 cent the first day, 2 cents the second day, 4 cents the third day, 8 cents the fourth day, and so on, doubling the amount you are paid each day.

- (a) How much will you be paid for the job in total under Option 1?
- (b) Complete Table 8.3 to determine the pay you will receive under Option 2 for the first 10 days.

Day	Pay on this day	Total amount paid to date
1	\$0.01	\$0.01
2	\$0.02	\$0.03
3		
4		
5		
6		
7		
8		
9		
10		

Table 8.3: Option 2 payments

- (c) Find a formula for the amount paid on day n , as well as for the total amount paid by day n . Use this formula to determine which option (1 or 2) you should take.
2. Suppose you drop a golf ball onto a hard surface from a height h . The collision with the ground causes the ball to lose energy and so it will not bounce back to its original height. The ball will then fall again to the ground, bounce back up, and continue. Assume that at each bounce the ball rises back to a height $\frac{3}{4}$ of the height from which it dropped. Let h_n be the height of the ball on the n th bounce, with $h_0 = h$. In this exercise we will determine the distance traveled by the ball and the time it takes to travel that distance.
- (a) Determine a formula for h_1 in terms of h .
- (b) Determine a formula for h_2 in terms of h .
- (c) Determine a formula for h_3 in terms of h .
- (d) Determine a formula for h_n in terms of h .
- (e) Write an infinite series that represents the total distance traveled by the ball. Then determine the sum of this series.
- (f) Next, let's determine the total amount of time the ball is in the air.
- (i) When the ball is dropped from a height H , if we assume the only force acting on it is the acceleration due to gravity, then the height of the ball at time t is given by
- $$H - \frac{1}{2}gt^2.$$
- Use this formula to determine the time it takes for the ball to hit the ground after being dropped from height H .
- (ii) Use your work in the preceding item, along with that in (a)-(e) above to determine the total amount of time the ball is in the air.
3. Suppose you play a game with a friend that involves rolling a standard six-sided die. Before a player can participate in the game, he or she must roll a six with the die. Assume that you roll first and that you and your friend take alternate rolls. In this exercise we will determine the probability that you roll the first six.
- (a) Explain why the probability of rolling a six on any single roll (including your first turn) is $\frac{1}{6}$.
- (b) If you don't roll a six on your first turn, then in order for you to roll the first six on your second turn, both you and your friend had to fail to roll a six on your first turns, and then you had to succeed in rolling a six on your second

turn. Explain why the probability of this event is

$$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right).$$

- (c) Now suppose you fail to roll the first six on your second turn. Explain why the probability is

$$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)$$

that you to roll the first six on your third turn.

- (d) The probability of you rolling the first six is the probability that you roll the first six on your first turn plus the probability that you roll the first six on your second turn plus the probability that your roll the first six on your third turn, and so on. Explain why this probability is

$$\frac{1}{6} + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right) + \cdots .$$

Find the sum of this series and determine the probability that you roll the first six.

4. The goal of a federal government stimulus package is to positively affect the economy. Economists and politicians quote numbers like “ k million jobs and a net stimulus to the economy of n billion of dollars.” Where do they get these numbers? Let’s consider one aspect of a stimulus package: tax cuts. Economists understand that tax cuts or rebates can result in long-term spending that is many times the amount of the rebate. For example, assume that for a typical person, 75% of her entire income is spent (that is, put back into the economy). Further, assume the government provides a tax cut or rebate that totals P dollars for each person.

- (a) The tax cut of P dollars is income for its recipient. How much of this tax cut will be spent?
- (b) In this simple model, we will say that the spent portion of the tax cut/rebate from part (a) then becomes income for another person who, in turn, spends 75% of this income. After this “second round” of spent income, how many total dollars have been added to the economy as a result of the original tax cut/rebate?
- (c) This second round of spending becomes income for another group who spend 75% of this income, and so on. In economics this is called the *multiplier effect*. Explain why an original tax cut/rebate of P dollars will result in multiplied spending of

$$0.75P(1 + 0.75 + 0.75^2 + \cdots).$$

dollars.

- (d) Based on these assumptions, how much stimulus will a 200 billion dollar tax cut/rebate to consumers add to the economy, assuming consumer spending remains consistent forever.
5. Like stimulus packages, home mortgages and foreclosures also impact the economy. A problem for many borrowers is the adjustable rate mortgage, in which the interest rate can change (and usually increases) over the duration of the loan, causing the monthly payments to increase beyond the ability of the borrower to pay. Most financial analysts recommend fixed rate loans, ones for which the monthly payments remain constant throughout the term of the loan. In this exercise we will analyze fixed rate loans.

When most people buy a large ticket item like car or a house, they have to take out a loan to make the purchase. The loan is paid back in monthly installments until the entire amount of the loan, plus interest, is paid. With a loan, we borrow money, say P dollars (called the principal), and pay off the loan at an interest rate of $r\%$. To pay back the loan we make regular monthly payments, some of which goes to pay off the principal and some of which is charged as interest. In most cases, the interest is computed based on the amount of principal that remains at the beginning of the month. We assume a fixed rate loan, that is one in which we make a constant monthly payment M on our loan, beginning in the original month of the loan.

Suppose you want to buy a house. You have a certain amount of money saved to make a down payment, and you will borrow the rest to pay for the house. Of course, for the privilege of loaning you the money, the bank will charge you interest on this loan, so the amount you pay back to the bank is more than the amount you borrow. In fact, the amount you ultimately pay depends on three things: the amount you borrow (called the *principal*), the interest rate, and the length of time you have to pay off the loan plus interest (called the *duration* of the loan). For this example, we assume that the interest rate is fixed at $r\%$.

To pay off the loan, each month you make a payment of the same amount (called *installments*). Suppose we borrow P dollars (our principal) and pay off the loan at an interest rate of $r\%$ with regular monthly installment payments of M dollars. So in month 1 of the loan, before we make any payments, our principal is P dollars. Our goal in this exercise is to find a formula that relates these three parameters to the time duration of the loan.

We are charged interest every month at an annual rate of $r\%$, so each month we pay $\frac{r}{12}\%$ interest on the principal that remains. Given that the original principal is P dollars, we will pay $(\frac{0.0r}{12})P$ dollars in interest on our first payment. Since we paid M dollars in total for our first payment, the remainder of the payment $(M - (\frac{r}{12})P)$ goes to pay down the principal. So the principal remaining after the first payment (let's call it P_1) is the original principal minus what we paid on the principal, or

$$P_1 = P - \left(M - \left(\frac{r}{12} \right) P \right) = \left(1 + \frac{r}{12} \right) P - M.$$

As long as P_1 is positive, we still have to keep making payments to pay off the loan.

- (a) Recall that the amount of interest we pay each time depends on the principal that remains. How much interest, in terms of P_1 and r , do we pay in the second installment?
- (b) How much of our second monthly installment goes to pay off the principal? What is the principal P_2 , or the balance of the loan, that we still have to pay off after making the second installment of the loan? Write your response in the form $P_2 = ()P_1 - ()M$, where you fill in the parentheses.
- (c) Show that $P_2 = \left(1 + \frac{r}{12}\right)^2 P - \left[1 + \left(1 + \frac{r}{12}\right)\right] M$.
- (d) Let P_3 be the amount of principal that remains after the third installment. Show that

$$P_3 = \left(1 + \frac{r}{12}\right)^3 P - \left[1 + \left(1 + \frac{r}{12}\right) + \left(1 + \frac{r}{12}\right)^2\right] M.$$

- (e) If we continue in the manner described in the problems above, then the remaining principal of our loan after n installments is

$$P_n = \left(1 + \frac{r}{12}\right)^n P - \left[\sum_{k=0}^{n-1} \left(1 + \frac{r}{12}\right)^k\right] M. \quad (8.7)$$

This is a rather complicated formula and one that is difficult to use. However, we can simplify the sum if we recognize part of it as a partial sum of a geometric series. Find a formula for the sum

$$\sum_{k=0}^{n-1} \left(1 + \frac{r}{12}\right)^k. \quad (8.8)$$

and then a general formula for P_n that does not involve a sum.

- (f) It is usually more convenient to write our formula for P_n in terms of years rather than months. Show that $P(t)$, the principal remaining after t years, can be written as

$$P(t) = \left(P - \frac{12M}{r}\right) \left(1 + \frac{r}{12}\right)^{12t} + \frac{12M}{r}. \quad (8.9)$$

- (g) Now that we have analyzed the general loan situation, we apply formula (8.9) to an actual loan. Suppose we charge \$1,000 on a credit card for holiday expenses. If our credit card charges 20% interest and we pay only the minimum payment of \$25 each month, how long will it take us to pay off the \$1,000 charge? How much in total will we have paid on this \$1,000 charge? How much total interest will we pay on this loan?

- (h) Now we consider larger loans, e.g. automobile loans or mortgages, in which we borrow a specified amount of money over a specified period of time. In this situation, we need to determine the amount of the monthly payment we need to make to pay off the loan in the specified amount of time. In this situation, we need to find the monthly payment M that will take our outstanding principal to 0 in the specified amount of time. To do so, we want to know the value of M that makes $P(t) = 0$ in formula (8.9). If we set $P(t) = 0$ and solve for M , it follows that

$$M = \frac{rP \left(1 + \frac{r}{12}\right)^{12t}}{12 \left(\left(1 + \frac{r}{12}\right)^{12t} - 1\right)}.$$

- (i) Suppose we want to borrow \$15,000 to buy a car. We take out a 5 year loan at 6.25%. What will our monthly payments be? How much in total will we have paid for this \$15,000 car? How much total interest will we pay on this loan?
- (ii) Suppose you charge your books for winter semester on your credit card. The total charge comes to \$525. If your credit card has an interest rate of 18% and you pay \$20 per month on the card, how long will it take before you pay off this debt? How much total interest will you pay?
- (iii) Say you need to borrow \$100,000 to buy a house. You have several options on the loan:
- 30 years at 6.5%
 - 25 years at 7.5%
 - 15 years at 8.25%.
- (a) What are the monthly payments for each loan?
- (b) Which mortgage is ultimately the best deal (assuming you can afford the monthly payments)? In other words, for which loan do you pay the least amount of total interest?

8.3 Series of Real Numbers

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is an infinite series?
- What is the n th partial sum of an infinite series?
- How do we add up an infinite number of numbers? In other words, what does it mean for an infinite series of real numbers to converge?
- What does it mean for an infinite series of real numbers to diverge?

Introduction

In Section 8.2, we encountered several situations where we naturally considered an infinite sum of numbers called a geometric series. For example, by writing

$$N = 0.1212121212 \dots = \frac{12}{100} + \frac{12}{100} \cdot \frac{1}{100} + \frac{12}{100} \cdot \frac{1}{100^2} + \dots$$

as a geometric series, we found a way to write the repeating decimal expansion of N as a single fraction: $N = \frac{4}{33}$. There are many other situations in mathematics where infinite sums of numbers arise, but often these are not geometric. In this section, we begin exploring these other types of infinite sums. Preview Activity 8.3 provides a context in which we see how one such sum is related to the famous number e .

Preview Activity 8.3. Have you ever wondered how your calculator can produce a numeric approximation for complicated numbers like e , π or $\ln(2)$? After all, the only operations a calculator can really perform are addition, subtraction, multiplication, and division, the operations that make up polynomials. This activity provides the first steps in understanding how this process works. Throughout the activity, let $f(x) = e^x$.

- (a) Find the tangent line to f at $x = 0$ and use this linearization to approximate e . That is, find a formula $L(x)$ for the tangent line, and compute $L(1)$, since $L(1) \approx f(1) = e$.
- (b) The linearization of e^x does not provide a good approximation to e since 1 is not very close to 0. To obtain a better approximation, we alter our approach a bit. Instead of using a straight line to approximate e , we put an appropriate bend in our estimating function to make it better fit the graph of e^x for x close to 0. With the linearization, we had both $f(x)$ and $f'(x)$ share the same value

as the linearization at $x = 0$. We will now use a quadratic approximation $P_2(x)$ to $f(x) = e^x$ centered at $x = 0$ which has the property that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$.

- (i) Let $P_2(x) = 1 + x + \frac{x^2}{2}$. Show that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. Then, use $P_2(x)$ to approximate e by observing that $P_2(1) \approx f(1)$.
- (ii) We can continue approximating e with polynomials of larger degree whose higher derivatives agree with those of f at 0. This turns out to make the polynomials fit the graph of f better for more values of x around 0. For example, let $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$. Show that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. Use $P_3(x)$ to approximate e in a way similar to how you did so with $P_2(x)$ above.

✕

Preview Activity 8.3 shows that an approximation to e using a linear polynomial is 2, an approximation to e using a quadratic polynomial is 2.5, and an approximation using a cubic polynomial is 2.6667. As we will see later, if we continue this process we can obtain approximations from quartic (degree 4), quintic (degree 5), and higher degree polynomials giving us the following approximations to e :

linear	$1 + 1$	2
quadratic	$1 + 1 + \frac{1}{2}$	2.5
cubic	$1 + 1 + \frac{1}{2} + \frac{1}{6}$	$2.\overline{6}$
quartic	$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$	2.708 $\overline{3}$
quintic	$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$	2.71 $\overline{6}$

We see an interesting pattern here. The number e is being approximated by the sum

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots + \frac{1}{n!} \quad (8.11)$$

for increasing values of n . And just as we did with Riemann sums, we can use the summation notation as a shorthand⁴ for writing the sum in Equation (8.11) so that

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots + \frac{1}{n!} = \sum_{k=0}^n \frac{1}{k!}. \quad (8.12)$$

We can calculate this sum using as large an n as we want, and the larger n is the more accurate the approximation (8.12) is. Ultimately, this argument shows that we can write the

⁴Note that $0!$ appears in Equation (8.12). By definition, $0! = 1$.

number e as the infinite sum

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}. \quad (8.13)$$

This sum is an example of a *series* (or an *infinite series*). Note that the series (8.13) is the sum of the terms of the (infinite) sequence $\{\frac{1}{n!}\}$. In general, we use the following notation and terminology.

Definition 8.3. An *infinite series* of real numbers is the sum of the entries in an infinite sequence of real numbers. In other words, an infinite series is sum of the form

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k,$$

where a_1, a_2, \dots , are real numbers.

We will normally use summation notation to identify a series. If the series adds the entries of a sequence $\{a_n\}_{n \geq 1}$, then we will write the series as

$$\sum_{k \geq 1} a_k$$

or

$$\sum_{k=1}^{\infty} a_k.$$

Note well: each of these notations is simply shorthand for the infinite sum $a_1 + a_2 + \cdots + a_n + \cdots$.

Is it even possible to sum an infinite list of numbers? This question is one whose answer shouldn't come as a surprise. After all, we have used the definite integral to add up continuous (infinite) collections of numbers, so summing the entries of a sequence might be even easier. Moreover, we have already examined the special case of geometric series in the previous section. The next activity provides some more insight into how we make sense of the process of summing an infinite list of numbers.

Activity 8.7.

Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

While it is physically impossible to add an infinite collection of numbers, we can, of course, add any finite collection of them. In what follows, we investigate how understanding how to find the n th partial sum (that is, the sum of the first n terms) enables us to make sense of the infinite sum.

(a) Sum the first two numbers in this series. That is, find a numeric value for

$$\sum_{k=1}^2 \frac{1}{k^2}$$

(b) Next, add the first three numbers in the series.

(c) Continue adding terms in this series to complete Table 8.4. Carry each sum to at least 8 decimal places.

$\sum_{k=1}^1 \frac{1}{k^2} = 1$	$\sum_{k=1}^6 \frac{1}{k^2} =$
$\sum_{k=1}^2 \frac{1}{k^2} =$	$\sum_{k=1}^7 \frac{1}{k^2} =$
$\sum_{k=1}^3 \frac{1}{k^2} =$	$\sum_{k=1}^8 \frac{1}{k^2} =$
$\sum_{k=1}^4 \frac{1}{k^2} =$	$\sum_{k=1}^9 \frac{1}{k^2} =$
$\sum_{k=1}^5 \frac{1}{k^2} =$	$\sum_{k=1}^{10} \frac{1}{k^2} =$

Table 8.4: Sums of some of the first terms of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$

(d) The sums in the table in (c) form a sequence whose n th term is $S_n = \sum_{k=1}^n \frac{1}{k^2}$. Based on your calculations in the table, do you think the sequence $\{S_n\}$ converges or diverges? Explain. How do you think this sequence $\{S_n\}$ is related to the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$?

◁

The example in Activity 8.7 illustrates how we define the sum of an infinite series. We can add up the first n terms of the series to obtain a new sequence of numbers (called the *sequence of partial sums*). Provided that sequence converges, the corresponding infinite series is said to converge, and we say that we can find the sum of the series.

Definition 8.4. The n th *partial sum* of the series $\sum_{k=1}^{\infty} a_k$ is the finite sum $S_n = \sum_{k=1}^n a_k$.

In other words, the n th partial sum S_n of a series is the sum of the first n terms in the series, or

$$S_n = a_1 + a_2 + \cdots + a_n.$$

We then investigate the behavior of a given series by examining the sequence

$$S_1, S_2, \dots, S_n, \dots$$

of its partial sums. If the sequence of partial sums converges to some finite number, then we say that the corresponding series *converges*. Otherwise, we say the series *diverges*. From our work in Activity 8.7, the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

appears to converge to some number near 1.54977. We formalize the concept of convergence and divergence of an infinite series in the following definition.

Definition 8.5. The infinite series

$$\sum_{k=1}^{\infty} a_k$$

converges (or is *convergent*) if the sequence $\{S_n\}$ of partial sums converges, where

$$S_n = \sum_{k=1}^n a_k.$$

If $\lim_{n \rightarrow \infty} S_n = S$, then we call S the sum of the series $\sum_{k=1}^{\infty} a_k$. That is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n = S.$$

If the sequence of partial sums does not converge, then the series

$$\sum_{k=1}^{\infty} a_k$$

diverges (or is *divergent*).

The early terms in a series do not contribute to whether or not the series converges or diverges. Rather, the convergence or divergence of a series

$$\sum_{k=1}^{\infty} a_k$$

is determined by what happens to the terms a_k for very large values of k . To see why, suppose that m is some constant larger than 1. Then

$$\sum_{k=1}^{\infty} a_k = (a_1 + a_2 + \dots + a_m) + \sum_{k=m+1}^{\infty} a_k.$$

Since $a_1 + a_2 + \cdots + a_m$ is a finite number, the series $\sum_{k=1}^{\infty} a_k$ will converge if and only if the series $\sum_{k=m+1}^{\infty} a_k$ converges. Because the starting index of the series doesn't affect whether the series converges or diverges, we will often just write

$$\sum a_k$$

when we are interested in questions of convergence/divergence and not necessarily the exact sum of a series.

In Section 8.2 we encountered the special family of infinite geometric series whose convergence or divergence we completely determined. Recall that a geometric series is a special series of the form $\sum_{k=0}^{\infty} ar^k$ where a and r are real numbers (and $r \neq 1$). We found that the n th partial sum S_n of a geometric series is given by the convenient formula

$$S_n = \frac{1 - r^n}{1 - r},$$

and thus a geometric series converges if $|r| < 1$. Geometric series diverge for all other values of r . While we have completely determined the convergence or divergence of geometric series, it is generally a difficult question to determine if a given nongeometric series converges or diverges. There are several tests we can use that we will consider in the following sections.

The Divergence Test

The first question we ask about any infinite series is usually “Does the series converge or diverge?” There is a straightforward way to check that certain series diverge; we explore this test in the next activity.

Activity 8.8.

If the series $\sum a_k$ converges, then an important result necessarily follows regarding the sequence $\{a_n\}$. This activity explores this result.

Assume that the series $\sum_{k=1}^{\infty} a_k$ converges and has sum equal to L .

- What is the n th partial sum S_n of the series $\sum_{k=1}^{\infty} a_k$?
- What is the $(n - 1)$ st partial sum S_{n-1} of the series $\sum_{k=1}^{\infty} a_k$?
- What is the difference between the n th partial sum and the $(n - 1)$ st partial sum of the series $\sum_{k=1}^{\infty} a_k$?
- Since we are assuming that $\sum_{k=1}^{\infty} a_k = L$, what does that tell us about $\lim_{n \rightarrow \infty} S_n$? Why? What does that tell us about $\lim_{n \rightarrow \infty} S_{n-1}$? Why?
- Combine the results of the previous two parts of this activity to determine $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (S_n - S_{n-1})$.

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The result of Activity 8.8 is the following important conditional statement:

If the series $\sum_{k=1}^{\infty} a_k$ converges, then the sequence $\{a_k\}$ of k th terms converges to 0.

It is logically equivalent to say that if the sequence $\{a_k\}$ of n terms does not converge to 0, then the series $\sum_{k=1}^{\infty} a_k$ cannot converge. This statement is called the Divergence Test.

The Divergence Test. If $\lim_{k \rightarrow \infty} a_k \neq 0$, then the series $\sum a_k$ diverges.

Activity 8.9.

Determine if the Divergence Test applies to the following series. If the test does not apply, explain why. If the test does apply, what does it tell us about the series?

- (a) $\sum \frac{k}{k+1}$
- (b) $\sum (-1)^k$
- (c) $\sum \frac{1}{k}$

<

Note well: be very careful with the Divergence Test. This test only tells us what happens to a series if the terms of the corresponding sequence do not converge to 0. If the sequence of the terms of the series does converge to 0, the Divergence Test does not apply: indeed, as we will soon see, a series whose terms go to zero may either converge or diverge.

The Integral Test

The Divergence Test settles the questions of divergence or convergence of series $\sum a_k$ in which $\lim_{k \rightarrow \infty} a_k \neq 0$. Determining the convergence or divergence of series $\sum a_k$ in which $\lim_{k \rightarrow \infty} a_k = 0$ turns out to be more complicated. Often, we have to investigate the sequence of partial sums or apply some other technique.

As an example, consider the *harmonic series*⁵

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

⁵This series is called harmonic because each term in the series after the first is the harmonic mean of the term before it and the term after it. The harmonic mean of two numbers a and b is $\frac{2ab}{a+b}$. See "What's Harmonic about the Harmonic Series", by David E. Kullman (in the *College Mathematics Journal*, Vol. 32, No. 3 (May, 2001), 201-203) for an interesting discussion of the harmonic mean.

Table 8.3 shows some partial sums of this series. This information doesn't seem to

$\sum_{k=1}^1 \frac{1}{k}$	1	$\sum_{k=1}^6 \frac{1}{k}$	2.450000000
$\sum_{k=1}^2 \frac{1}{k}$	1.5	$\sum_{k=1}^7 \frac{1}{k}$	2.592857143
$\sum_{k=1}^3 \frac{1}{k}$	1.833333333	$\sum_{k=1}^8 \frac{1}{k}$	2.717857143
$\sum_{k=1}^4 \frac{1}{k}$	2.083333333	$\sum_{k=1}^9 \frac{1}{k}$	2.828968254
$\sum_{k=1}^5 \frac{1}{k}$	2.283333333	$\sum_{k=1}^{10} \frac{1}{k}$	2.928968254

Table 8.5: Sums of some of the first terms of the sequence $\sum_{k=1}^{\infty} \frac{1}{k}$.

be enough to tell us if the series $\sum_{k=1}^{\infty} \frac{1}{k}$ converges or diverges. The partial sums could eventually level off to some fixed number or continue to grow without bound. Even if we look at larger partial sums, such as $\sum_{n=1}^{1000} \frac{1}{k} \approx 7.485470861$, the result doesn't particularly sway us one way or another. The Integral Test is one way to determine whether or not the harmonic series converges, and we explore this further in the next activity.

Activity 8.10.

Consider the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Recall that the harmonic series will converge provided that its sequence of partial sums converges. The n th partial sum S_n of the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is

$$\begin{aligned} S_n &= \sum_{k=1}^n \frac{1}{k} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ &= 1(1) + (1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{3}\right) + \cdots + (1)\left(\frac{1}{n}\right). \end{aligned}$$

Through this last expression for S_n , we can visualize this partial sum as a sum of areas of rectangles with heights $\frac{1}{m}$ and bases of length 1, as shown in Figure 8.3, which uses the 9th partial sum. The graph of the continuous function f defined by $f(x) = \frac{1}{x}$ is overlaid on this plot.

- (a) Explain how this picture represents a particular Riemann sum.
- (b) What is the definite integral that corresponds to the Riemann sum you considered in (a)?

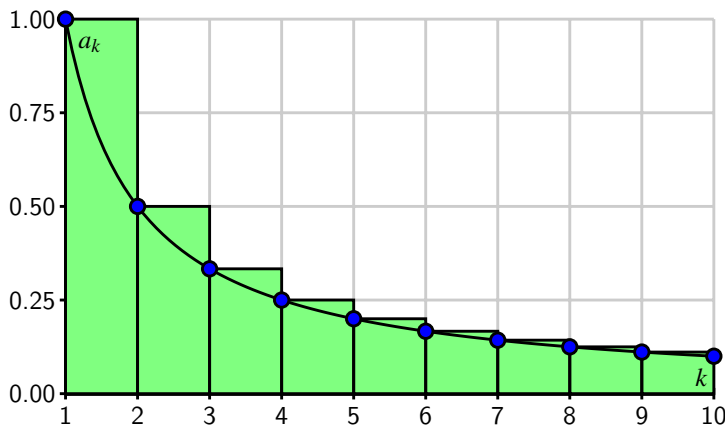


Figure 8.3: A picture of the 9th partial sum of the harmonic series as a sum of areas of rectangles.

- (c) Which is larger, the definite integral in (b), or the corresponding partial sum S_9 of the series? Why?
- (d) If instead of considering the 9th partial sum, we consider the n th partial sum, and we let n go to infinity, we can then compare the series $\sum_{k=1}^{\infty} \frac{1}{k}$ to the improper integral $\int_1^{\infty} \frac{1}{x} dx$. Which of these quantities is larger? Why?
- (e) Does the improper integral $\int_1^{\infty} \frac{1}{x} dx$ converge or diverge? What does that result, together with your work in (d), tell us about the series $\sum_{k=1}^{\infty} \frac{1}{k}$?

◁

The ideas from Activity 8.10 hold more generally. Suppose that f is a continuous decreasing function and that $a_k = f(k)$ for each value of k . Consider the corresponding series $\sum_{k=1}^{\infty} a_k$. The partial sum

$$S_n = \sum_{k=1}^n a_k$$

can always be viewed as a left hand Riemann sum of $f(x)$ using rectangles with heights given by the values a_k and bases of length 1. A representative picture is shown at left in Figure 8.4. Since f is a decreasing function, we have that

$$S_n > \int_1^n f(x) dx.$$

Taking limits as n goes to infinity shows that

$$\sum_{k=1}^{\infty} a_k > \int_1^{\infty} f(x) dx.$$

Therefore, if the improper integral $\int_1^{\infty} f(x) dx$ diverges, so does the series $\sum_{k=1}^{\infty} a_k$.

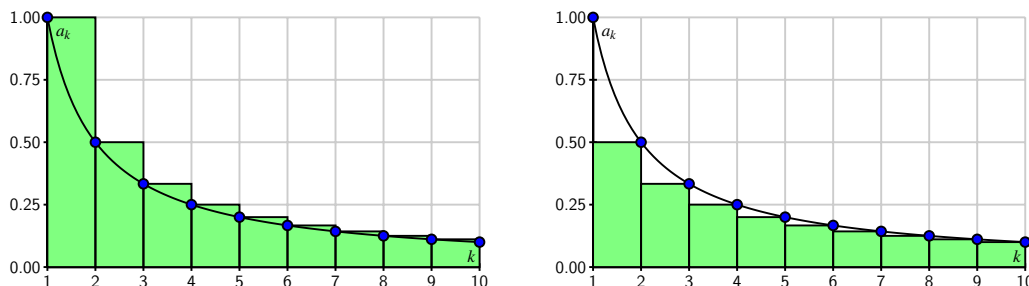


Figure 8.4: Comparing an improper integral to a series

What's more, if we look at the right hand Riemann sums of f on $[1, n]$ as shown at right in Figure 8.4, we see that

$$\int_1^{\infty} f(x) dx > \sum_{k=2}^{\infty} a_k.$$

So if $\int_1^{\infty} f(x) dx$ converges, then so does $\sum_{k=2}^{\infty} a_k$, which also means that the series $\sum_{k=1}^{\infty} a_k$ converges. Our preceding discussion has demonstrated the truth of the Integral Test.

The Integral Test. Let f be a real valued function and assume f is decreasing and positive for all x larger than some number c . Let $a_k = f(k)$ for each positive integer k .

1. If the improper integral $\int_c^{\infty} f(x) dx$ converges, then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If the improper integral $\int_c^{\infty} f(x) dx$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Note that the Integral Test compares a given infinite series to a natural, corresponding improper integral and basically says that the infinite series and corresponding improper integral both have the same convergence status. In the next activity, we apply the Integral Test to determine the convergence or divergence of a class of important series.

Activity 8.11.

The series $\sum \frac{1}{k^p}$ are special series called p -series. We have already seen that the p -series with $p = 1$ (the harmonic series) diverges. We investigate the behavior of other p -series in this activity.

- Evaluate the improper integral $\int_1^\infty \frac{1}{x^2} dx$. Does the series $\sum_{k=1}^\infty \frac{1}{k^2}$ converge or diverge? Explain.
- Evaluate the improper integral $\int_1^\infty \frac{1}{x^p} dx$ where $p > 1$. For which values of p can we conclude that the series $\sum_{k=1}^\infty \frac{1}{k^p}$ converges?
- Evaluate the improper integral $\int_1^\infty \frac{1}{x^p} dx$ where $p < 1$. What does this tell us about the corresponding p -series $\sum_{k=1}^\infty \frac{1}{k^p}$?
- Summarize your work in this activity by completing the following statement.

The p -series $\sum_{k=1}^\infty \frac{1}{k^p}$ converges if and only if _____.

◁

The Limit Comparison Test

The Integral Test allows us to determine the convergence of an entire family of series: the p -series. However, we have seen that it is, in general, difficult to integrate functions, so the Integral Test is not one that we can use all of the time. In fact, even for a relatively simple series like $\sum \frac{k^2+1}{k^4+2k+2}$, the Integral Test is not an option. In this section we will develop a test that we can use to apply to series of rational functions like this by comparing their behavior to the behavior of p -series.

Activity 8.12.

Consider the series $\sum \frac{k+1}{k^3+2}$. Since the convergence or divergence of a series only depends on the behavior of the series for large values of k , we might examine the terms of this series more closely as k gets large.

- By computing the value of $\frac{k+1}{k^3+2}$ for $k = 100$ and $k = 1000$, explain why the terms $\frac{k+1}{k^3+2}$ are essentially $\frac{k}{k^3}$ when k is large.
- Let's formalize our observations in (a) a bit more. Let $a_k = \frac{k+1}{k^3+2}$ and $b_k = \frac{k}{k^3}$. Calculate

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k}.$$

What does the value of the limit tell you about a_k and b_k for large values of k ? Compare your response from part (a).

- Does the series $\sum \frac{k}{k^3}$ converge or diverge? Why? What do you think that tells us about the convergence or divergence of the series $\sum \frac{k+1}{k^3+2}$? Explain.

◁

Activity 8.12 illustrates how we can compare one series with positive terms to another whose behavior (that is, whether the series converges or diverges) we know. More generally, suppose we have two series $\sum a_k$ and $\sum b_k$ with positive terms and we know the behavior of the series $\sum a_k$. Recall that the convergence or divergence of a series depends only on what happens to the terms of the series for large values of k , so if we know that a_k and b_k are essentially proportional to each other for large k , then the two series $\sum a_k$ and $\sum b_k$ should behave the same way. In other words, if there is a positive finite constant c such that

$$\lim_{k \rightarrow \infty} \frac{b_k}{a_k} = c,$$

then $b_k \approx ca_k$ for large values of k . So

$$\sum b_k \approx \sum ca_k = c \sum a_k.$$

Since multiplying by a nonzero constant does not affect the convergence or divergence of a series, it follows that the series $\sum a_k$ and $\sum b_k$ either both converge or both diverge. The formal statement of this fact is called the Limit Comparison Test.

The Limit Comparison Test. Let $\sum a_k$ and $\sum b_k$ be series with positive terms. If

$$\lim_{k \rightarrow \infty} \frac{b_k}{a_k} = c$$

for some positive (finite) constant c , then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.

In essence, the Limit Comparison Test shows that if we have a series $\sum \frac{p(k)}{q(k)}$ of rational functions where $p(k)$ is a polynomial of degree m and $q(k)$ a polynomial of degree l , then the series $\sum \frac{p(k)}{q(k)}$ will behave like the series $\sum \frac{k^m}{k^l}$. So this test allows us to quickly and easily determine the convergence or divergence of series whose summands are rational functions.

Activity 8.13.

Use the Limit Comparison Test to determine the convergence or divergence of the series

$$\sum \frac{3k^2 + 1}{5k^4 + 2k + 2}.$$

by comparing it to the series $\sum \frac{1}{k^2}$.

◁

The Ratio Test

The Limit Comparison Test works well if we can find a series with known behavior to compare. But such series are not always easy to find. In this section we will examine a test that allows us to examine the behavior of a series by comparing it to a geometric series, without knowing in advance which geometric series we need.

Activity 8.14.

Consider the series defined by

$$\sum_{k=1}^{\infty} \frac{2^k}{3^k - k}. \quad (8.14)$$

This series is not a geometric series, but this activity will illustrate how we might compare this series to a geometric one. Recall that a series $\sum a_k$ is geometric if the ratio $\frac{a_{k+1}}{a_k}$ is always the same. For the series in (8.14), note that $a_k = \frac{2^k}{3^k - k}$.

- (a) To see if $\sum \frac{2^k}{3^k - k}$ is comparable to a geometric series, we analyze the ratios of successive terms in the series. Complete Table 8.6, listing your calculations to at least 8 decimal places.

k	$\frac{a_{k+1}}{a_k}$
5	
10	
20	
21	
22	
23	
24	
25	

Table 8.6: Ratios of successive terms in the series $\sum \frac{2^k}{3^k - k}$

- (b) Based on your calculations in Table 8.6, what can we say about the ratio $\frac{a_{k+1}}{a_k}$ if k is large?
- (c) Do you agree or disagree with the statement: “the series $\sum \frac{2^k}{3^k - k}$ is approximately geometric when k is large”? If not, why not? If so, do you think the series $\sum \frac{2^k}{3^k - k}$ converges or diverges? Explain.

<

We can generalize the argument in Activity 8.14 in the following way. Consider the series $\sum a_k$. If

$$\frac{a_{k+1}}{a_k} \approx r$$

for large values of k , then $a_{k+1} \approx ra_k$ for large k and the series $\sum a_k$ is approximately the geometric series $\sum ar^k$ for large k . Since the geometric series with ratio r converges only for $-1 < r < 1$, we see that the series $\sum a_k$ will converge if

$$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = r$$

for a value of r such that $|r| < 1$. This result is known as the Ratio Test.

The Ratio Test. Let $\sum a_k$ be an infinite series. Suppose

$$\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = r.$$

1. If $0 \leq r < 1$, then the series $\sum a_k$ converges.
2. If $1 < r$, then the series $\sum a_k$ diverges.
3. If $r = 1$, then the test is inconclusive.

Note well: The Ratio Test takes a given series and looks at the limit of the ratio of consecutive terms; in so doing, the test is essentially asking, “is this series approximately geometric?” If the series can be thought of as essentially geometric, the test uses the limiting common ratio to determine if the given series converges.

We have now encountered several tests for determining convergence or divergence of series. The Divergence Test can be used to show that a series diverges, but never to prove that a series converges. We used the Integral Test to determine the convergence status of an entire class of series, the p -series. The Limit Comparison Test works well for series that involve rational functions and which can therefore be compared to p -series. Finally, the Ratio Test allows us to compare our series to a geometric series; it is particularly useful for series that involve n th powers and factorials. Two other tests, the Direct Comparison Test and the Root Test, are discussed in the exercises. Now it is time for some practice.

Activity 8.15.

Determine whether each of the following series converges or diverges. Explicitly state which test you use.

(a) $\sum \frac{k}{2^k}$

$$(b) \sum \frac{k^3 + 2}{k^2 + 1}$$

$$(c) \sum \frac{10^k}{k!}$$

$$(d) \sum \frac{k^3 - 2k^2 + 1}{k^6 + 4}$$

◁

Summary

In this section, we encountered the following important ideas:

- An infinite series is a sum of the elements in an infinite sequence. In other words, an infinite series is a sum of the form

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$$

where a_k is a real number for each positive integer k .

- The n th partial sum S_n of the series $\sum_{k=1}^{\infty} a_k$ is the sum of the first n terms of the series. That is,

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^n a_k.$$

- The sequence $\{S_n\}$ of partial sums of a series $\sum_{k=1}^{\infty} a_k$ tells us about the convergence or divergence of the series. In particular
 - The series $\sum_{k=1}^{\infty} a_k$ converges if the sequence $\{S_n\}$ of partial sums converges. In this case we say that the series is the limit of the sequence of partial sums and write

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n.$$

- The series $\sum_{k=1}^{\infty} a_k$ diverges if the sequence $\{S_n\}$ of partial sums diverges.

Exercises

- In this exercise we investigate the sequence $\left\{\frac{b^n}{n!}\right\}$ for any constant b .
 - Use the Ratio Test to determine if the series $\sum \frac{10^k}{k!}$ converges or diverges.
 - Now apply the Ratio Test to determine if the series $\sum \frac{b^k}{k!}$ converges for any constant b .

- (c) Use your result from (b) to decide whether the sequence $\left\{\frac{b^n}{n!}\right\}$ converges or diverges. If the sequence $\left\{\frac{b^n}{n!}\right\}$ converges, to what does it converge? Explain your reasoning.
2. There is a test for convergence similar to the Ratio Test called the *Root Test*. Suppose we have a series $\sum a_k$ of positive terms so that $a_n \rightarrow 0$ as $n \rightarrow \infty$.

(a) Assume

$$\sqrt[n]{a_n} \rightarrow r$$

as n goes to infinity. Explain why this tells us that $a_n \approx r^n$ for large values of n .

- (b) Using the result of part (a), explain why $\sum a_k$ looks like a geometric series when n is big. What is the ratio of the geometric series to which $\sum a_k$ is comparable?
- (c) Use what we know about geometric series to determine that values of r so that $\sum a_k$ converges if $\sqrt[n]{a_n} \rightarrow r$ as $n \rightarrow \infty$.

3. The associative and distributive laws of addition allow us to add finite sums in any order we want. That is, if $\sum_{k=0}^n a_k$ and $\sum_{k=0}^n b_k$ are finite sums of real numbers, then

$$\sum_{k=0}^n a_k + \sum_{k=0}^n b_k = \sum_{k=0}^n (a_k + b_k).$$

However, we do need to be careful extending rules like this to infinite series.

- (a) Let $a_n = 1 + \frac{1}{2^n}$ and $b_n = -1$ for each nonnegative integer n .
- (i) Explain why the series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ both diverge.
- (ii) Explain why the series $\sum_{k=0}^{\infty} (a_k + b_k)$ converges.
- (iii) Explain why

$$\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k \neq \sum_{k=0}^{\infty} (a_k + b_k).$$

This shows that it is possible to have two divergent series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ but yet have the series $\sum_{k=0}^{\infty} (a_k + b_k)$ converge.

- (b) While part (a) shows that we cannot add series term by term in general, we can under reasonable conditions. The problem in part (a) is that we tried to add divergent series. In this exercise we will show that if $\sum a_k$ and $\sum b_k$ are convergent series, then $\sum (a_k + b_k)$ is a convergent series and

$$\sum (a_k + b_k) = \sum a_k + \sum b_k.$$

- (i) Let A_n and B_n be the n th partial sums of the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$,

respectively. Explain why

$$A_n + B_n = \sum_{k=1}^n (a_k + b_k).$$

(ii) Use the previous result and properties of limits to show that

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

(Note that the starting point of the sum is irrelevant in this problem, so it doesn't matter where we begin the sum.)

(c) Use the prior result to find the sum of the series $\sum_{k=0}^{\infty} \frac{2^k + 3^k}{5^k}$.

4. In the Limit Comparison Test we compared the behavior of a series to one whose behavior we know. In that test we use the limit of the ratio of corresponding terms of the series to determine if the comparison is valid. In this exercise we see how we can compare two series directly, term by term, without using a limit of sequence. First we consider an example.

(a) Consider the series

$$\sum \frac{1}{k^2} \quad \text{and} \quad \sum \frac{1}{k^2 + k}.$$

We know that the series $\sum \frac{1}{k^2}$ is a p -series with $p = 2 > 1$ and so $\sum \frac{1}{k^2}$ converges. In this part of the exercise we will see how to use information about $\sum \frac{1}{k^2}$ to determine information about $\sum \frac{1}{k^2 + k}$. Let $a_k = \frac{1}{k^2}$ and $b_k = \frac{1}{k^2 + k}$.

(i) Let S_n be the n th partial sum of $\sum \frac{1}{k^2}$ and T_n the n th partial sum of $\sum \frac{1}{k^2 + k}$. Which is larger, S_1 or T_1 ? Why?

(ii) Recall that

$$S_2 = S_1 + a_2 \quad \text{and} \quad T_2 = T_1 + b_2.$$

Which is larger, a_2 or b_2 ? Based on that answer, which is larger, S_2 or T_2 ?

(iii) Recall that

$$S_3 = S_2 + a_3 \quad \text{and} \quad T_3 = T_2 + b_3.$$

Which is larger, a_3 or b_3 ? Based on that answer, which is larger, S_3 or T_3 ?

(iv) Which is larger, a_n or b_n ? Explain. Based on that answer, which is larger, S_n or T_n ?

(v) Based on your response to the previous part of this exercise, what relationship do you expect there to be between $\sum \frac{1}{k^2}$ and $\sum \frac{1}{k^2 + k}$? Do you expect

$\sum \frac{1}{k^2+k}$ to converge or diverge? Why?

- (b) The example in the previous part of this exercise illustrates a more general result. Explain why the Direct Comparison Test, stated here, works.

The Direct Comparison Test. If

$$0 \leq b_k \leq a_k$$

for every k , then we must have

$$0 \leq \sum b_k \leq \sum a_k$$

1. If $\sum a_k$ converges, then $\sum b_k$ converges.
2. If $\sum b_k$ diverges, then $\sum a_k$ diverges.

Important Note: This comparison test applies only to series with nonnegative terms.

- (i) Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum \frac{1}{k-1}$. Hint: Compare to the harmonic series.
- (ii) Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum \frac{k}{k^3+1}$.

8.4 Alternating Series

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is an alternating series?
- What does it mean for an alternating series to converge?
- Under what conditions does an alternating series converge? Why?
- How well does the n th partial sum of a convergent alternating series approximate the actual sum of the series? Why?
- What is the difference between absolute convergence and conditional convergence?

Introduction

In our study of series so far, almost every series that we've considered has exclusively nonnegative terms. Of course, it is possible to consider series that have some negative terms. For instance, if we consider the geometric series

$$2 - \frac{4}{3} + \frac{8}{9} - \cdots + 2 \left(\frac{-2}{3} \right)^n + \cdots,$$

which has $a = 2$ and $r = -\frac{2}{3}$, we see that not only does every other term alternate in sign, but also that this series converges to

$$S = \frac{a}{1-r} = \frac{2}{1 - (-\frac{2}{3})} = \frac{6}{5}.$$

In Preview Activity 8.4 and our following discussion, we investigate the behavior of similar series where consecutive terms have opposite signs.

Preview Activity 8.4. Preview Activity 8.3 showed how we can approximate the number e with linear, quadratic, and other polynomial approximations. We use a similar approach in this activity to obtain linear and quadratic approximations to $\ln(2)$. Along the way, we encounter a type of series that is different than most of the ones we have seen so far. Throughout this activity, let $f(x) = \ln(1+x)$.

- (a) Find the tangent line to f at $x = 0$ and use this linearization to approximate $\ln(2)$. That is, find $L(x)$, the tangent line approximation to $f(x)$, and use the fact that $L(1) \approx f(1)$ to estimate $\ln(2)$.

- (b) The linearization of $\ln(1+x)$ does not provide a very good approximation to $\ln(2)$ since 1 is not that close to 0. To obtain a better approximation, we alter our approach; instead of using a straight line to approximate $\ln(2)$, we use a quadratic function to account for the concavity of $\ln(1+x)$ for x close to 0. With the linearization, both the function's value and slope agree with the linearization's value and slope at $x = 0$. We will now make a quadratic approximation $P_2(x)$ to $f(x) = \ln(1+x)$ centered at $x = 0$ with the property that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$.
- (i) Let $P_2(x) = x - \frac{x^2}{2}$. Show that $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$. Use $P_2(x)$ to approximate $\ln(2)$ by using the fact that $P_2(1) \approx f(1)$.
- (ii) We can continue approximating $\ln(2)$ with polynomials of larger degree whose derivatives agree with those of f at 0. This makes the polynomials fit the graph of f better for more values of x around 0. For example, let $P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$. Show that $P_3(0) = f(0)$, $P_3'(0) = f'(0)$, $P_3''(0) = f''(0)$, and $P_3'''(0) = f'''(0)$. Taking a similar approach to preceding questions, use $P_3(x)$ to approximate $\ln(2)$.
- (iii) If we used a degree 4 or degree 5 polynomial to approximate $\ln(1+x)$, what approximations of $\ln(2)$ do you think would result? Use the preceding questions to conjecture a pattern that holds, and state the degree 4 and degree 5 approximation.

✕

Preview Activity 8.4 gives us several approximations to $\ln(2)$, the linear approximation is 1 and the quadratic approximation is $1 - \frac{1}{2} = \frac{1}{2}$. If we continue this process we will obtain approximations from cubic, quartic (degree 4), quintic (degree 5), and higher degree polynomials giving us the following approximations to $\ln(2)$:

linear	1	1
quadratic	$1 - \frac{1}{2}$	0.5
cubic	$1 - \frac{1}{2} + \frac{1}{3}$	$0.8\bar{3}$
quartic	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$	$0.58\bar{3}$
quintic	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$	$0.78\bar{3}$

The pattern here shows the fact that the number $\ln(2)$ can be approximated by the partial sums of the infinite series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \quad (8.15)$$

where the alternating signs are determined by the factor $(-1)^{k+1}$.

Using computational technology, we find that 0.6881721793 is the sum of the first 100 terms in this series. As a comparison, $\ln(2) \approx 0.6931471806$. This shows that even

though the series (8.15) converges to $\ln(2)$, it must do so quite slowly, since the sum of the first 100 terms isn't particularly close to $\ln(2)$. We will investigate the issue of how quickly an alternating series converges later in this section. Again, note particularly that the series (8.15) is different from the series we have considered earlier in that some of the terms are negative. We call such a series an *alternating series*.

Definition 8.6. An alternating series is a series of the form

$$\sum_{k=0}^{\infty} (-1)^k a_k,$$

where $a_k \geq 0$ for each k .

We have some flexibility in how we write an alternating series; for example, the series

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k,$$

whose index starts at $k = 1$, is also alternating. As we will soon see, there are several very nice results that hold about alternating series, while alternating series can also demonstrate some unusual behavior.

It is important to remember that most of the series tests we have seen in previous sections apply only to series with nonnegative terms. Thus, alternating series require a different test. To investigate this idea, we return to the example in Preview Activity 8.4.

Activity 8.16.

Remember that, by definition, a series converges if and only if its corresponding sequence of partial sums converges.

- (a) Complete Table 8.7 by calculating the first few partial sums (to 10 decimal places) of the alternating series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}.$$

- (b) Plot the sequence of partial sums from part (a) in the plane. What do you notice about this sequence?

◀

Activity 8.16 exemplifies the general behavior that any convergent alternating series will demonstrate. In this example, we see that the partial sums of the alternating harmonic series oscillate around a fixed number that turns out to be the sum of the series.

$$\begin{array}{l}
\sum_{k=1}^1 (-1)^{k+1} \frac{1}{k} = \\
\sum_{k=1}^2 (-1)^{k+1} \frac{1}{k} = \\
\sum_{k=1}^3 (-1)^{k+1} \frac{1}{k} = \\
\sum_{k=1}^4 (-1)^{k+1} \frac{1}{k} = \\
\sum_{k=1}^5 (-1)^{k+1} \frac{1}{k} = \\
\sum_{k=1}^6 (-1)^{k+1} \frac{1}{k} = \\
\sum_{k=1}^7 (-1)^{k+1} \frac{1}{k} = \\
\sum_{k=1}^8 (-1)^{k+1} \frac{1}{k} = \\
\sum_{k=1}^9 (-1)^{k+1} \frac{1}{k} = \\
\sum_{k=1}^{10} (-1)^{k+1} \frac{1}{k} =
\end{array}$$

Table 8.7: Partial sums of the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$

Recall that if $\lim_{k \rightarrow \infty} a_k \neq 0$, then the series $\sum a_k$ diverges by the Divergence Test. From this point forward, we will thus only consider alternating series

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

in which the sequence a_k consists of positive numbers that decrease to 0. For such a series, the n th partial sum S_n satisfies

$$S_n = \sum_{k=1}^n (-1)^{k+1} a_k.$$

Notice that

- $S_1 = a_1$
- $S_2 = a_1 - a_2$, and since $a_1 > a_2$ we have

$$0 < S_2 < S_1.$$

- $S_3 = S_2 + a_3$ and so $S_2 < S_3$. But $a_3 < a_2$, so $S_3 < S_1$. Thus,

$$0 < S_2 < S_3 < S_1.$$

- $S_4 = S_3 - a_4$ and so $S_4 < S_3$. But $a_4 < a_3$, so $S_2 < S_4$. Thus,

$$0 < S_2 < S_4 < S_3 < S_1.$$

- $S_5 = S_4 + a_5$ and so $S_4 < S_5$. But $a_5 < a_4$, so $S_5 < S_3$. Thus,

$$0 < S_2 < S_4 < S_5 < S_3 < S_1.$$

This pattern continues as illustrated in Figure 8.5 (with n odd) so that each partial sum lies between the previous two partial sums. Note further that the absolute value of the difference between the $(n - 1)$ st partial sum S_{n-1} and the n th partial sum S_n is

$$|S_n - S_{n-1}| = a_n.$$

Since the sequence $\{a_n\}$ converges to 0, the distance between successive partial sums becomes as close to zero as we'd like, and thus the sequence of partial sums converges (even though we don't know the exact value to which the sequence of partial sums converges).

The preceding discussion has demonstrated the truth of the Alternating Series Test.

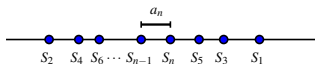


Figure 8.5: Partial sums of an alternating series

The Alternating Series Test. Given an alternating series

$$\sum (-1)^k a_k,$$

if the sequence $\{a_k\}$ of positive terms decreases to 0 as $k \rightarrow \infty$, then the alternating series converges.

Note particularly that if the limit of the sequence $\{a_k\}$ is not 0, then the alternating series diverges.

Activity 8.17.

Which series converge and which diverge? Justify your answers.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 + 2}$

(b) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2k}{k + 5}$

$$(c) \sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$$

<

The argument for the Alternating Series Test also provides us with a method to determine how close the n th partial sum S_n is to the actual sum of a convergent alternating series. To see how this works, let S be the sum of a convergent alternating series, so

$$S = \sum_{k=1}^{\infty} (-1)^k a_k.$$

Recall that the sequence of partial sums oscillates around the sum S so that

$$|S - S_n| < |S_{n+1} - S_n| = a_{n+1}.$$

Therefore, the value of the term a_{n+1} provides an error estimate for how well the partial sum S_n approximates the actual sum S . We summarize this fact in the statement of the Alternating Series Estimation Theorem.

Alternating Series Estimation Theorem. If the alternating series

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

converges and has sum S , and

$$S_n = \sum_{k=1}^n (-1)^{k+1} a_k$$

is the n th partial sum of the alternating series, then

$$\left| \sum_{k=1}^{\infty} (-1)^{k+1} a_k - S_n \right| \leq a_{n+1}.$$

Example 8.1. Let's determine how well the 100th partial sum S_{100} of

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

approximates the sum of the series.

Solution.

If we let S be the sum of the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$, then we know that

$$|S_{100} - S| < a_{101}.$$

Now

$$a_{101} = \frac{1}{101} \approx 0.0099,$$

so the 100th partial sum is within 0.0099 of the sum of the series. We have discussed the fact (and will later verify) that

$$S = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2),$$

and so $S \approx 0.693147$ while

$$S_{100} = \sum_{k=1}^{100} \frac{(-1)^{k+1}}{k} \approx 0.6881721793.$$

We see that the actual difference between S and S_{100} is approximately 0.0049750013, which is indeed less than 0.0099.

Activity 8.18.

Determine the number of terms it takes to approximate the sum of the convergent alternating series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$$

to within 0.0001.

◁

Absolute and Conditional Convergence

A series such as

$$1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \dots \quad (8.16)$$

whose terms are neither all nonnegative nor alternating is different from any series that we have considered to date. The behavior of these series can be rather complicated, but there is an important connection between these arbitrary series that have some negative terms and series with all nonnegative terms that we illustrate with the next activity.

Activity 8.19.



(a) Explain why the series

$$1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \cdots$$

must have a sum that is less than the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

(b) Explain why the series

$$1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \cdots$$

must have a sum that is greater than the series

$$\sum_{k=1}^{\infty} -\frac{1}{k^2}.$$

(c) Given that the terms in the series

$$1 - \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} - \frac{1}{81} - \frac{1}{100} + \cdots$$

converge to 0, what do you think the previous two results tell us about the convergence status of this series?

◁

As the example in Activity 8.19 suggests, if we have a series $\sum a_k$, (some of whose terms may be negative) such that $\sum |a_k|$ converges, it turns out to always be the case that the original series, $\sum a_k$, must also converge. That is, if $\sum |a_k|$ converges, then so must $\sum a_k$.

As we just observed, this is the case for the series (8.16), since the corresponding series of the absolute values of its terms is the convergent p -series $\sum \frac{1}{k^2}$. At the same time, there are series like the alternating harmonic series $\sum (-1)^{k+1} \frac{1}{k}$ that converge, while the corresponding series of absolute values, $\sum \frac{1}{k}$, diverges. We distinguish between these behaviors by introducing the following language.

Definition 8.7. Consider a series $\sum a_k$.

1. The series $\sum a_k$ *converges absolutely* (or is *absolutely convergent*) provided that $\sum |a_k|$ converges.
2. The series $\sum a_k$ *converges conditionally* (or is *conditionally convergent*) provided that $\sum |a_k|$ diverges and $\sum a_k$ converges.

In this terminology, the series (8.16) converges absolutely while the alternating harmonic series is conditionally convergent.

Activity 8.20.

(a) Consider the series $\sum (-1)^k \frac{\ln(k)}{k}$.

- (i) Does this series converge? Explain.
- (ii) Does this series converge absolutely? Explain what test you use to determine your answer.

(b) Consider the series $\sum (-1)^k \frac{\ln(k)}{k^2}$.

- (i) Does this series converge? Explain.
- (ii) Does this series converge absolutely? Hint: Use the fact that $\ln(k) < \sqrt{k}$ for large values of k and then compare to an appropriate p -series.

◁

Conditionally convergent series turn out to be very interesting. If the sequence $\{a_n\}$ decreases to 0, but the series $\sum a_k$ diverges, the conditionally convergent series $\sum (-1)^k a_k$ is right on the borderline of being a divergent series. As a result, any conditionally convergent series converges very slowly. Furthermore, some very strange things can happen with conditionally convergent series, as illustrated in some of the exercises.

Summary of Tests for Convergence of Series

We have discussed several tests for convergence/divergence of series in our sections and in exercises. We close this section of the text with a summary of all the tests we have encountered, followed by an activity that challenges you to decide which convergence test to apply to several different series.

Geometric Series	The geometric series $\sum ar^k$ with ratio r converges for $-1 < r < 1$ and diverges for $ r \geq 1$.	The sum of the convergent geometric series $\sum_{k=0}^{\infty} ar^k$ is $\frac{a}{1-r}$.
Divergence Test	If the sequence a_n does not converge to 0, then the series $\sum a_k$ diverges.	This is the first test to apply because the conclusion is simple. However, if $\lim_{n \rightarrow \infty} a_n = 0$, no conclusion can be drawn.

Integral Test	<p>Let f be a positive, decreasing function on an interval $[c, \infty)$ and let $a_k = f(k)$ for each positive integer $k \geq c$.</p> <ul style="list-style-type: none"> • If $\int_c^\infty f(t) dt$ converges, then $\sum a_k$ converges. • If $\int_c^\infty f(t) dt$ diverges, then $\sum a_k$ diverges. 	Use this test when $f(x)$ is easy to integrate.
Direct Comparison Test (see Ex 4 in Section 8.3)	<p>Let $0 \leq a_k \leq b_k$ for each positive integer k.</p> <ul style="list-style-type: none"> • If $\sum b_k$ converges, then $\sum a_k$ converges. • If $\sum a_k$ diverges, then $\sum b_k$ diverges. 	Use this test when you have a series with known behavior that you can compare to - this test can be difficult to apply.
Limit Comparison Test	<p>Let a_n and b_n be sequences of positive terms. If</p> $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = L$ <p>for some positive finite number L, then the two series $\sum a_k$ and $\sum b_k$ either both converge or both diverge.</p>	Easier to apply in general than the comparison test, but you must have a series with known behavior to compare. Useful to apply to series of rational functions.
Ratio Test	<p>Let $a_k \neq 0$ for each k and suppose</p> $\lim_{k \rightarrow \infty} \frac{ a_{k+1} }{ a_k } = r.$ <ul style="list-style-type: none"> • If $r < 1$, then the series $\sum a_k$ converges absolutely. • If $r > 1$, then the series $\sum a_k$ diverges. • If $r = 1$, then test is inconclusive. 	This test is useful when a series involves factorials and powers.
Root Test (see Exercise 2 in Section 8.3)	<p>Let $a_k \geq 0$ for each k and suppose</p> $\lim_{k \rightarrow \infty} \sqrt[k]{a_k} = r.$ <ul style="list-style-type: none"> • If $r < 1$, then the series $\sum a_k$ converges. • If $r > 1$, then the series $\sum a_k$ diverges. • If $r = 1$, then test is inconclusive. 	In general, the Ratio Test can usually be used in place of the Root Test. However, the Root Test can be quick to use when a_k involves k th powers.
Alternating Series Test	<p>If a_n is a positive, decreasing sequence so that $\lim_{n \rightarrow \infty} a_n = 0$, then the alternating series $\sum (-1)^{k+1} a_k$ converges.</p>	This test applies only to alternating series - we assume that the terms a_n are all positive and that the sequence $\{a_n\}$ is decreasing.
Alternating Series Estimation Theorem	<p>Let $S_n = \sum_{k=1}^n (-1)^{k+1} a_k$ be the nth partial sum of the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$. Assume $a_n > 0$ for each positive integer n, the sequence a_n decreases to 0 and $\lim_{n \rightarrow \infty} S_n = S$. Then it follows that $S - S_n < a_{n+1}$.</p>	This bound can be used to determine the accuracy of the partial sum S_n as an approximation of the sum of a convergent alternating series.

Activity 8.21.

For (a)-(j), use appropriate tests to determine the convergence or divergence of the following series. Throughout, if a series is a convergent geometric series, find its sum.

$$(a) \sum_{k=3}^{\infty} \frac{2}{\sqrt{k-2}}$$

$$(b) \sum_{k=1}^{\infty} \frac{k}{1+2k}$$

$$(c) \sum_{k=0}^{\infty} \frac{2k^2+1}{k^3+k+1}$$

$$(d) \sum_{k=0}^{\infty} \frac{100^k}{k!}$$

$$(e) \sum_{k=1}^{\infty} \frac{2^k}{5^k}$$

$$(f) \sum_{k=1}^{\infty} \frac{k^3-1}{k^5+1}$$

$$(g) \sum_{k=2}^{\infty} \frac{3^{k-1}}{7^k}$$

$$(h) \sum_{k=2}^{\infty} \frac{1}{k^k}$$

$$(i) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k+1}}$$

$$(j) \sum_{k=2}^{\infty} \frac{1}{k \ln(k)}$$

(k) Determine a value of n so that the n th partial sum S_n of the alternating series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$
 approximates the sum to within 0.001.

◁

Summary

In this section, we encountered the following important ideas:

- An alternating series is a series whose terms alternate in sign. In other words, an alternating series is a series of the form

$$\sum (-1)^k a_k$$

where a_k is a positive real number for each k .

- An alternating series $\sum_{k=1}^{\infty} (-1)^k a_k$ converges if and only if its sequence $\{S_n\}$ of partial sums converges, where

$$S_n = \sum_{k=1}^n (-1)^k a_k.$$

- The sequence of partial sums of a convergent alternating series oscillates around and converge to the sum of the series if the sequence of n th terms converges to 0. That is why the Alternating Series Test shows that the alternating series $\sum_{k=1}^{\infty} (-1)^k a_k$ converges whenever the sequence $\{a_n\}$ of n th terms decreases to 0.
- The difference between the $n - 1$ st partial sum S_{n-1} and the n th partial sum S_n of a convergent alternating series $\sum_{k=1}^{\infty} (-1)^k a_k$ is $|S_n - S_{n-1}| = a_n$. Since the partial sums oscillate around the sum S of the series, it follows that

$$|S - S_n| < a_n.$$

So the n th partial sum of a convergent alternating series $\sum_{k=1}^{\infty} (-1)^k a_k$ approximates the actual sum of the series to within a_n .

Exercises

1. Conditionally convergent series converge very slowly. As an example, consider the famous formula⁶

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}. \quad (8.17)$$

In theory, the partial sums of this series could be used to approximate π .

- (a) Show that the series in (8.17) converges conditionally.
- (b) Let S_n be the n th partial sum of the series in (8.17). Calculate the error in

⁶We will derive this formula in upcoming work.

approximating $\frac{\pi}{4}$ with S_{100} and explain why this is not a very good approximation.

- (c) Determine the number of terms it would take in the series (8.17) to approximate $\frac{\pi}{4}$ to 10 decimal places. (The fact that it takes such a large number of terms to obtain even a modest degree of accuracy is why we say that conditionally convergent series converge very slowly.)
2. We have shown that if $\sum(-1)^{k+1}a_k$ is a convergent alternating series, then the sum S of the series lies between any two consecutive partial sums S_n . This suggests that the average $\frac{S_n+S_{n+1}}{2}$ is a better approximation to S than is S_n .

(a) Show that $\frac{S_n+S_{n+1}}{2} = S_n + \frac{1}{2}(-1)^{n+2}a_{n+1}$.

- (b) Use this revised approximation in (a) with $n = 20$ to approximate $\ln(2)$ given that

$$\ln(2) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}.$$

Compare this to the approximation using just S_{20} . For your convenience, $S_{20} = \frac{155685007}{232792560}$.

3. In this exercise, we examine one of the conditions of the Alternating Series Test. Consider the alternating series

$$1 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{9} + \frac{1}{4} - \frac{1}{16} + \cdots,$$

where the terms are selected alternately from the sequences $\{\frac{1}{n}\}$ and $\{-\frac{1}{n^2}\}$.

- (a) Explain why the n th term of the given series converges to 0 as n goes to infinity.
- (b) Rewrite the given series by grouping terms in the following manner:

$$(1 - 1) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{9}\right) + \left(\frac{1}{4} - \frac{1}{16}\right) + \cdots.$$

Use this regrouping to determine if the series converges or diverges.

- (c) Explain why the condition that the sequence $\{a_n\}$ *decreases* to a limit of 0 is included in the Alternating Series Test.
4. Conditionally convergent series exhibit interesting and unexpected behavior. In this exercise we examine the conditionally convergent alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ and discover that addition is not commutative for conditionally convergent series. We will also encounter Riemann's Theorem concerning rearrangements of conditionally

convergent series. Before we begin, we remind ourselves that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2),$$

a fact which will be verified in a later section.

(a) First we make a quick analysis of the positive and negative terms of the alternating harmonic series.

(i) Show that the series $\sum_{k=1}^{\infty} \frac{1}{2k}$ diverges.

(ii) Show that the series $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ diverges.

(iii) Based on the results of the previous parts of this exercise, what can we say about the sums $\sum_{k=C}^{\infty} \frac{1}{2k}$ and $\sum_{k=C}^{\infty} \frac{1}{2k+1}$ for any positive integer C ? Be specific in your explanation.

(b) Recall addition of real numbers is commutative; that is

$$a + b = b + a$$

for any real numbers a and b . This property is valid for any sum of finitely many terms, but does this property extend when we add infinitely many terms together?

The answer is no, and something even more odd happens. Riemann's Theorem (after the nineteenth-century mathematician Georg Friedrich Bernhard Riemann) states that a conditionally convergent series can be rearranged to converge to *any* prescribed sum. More specifically, this means that if we choose any real number S , we can rearrange the terms of the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ so that the sum is S . To understand how Riemann's Theorem works, let's assume for the moment that the number S we want our rearrangement to converge to is positive. Our job is to find a way to order the sum of terms of the alternating harmonic series to converge to S .

(i) Explain how we know that, regardless of the value of S , we can find a partial sum P_1

$$P_1 = \sum_{k=1}^{n_1} \frac{1}{2k+1} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n_1+1}$$

of the positive terms of the alternating harmonic series that equals or

exceeds S . Let

$$S_1 = P_1.$$

- (ii) Explain how we know that, regardless of the value of S_1 , we can find a partial sum N_1

$$N_1 = - \sum_{k=1}^{m_1} \frac{1}{2k} = -\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots - \frac{1}{2m_1}$$

so that

$$S_2 = S_1 + N_1 \leq S.$$

- (iii) Explain how we know that, regardless of the value of S_2 , we can find a partial sum P_2

$$P_2 = \sum_{k=n_1+1}^{n_2} \frac{1}{2k+1} = \frac{1}{2(n_1+1)+1} + \frac{1}{2(n_1+2)+1} + \cdots + \frac{1}{2n_2+1}$$

of the remaining positive terms of the alternating harmonic series so that

$$S_3 = S_2 + P_2 \geq S.$$

- (iv) Explain how we know that, regardless of the value of S_3 , we can find a partial sum

$$N_2 = - \sum_{k=m_1+1}^{m_2} \frac{1}{2k} = -\frac{1}{2(m_1+1)} - \frac{1}{2(m_1+2)} - \cdots - \frac{1}{2m_2}$$

of the remaining negative terms of the alternating harmonic series so that

$$S_4 = S_3 + N_2 \leq S.$$

- (v) Explain why we can continue this process indefinitely and find a sequence $\{S_n\}$ whose terms are partial sums of a rearrangement of the terms in the alternating harmonic series so that $\lim_{n \rightarrow \infty} S_n = S$.

8.5 Taylor Polynomials and Taylor Series

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a Taylor polynomial? For what purposes are Taylor polynomials used?
- What is a Taylor series?
- How are Taylor polynomials and Taylor series different? How are they related?
- How do we determine the accuracy when we use a Taylor polynomial to approximate a function?

Introduction

In our work to date in Chapter 8, essentially every sum we have considered has been a sum of numbers. In particular, each infinite series that we have discussed has been a series of real numbers, such as

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \cdots = \sum_{k=0}^{\infty} \frac{1}{2^k}. \quad (8.18)$$

In the remainder of this chapter, we will expand our notion of series to include series that involve a variable, say x . For instance, if in the geometric series in Equation (8.18) we replace the ratio $r = \frac{1}{2}$ with the variable x , then we have the infinite (still geometric) series

$$1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k. \quad (8.19)$$

Here we see something very interesting: since a geometric series converges whenever its ratio r satisfies $|r| < 1$, and the sum of a convergent geometric series is $\frac{a}{1-r}$, we can say that for $|x| < 1$,

$$1 + x + x^2 + \cdots + x^k + \cdots = \frac{1}{1-x}. \quad (8.20)$$

Note well what Equation (8.20) states: the non-polynomial function $\frac{1}{1-x}$ on the right is equal to the infinite polynomial expression on the left. Moreover, it appears natural to truncate the infinite sum on the left (whose terms get very small as k gets large) and say, for example, that

$$1 + x + x^2 + x^3 \approx \frac{1}{1-x}$$

for small values of x . This shows one way that a polynomial function can be used to approximate a non-polynomial function; such approximations are one of the main themes in this section and the next.

In Preview Activity 8.5, we begin our explorations of approximating non-polynomial functions with polynomials, from which we will also develop ideas regarding infinite series that involve a variable, x .

Preview Activity 8.5. Preview Activity 8.3 showed how we can approximate the number e using linear, quadratic, and other polynomial functions; we then used similar ideas in Preview Activity 8.4 to approximate $\ln(2)$. In this activity, we review and extend the process to find the “best” quadratic approximation to the exponential function e^x around the origin. Let $f(x) = e^x$ throughout this activity.

- (a) Find a formula for $P_1(x)$, the linearization of $f(x)$ at $x = 0$. (We label this linearization P_1 because it is a first degree polynomial approximation.) Recall that $P_1(x)$ is a good approximation to $f(x)$ for values of x close to 0. Plot f and P_1 near $x = 0$ to illustrate this fact.
- (b) Since $f(x) = e^x$ is not linear, the linear approximation eventually is not a very good one. To obtain better approximations, we want to develop a different approximation that “bends” to make it more closely fit the graph of f near $x = 0$. To do so, we add a quadratic term to $P_1(x)$. In other words, we let

$$P_2(x) = P_1(x) + c_2x^2$$

for some real number c_2 . We need to determine the value of c_2 that makes the graph of $P_2(x)$ best fit the graph of $f(x)$ near $x = 0$.

Remember that $P_1(x)$ was a good linear approximation to $f(x)$ near 0; this is because $P_1(0) = f(0)$ and $P_1'(0) = f'(0)$. It is therefore reasonable to seek a value of c_2 so that

$$\begin{aligned} P_2(0) &= f(0), \\ P_2'(0) &= f'(0), \text{ and} \\ P_2''(0) &= f''(0). \end{aligned}$$

Remember, we are letting $P_2(x) = P_1(x) + c_2x^2$.

- (i) Calculate $P_2(0)$ to show that $P_2(0) = f(0)$.
- (ii) Calculate $P_2'(0)$ to show that $P_2'(0) = f'(0)$.
- (iii) Calculate $P_2''(x)$. Then find a value for c_2 so that $P_2''(0) = f''(0)$.
- (iv) Explain why the condition $P_2''(0) = f''(0)$ will put an appropriate “bend” in the graph of P_2 to make P_2 fit the graph of f around $x = 0$.

Taylor Polynomials

Preview Activity 8.5 illustrates the first steps in the process of approximating complicated functions with polynomials. Using this process we can approximate trigonometric, exponential, logarithmic, and other nonpolynomial functions as closely as we like (for certain values of x) with polynomials. This is extraordinarily useful in that it allows us to calculate values of these functions to whatever precision we like using only the operations of addition, subtraction, multiplication, and division, which are operations that can be easily programmed in a computer.

We next extend the approach in Preview Activity 8.5 to arbitrary functions at arbitrary points. Let f be a function that has as many derivatives at a point $x = a$ as we need. Since first learning it in Section 1.8, we have regularly used the linear approximation $P_1(x)$ to f at $x = a$, which in one sense is the best linear approximation to f near a . Recall that $P_1(x)$ is the tangent line to f at $(a, f(a))$ and is given by the formula

$$P_1(x) = f(a) + f'(a)(x - a).$$

If we proceed as in Preview Activity 8.5, we then want to find the best quadratic approximation

$$P_2(x) = P_1(x) + c_2(x - a)^2$$

so that $P_2(x)$ more closely models $f(x)$ near $x = a$. Consider the following calculations of the values and derivatives of $P_2(x)$:

$$P_2(x) = P_1(x) + c_2(x - a)^2$$

$$P_2'(x) = P_1'(x) + 2c_2(x - a)$$

$$P_2''(x) = 2c_2$$

$$P_2(a) = P_1(a) = f(a)$$

$$P_2'(a) = P_1'(a) = f'(a)$$

$$P_2''(a) = 2c_2.$$

To make $P_2(x)$ fit $f(x)$ better than $P_1(x)$, we want $P_2(x)$ and $f(x)$ to have the same concavity at $x = a$. That is, we want to have

$$P_2''(a) = f''(a).$$

This implies that

$$2c_2 = f''(a)$$

and thus

$$c_2 = \frac{f''(a)}{2}.$$

Therefore, the quadratic approximation $P_2(x)$ to f centered at $x = 0$ is

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2.$$

This approach extends naturally to polynomials of higher degree. In this situation, we define polynomials

$$P_3(x) = P_2(x) + c_3(x - a)^3,$$

$$P_4(x) = P_3(x) + c_4(x - a)^4,$$

$$P_5(x) = P_4(x) + c_5(x - a)^5,$$

and so on, with the general one being

$$P_n(x) = P_{n-1}(x) + c_n(x - a)^n.$$

The defining property of these polynomials is that for each n , $P_n(x)$ must have its value and all its first n derivatives agree with those of f at $x = a$. In other words we require that

$$P_n^{(k)}(a) = f^{(k)}(a)$$

for all k from 0 to n .

To see the conditions under which this happens, suppose

$$P_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n.$$

Then

$$P_n^{(0)}(a) = c_0$$

$$P_n^{(1)}(a) = c_1$$

$$P_n^{(2)}(a) = 2c_2$$

$$P_n^{(3)}(a) = (2)(3)c_3$$

$$P_n^{(4)}(a) = (2)(3)(4)c_4$$

$$P_n^{(5)}(a) = (2)(3)(4)(5)c_5$$

and, in general,

$$P_n^{(k)}(a) = (2)(3)(4) \cdots (k-1)(k)c_k = k!c_k.$$

So having

$$P_n^{(k)}(a) = f^{(k)}(a)$$

means that

$$k!c_k = f^{(k)}(a)$$

and therefore

$$c_k = \frac{f^{(k)}(a)}{k!}$$

for each value of k . In this expression for c_k , we have found the formula for the degree n polynomial approximation of f that we seek.

The n th order Taylor polynomial of f centered at $x = a$ is given by

$$\begin{aligned} P_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k. \end{aligned}$$

This degree n polynomial approximates $f(x)$ near $x = a$ and has the property that $P_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0 \dots n$.

Example 8.2. Determine the third order Taylor polynomial for $f(x) = e^x$, as well as the general n th order Taylor polynomial for f centered at $x = 0$.

Solution. We know that $f'(x) = e^x$ and so $f''(x) = e^x$ and $f'''(x) = e^x$. Thus,

$$f(0) = f'(0) = f''(0) = f'''(0) = 1.$$

So the third order Taylor polynomial of $f(x) = e^x$ centered at $x = 0$ is

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

In general, for the exponential function f we have $f^{(k)}(x) = e^x$ for every positive integer k . Thus, the k th term in the n th order Taylor polynomial for $f(x)$ centered at $x = 0$ is

$$\frac{f^{(k)}(0)}{k!}(x-0)^k = \frac{1}{k!}x^k.$$

Therefore, the n th order Taylor polynomial for $f(x) = e^x$ centered at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{1}{n!}x^n = \sum_{k=0}^n \frac{x^k}{k!}.$$

Activity 8.22.

We have just seen that the n th order Taylor polynomial centered at $a = 0$ for the exponential function e^x is

$$\sum_{k=0}^n \frac{x^k}{k!}.$$

In this activity, we determine small order Taylor polynomials for several other familiar functions, and look for general patterns that will help us find the Taylor series expansions a bit later.

- (a) Let $f(x) = \frac{1}{1-x}$.
- Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\frac{1}{1-x}$ centered at 0.
 - Based on your results from part (i), determine a general formula for $f^{(k)}(0)$.
- (b) Let $f(x) = \cos(x)$.
- Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\cos(x)$ centered at 0.
 - Based on your results from part (i), find a general formula for $f^{(k)}(0)$. (Think about how k being even or odd affects the value of the k th derivative.)
- (c) Let $f(x) = \sin(x)$.
- Calculate the first four derivatives of $f(x)$ at $x = 0$. Then find the fourth order Taylor polynomial $P_4(x)$ for $\sin(x)$ centered at 0.
 - Based on your results from part (i), find a general formula for $f^{(k)}(0)$. (Think about how k being even or odd affects the value of the k th derivative.)

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It is possible that an n th order Taylor polynomial is not a polynomial of degree n ; that is, the order of the approximation can be different from the degree of the polynomial. For example, in Activity 8.22 we found that the second order Taylor polynomial $P_2(x)$ centered at 0 for $\sin(x)$ is $P_2(x) = x$. In this case, the second order Taylor polynomial is a degree 1 polynomial.

Taylor Series

In Activity 8.22 we saw that the fourth order Taylor polynomial $P_4(x)$ for $\sin(x)$ centered at 0 is

$$P_4(x) = x - \frac{x^3}{3!}.$$

The pattern we found for the derivatives $f^{(k)}(0)$ describe the higher-order Taylor polynomials, e.g.,

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \quad P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}, \quad P_9(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!},$$

and so on. It is instructive to consider the graphical behavior of these functions; Figure 8.6 shows the graphs of a few of the Taylor polynomials centered at 0 for the sine function.

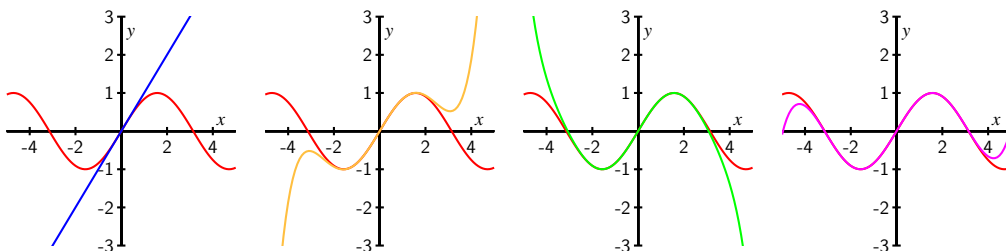


Figure 8.6: The order 1, 5, 7, and 9 Taylor polynomials centered at $x = 0$ for $f(x) = \sin(x)$.

Notice that $P_1(x)$ is close to the sine function only for values of x that are close to 0, but as we increase the degree of the Taylor polynomial the Taylor polynomials provide a better fit to the graph of the sine function over larger intervals. This illustrates the general behavior of Taylor polynomials: for any sufficiently well-behaved function, the sequence $\{P_n(x)\}$ of Taylor polynomials converges to the function f on larger and larger intervals (though those intervals may not necessarily increase without bound). If the Taylor polynomials ultimately converge to f on its entire domain, we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

Definition 8.8. Let f be a function all of whose derivatives exist at $x = a$. The **Taylor series** for f centered at $x = a$ is the series $T_f(x)$ defined by

$$T_f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

In the special case where $a = 0$ in Definition 8.8, the Taylor series is also called the *Maclaurin series* for f . From Example 8.2 we know the n th order Taylor polynomial centered at 0 for the exponential function e^x ; thus, the Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Activity 8.23.

In Activity 8.22 we determined small order Taylor polynomials for a few familiar functions, and also found general patterns in the derivatives evaluated at 0. Use that information to write the Taylor series centered at 0 for the following functions.

- (a) $f(x) = \frac{1}{1-x}$
- (b) $f(x) = \cos(x)$ (You will need to carefully consider how to indicate that many of the coefficients are 0. Think about a general way to represent an even integer.)
- (c) $f(x) = \sin(x)$ (You will need to carefully consider how to indicate that many of the coefficients are 0. Think about a general way to represent an odd integer.)

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The next activity further considers the important issue of the x -values for which the Taylor series of a function converges to the function itself.

Activity 8.24.

- (a) Plot the graphs of several of the Taylor polynomials centered at 0 (of order at least 5) for e^x and convince yourself that these Taylor polynomials converge to e^x for every value of x .
- (b) Draw the graphs of several of the Taylor polynomials centered at 0 (of order at least 6) for $\cos(x)$ and convince yourself that these Taylor polynomials converge to $\cos(x)$ for every value of x . Write the Taylor series centered at 0 for $\cos(x)$.
- (c) Draw the graphs of several of the Taylor polynomials centered at 0 for $\frac{1}{1-x}$. Based on your graphs, for what values of x do these Taylor polynomials appear to converge to $\frac{1}{1-x}$? How is this situation different from what we observe with e^x and $\cos(x)$? In addition, write the Taylor series centered at 0 for $\frac{1}{1-x}$.

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The Maclaurin series for e^x , $\sin(x)$, $\cos(x)$, and $\frac{1}{1-x}$ will be used frequently, so we should be certain to know and recognize them well.

The Interval of Convergence of a Taylor Series

In the previous section (in Figure 8.6 and Activity 8.24) we observed that the Taylor polynomials centered at 0 for e^x , $\cos(x)$, and $\sin(x)$ converged to these functions for all values of x in their domain, but that the Taylor polynomials centered at 0 for $\frac{1}{1-x}$ converged to $\frac{1}{1-x}$ for only some values of x . In fact, the Taylor polynomials centered at 0 for $\frac{1}{1-x}$ converge to $\frac{1}{1-x}$ on the interval $(-1, 1)$ and diverge for all other values of x . So the Taylor series for a function $f(x)$ does not need to converge for all values of x in the domain of f .

Our observations to date suggest two natural questions: can we determine the values

of x for which a given Taylor series converges? Moreover, given the Taylor series for a function f , does it actually converge to $f(x)$ for those values of x for which the Taylor series converges?

Example 8.3. Graphical evidence suggests that the Taylor series centered at 0 for e^x converges for all values of x . To verify this, use the Ratio Test to determine all values of x for which the Taylor series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (8.21)$$

converges absolutely.

Solution. In previous work, we used the Ratio Test on series of numbers that did not involve a variable; recall, too, that the Ratio Test only applies to series of nonnegative terms. In this example, we have to address the presence of the variable x . Because we are interested in absolute convergence, we apply the Ratio Test to the series

$$\sum_{k=0}^{\infty} \left| \frac{x^k}{k!} \right| = \sum_{k=0}^{\infty} \frac{|x|^k}{k!}.$$

Now, observe that

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} &= \lim_{k \rightarrow \infty} \frac{\frac{|x|^{k+1}}{(k+1)!}}{\frac{|x|^k}{k!}} \\ &= \lim_{k \rightarrow \infty} \frac{|x|^{k+1} k!}{|x|^k (k+1)!} \\ &= \lim_{k \rightarrow \infty} \frac{|x|}{k+1} \\ &= 0 \end{aligned}$$

for any value of x . So the Taylor series (8.21) converges absolutely for every value of x , and thus converges for every value of x .

One key question remains: while the Taylor series for e^x converges for all x , what we have done does not tell us that this Taylor series actually converges to e^x for each x . We'll return to this question when we consider the error in a Taylor approximation near the end of this section.

We can apply the main idea from Example 8.3 in general. To determine the values of x for which a Taylor series

$$\sum_{k=0}^{\infty} c_k(x-a)^k,$$

centered at $x = a$ will converge, we apply the Ratio Test with $a_k = |c_k(x-a)^k|$ and recall that the series to which the Ratio Test is applied converges if $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} < 1$.

Observe that

$$\frac{a_{k+1}}{a_k} = |x-a| \frac{|c_{k+1}|}{|c_k|},$$

so when we apply the Ratio Test, we get that

$$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} |x-a| \frac{c_{k+1}}{c_k}.$$

Note further that $c_k = \frac{f^{(k)}(a)}{k!}$, and say that

$$\lim_{k \rightarrow \infty} \frac{c_{k+1}}{c_k} = L.$$

Thus, we have found that

$$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = |x-a| \cdot L.$$

There are three important possibilities for L : L can be 0, a finite positive value, or infinite. Based on this value of L , we can therefore determine for which values of x the original Taylor series converges.

- If $L = 0$, then the Taylor series converges on $(-\infty, \infty)$.
- If L is infinite, then the Taylor series converges only at $x = a$.
- If L is finite and nonzero, then the Taylor series converges absolutely for all x that satisfy

$$|x-a| \cdot L < 1.$$

In other words, the series converges absolutely for all x such that

$$|x-a| < \frac{1}{L},$$

which is also the interval

$$\left(a - \frac{1}{L}, a + \frac{1}{L}\right).$$

Because the Ratio Test is inconclusive when the $|x-a| \cdot L = 1$, the endpoints $a \pm \frac{1}{L}$ have to be checked separately.

It is important to notice that the set of x values at which a Taylor series converges

is always an interval centered at $x = a$. For this reason, the set on which a Taylor series converges is called the *interval of convergence*. Half the length of the interval of convergence is called the *radius of convergence*. If the interval of convergence of a Taylor series is infinite, then we say that the radius of convergence is infinite.

Activity 8.25.

- (a) Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \frac{1}{1-x}$ centered at $x = 0$.
- (b) Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \cos(x)$ centered at $x = 0$.
- (c) Use the Ratio Test to explicitly determine the interval of convergence of the Taylor series for $f(x) = \sin(x)$ centered at $x = 0$.

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The Ratio Test tells us how we can determine the set of x values for which a Taylor series converges absolutely. However, just because a Taylor series for a function f converges, we cannot be certain that the Taylor series actually converges to $f(x)$ on its interval of convergence. To show why and where a Taylor series does in fact converge to the function f , we next consider the error that is present in Taylor polynomials.

Error Approximations for Taylor Polynomials

We now know how to find Taylor polynomials for functions such as $\sin(x)$, as well as how to determine the interval of convergence of the corresponding Taylor series. We next develop an error bound that will tell us how well an n th order Taylor polynomial $P_n(x)$ approximates its generating function $f(x)$. This error bound will also allow us to determine whether a Taylor series on its interval of convergence actually equals the function f from which the Taylor series is derived. Finally, we will be able to use the error bound to determine the order of the Taylor polynomial $P_n(x)$ for a function f that we need to ensure that $P_n(x)$ approximates $f(x)$ to any desired degree of accuracy.

In all of this, we need to compare $P_n(x)$ to $f(x)$. For this argument, we assume throughout that we center our approximations at 0 (a similar argument holds for approximations centered at a). We define the exact error, $E_n(x)$, that results from approximating $f(x)$ with $P_n(x)$ by

$$E_n(x) = f(x) - P_n(x).$$

We are particularly interested in $|E_n(x)|$, the distance between P_n and f . Note that since

$$P_n^{(k)}(0) = f^{(k)}(0)$$

for $0 \leq k \leq n$, we know that

$$E_n^{(k)}(0) = 0$$

for $0 \leq k \leq n$. Furthermore, since $P_n(x)$ is a polynomial of degree less than or equal to n , we know that

$$P_n^{(n+1)}(x) = 0.$$

Thus, since $E_n^{(n+1)}(x) = f^{(n+1)}(x) - P_n^{(n+1)}(x)$, it follows that

$$E_n^{(n+1)}(x) = f^{(n+1)}(x)$$

for all x .

Suppose that we want to approximate $f(x)$ at a number c close to 0 using $P_n(c)$. If we assume $|f^{(n+1)}(t)|$ is bounded by some number M on $[0, c]$, so that

$$|f^{(n+1)}(t)| \leq M$$

for all $0 \leq t \leq c$, then we can say that

$$|E_n^{(n+1)}(t)| = |f^{(n+1)}(t)| \leq M$$

for all t between 0 and c . Equivalently,

$$-M \leq E_n^{(n+1)}(t) \leq M \tag{8.22}$$

on $[0, c]$. Next, we integrate the three terms in the inequality (8.22) from $t = 0$ to $t = x$, and thus find that

$$\int_0^x -M dt \leq \int_0^x E_n^{(n+1)}(t) dt \leq \int_0^x M dt$$

for every value of x in $[0, c]$. Since $E_n^{(n)}(0) = 0$, the First FTC tells us that

$$-Mx \leq E_n^{(n)}(x) \leq Mx$$

for every x in $[0, c]$.

Integrating the most recent inequality, we obtain

$$\int_0^x -Mt dt \leq \int_0^x E_n^{(n)}(t) dt \leq \int_0^x Mt dt$$

and thus

$$-M \frac{x^2}{2} \leq E_n^{(n-1)}(x) \leq M \frac{x^2}{2}$$

for all x in $[0, c]$.

Integrating n times, we arrive at

$$-M \frac{x^{n+1}}{(n+1)!} \leq E_n(x) \leq M \frac{x^{n+1}}{(n+1)!}$$

for all x in $[0, c]$. This enables us to conclude that

$$|E_n(x)| \leq M \frac{|x|^{n+1}}{(n+1)!}$$

for all x in $[0, c]$, which shows an important bound on the approximation's error, E_n .

Our work above was based on the approximation centered at $a = 0$; the argument may be generalized to hold for any value of a , which results in the following theorem.

The Lagrange Error Bound for $P_n(x)$. Let f be a continuous function with $n + 1$ continuous derivatives. Suppose that M is a positive real number such that $|f^{(n+1)}(x)| \leq M$ on the interval $[a, c]$. If $P_n(x)$ is the n th order Taylor polynomial for $f(x)$ centered at $x = a$, then

$$|P_n(c) - f(c)| \leq M \frac{|c - a|^{n+1}}{(n+1)!}.$$

This error bound may now be used to tell us important information about Taylor polynomials and Taylor series, as we see in the following examples and activities.

Example 8.4. Determine how well the 10th order Taylor polynomial $P_{10}(x)$ for $\sin(x)$, centered at 0, approximates $\sin(2)$.

Solution. To answer this question we use $f(x) = \sin(x)$, $c = 2$, $a = 0$, and $n = 10$ in the Lagrange error bound formula. To use the bound, we also need to find an appropriate value for M . Note that the derivatives of $f(x) = \sin(x)$ are all equal to $\pm \sin(x)$ or $\pm \cos(x)$. Thus,

$$|f^{(n+1)}(x)| \leq 1$$

for any n and x . Therefore, we can choose M to be 1. Then

$$|P_{10}(2) - f(2)| \leq (1) \frac{|2 - 0|^{11}}{(11)!} = \frac{2^{11}}{(11)!} \approx 0.00005130671797.$$

So $P_{10}(2)$ approximates $\sin(2)$ to within at most 0.00005130671797. A computer algebra system tells us that

$$P_{10}(2) \approx 0.9093474427 \quad \text{and} \quad \sin(2) \approx 0.9092974268$$

with an actual difference of about 0.0000500159.

Activity 8.26.

Let $P_n(x)$ be the n th order Taylor polynomial for $\sin(x)$ centered at $x = 0$. Determine how large we need to choose n so that $P_n(2)$ approximates $\sin(2)$ to 20 decimal places.

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Example 8.5. Show that the Taylor series for $\sin(x)$ actually converges to $\sin(x)$ for all x .

Solution. Recall from the previous example that since $f(x) = \sin(x)$, we know

$$|f^{(n+1)}(x)| \leq 1$$

for any n and x . This allows us to choose $M = 1$ in the Lagrange error bound formula. Thus,

$$|P_n(x) - \sin(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \quad (8.23)$$

for every x .

We showed in earlier work with the Taylor series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for every value of x .

Since the terms of any convergent series must approach zero, it follows that

$$\lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)!} = 0$$

for every value of x . Thus, taking the limit as $n \rightarrow \infty$ in the inequality (8.23), it follows that

$$\lim_{n \rightarrow \infty} |P_n(x) - \sin(x)| = 0.$$

As a result, we can now write

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for every real number x .

Activity 8.27.

- Show that the Taylor series centered at 0 for $\cos(x)$ converges to $\cos(x)$ for every real number x .
- Next we consider the Taylor series for e^x .

- (i) Show that the Taylor series centered at 0 for e^x converges to e^x for every nonnegative value of x .
- (ii) Show that the Taylor series centered at 0 for e^x converges to e^x for every negative value of x .
- (iii) Explain why the Taylor series centered at 0 for e^x converges to e^x for every real number x . Recall that we earlier showed that the Taylor series centered at 0 for e^x converges for all x , and we have now completed the argument that the Taylor series for e^x actually converges to e^x for all x .
- (c) Let $P_n(x)$ be the n th order Taylor polynomial for e^x centered at 0. Find a value of n so that $P_n(5)$ approximates e^5 correct to 8 decimal places.

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Summary

In this section, we encountered the following important ideas:

- We can use Taylor polynomials to approximate complicated functions. This allows us to approximate values of complicated functions using only addition, subtraction, multiplication, and division of real numbers. The n th order Taylor polynomial centered at $x = a$ of a function f is

$$\begin{aligned} P_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k. \end{aligned}$$

- The Taylor series centered at $x = a$ for a function f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

- The n th order Taylor polynomial centered at a for f is the n th partial sum of its Taylor series centered at a . So the n th order Taylor polynomial for a function f is an approximation to f on the interval where the Taylor series converges; for the values of x for which the Taylor series converges to f we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

- The Lagrange Error Bound shows us how to determine the accuracy in using a Taylor polynomial to approximate a function. More specifically, if $P_n(x)$ is the n th order Taylor

polynomial for f centered at $x = a$ and if M is an upper bound for $|f^{(n+1)}(x)|$ on the interval $[a, c]$, then

$$|P_n(c) - f(c)| \leq M \frac{|c - a|^{n+1}}{(n + 1)!}.$$

Exercises

- In this exercise we investigate the Taylor series of polynomial functions.
 - Find the 3rd order Taylor polynomial centered at $a = 0$ for $f(x) = x^3 - 2x^2 + 3x - 1$. Does your answer surprise you? Explain.
 - Without doing any additional computation, find the 4th, 12th, and 100th order Taylor polynomials (centered at $a = 0$) for $f(x) = x^3 - 2x^2 + 3x - 1$. Why should you expect this?
 - Now suppose $f(x)$ is a degree m polynomial. Completely describe the n th order Taylor polynomial (centered at $a = 0$) for each n .
- The examples we have considered in this section have all been for Taylor polynomials and series centered at 0, but Taylor polynomials and series can be centered at any value of a . We look at examples of such Taylor polynomials in this exercise.
 - Let $f(x) = \sin(x)$. Find the Taylor polynomials up through order four of f centered at $x = \frac{\pi}{2}$. Then find the Taylor series for $f(x)$ centered at $x = \frac{\pi}{2}$. Why should you have expected the result?
 - Let $f(x) = \ln(x)$. Find the Taylor polynomials up through order four of f centered at $x = 1$. Then find the Taylor series for $f(x)$ centered at $x = 1$.
 - Use your result from (b) to determine which Taylor polynomial will approximate $\ln(2)$ to two decimal places. Explain in detail how you know you have the desired accuracy.
- We can use known Taylor series to obtain other Taylor series, and we explore that idea in this exercise, as a preview of work in the following section.
 - Calculate the first four derivatives of $\sin(x^2)$ and hence find the fourth order Taylor polynomial for $\sin(x^2)$ centered at $a = 0$.
 - Part (a) demonstrates the brute force approach to computing Taylor polynomials and series. Now we find an easier method that utilizes a known Taylor series. Recall that the Taylor series centered at 0 for $f(x) = \sin(x)$ is

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k + 1)!}. \quad (8.24)$$

- (i) Substitute x^2 for x in the Taylor series (8.24). Write out the first several terms and compare to your work in part (a). Explain why the substitution in this problem should give the Taylor series for $\sin(x^2)$ centered at 0.
- (ii) What should we expect the interval of convergence of the series for $\sin(x^2)$ to be? Explain in detail.
4. Based on the examples we have seen, we might expect that the Taylor series for a function f always converges to the values $f(x)$ on its interval of convergence. We explore that idea in more detail in this exercise. Let $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
- (a) Show, using the definition of the derivative, that $f'(0) = 0$.
- (b) It can be shown that $f^{(n)}(0) = 0$ for all $n \geq 2$. Assuming that this is true, find the Taylor series for f centered at 0.
- (c) What is the interval of convergence of the Taylor series centered at 0 for f ? Explain. For which values of x the interval of convergence of the Taylor series does the Taylor series converge to $f(x)$?
-

8.6 Power Series

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a power series?
- What are some important uses of power series?
- What is the connection between power series and Taylor series?

Introduction

We have noted at several points in our work with Taylor polynomials and Taylor series that polynomial functions are the simplest possible functions in mathematics, in part because they essentially only require addition and multiplication to evaluate. Moreover, from the point of view of calculus, polynomials are especially nice: we can easily differentiate or integrate any polynomial. In light of our work in Section 8.5, we now know that several important non-polynomials have polynomial-like expansions. For example, for any real number x ,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

As we continue our study of infinite series, there are two settings where other series like the one for e^x arise: one is where we are simply given an expression like

$$1 + 2x + 3x^2 + 4x^3 + \cdots$$

and we seek the values of x for which the expression makes sense, while another is where we are trying to find an unknown function f , and we think about the possibility that the function has expression

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k + \cdots ,$$

and we try to determine the values of the constants a_0, a_1, \dots . The latter situation is explored in Preview Activity 8.6.

Preview Activity 8.6. In Chapter 7, we learned some of the many important applications of differential equations, and learned some approaches to solve or analyze them. Here, we consider an important approach that will allow us to solve a wider variety of differential equations.

Let's consider the familiar differential equation from exponential population growth

given by

$$y' = ky, \quad (8.25)$$

where k is the constant of proportionality. While we can solve this differential equation using methods we have already learned, we take a different approach now that can be applied to a much larger set of differential equations. For the rest of this activity, let's assume that $k = 1$. We will use our knowledge of Taylor series to find a solution to the differential equation (8.25).

To do so, we assume that we have a solution $y = f(x)$ and that $f(x)$ has a Taylor series that can be written in the form

$$y = f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

where the coefficients a_k are undetermined. Our task is to find the coefficients.

- (a) Assume that we can differentiate a power series term by term. By taking the derivative of $f(x)$ with respect to x and substituting the result into the differential equation (8.25), show that the equation

$$\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} a_k x^k$$

must be satisfied in order for $f(x) = \sum_{k=0}^{\infty} a_k x^k$ to be a solution of the DE.

- (b) Two series are equal if and only if they have the same coefficients on like power terms. Use this fact to find a relationship between a_1 and a_0 .
- (c) Now write a_2 in terms of a_1 . Then write a_2 in terms of a_0 .
- (d) Write a_3 in terms of a_2 . Then write a_3 in terms of a_0 .
- (e) Write a_4 in terms of a_3 . Then write a_4 in terms of a_0 .
- (f) Observe that there is a pattern in (b)-(e). Find a general formula for a_k in terms of a_0 .
- (g) Write the series expansion for y using only the unknown coefficient a_0 . From this, determine what familiar functions satisfy the differential equation (8.25). (**Hint:** Compare to a familiar Taylor series.)

Power Series

As Preview Activity 8.6 shows, it can be useful to treat an unknown function as if it has a Taylor series, and then determine the coefficients from other information. In other words, we define a function as an infinite series of powers of x and then determine the coefficients based on something besides a formula for the function. This method of using series illustrated in Preview Activity 8.6 to solve differential equations is a powerful and important one that allows us to approximate solutions to many different types of differential equations even if we cannot explicitly solve them. This approach is different from defining a Taylor series based on a given function, and these functions we define with arbitrary coefficients are given a special name.

Definition 8.9. A *power series* centered at $x = a$ is a function of the form

$$\sum_{k=0}^{\infty} c_k(x - a)^k \quad (8.26)$$

where $\{c_k\}$ is a sequence of real numbers and x is an independent variable.

We can substitute different values for x and test whether the resulting series converges or diverges. Thus, a power series defines a function f whose domain is the set of x values for which the power series converges. We therefore write

$$f(x) = \sum_{k=0}^{\infty} c_k(x - a)^k.$$

It turns out that⁷, on its interval of convergence, a power series is the Taylor series of the function that is the sum of the power series, so all of the techniques we developed in the previous section can be applied to power series as well.

Example 8.6. Consider the power series defined by

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{2^k}.$$

What are $f(1)$ and $f\left(\frac{3}{2}\right)$? Find a general formula for $f(x)$ and determine the values for which this power series converges.

⁷See Exercise 2 in this section.

Solution. If we evaluate f at $x = 1$ we obtain the series

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

which is a geometric series with ratio $\frac{1}{2}$. So we can sum this series and find that

$$f(1) = \frac{1}{1 - \frac{1}{2}} = 2.$$

Similarly,

$$f(3/2) = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k = \frac{1}{1 - \frac{3}{4}} = 4.$$

In general, $f(x)$ is a geometric series with ratio $\frac{x}{2}$ and

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k = \frac{1}{1 - \frac{x}{2}} = \frac{2}{2 - x}$$

provided that $-1 < \frac{x}{2} < 1$ (so that the ratio is less than 1 in absolute value). Thus, the power series that defines f converges for $-2 < x < 2$.

As with Taylor series, we define the interval of convergence of a power series (8.26) to be the set of values of x for which the series converges. In the same way as we did with Taylor series, we typically use the Ratio Test to find the values of x for which the power series converges absolutely, and then check the endpoints separately if the radius of convergence is finite.

Example 8.7. Let $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$. Determine the interval of convergence of this power series.

Solution. First we will draw graphs of some of the partial sums of this power series to get an idea of the interval of convergence. Let

$$S_n(x) = \sum_{k=1}^n \frac{x^k}{k^2}$$

for each $n \geq 1$. Figure 8.7 shows plots of $S_{10}(x)$ (in red), $S_{25}(x)$ (in blue), and $S_{50}(x)$ (in green). The behavior of S_{50} particularly highlights that it appears to be converging to

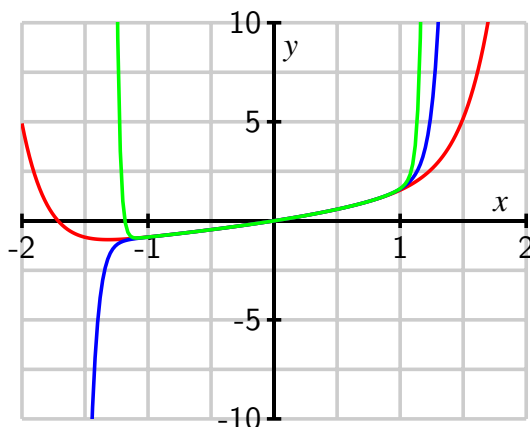


Figure 8.7: Graphs of partial sums of the power series $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$

a particular curve on the interval $(-1, 1)$, while growing without bound outside of that interval. This suggests that the interval of convergence might be $-1 < x < 1$. To more fully understand this power series, we apply the Ratio Test to determine the values of x for which the power series converges absolutely. For the given series, we have

$$a_k = \frac{x^k}{k^2},$$

so

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k \rightarrow \infty} \frac{\frac{|x|^{k+1}}{(k+1)^2}}{\frac{|x|^k}{k^2}} \\ &= \lim_{k \rightarrow \infty} |x| \left(\frac{k}{k+1} \right)^2 \\ &= |x| \lim_{k \rightarrow \infty} \left(\frac{k}{k+1} \right)^2 \\ &= |x|. \end{aligned}$$

Therefore, the Ratio Test tells us that the given power series $f(x)$ converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. Since the Ratio Test is inconclusive when $|x| = 1$, we need to check $x = 1$ and $x = -1$ individually.

When $x = 1$, observe that

$$f(1) = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

This is a p -series with $p > 1$, which we know converges. When $x = -1$, we have

$$f(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}.$$

This is an alternating series, and since the sequence $\left\{\frac{1}{n^2}\right\}$ decreases to 0, the power series converges when $x = -1$ by the Alternating Series Test. Thus, the interval of convergence of this power series is $-1 \leq x \leq 1$.

Activity 8.28.

Determine the interval of convergence of each power series.

(a) $\sum_{k=1}^{\infty} \frac{(x-1)^k}{3k}$

(b) $\sum_{k=1}^{\infty} kx^k$

(c) $\sum_{k=1}^{\infty} \frac{k^2(x+1)^k}{4^k}$

(d) $\sum_{k=1}^{\infty} \frac{x^k}{(2k)!}$

(e) $\sum_{k=1}^{\infty} k!x^k$

◀

Manipulating Power Series

Recall that we know several power series expressions for important functions such as $\sin(x)$ and e^x . Often, we can take a known power series expression for such a function and use that series expansion to find a power series for a different, but related, function. The next activity demonstrates one way to do this.

Activity 8.29.

Our goal in this activity is to find a power series expansion for $f(x) = \frac{1}{1+x^2}$ centered at $x = 0$.

While we could use the methods of Section 8.5 and differentiate $f(x) = \frac{1}{1+x^2}$ several times to look for patterns and find the Taylor series for $f(x)$, we seek an alternate approach because of how complicated the derivatives of $f(x)$ quickly become.

- What is the Taylor series expansion for $g(x) = \frac{1}{1-x}$? What is the interval of convergence of this series?
- How is $g(-x^2)$ related to $f(x)$? Explain, and hence substitute $-x^2$ for x in the power series expansion for $g(x)$. Given the relationship between $g(-x^2)$ and $f(x)$, how is the resulting series related to $f(x)$?
- For which values of x will this power series expansion for $f(x)$ be valid? Why?

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In a previous section we determined several important Maclaurin series and their intervals of convergence. Here, we list these key functions and remind ourselves of their corresponding expansions.

$$\begin{aligned}\sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} && \text{for } -\infty < x < \infty \\ \cos(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} && \text{for } -\infty < x < \infty \\ e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} && \text{for } -\infty < x < \infty \\ \frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k && \text{for } -1 < x < 1\end{aligned}$$

As we saw in Activity 8.29, we can use these known series to find other power series expansions for related functions such as $\sin(x^2)$, e^{5x^3} , and $\cos(x^5)$. Another important way that we can manipulate power series is illustrated in the next activity.

Activity 8.30.

Let f be the function given by the power series expansion

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

- Assume that we can differentiate a power series term by term, just like we can differentiate a (finite) polynomial. Use the fact that

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots$$

to find a power series expansion for $f'(x)$.

- (b) Observe that $f(x)$ and $f'(x)$ have familiar Taylor series. What familiar functions are these? What known relationship does our work demonstrate?
- (c) What is the series expansion for $f''(x)$? What familiar function is $f''(x)$?

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It turns out that our work in Activity 8.29 holds more generally. The corresponding theorem, which we will not prove, states that we can differentiate a power series for a function f term by term and obtain the series expansion for f' , and similarly we can integrate a series expansion for a function f term by term and obtain the series expansion for $\int f(x) dx$. For both, the radius of convergence of the resulting series is the same as the original, though it is possible that the convergence status of the resulting series may differ at the endpoints. The formal statement of the Power Series Differentiation and Integration Theorem follows.

Power Series Differentiation and Integration Theorem. Suppose $f(x)$ has a power series expansion

$$f(x) = \sum_{k=0}^{\infty} c_k x^k$$

so that the series converges absolutely to $f(x)$ on the interval $-r < x < r$. Then, the power series $\sum_{k=1}^{\infty} k c_k x^{k-1}$ obtained by differentiating the power series for $f(x)$ term by term converges absolutely to $f'(x)$ on the interval $-r < x < r$. That is,

$$f'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}, \text{ for } |x| < r.$$

Similarly, the power series $\sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}$ obtained by integrating the power series for $f(x)$ term by term converges absolutely on the interval $-r < x < r$, and

$$\int f(x) dx = \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1} + C, \text{ for } |x| < r.$$

This theorem validates the steps we took in Activity 8.30. It is important to note that this result about differentiating and integrating power series tells us that we can differentiate and integrate term by term on the interior of the interval of convergence, but it does not tell us what happens at the endpoints of this interval. We always need to check what happens at the endpoints separately. More importantly, we can use the approach

of differentiating or integrating a series term by term to find new series.

Example 8.8. Find a series expansion centered at $x = 0$ for $\arctan(x)$, as well as its interval of convergence.

Solution. While we could differentiate $\arctan(x)$ repeatedly and look for patterns in the derivative values at $x = 0$ in an attempt to find the Maclaurin series for $\arctan(x)$ from the definition, it turns out to be far easier to use a known series in an insightful way. In Activity 8.29, we found that

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

for $-1 < x < 1$. Recall that

$$\frac{d}{dx} [\arctan(x)] = \frac{1}{1+x^2},$$

and therefore

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C.$$

It follows that we can integrate the series for $\frac{1}{1+x^2}$ term by term to obtain the power series expansion for $\arctan(x)$. Doing so, we find that

$$\begin{aligned} \arctan(x) &= \int \left(\sum_{k=0}^{\infty} (-1)^k x^{2k} \right) dx \\ &= \sum_{k=0}^{\infty} \left(\int (-1)^k x^{2k} dx \right) \\ &= \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \right) + C. \end{aligned}$$

The Power Series Differentiation and Integration Theorem tells us that this equality is valid for at least $-1 < x < 1$.

To find the value of the constant C , we can use the fact that $\arctan(0) = 0$. So

$$0 = \arctan(0) = \left(\sum_{k=0}^{\infty} (-1)^k \frac{0^{2k+1}}{2k+1} \right) + C = C,$$

and we must have $C = 0$. Therefore,

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \tag{8.27}$$

for at least $-1 < x < 1$.

It is a straightforward exercise to check that the power series

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

converges both when $x = -1$ and when $x = 1$; in each case, we have an alternating series with terms $\frac{1}{2k+1}$ that decrease to 0, and thus the interval of convergence for the series expansion for $\arctan(x)$ in Equation (8.27) is $-1 \leq x \leq 1$.

Activity 8.31.

Find a power series expansion for $\ln(1+x)$ centered at $x = 0$ and determine its interval of convergence. (**Hint:** Use the Taylor series expansion for $\frac{1}{1+x}$ centered at $x = 0$.)

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Summary

In this section, we encountered the following important ideas:

- A power series is a series of the form

$$\sum_{k=0}^{\infty} a_k x^k.$$

- We can often assume a solution to a given problem can be written as a power series, then use the information in the problem to determine the coefficients in the power series. This method allows us to approximate solutions to certain problems using partial sums of the power series; that is, we can find approximate solutions that are polynomials.
 - The connection between power series and Taylor series is that they are essentially the same thing: on its interval of convergence a power series is the Taylor series of its sum.
-

Exercises

1. We can use power series to approximate definite integrals to which known techniques of integration do not apply. We will illustrate this in this exercise with the definite integral $\int_0^1 \sin(x^2) ds$.
- Use the Taylor series for $\sin(x)$ to find the Taylor series for $\sin(x^2)$. What is the interval of convergence for the Taylor series for $\sin(x^2)$? Explain.
 - Integrate the Taylor series for $\sin(x^2)$ term by term to obtain a power series expansion for $\int \sin(x^2) dx$.
 - Use the result from part (b) to explain how to evaluate $\int_0^1 \sin(x^2) dx$. Determine the number of terms you will need to approximate $\int_0^1 \sin(x^2) dx$ to 3 decimal places.
2. There is an important connection between power series and Taylor series. Suppose f is defined by a power series centered at 0 so that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

- Determine the first 4 derivatives of f evaluated at 0 in terms of the coefficients a_k .
- Show that $f^{(n)}(0) = n!a_n$ for each positive integer n .
- Explain how the result of (b) tells us the following:

On its interval of convergence, a power series is the Taylor series of its sum.

3. In this exercise we will begin with a strange power series and then find its sum. The Fibonacci sequence $\{f_n\}$ is a famous sequence whose first few terms are

$$f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, \dots,$$

where each term in the sequence after the first two is the sum of the preceding two terms. That is, $f_0 = 0$, $f_1 = 1$ and for $n \geq 2$ we have

$$f_n = f_{n-1} + f_{n-2}.$$

Now consider the power series

$$F(x) = \sum_{k=0}^{\infty} f_k x^k.$$

We will determine the sum of this power series in this exercise.

(a) Explain why each of the following is true.

$$(i) \quad xF(x) = \sum_{k=1}^{\infty} f_{k-1}x^k$$

$$(ii) \quad x^2F(x) = \sum_{k=2}^{\infty} f_{k-2}x^k$$

(b) Show that

$$F(x) - xF(x) - x^2F(x) = x.$$

(c) Now use the equation

$$F(x) - xF(x) - x^2F(x) = x$$

to find a simple form for $F(x)$ that doesn't involve a sum.

(d) Use a computer algebra system or some other method to calculate the first 8 derivatives of $\frac{x}{1-x-x^2}$ evaluated at 0. Why shouldn't the results surprise you?

4. Airy's equation⁸

$$y'' - xy = 0, \tag{8.28}$$

can be used to model an undamped vibrating spring with spring constant x (note that y is an unknown function of x). So the solution to this differential equation will tell us the behavior of a spring-mass system as the spring ages (like an automobile shock absorber). Assume that a solution $y = f(x)$ has a Taylor series that can be written in the form

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

where the coefficients are undetermined. Our job is to find the coefficients.

(a) Differentiate the series for y term by term to find the series for y' . Then repeat to find the series for y'' .

(b) Substitute your results from part (a) into the Airy equation and show that we can write Equation (8.28) in the form

$$\sum_{k=2}^{\infty} (k-1)ka_k x^{k-2} - \sum_{k=0}^{\infty} a_k x^{k+1} = 0. \tag{8.29}$$

(c) At this point, it would be convenient if we could combine the series on the left in (8.29), but one written with terms of the form x^{k-2} and the other with terms

⁸The general differential equations of the form $y'' \pm k^2xy = 0$ is called Airy's equation. These equations arise in many problems, such as the study of diffraction of light, diffraction of radio waves around an object, aerodynamics, and the buckling of a uniform column under its own weight.

in the form x^{k+1} . Explain why

$$\sum_{k=2}^{\infty} (k-1)k a_k x^{k-2} = \sum_{k=0}^{\infty} (k+1)(k+2) a_{k+2} x^k. \quad (8.30)$$

(d) Now show that

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k. \quad (8.31)$$

(e) We can now substitute (8.30) and (8.31) into (8.29) to obtain

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0. \quad (8.32)$$

Combine the like powers of x in the two series to show that our solution must satisfy

$$2a_2 + \sum_{k=1}^{\infty} [(k+1)(k+2)a_{k+2} - a_{k-1}] x^k = 0. \quad (8.33)$$

(f) Use equation (8.33) to show the following:

- (i) $a_{3k+2} = 0$ for every positive integer k ,
- (ii) $a_{3k} = \frac{1}{(2)(3)(5)(6)\cdots(3k-1)(3k)} a_0$ for $k \geq 1$,
- (iii) $a_{3k+1} = \frac{1}{(3)(4)(6)(7)\cdots(3k)(3k+1)} a_1$ for $k \geq 1$.

(g) Use the previous part to conclude that the general solution to the Airy equation (8.28) is

$$y = a_0 \left(1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2)(3)(5)(6)\cdots(3k-1)(3k)} \right) + a_1 \left(x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{(3)(4)(6)(7)\cdots(3k)(3k+1)} \right).$$

Any values for a_0 and a_1 then determine a specific solution that we can approximate as closely as we like using this series solution.