Searching Games

A bound for the Responder

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Searching Games with Two Secret Numbers

Goal: The *Responder* thinks of two numbers from 0 to $n-1$ and the *Questioner* searches and identifies those 2 numbers by asking questions of the form "How many of your secret numbers are in the set $Q \subset \{0, \ldots, n-1\}$".

Setup: The *Questioner* Places a bet on how many questions k they need to ask in order to identify the two numbers.

Game Play: The *Questioner* needs to ask a question one at a time and wait for a response from the *Responder*. *Note:* If agreed upon, the *Responder* can lie to at most one question during the game.

Winning: The *Questioner* wins if they identify the two numbers within k questions, Otherwise the *Responder wins*.

Some Notation

In order to show some example games here is some notation.

Pairs of numbers that represent the numbers that the responder is thinking of (*Possibility Pairs*) will be in curly braces.

Example:

- If the Secret Numbers are 1 and 5, then they are shown as $\{1, 5\}$.
- If the Secret Numbers are 0 and 7, then they are shown as $\{0, 7\}$.
- If the Secret Numbers are 4 and 9, then they are shown as $\{4, 9\}$.

Searching Game with 2 Unknowns and No Lies

First we will look at a game where no lies are allowed by the responder. Consider a game on $n = 5$ numbers, and the questioner bets they only need $k = 2$ questions.

Here is how a game might play out (r is the response indicating how many numbers are in the set):

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$

Possible Answers:

Possible	*NotPossible*
$\{0,1\}, \{0,2\}, \{0,3\}$ $\{0,4\}, \{1,2\}, \{1,3\}$ $\{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$	

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- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$

Outcome after Q1:

$Possible$	$NotPossible$
$\{0, 1\}, \{0, 2\}, \{0, 4\}$ $\{1, 3\}, \{2, 3\}, \{3, 4\}$	$\{0, 3\}, \{1, 2\}, \{1, 4\}$ $\{2, 4\}$

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Here is how a game might play out (r is the response indicating how many numbers are in the set):

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$

Outcome after Q1:

Possible	*NotPossible*
$\{0,1\}, \{0,2\}, \{0,4\}$ $\{1,3\}, \{2,3\}, \{3,4\}$	$\{0,3\}, \{1,2\}, \{1,4\}$ $\{2,4\}$

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Here is how a game might play out (r is the response indicating how many numbers are in the set):

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$

Outcome after Q2:

Possible	*NotPossible*
$\{0, 1\}, \{0, 4\}, \{2, 3\}$	**{0,2}, {1,3}, {3,4},** $\{0, 3\}, \{1, 2\}, \{1, 4\}$ $\{2, 4\}$

Searching Game Result

After question 2 in the game we just looked at we have the following outcome,

Possible	*NotPossible*
$\{0,1\},\{0,4\},\{2,3\}$	$\{0,2\},\{1,3\},\{3,4\},$ $\{0,3\},\{1,2\},\{1,4\}$ $\{2,4\}$

Since there is more than one pair in the 'Possible' section, we say that the questioner was not able to search and find with certainty the two secret numbers the responder was thinking about.

Result: The Responder Wins!

Searching Game with 2 Unknowns and No Lies

Now suppose we look at a rerun of the previous game; so $n = 5$ numbers, and no lies are allowed. This time however the questioner is confident he will win with betting on only needing $k = 3$ questions.

Here is how a game might play out (Again r is the response indicating how many numbers are in the set):

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$

Possible Answers:

Possible	*NotPossible*
$\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,2\}$ $\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}$	

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Here is how a game might play out (Again r is the response indicating how many numbers are in the set):

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$

Outcome after Q1:

Possible	*NotPossible*
$\{0,1\}, \{0,2\}, \{0,4\}, \{1,3\}, \{2,3\}$ $\{3,4\}$	**{0,3}, {1,2}, {1,4}, {2,4}**

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Here is how a game might play out (Again r is the response indicating how many numbers are in the set):

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$

Outcome after Q1:

Possible	*NotPossible*
$\{0,1\}, \{0,2\}, \{0,4\}, \{1,3\}, \{2,3\}$ $\{3,4\}$	**{0,3}, {1,2}, {1,4}, {2,4}**

Searching Game with 2 Unknowns and No Lies

Now suppose we look at a rerun of the previous game; so $n = 5$ numbers, and no lies are allowed. This time however the questioner is confident he will win with betting on only needing $k = 3$ questions.

Here is how a game might play out (Again r is the response indicating how many numbers are in the set):

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$

Outcome after Q2:

Possible	*NotPossible*
$\{0, 1\}, \{0, 4\}, \{2, 3\}$	$\{\mathbf{0,2}\}$, $\{\mathbf{1,3}\}$, $\{\mathbf{3,4}\}, \{0, 3\}, \{1, 2\}$ $\{1, 4\}, \{2, 4\}$

Searching Game with 2 Unknowns and No Lies

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Here is how a game might play out (Again r is the response indicating how many numbers are in the set):

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$
- "How many secret numbers are in the set $\{2, 3\}$?" r $= 2$

Outcome after Q2:

Possible	*NotPossible*
$\{0, 1\}, \{0, 4\}, \{2, 3\}$	$\{\mathbf{0,2}\}$, $\{\mathbf{1,3}\}$, $\{\mathbf{3,4}\}, \{0, 3\}, \{1, 2\}$ $\{1, 4\}, \{2, 4\}$

Searching Game with 2 Unknowns and No Lies

Now suppose we look at a rerun of the previous game; so $n = 5$ numbers, and no lies are allowed. This time however the questioner is confident he will win with betting on only needing $k = 3$ questions.

Here is how a game might play out (Again r is the response indicating how many numbers are in the set):

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$
- "How many secret numbers are in the set $\{2, 3\}$?" r $= 2$

Outcome after Q3:

$Possible$	$NotPossible$
$\{2, 3\}$	$\{\mathbf{0,1}\}, \{0, 2\}, \{\mathbf{0,4}\}, \{0, 3\}, \{1, 2\}$ $\{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}$

Rematch Searching Game Result

After question 3 in the game we just looked at we have the following outcome,

Possible	*NotPossible*
$\{2,3\}$	$\{\mathbf{0,1}\},\{0,2\},\{\mathbf{0,4}\},\{0,3\},\{1,2\}$ $\{1,3\},\{1,4\},\{2,4\},\{3,4\}$

Since there is exactly one pair in the 'Possible' section, we say that the questioner was is able to search and find with certainty the two secret numbers the responder was thinking about.

Result: The Questioner Wins! ...this round...

A Lying Responder

Now suppose that the responder is allowed to lie to at most one question during a game. What does this mean?
Consider the following question and scenarios.

Example: Question: "How many secret numbers are in the set $\{1, 2, 4\}$?"

- If the responder's secret numbers are $\{1, 2\}$, the truthful response would be $r = 2$, but a lie would be a response of $r = 1$ or 0.
- If the responder's secret numbers are $\{0, 9\}$ then the truthful response would be $r = 0$, and a lie would be a response of $r = 1$ or 2.
- If the responder's secret numbers are $\{4, 5\}$ then the truthful response would be $r = 1$ and a lie would be a response of $r = 0$ or 2.

NotPossible to *LieSet*

We will consider what will happen when the responder is allowed to lie at most one time.

We will use the following to describe the game.

- A pair is in the Truth Set if it is consistent with all the previous questions and responses.
- A pair is in the Lie Set if it is not consistent with at least one of the previous questions (Responder could have lied).

TruthSet	*LieSet*
$\{0,1\},\{0,4\},\{2,3\}$	$\{0,2\},\{1,3\},\{3,4\},\{0,3\}$ $\{1,2\},\{1,4\},\{2,4\}$

Searching Game 2 Unknowns and Up to 1 Lie

Consider a game on $n = 5$ numbers, where the questioner bet's they will need $k = 3$ questions.

Here is an example of how a game my play out:

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$

Possible Answers:

TruthSet	*LieSet*
$\{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{1,2\}$ $\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$	

Searching Game 2 Unknowns and Up to 1 Lie

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Here is an example of how a game my play out:

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$

Outcome after Q1:

TruthSet	*LieSet*
$\{0, 1\}, \{0, 2\}, \{0, 4\}, \{1, 3\}, \{2, 3\}$ $\{3, 4\}$	$\{0, 3\}, \{1, 2\}, \{1, 4\}, \{2, 4\}$

Searching Game 2 Unknowns and Up to 1 Lie

Consider a game on $n = 5$ numbers, where the questioner bet's they will need $k = 3$ questions.

Here is an example of how a game my play out:

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$

Outcome after Q1:

TruthSet	*LieSet*
$\{0,1\}, \{0,2\}, \{0,4\}, \{1,3\}, \{2,3\}$ $\{3,4\}$	$\{0,3\}, \{1,2\}, \{1,4\}, \{2,4\}$

Searching Game 2 Unknowns and Up to 1 Lie

Consider a game on $n = 5$ numbers, where the questioner bet's they will need $k = 3$ questions.

Here is an example of how a game my play out:

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$

Outcome after Q2:

TruthSet	*LieSet*
$\{0, 1\}, \{0, 4\}, \{2, 3\}$	$\mathbf{\{0,2\}}$, $\mathbf{\{1,3\}}$, $\mathbf{\{3,4\}}$, $\{0, 3\}, \{1, 2\}, \{2, 4\}$

Searching Game 2 Unknowns and Up to 1 Lie

Consider a game on $n = 5$ numbers, where the questioner bet's they will need $k = 3$ questions.

Here is an example of how a game my play out:

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$
- "How many secret numbers are in the set $\{2, 3\}$?" r = 2

Outcome after Q2:

TruthSet	*LieSet*
$\{0, 1\}, \{0, 4\}, \{2, 3\}$	**{0,2}, {1,3}, {3,4}**, $\{0, 3\}, \{1, 2\}, \{2, 4\}$

Searching Game 2 Unknowns and Up to 1 Lie

Consider a game on $n = 5$ numbers, where the questioner bet's they will need $k = 3$ questions.

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- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$
- "How many secret numbers are in the set $\{2, 3\}$?" r $= 2$

Outcome after Q3:

TruthSet	*LieSet*
$\{2, 3\}$	$\{\mathbf{0,1}\}$, $\{\mathbf{0,4}\}$

Searching Game 2 Unknowns and Up to 1 Lie

Consider a game on $n = 5$ numbers, where the questioner bet's they will need $k = 3$ questions.

Here is an example of how a game my play out:

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$
- "How many secret numbers are in the set $\{2, 3\}$?" r = 2
- "How many secret numbers are in the set $\{3, 4\}$?" r = 2

Outcome after Q3:

TruthSet	*LieSet*
$\{2, 3\}$	{**0,1**}, {**0,4**}

Searching Game 2 Unknowns and Up to 1 Lie

Consider a game on $n = 5$ numbers, where the questioner bet's they will need $k = 3$ questions.

Here is an example of how a game my play out:

- "How many secret numbers are in the set $\{1, 2, 4\}$?" $r = 1$
- "How many secret numbers are in the set $\{0, 2\}$?" $r = 1$
- "How many secret numbers are in the set $\{2, 3\}$?" r = 2
- "How many secret numbers are in the set $\{3, 4\}$?" r = 2

Outcome after Q4:

TruthSet	*LieSet*
	{2,3}

Searching Game Result

After question 4 in the game we just looked at we have the following outcome,

TruthSet	*LieSet*
	{**2,3**}

There is no possibility pair left in the Truth Set, and 1 left in the lie set. Recall that being in the lie Set means that the pair is inconsistent with at least one question and response that was given. However since it is the only pair left in the whole game. Therefore 2 and 3 are the secret numbers as they are the only ones left in the game.

Result: The Questioner Wins! ...hmmm...

Truth Set and Lie Set

Definition

A possibility pair x is in the *truth set* T if it satisfies all the previously given responses. A possibility pair is in the *lie set* L if it satisfies all but one previous response.

- Question/Response
- Stay or Move sets
- Helps Identifies Possible Solutions

States

Definition

After every question's response there is an associated *state* denoted as $(|T|, |L|)$, where T and L represent the truth and lie sets.

Example 1.

TruthSet	*LieSet*
$\{2, 3\}$	$\{0, 1\}, \{0, 4\}$

The state here is (1,2).

States

Definition

After every question's response there is an associated *state* denoted as $(|T|, |L|)$, where T and L represent the truth and lie sets.

Example 2.

TruthSet	*LieSet*
	$\{2, 3\}$

The state here is (0,1).

Weights and 1/3 Lemma

Definition

The *weight* of a state $(|T|, |L|)$ with j questions remaining is $w_j(|T|, |L|) = |T|(j+1) + |L|$.

Example 1.

TruthSet	*LieSet*
$\{2,3\}$	$\{0,1\},\{0,4\}$

$w_3(|T|, |L|) = 1(4) + 2 = 6.$

Weights and 1/3 Lemma

Definition

The *weight* of a state $(|T|, |L|)$ with j questions remaining is $w_j(|T|, |L|) = |T|(j+1) + |L|$.

Example 2.

TruthSet	*LieSet*
	$\{2, 3\}$

$w_3(|T|, |L|) = 0(4){+}1 = 1.$

Proof by Picture

Remark (1/3 Observation)

The weight of a state with $j-1$ questions remaining will be at least one third of the previous weight.

PossibilityPairs

$w_j(|T|, |L|)$

T	L

$r = 0$

T	L

$r = 1$

T	L

$r = 2$

$(\frac{1}{3}),\ w_j(|T|, |L|)$

A bound on n and k (I)

Theorem

The responder wins a Searching game G with at most 1 lie on n numbers and k questions if $\binom{n}{2}(k+1) > 3^k$.

A bound on n and k (I)

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- Apply the $1/3$ Observation k times

A bound on n and k (I)

Theorem

The responder wins a Searching game G with at most 1 lie on n numbers and k questions if $\binom{n}{2}(k+1) > 3^k$.

- Apply the 1/3 Observation k times
- $w_0(|T|,|L|) > 3^{k-k} = 1$

Question

Can the questioner win the game if $w_j(|T|,|L|) \geq 2$*?*

A bound on n and k (II)

Here we will be outlining the proof of the Theorem. Consider the following states

A bound on n and k (II)

Here we will be outlining the proof of the Theorem. Consider the following states Example State: (0,1) and 0 questions remain implies a win from the questioner since there are no pairs left in the *TruthSet* and one number left in the *LieSet*.

TruthSet	*LieSet*
	$\{2, 3\}$

$w_0(|T|, |L|) = 0(0+1)+1 = 1.$

A bound on n and k (II)

Here we will be outlining the proof of the Theorem. Consider the following states
Example State: (2,1) and 0 questions remain implies a loss for the questioner since there are two pairs left in the *TruthSet* and 1 pair left in the *LieSet*.

TruthSet	*LieSet*
$\{0,1\},\{0,2\}$	$\{2,3\}$

$w_0(|T|,|L|) = 2(0+1)+1 = 3.$

A bound on n and k (II)

Here we will be outlining the proof of the Theorem. Consider the following states Example State: (1,0) and 1 question remains implies a win for the questioner since the 1 means there is only pair left in the *TruthSet* and none in the *LieSet*.

TruthSet	*LieSet*
$\{0, 1\}$	

$w_0(|T|, |L|) = 1(1+1)+0 = 2.$

A bound on n and k (III)

Goal:

- Show (1,0) cannot happen
- Assume it does

Implication:

- $(1, 0)$ comes after state $(1, c)$, $c \in \mathbb{N}$
- $j > 3^{j-1}$, False $\forall j \geq 0$.

TruthSet	*LieSet*
$\{0, 1\}$	$\{1, 2\}, \{1, 3\}, \{2, 3\}$

TruthSet	*LieSet*
$\{0, 1\}$	

Theorem

The responder wins a searching game G with at most 1 lie on n numbers and k questions if $\binom{n}{2}(k+1) > 3^k$.

Current and Future Work

In this research we provide a bound that guarantees a win for the responder. In the future it would be good to provide a lower bound that would do the same.

Currently working on...

- investigating offline searching games.
- identifying offline strategy that allows questioner to win.
- is the strategy optional?

Future work on offline searching games would involve looking for an optimal strategy for the responder.

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