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Fighting Fires in Siberia French Federation of Mathematical Games 2013 Competition

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1 Introduction

1.1 The Mathematical Game

In late 2012, Prof. Bernard Beauzamy of the Société de Calcul Mathématique SA proposed a "mathematical game" inspired by the massive fires that occurred in Siberia during the summer of 2012. This article is our entry for the competition.

In this contest, Siberia is divided into regions called "trapezes", bounded by latitute and longitude lines (see figures in the appendix), and for each region a percentage is given of urban, cultivated, and virgin zones. Probabilities are defined that a fire would start in a region over a given period of time, which would lead to damage, with urban areas facing signficantly larger damages than cultivated or virgin areas. Also provided are rules for fire-fighting that involve the placement of fire brigades and airplanes in the center of each trapeze, rules for the spread and dousing of fires, and more rules for the speed of fire-fighting equipment and the availability of water. The objective is to distribute the fire-fighters among 94 different trapezes, and to define strategies to send out the fire-fighters depending on the location of the fires, in order to minimize total damage and total fire-fighting costs. Two key points to keep in mind are: (1) we are allowed to make assumptions beyond what was given by Prof. Beauzamy, provided that these assumptions "make sense and are explicitly set", and (2) there will be scenarios where the fire-fighting capabilities will prove to be insufficient.

It is noted in the description from Prof. Beauzamy that "This Competitive Game is a part of a collaborative project which SCM proposes to the Government of Novosibirsk Oblast, dealing with natural risks. This project received the support of the French Embassy in Moscow."

1.2 Outline of this Article & Summary of Strategy

In Section 2 of this paper, we describe the details provided by Prof. Beauzamy, followed by some preliminary calculations that we will need, such as the distance between two points on the Earth. We follow with additional, reasonable assumptions made by our team. In Section 3, we give our placement of brigades and planes and our rationale for the placement. In Section 4, we determine subregions of trapezes where fires can be put out without getting help from neighboring trapezes, and then we extend these subregions by describing strategies to extinguish fires with assistance from another trapeze. In Section 5, we answer the three questions required in the contest. We conclude with two references and an appendix of information about the trapezes.

A summary of our strategy is as follows. First, we make the additional assumption that urban areas are located in the center of trapezes. Then, we calculate the size of urban disks for each trapeze and distribute the fire fighters so that nearly all urban fires can be extinguished immediately upon the arrival of the brigades or planes in the trapeze - that is, no additional water or fire-fighting resources are needed beyond what the fire-fighters take with them initially. We then expand the protected regions by exploring situations where extra water is used or in which fire-fighters from neighboring regions are called in.

2 Structure and Assumptions

2.1 Given by Prof. Beauzamy

In the problem description, we are given the following information:

- Each trapeze is bounded by latitute and longitude lines that are multiples of 5 degrees. For example, the trapeze labeled "D4" is bounded by longitudes 75E and 80E and by latitudes 55N and 60N. Trapezes that are bounded by the same latitudes have the same area, but northern trapezes have less area than southern.

- For each trapeze, the percentage of area that is urban, cultivated, or virgin is given, but the distribution of these areas within each trapeze are not. The percentages are given in the appendix.

- The probability of whether or not a fire starts in a trapeze was given, and depends on the percentages of urban and cultivated areas in the trapeze. In particular, for trapezes that are 20% or more urbanized, the probability is defined to be 0.00002 per square km and per day. For less urbanized areas that nevertheless are 20% or more cultivated, the probability is 0.000005 per square km and per day. For all other trapezes, the probability is 0.000001 per square km and per day. We interpret these probabilities as applying to the whole trapeze, regardless of the location of the urban, cultivated, and virgin areas. Fires can appear anywhere in the trapeze and are independent of each other.

- We will focus on fires occuring over a six month period: April 1 to September 30.

- Damage calculations for burnt areas are 40 roubles per square meter in urban areas, 4 roubles per square meter in cultivated areas, and 0.4 roubles per square meter in virgin areas.

- Fires start at a single point and propagate in a circle with increasing radius, at a rate of 2 km per hour, that is, 0.56 meters per second. The active zone (where the fire is currently burning) is an annulus of width 10 meters. We consider the center of the zone as having burned and only a corona to be under fire. If the firefighters extinguish the corona, they have extinguished the fire.

- There are two types of fire-fighters: brigades and planes. A brigade has three vehicles, each of which can carry 6000 liters of water, enough to extinguish 1000 square meters of the corona. Thus, one brigade can extinguish 3000 square meters of fire. One plane carries enough water to extinguish 1000 square meters of fire. Brigades travel at a speed of 40 km per hour; planes are much faster: 300 km/hr.

- Each vehicle in a brigade consumes 0.3 liters of diesel fuel for every 1 km traveled, and each plane consumes 2 liters of kerosene for every 1 km traveled. Both fuels costs 30 roubles per liter.

- When the firefighters arrive at the fire and use the water they are carrying, if the fire is not extinguished, then they will need additional water. In that case, the fire-fighters must travel to a lake or river. The distance to the nearest lake or river is defined to be 100 km away from the fire, regardless of the fire's location. In the next section, we modfiy this requirement in a reasonable way.

- There are 5000 brigades and 100 specialized planes available to fight fires. Brigades and planes have their headquarters at the centers of trapezes. The key question is to position the brigades and planes throughout the 94 trapezes in order to minimize the combined damage costs and fire-fighting costs. Once this is determined, we are asked to address three questions that we consider at the end of this report:

  Question 1 : What is the expectation of the total cost of burnings?

  Question 2 : What is the expectation of the total budget for expenditures, that is diesel for vehicles and kerosene for planes?

  Question 3 : What is the probability that the fight capacities prove to be insufficient?

2.2 Preliminary Formulas and Calculations

Computing Distances Using Longitude and Latitude

In order to find the distance from one point to another, we will use the Haversine Formula [1]. Suppose that one point has longitude ω_1 and latitude μ_1 and a second point has longitude ω_2 and latitude μ_2. Then the distance d between the points is:

$$d = 2R \cdot \arcsin\left(\min\left\{1, \sqrt{\sin^2 \frac{|\mu_1 - \mu_2|}{2} + \cos(\mu_1) \cdot \cos(\mu_2) \cdot \sin^2 \frac{|\omega_1 - \omega_2|}{2}}\right\}\right)$$

where R is the radius of the Earth (R is approximately 6378.1 kilometers).

Computing Areas Using Longitude and Latitude

We use this formula for two points that are the corners of a 'wedge' of the earth as found in [2]. We again define the longitude of the points to be ω_1 and ω_2 and the latitudes of the points be μ_1 and μ_2, respectively. Then the area A in square kilometers is

$$A = \frac{\pi}{180} \cdot R^2 \cdot |\sin(\mu_1) - \sin(\mu_2)| \cdot |\omega_1 - \omega_2|$$

where R is again the radius of the earth in kilometers.

Growth rate of a fire

As noted above, the burning area of the fire is a corona, but when a fire is ignited, we will think of it as initially a small disk, until the radius is 10 meters. We compute the area of a corona with inner radius r and outer radius $r+10$, with lengths measured in meters: $\pi((r+10)^2 - r^2) = \pi(20r + 100) = 20\pi r + 100\pi$. Since r is increasing at the rate of 0.56 meters per second, the area of the burning corona is increasing at the rate of 35.1858 square meters per second. We will use the result that t seconds after a fire begins, the burning area is $35.1858t$ square meters, ignoring the 100π term, which is consistent with the approximation formula given by Prof. Beauzamy.

Number of brigades needed for a given distance from the center of the trapeze

Suppose that a fire breaks out that is d kilometers from the center of a trapeze. Since brigades travel at 40 km/hr, it will take $d/40$ hours for the brigades to arrive at the fire. When the brigades arrive, the burning area is $(35.1858)\ (d/40)\ (3600) = 3166.72d$ square meters. Since each brigade carries enough water to extinguish 3000 square meters, then $1.056d$ brigades are needed to extinguish the fire (rounding up to the nearest whole number).

In the same way, given a fixed number of brigades b, then the maximum distance from the center for which the fire can be extinguished without any extra fire-fighters needed is $b/1.056 = 0.947b$ kilometers.

Number of planes needed for a given distance from the center of the trapeze

In a similar fashion, we can calculate the number of planes needed if a fire breaks out d kilometers from the center of a trapeze. Since planes travel at 300 km/hr, it will take $d/300$ hours for the planes to arrive at the fire. When the planes arrive, the burning area is $(35.1858)\ (d/300)\ (3600) = 422.23d$ square meters. Since each plane carries enough water to extinguish 1000 square meters, then $0.422d$ planes are needed (rounding up to the nearest whole number).

In the same way, given a fixed number of planes p, then the maximum distance from the center for which the fire can be extinguished without any additional fire-fighters needed is $2.368p$ kilometers.

Probability that a fire would start during a day in a specific trapeze

We demonstrate this calculation with an example. Trapeze D4 has area 166401 square kilometers, and it is 25% urban. The probability that a fire is initiated in D4 is defined to be 0.00002 per square kilometers and per day. In one day, each square kilometer has a 0.99998 probability of being fire-free, so the probability that every square kilometer is fire-free is $(0.9998)^{166401} = 0.03586$. So, in a given day, the probability that a fire occurs in D4 is 0.96414 – very likely! This is reasonable – in a trapeze which is 25% urban, it will be a rare day that a fire does not occur. The probability for each trapeze can be found in the appendix.

Amount of water needed to put out a fire

In the situation where a fire starts at time $t = 0$ and is extinguished at time $t = T$ seconds, then the amount of water needed to put out the fire depends only on T. To see this, suppose that fire-fighters (brigades or planes) arrive with water at times t_1, t_2, t_3, ... $t_n = T$ seconds, using volumes of water w_1, w_2, w_3, ..., w_n liters. At time t_1, the size of the burning area is $35.1858t_1$ square meters. The volume of water, w_1 liters, will extinguish $w_1/6$ square meters, so the new size of the burning area is $35.1858t_1 - w_1/6$. At time t_2, the fire has grown again by $35.1858(t_2 - t_1)$. When the water w_2 is applied, the new size of the burning area is $35.1858t_1 - w_1/6 + 35.1858(t_2 - t_1) - w_2/6 = 35.1858t_2 - (w_1 + w_2)/6$. Continuing in this way, at time T, the size of the burning area is $35.1858T - (w_1 + w_2 + w_3 + ... + w_n)/6$, which equals 0 when the fire is put out. Thus, we find that the total amount of water needed to put out the fire is $211.1148T$ liters.

2.2.1 Summary of Key Results

- Burning area after t seconds: $35.1858t$.
- Time needed for fire-fighters to arrive at a fire that is d km away: for brigades, $d/40$ hours, or $90d$ seconds. For planes, $d/300$ hours, or $12d$

seconds.

- Number of brigades needed to extinguish a fire d km away from the center of the trapeze: $1.056d$ brigades (round up to next integer)
- Maximum distance from the center for which a fire can be extinguished using b brigades: $0.947b$ kilometers.
- Number of planes needed to extinguish a fire d km away from the center of the trapeze: $0.422d$ planes (round up to next integer).
- Maximum distance from the center for which a fire can be extinguished using p planes: $2.368p$ kilometers.
- Amount of water needed to put out a fire that is extinguished after T seconds: $211.1148T$ liters.

2.3 Additional Assumptions by Our Team

Although trapeze T5 is indicated on the map that given by Prof. Beauzamy, T5 is not included in the given data; as such we have not included T5 in our solution.

In each trapeze, the percentage of area that is urban, cultivated, or virgin is given, but the location of these areas within each trapeze is not. As the fire stations are located in the center of each trapeze, it is reasonable that the fire stations would be in the urban areas. So, we add the addtional assumption that in each trapeze, the urban area is located in the middle as a circle centered at the fire station. Then, it is reasonable to further assume that the cultivated region surrounded the urban region is an annulus. The remaining area in each trapeze is virgin territory. This arrangement gives the additional benefit of placing the urban zones – which have the highest cost of burnings – as close as possible to the fire stations. See Figure 1.

We add the additional assumption that fire-fighters can get more water at the center of each trapeze, as wells as by traveling 100 km from the fire to nearby wells, lakes, and rivers. This is consistent with an e-mail communication we had with Prof. Beauzamy, where he wrote, "at the center of the cell, all facilities are provided, food for the firemen and water for the tanks".

In interpreting the probabilities given in section 2.1, we made the assumption that for each trapeze, there would be at most one new fire per day.

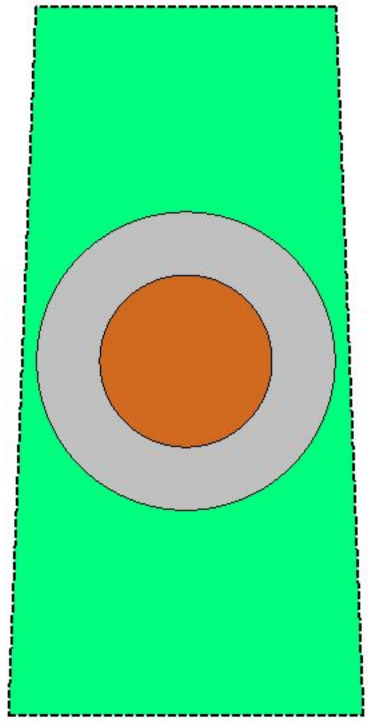

Figure 1: Distribution of urban (brown), cultivated (silver), and virgin (green) regions in a trapeze.

So, in the D4 example above, there is more than a 96% chance that a fire occurs each day, but if a fire occurs, then there is only one new fire in D4 that day. All new fires begin at 12 Noon GMT, regardless of the local time in the trapeze.

3 Location of Brigades and Airplanes

3.1 Key Issues

Given the way longitude and latitude lines are distributed on the earth, there are six sizes of trapezes in this game. Furthest north are trapezes between 70 and 75 degrees North latitude. These trapezes have areas of 93,128 square km. Furthest south are trapezes between 45 and 50 degrees North latitude.

These trapezes are significantly larger: 209,229 square km.

In terms of the distribution of urban, cultivated, and virgin lands, there are five types of trapezes: 1% urban, 2% cultivated, 97% virgin; 3%-7%-90%; 10%-20%-70%; 15%-25%-60%; and 25%-35%-45%. Interestingly, the regions with larger urban percentages also tend to be larger in area. Because urban damage per unit area is 10 times that of cultivated area, and 100 times that of virgin area, we explored if it was possible to place brigades and planes so that *all* urban fires could be extingished when the brigades or planes arrived, so that it was not necessary to get extra water or to use extra brigades or planes. We describe a distribution of brigades and planes in the next section that nearly meets this goal.

3.2 Our Placement of Fire-Fighters

We have distributed the fire-fighters so that for a specific trapeze, there are either brigades or planes, but not both. Our distribution is based on the goal that most urban fires will be extinguished when the brigades or planes arrive. This result is derived in the next section, followed by further discussion of other fires. See Tables 1 and 2 for our distribution of brigades and planes.

Trapezes	Number of Brigades
N2, O2, P2, Q2, R2, S2, T2, U2, V2, W2, X2, Y2	13
P3	15
Q3, R3, S3, T3, U3, V3, W3	16
A2, B2, C2, D2, E2, F2, G2, H2, I2, J2, K2, L2, M2	28
A3, B3, C3, D3, E3, F3, G3, H3, I3, J3, K3, L3, M3, N3, O3	65
E4, F4, G4, H4, I4, J4, K4, L4, M4, N4, O4, P4	94
E5, F5, G5, H5, I5, J5, K5, L5, N5, O5, P5, Q5	100
O6, P6, Q6	106
A4, B4, C4, D4, T4, U4	122

Table 1: Our Distribution of Brigades

Trapezes	Number of Planes
B1, C1, M1, N1	7
D1, E1, F1, G1, H1, I1, J1, K1, L1	8

Table 2: Our Distribution of Planes

4 Fighting Fires in Siberia

4.1 Urban, Cultivated, and Virgin Areas in Each Trapeze

Trapeze Level	Latitude (North)	Area (square km)
1	70 - 75	93128
2	65 - 70	118517
3	60 - 65	143003
4	55 - 60	166401
5	50 - 55	188533
6	45 - 50	209230

Table 3: Areas of Trapezes

Because of the nature of latitude lines, the trapezes in Siberia vary in size. In particular, the northern trapezes are smaller in area than the southern trapezes, using our area formula in section 2.2. See Table 3.

We can use the percentages given to calculate the amount of urban area in each trapeze. For instance, the D4 trapeze is 25% urban, so the urban area in that trapeze is 41600 square kilometers. As indicated in section 2.3, we are adding the assumption that the urban area is a circle in the center of the trapeze. A circle with area 41600 square kilometers has radius 115.1 kilometers (using the standard area formula and ignoring any error due to the curve of the earth). So, we define the "urban radius" of D4 to be 115.1 kilometers. The urban radii for all 94 trapezes are given in the appendix.

As urban fires cause the most financial damage, is it possible to place brigades and planes so that all urban fires can be extinguished if all brigades are sent immediately to put out a new fire? To answer this question, we use the urban radii calculations as well as the "number of brigades needed" formula from section 2.2: $1.056d$, where we now use the urban radius for d. There are nine different possible urban radii, and Table 4 indicates the minimum number of brigades needed to extinguish all urban fires without extra water.

If only brigades were given, in order to distribute them in each trapeze using the minimums shown in the table, we would need 5633 brigades. Unfortunately, we only have 5000 brigades given. To distribute the brigades, we used the minimums in the table for the trapezes with the largest area,

Urban Radius (km)	Minimum Brigades Needed	Trapeze Example	% of U-C-V
17.22	19	D1	1 - 2 - 97
19.42	21	P2	1 - 2 - 97
21.34	23	P3	1 - 2 - 97
33.64	36	A2	3 - 7- 90
67.47	72	A3	10 - 20 - 70
89.14	95	E4	15 - 25 - 60
94.88	101	E5	15 - 25- 60
99.95	106	P6	15 - 25 - 60
115.07	122	A4	25 - 30 - 45

Table 4: Minimum Brigades Needed

placed planes rather than brigades in the most northern trapezes, and went below the minimum in smaller urban regions, as indicated in Table 1.

For the most northern trapezes (B1 through N1), we evenly distributed the 100 planes, with eight planes in the middle trapezes and seven planes on the two ends. Despite their speed, planes do not carry much water and are therefore not terribly useful to extinguish fires. However, these trapezes have small urban areas. This means that fires are less likely to occur and when they do occur, they are less costly. For each northern trapeze, the urban radius is 17.22 km. Seven planes can cover all fires within 16.58 km of the center, while eight planes can handle fires out to 18.95 km, using our earlier formula.

4.2 One-Alarm Fires Without Extra Water

Based on the explanation above, most urban fires will be extinguished when the brigades or planes arrive. This is particular true in large urban areas found in trapezes like E4, E5, O6, and A4. For trapezes with smaller urban areas, urban fires centered near the edge of the urban area cannot be put out without additional resources.

For each trapeze, we define the *one-alarm radius* to be the maximum distance from the center for which a fire can be extinguished utilizing all fire fighters in that trapeze. The value of the one-alarm radius will be close to the value of the urban radius for each trapeze. For example, D4 has a one-alarm radius of 115.58 km, which is very close to its urban radius of 115.07 km.

In comparison, A2 has a one-alarm radius of only 26.53 km, compared to its urban radius of 33.64 km.

4.3 One-Alarm Fires With Extra Water

In this subsection, we will consider the question of whether we can extend the distance that a single trapeze's brigades can cover by adding the additional step of getting water. As explained above, the amount of water that can be carried by planes is not sufficient to offset travel time to the fire, and so we will only consider brigades here. Based on the growth rate of a fire d kilometers away from a center with b brigades, we can put out a fire after getting water n times if

$$0 = 35.1858\left(\frac{d}{40}\right)3600 + 35.1858n \cdot \frac{\min\{2d, 200\}}{40} \cdot 3600 - 3000b(n+1).$$

Notice that the last two terms above represent the fire growth while getting water (either from the center or 100 kilometers away) and the total amount of fire that will be put out. We simplify the above equation to observe that, in the case of a fire being within 100 kilometers of the center,

$$d = \frac{40b(n+1)}{35.1858(1+2n)}.$$

Similarly, we find in the case of a fire being more than 100 kilometers from the center that

$$d = \frac{40b(n+1)}{35.1858} - 200n.$$

In either case, as n increases, d decreases, meaning that the more times that we get water, the smaller the area of the fire can be in order for us to extinguish the fire. Basically, in the time it takes to fetch more water, the growth in the fire overwhelms the new water that is introduced. Consequently, we cannot extend the distance that a single trapeze's brigades can cover by adding the additional step of getting water.

4.4 Two-Alarm Fires Without Extra Water

We will consider a two-alarm fire to be a fire where fire-fighters from two different trapeze centers are used to extinguish the fire. When a fire of this type is ignited in a trapeze, the location is outside of the one-alarm fire boundary, meaning that all of the fire-fighters in that trapeze cannot extinguish the fire when they arrive from the center.

Suppose that a fire is ignited that is distance d_1 from the center of its trapeze, and a longer distance d_2 from the center of a nearby trapeze. We will first consider the case where both trapezes have brigades, and assume that the trapeze where the fire is has b_1 brigades and the nearby trapeze has b_2 brigades. We will further consider the case where the combined water of the two brigades is sufficient to extinguish the fire without any need to find more water.

Let $T = (d_2/40)(3600) = 90d_2$ seconds. At some time before T seconds, the b_1 brigades will arrive and attempt to extinguish the fire, shrinking it enough so that when the b_2 brigades arrives at T seconds, they will douse the fire. In order for that to happen, $b_1 + b_2 \geq 1.06d_2$, based on a calculation above.

For example, with adjacent trapezes such as C4 and D4, $b_1 + b_2 = 244$, so $d_2 \leq 231.2$ km. Since for these adjacent trapezes, the horizontal distance between the centers is about 300 kilometers, it is clearly possible that brigades from D4 can be used to extinguish a fire in C4 that the brigades in C4 can only slow. See Figure 2.

In the case where both trapezes have planes (p_1 and p_2 planes, and distances d_1 and d_2), the calculation is similar. In this case, $T = (d_2/300)(3600) = 12d_2$ seconds, and $p_1 + p_2 \geq 0.422d_2$. In the case where both trapezes have 8 planes, we have that $d_2 \leq 37.9$. Because this bound is small, we cannot extinguish any fires with planes through this method. Basically, the planes do not carry enough water to make a difference.

For a pair of trapezes where one has planes and the other has brigades, then the trapezes would be adjacent vertically, and the distance between the two centers would be 556 km. This distance is too great for the fire-fighters from the adjacent trapeze to arrive in time to extinguish the fire, without any extra water being used.

So, in the case of adjacent trapezes that both have brigades, it is possible for the brigades to join forces to extinguish certain fires, provided that the distance from the fire to the center of the adjacent trapeze is not very large.

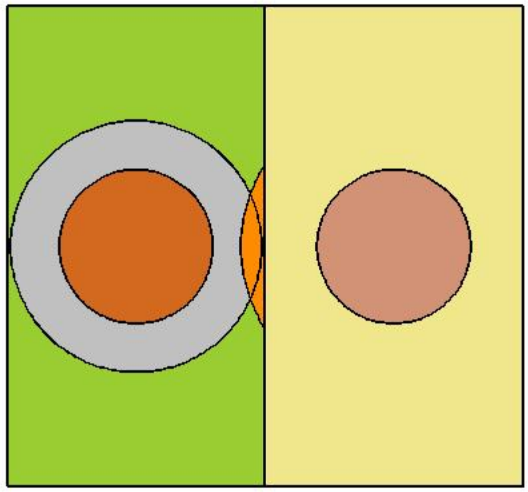

Figure 2: Two-Alarm Fire: Brigades from an adjacent trapeze can be used to extinguish fires in the orange region.

4.5 Two-Alarm Fires With Extra Water

In this subsection, we will consider the following questions: Is it possible to put out fires in one of the following ways: (1) Fire-fighters from the center slow the fire, then fire-fighters from an adjacent trapeze arrive and slow the fire more, allowing time for the first fire-fighters to go back to the center, get more water, and the extinguish the fire. (2) The same method, but the fire is not extinguished when the first fire-fighters arrive a second time, so the second fire-fighters also go to the headquarters *of the first fire-fighters*, get water, and use it to completely extinguish the fire. For both questions, like before we will assume that the first is a distance d_1 from the center of the trapeze, and that the center of the adjacent trapeze is a distance d_2 from the center of the fire. To begin, we will focus just on situations where both trapezes have brigades (b_1 and b_2 respectively).

For the first question, the fire is extinguished when $T = 3d_1/40 > d_2/40$. In the same manner as earlier calculations, balancing fire growth with water used, we arrive at the following inequality:

$$1.056(3d) \leq 2b_1 + b_2.$$

In situations where adjacent trapezes have the same number of brigades, this inequality yields the same urban radius calculated earlier, so no new areas can be protected. In the few cases when $b_2 > b_1$, then we can add to the protected area, provided that $3d_1 > d_2$. For instance, if we have a fire in E4 ($b_1 = 94$, urban radius = 89.14 km), and brigades from D4 ($b_2 = 122$) are called in to help, then the inequality reduces to $d_1 \leq 97.85$ (and $d_2 < 293.5$) which is area outside of the urban radius, provided the brigades from D4 are available.

The only cases where this situation works is to add protected areas to P3, E4, and N2.

For the second question, the fire is extinguished when $T = (d_2 + 2d_1)/40 > 3d_1/40$, and this leads to the inequality:

$$1.056(d_2 + 2d_1) \leq 2b_1 + 2b_2.$$

Again in the example of adjacent trapezes such C4 and D4, $b_1 + b_2 = 244$, and the inequality becomes $d_2 + 2d \leq 462.1$. This adds some protected area in a few situations. Figure 3 indicates the extra area that is covered in C4.

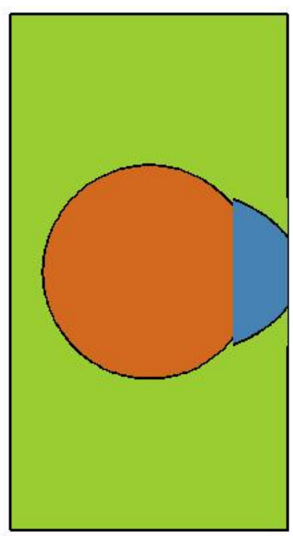

Figure 3: Two-Alarm Fire, Fetching Extra Water: Brigades from an adjacent trapeze can be used to extinguish fires in the blue region, if extra water is fetched.

For other scenarios (planes and planes, planes and brigades), no addition area is protected.

4.6 More Difficult Fires

We believe that there are other fires that can be extinguished by bringing in fire-fighters from a third or fourth trapeze, but the regions that contain these fires is very small, as is the likelihood that all fire-fighters will be available and not dealing with fires in their own trapezes. In the interest of simplicity, for the calculations in the next section, we will consider that fire-fighting capacities are insufficient to deal with these fires on a consistent basis.

5 Three Questions

Question 1 : What is the expectation of the total cost of burnings?
Question 2 : What is the expectation of the total budget for expenditures, that is diesel for vehicles and kerosene for planes?

The first two questions are related and so we will answer them jointly.

As we will describe below, it is almost certain that on any given day, there will be at least one fire that cannot be extinguished. Hence, the idea of "expected cost" is problematic. In spite of this, we can develop some results. To start, we compute the expected expenditures and damage costs for one-alarm fires.

Suppose that a one-alarm fire occurs in a trapeze. The average location of this fire is a distance 2/3 of the one-alarm radius. This results follows from calculus. Given a disk of radius R, the average distance of points on that disk to the center is given by:

$$\frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R r \cdot r dr d\theta = \frac{2}{3}R$$

That is, if a fire occurs within the one-alarm radius of the center of the trapeze, the average distance from the fire-fighting headquarters to the fire is 2/3 of the radius.

For each trapeze, we calculated the burning cost and fire-fighting expenditures for a fire located 2/3 of the one-alarm radius away from the center. If one fire occurs in each trapeze, 2/3 of the way from the center of the trapeze, we find expenditures of 217,142.33 roubles per day for transportation. These fires will have a total cost of burnings of 42,000,045.03 roubles per day.

In order to find the *expected* costs, we weight the fire-fighting and fire-damage costs for each trapeze by the probability of a fire within the one-alarm radius, and compute the sum. The probability of a fire occuring within the one-alarm radius of the cell is the probability of any fire occuring in the cell, multiplied by the percentage of the trapeze's area that is within the one-alarm radius. In this way, we find a total expected transportation expenditures of 18,487.77 roubles per day. Similarly, we find an expected total cost of burnings of 3,687,930.98 roubles per day. Over the 183-day period, we have expected transportation expenditures of about 3.4 million roubles, and expected burning costs of about 675 million roubles.

To get an upper-bound on total expected burning costs, we will assume that if a fire occurs that starts outside of the one-alarm area, *all* of the trapeze land outside the one-alarm area will burn. (As indicated above, there are certain cases where this is not true, but they are rare.) Fires outside the one-alarm area will be almost exclusively cultivated or virgin territory fires, with dramatically smaller per-km costs. However, these territories are vast. Again taking into account the probability that these fires will occur, the expected costs of burning per day for these fires is about 6.5 trillion roubles, or about 1.2 quadrillion roubles for the six-month period.

Question 3 : What is the probability that the fight capacities prove to be insufficient?

The probability is very, very high. For each trapeze and for each day, one of three mutually exclusive events can occur: there is no fire, there is a fire that can be extinguished, and there is a fire that cannot be extinguished. Consider the trapeze A4. As indicated in the appendix, the probability of a fire (per day) for B1 is 42.8%. The area of the one-alarm disk is about 1% of the total area of the trapeze. So, the chance of a fire outside the one alarm area is (42.8%)(1-1%)=42.4%. Then, the chance that for that trapeze, there is either no fire or there is a fire that can be extinguished is 57.6%.

Fire-fighting capacities will be sufficient if, for every trapeze, there is either no fire or there is a fire that is within the one-alarm radius. To compute this probability, we will *multiply* the probabilities from each trapeze. This product is approximately 4.3×10^{-30}. Substracting this product from 1 yields a number extremely close to 1. In other words, it is practically impossible to put out all fires in a given day.

Additional Comment: For all three of these questions, the conclusions would not change substantively by including the small areas covered by two-alarm fires. In order to simplify the calculations, we have not included these areas in our final results.

Summary: By focusing fire-fighting capabilities on urban areas, where damage costs are 10 times the costs in cultivated areas and 100 times the costs in virgin territitory, we believe our solution is better than other alternatives. Unfortunately, given our placement of fire-fighters, nearly all cultivated and virgin-territory fires will burn out of control. Unless there is a more timely way for fire-fighters to fetch water, or larger water-carrying capacities for brigades and planes, we do not believe there is a better solution that minimizes burning costs and travel expenditures.

6 References

1. Sinnott, R. "Virtues of the Haversine" *Sky and Telescope*, vol. 68, no. 2, 1984, p.159.

2. "Area of a Latitude-Longitude Rectangle",

   mathforum.org/library/drmath/view/63767.html

7 Acknowledgements

We express our appreciation to David Austin and Robert Talbert for their suggestions in writing this report.

8 Appendix: The Trapezes of Siberia

Images of the trapezes (using a Python module called "Basemap") with the one-alarm radii indicated:

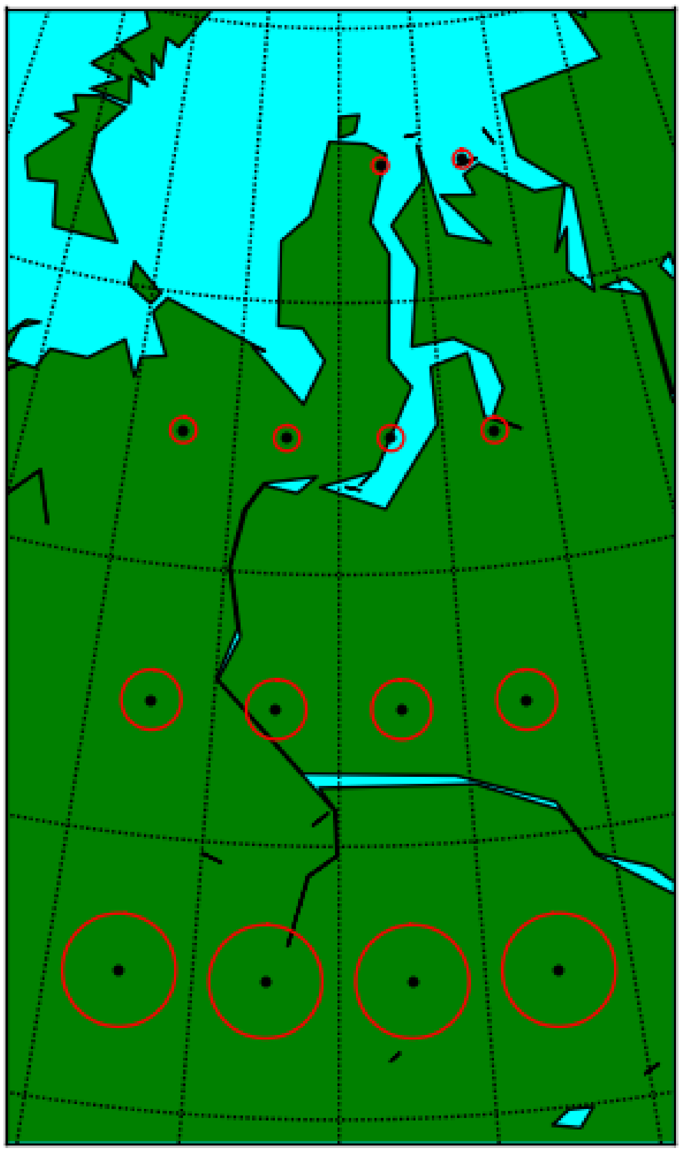

Figure 4: Trapezes in western Siberia.

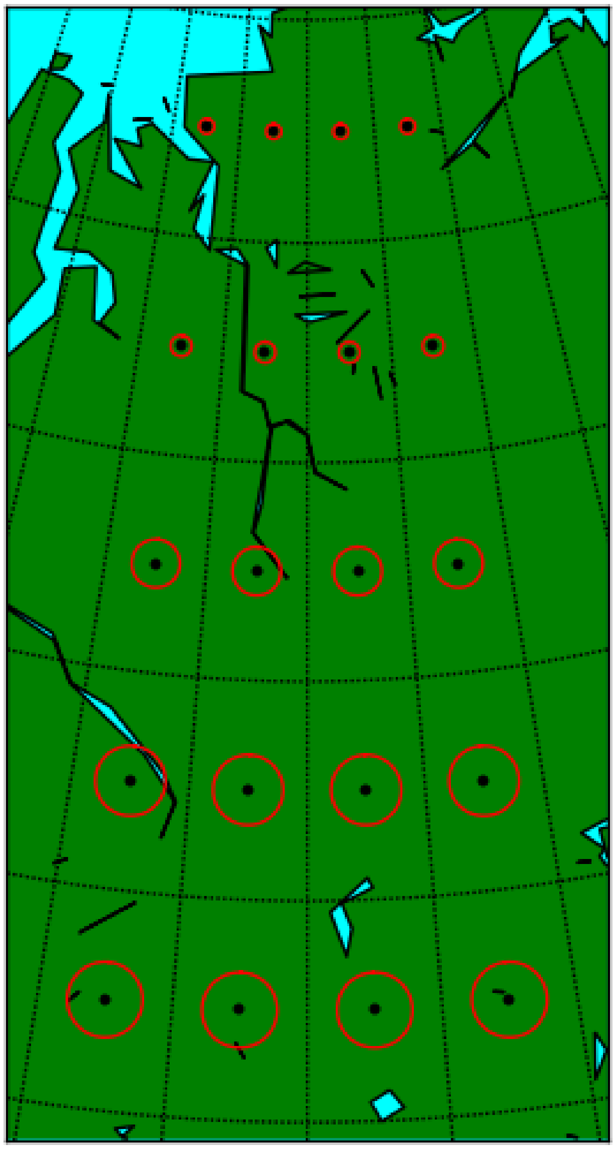

Figure 5: Trapezes in west central Siberia.

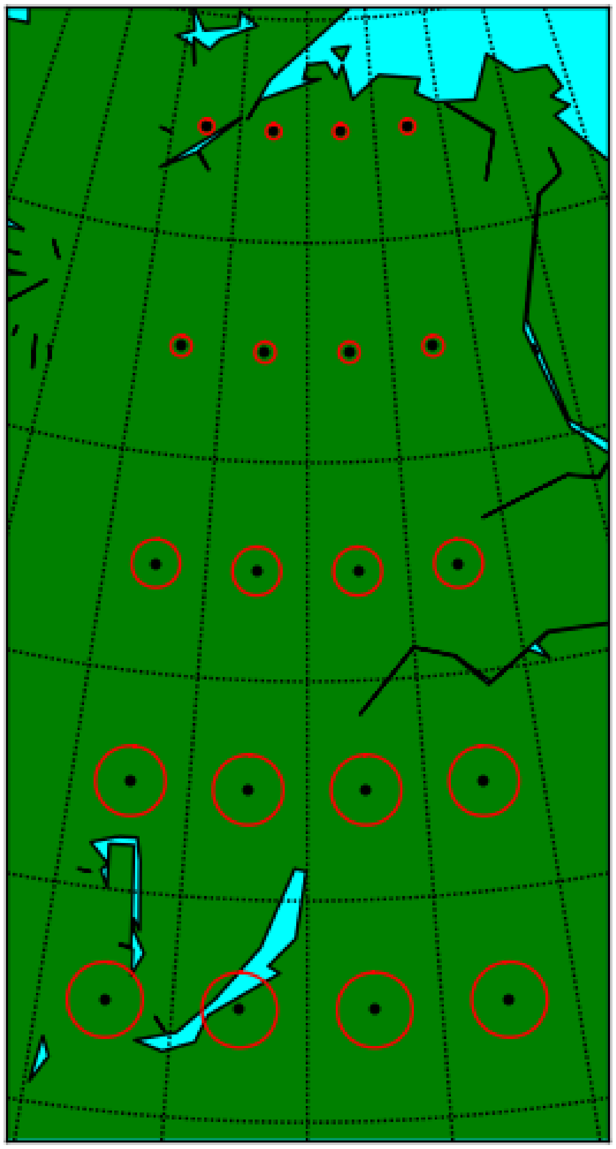

Figure 6: Trapezes in central Siberia.

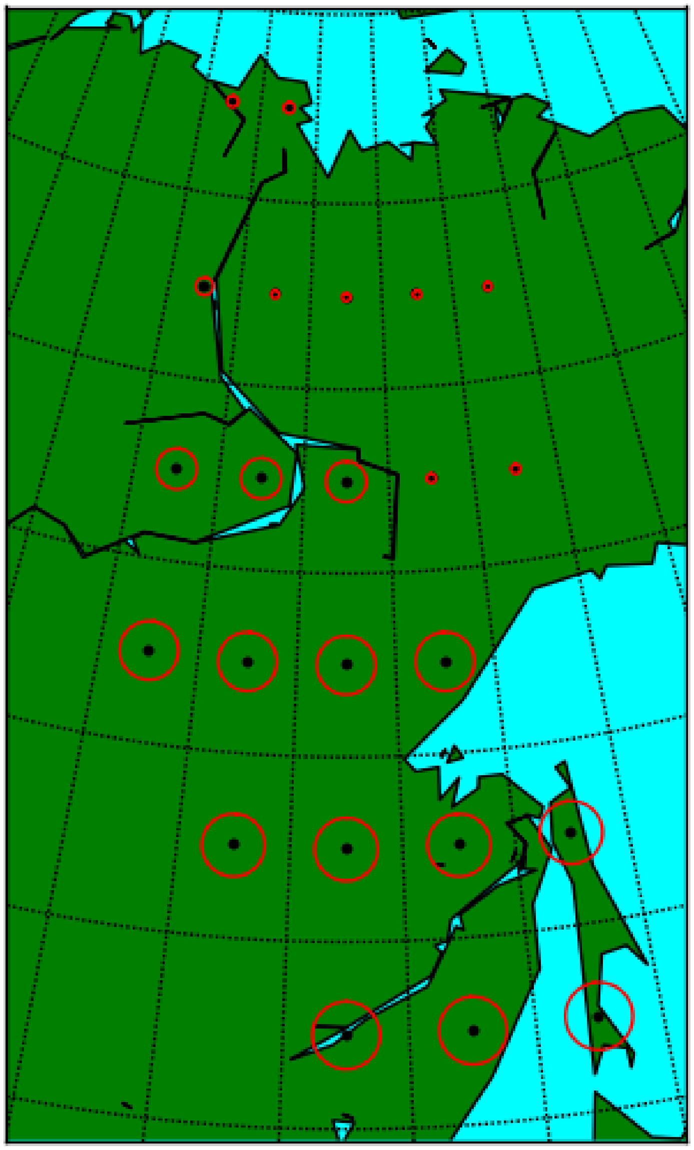

Figure 7: Trapezes in east central Siberia.

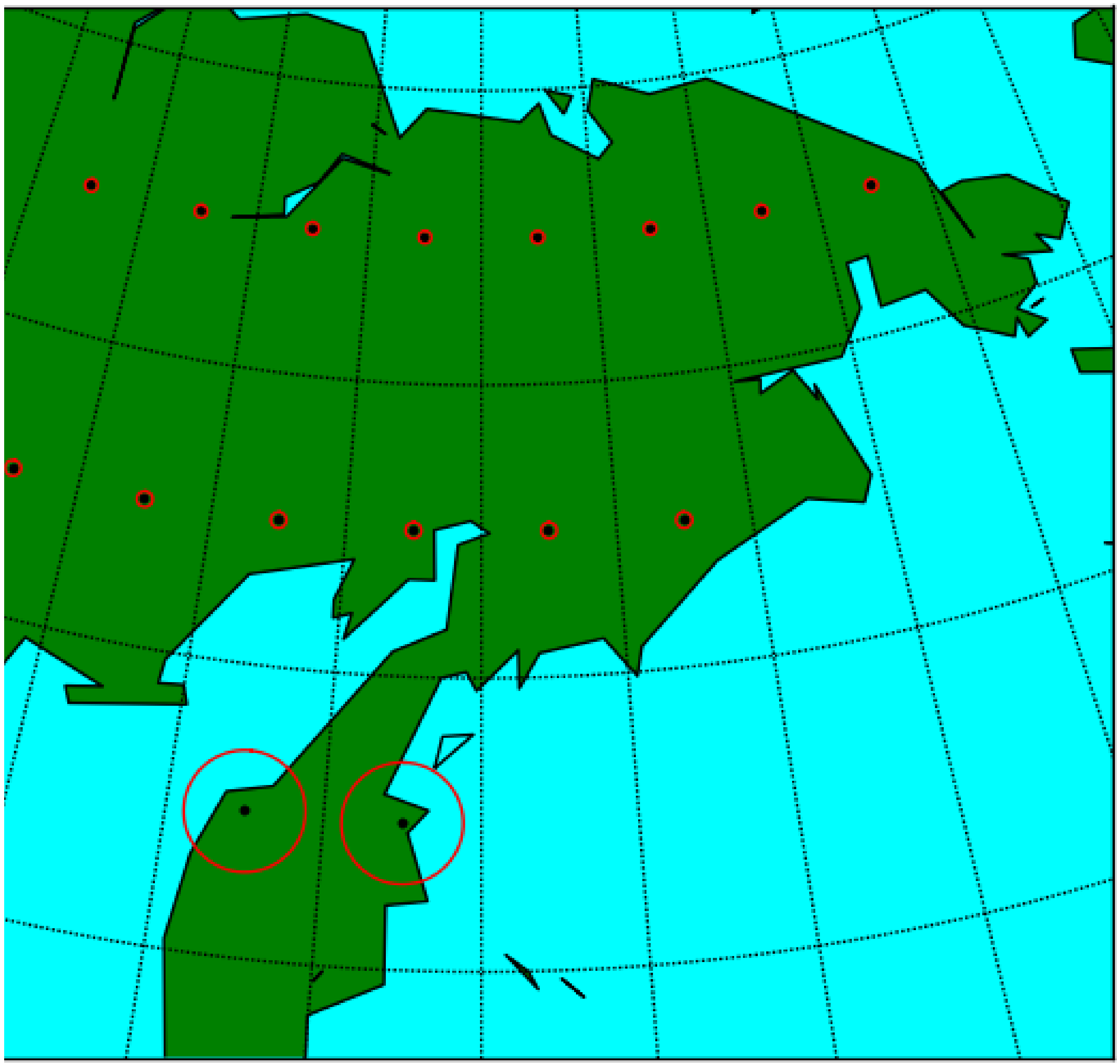

Figure 8: Trapezes in east Siberia.

Trapeze	Longitude (East)	Latitude (North)	% of U-C-V	Urban Radius (km)	Prob. of Fire (per day)
A2	60 - 65	65 - 70	3 - 7 - 90	33.64	50.9%
A3	60 - 65	60 - 65	10 - 20 - 70	67.47	51.1%
A4	60 - 65	55 - 60	25 - 30 - 45	115.07	96.4%
B1	65 - 70	70 - 75	1 - 2 - 97	17.22	42.8%
B2	65 - 70	65 - 70	3 - 7 - 90	33.64	50.9%
B3	65 - 70	60 - 65	10 - 20 - 70	67.47	51.1%
B4	65 - 70	55 - 60	25 - 30 - 45	115.07	96.4%
C1	70 - 75	70 - 75	1 - 2 - 97	17.22	42.8%
C2	70 - 75	65 - 70	3 - 7 - 90	33.64	50.9%
C3	70 - 75	60 - 65	10 - 20 - 70	67.47	51.1%
C4	70 - 75	55 - 60	25 - 30 - 45	115.07	96.4%
D1	75 - 80	70 - 75	1 - 2 - 97	17.22	42.8%
D2	75 - 80	65 - 70	3 - 7 - 90	33.64	50.9%
D3	75 - 80	60 - 65	10 - 20 - 70	67.47	51.1%
D4	75 - 80	55 - 60	25 - 30 - 45	115.07	96.4%
E1	80 - 85	70 - 75	1 - 2 - 97	17.22	42.8%
E2	80 - 85	65 - 70	3 - 7 - 90	33.64	50.9%
E3	80 - 85	60 - 65	10 - 20 - 70	67.47	51.1%
E4	80 - 85	55 - 60	15 - 25 - 60	89.14	56.5%
E5	80 - 85	50 - 55	15 - 25 - 60	94.88	61.0%
F1	85 - 90	70 - 75	1 - 2 - 97	17.22	42.8%
F2	85 - 90	65 - 70	3 - 7 - 90	33.64	50.9%
F3	85 - 90	60 - 65	10 - 20 - 70	67.47	51.1%
F4	85 - 90	55 - 60	15 - 25 - 60	89.14	56.5%
F5	85 - 90	50 - 55	15 - 25 - 60	94.88	61.0%
G1	90 - 95	70 - 75	1 - 2 - 97	17.22	42.8%
G2	90 - 95	65 - 70	3 - 7 - 90	33.64	50.9%
G3	90 - 95	60 - 65	10 - 20 - 70	67.47	51.1%
G4	90 - 95	55 - 60	15 - 25 - 60	89.14	56.5%
G5	90 - 95	50 - 55	15 - 25 - 60	94.88	61.0%

Table 5: 94 trapezes, Part I

Trapeze	Longitude (East)	Latitude (North)	% of U-C-V	Urban Radius (km)	Prob. of Fire (per day)
H1	95 - 100	70 - 75	1 - 2 - 97	17.22	42.8%
H2	95 - 100	65 - 70	3 - 7 - 90	33.64	50.9%
H3	95 - 100	60 - 65	10 - 20 - 70	67.47	51.1%
H4	95 - 100	55 - 60	15 - 25 - 60	89.14	56.5%
H5	95 - 100	50 - 55	15 - 25 - 60	94.88	61.0%
I1	100 - 105	70 - 75	1 - 2 - 97	17.22	42.8%
I2	100 - 105	65 - 70	3 - 7 - 90	33.64	50.9%
I3	100 - 105	60 - 65	10 - 20 - 70	67.47	51.1%
I4	100 - 105	55 - 60	15 - 25 - 60	89.14	56.5%
I5	100 - 105	50 - 55	15 - 25 - 60	94.88	61.0%
J1	105 - 110	70 - 75	1 - 2 - 97	17.22	42.8%
J2	105 - 110	65 - 70	3 - 7 - 90	33.64	50.9%
J3	105 - 110	60 - 65	10 - 20 - 70	67.47	51.1%
J4	105 - 110	55 - 60	15 - 25 - 60	89.14	56.5%
J5	105 - 110	50 - 55	15 - 25 - 60	94.88	61.0%
K1	110 - 115	70 - 75	1 - 2 - 97	17.22	42.8%
K2	110 - 115	65 - 70	3 - 7 - 90	33.64	50.9%
K3	110 - 115	60 - 65	10 - 20 - 70	67.47	51.1%
K4	110 - 115	55 - 60	15 - 25 - 60	89.14	56.5%
K5	110 - 115	50 - 55	15 - 25 - 60	94.88	61.0%
L1	115 - 120	70 - 75	1 - 2 - 97	17.22	42.8%
L2	115 - 120	65 - 70	3 - 7 - 90	33.64	50.9%
L3	115 - 120	60 - 65	10 - 20 - 70	67.47	51.1%
L4	115 - 120	55 - 60	15 - 25 - 60	89.14	56.5%
L5	115 - 120	50 - 55	15 - 25 - 60	94.88	61.0%
M1	120 - 125	70 - 75	1 - 2 - 97	17.22	42.8%
M2	120 - 125	65 - 70	3 - 7 - 90	33.64	50.9%
M3	120 - 125	60 - 65	10 - 20 - 70	67.47	51.1%
M4	120 - 125	55 - 60	15 - 25 - 60	89.14	56.5%
N1	125 - 130	70 - 75	1 - 2 - 97	17.22	42.8%
N2	125 - 130	65 - 70	1 - 2 - 97	19.42	50.9%
N3	125 - 130	60 - 65	10 - 20 - 70	67.47	51.1%
N4	125 - 130	55 - 60	15 - 25 - 60	89.14	56.5%

Table 6: 94 trapezes, Part II

Trapeze	Longitude (East)	Latitude (North)	% of U-C-V	Urban Radius (km)	Prob. of Fire (per day)
N5	125 - 130	50 - 55	15 - 25 - 60	94.88	61.0%
O2	130 - 135	65 - 70	1 - 2 - 97	19.42	50.9%
O3	130 - 135	60 - 65	10 - 20 - 70	67.47	51.1%
O4	130 - 135	55 - 60	15 - 25 - 60	89.14	56.5%
O5	130 - 135	50 - 55	15 - 25 - 60	94.88	61.0%
O6	130 - 135	45 - 50	15 - 25 - 60	99.95	64.9%
P2	135 - 140	65 - 70	1 - 2 - 97	19.42	50.9%
P3	135 - 140	60 - 65	1 - 2 - 97	21.34	57.6%
P4	135 - 140	55 - 60	15 - 25 - 60	89.14	56.5%
P5	135 - 140	50 - 55	15 - 25 - 60	94.88	61.0%
P6	135 - 140	45 - 50	15 - 25 - 60	99.95	64.9%
Q2	140 - 145	65 - 70	1 - 2 - 97	19.42	50.9%
Q3	140 - 145	60 - 65	1 - 2 - 97	21.34	57.6%
Q5	140 - 145	50 - 55	15 - 25 - 60	94.88	61.0%
Q6	140 - 145	45 - 50	15 - 25 - 60	99.95	64.9%
R2	145 - 150	65 - 70	1 - 2 - 97	19.42	50.9%
R3	145 - 150	60 - 65	1 - 2 - 97	21.34	57.6%
S2	150 - 155	65 - 70	1 - 2 - 97	19.42	50.9%
S3	150 - 155	60 - 65	1 - 2 - 97	21.34	57.6%
T2	155 - 160	65 - 70	1 - 2 - 97	19.42	50.9%
T3	155 - 160	60 - 65	1 - 2 - 97	21.34	57.6%
T4	155 - 160	55 - 60	25 - 30 - 45	115.07	96.4%
U2	160 - 165	65 - 70	1 - 2 - 97	19.42	50.9%
U3	160 - 165	60 - 65	1 - 2 - 97	21.34	57.6%
U4	160 - 165	55 - 60	25 - 30 - 45	115.07	96.4%
V2	165 - 170	65 - 70	1 - 2 - 97	19.42	50.9%
V3	165 - 170	60 - 65	1 - 2 - 97	21.34	57.6%
W2	170 - 175	65 - 70	1 - 2 - 97	19.42	50.9%
W3	170 - 175	60 - 65	1 - 2 - 97	21.34	57.6%
X2	175 - 180	65 - 70	1 - 2 - 97	19.42	50.9%
Y2	180 - 185	65 - 70	1 - 2 - 97	19.42	50.9%

Table 7: 94 trapezes, Part III