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Winnability of The Group Labeling Lights Out Game on Complete Bipartite Graphs

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Abstract

For an arbitrary graph, we can play Lights Out on it if we assign a number label to each of the vertices of a graph G, representing states of on/off in the original Lights Out game, with the edges connecting those vertices representing the buttons that are adjacent to each other. This project is focused on a slightly modified version of the game's original rules, with the labels for the vertices coming from the group Z_n. It is not always possible to win the game. We will be investigating the values of n for which this group labeling "Lights Out!" game is always winnable when played on complete bipartite graphs.

1 Introduction

The game Lights Out! was originally a hand-held, electronic game made by Tiger Electronics. The game consisted of a 5x5 grid of buttons that have two light states, on and off. We can think of each of these states as having a number label, 0 for off and 1 for on. If you push/toggle a button, it will cause the pushed button and all its adjacent buttons (buttons directly to the right/left and up/down from the button) to change their states. For example, pushing a button labeled 1 will change it to 0, and if the button had a neighbor with a label of 0, it would change to a label of 1. To win the game, we must be able to turn off all the lights, or in other words, to change the state of each of the buttons to 0.

The standard grid version of the game has been well studied,and mathematicians have used concepts from linear algebra as well as other methods to explore the winnability of the game[7, 4, 2]. As it turns out, the rules of this grid Lights Out game can be easily represented using graph theory. For any graph G with vertex set $V(G)$ and edge set

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$E(G)$, we can play Lights Out on it if we let all of the vertices represent buttons from our original game, each with a vertex label that represents the different states of the buttons. Additionally, the edges in our graph correspond to adjacent buttons. Toggling a vertex $v \in V(G)$, has the effect of changing the label of that vertex that was toggled, as well as the label of all of the vertices that are adjacent to it. To show an example of this game, consider the following graph G in Figure 1.

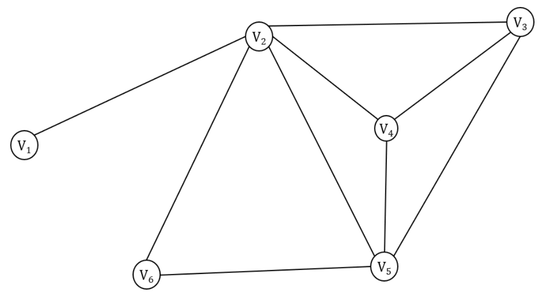

Figure 1: An arbitrary graph G, with vertices $v_1, v_2, v_3, v_4, v_5, v_6 \in V(G)$

If we assign a vertex label to each of the vertices of G, then we can play the standard game of Lights Out on that graph. Figure 2 on the next page shows how the game is played with an arbitrary vertex labeling of G by toggling vertex v_4, which is denoted with a black box.

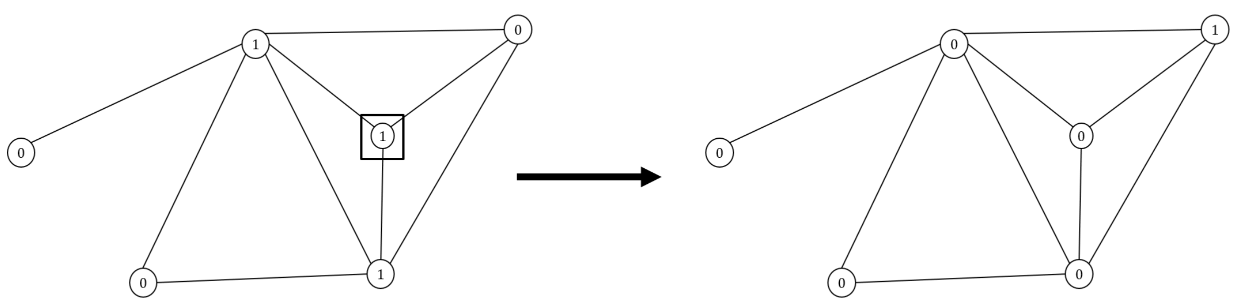

Figure 2: Graph G before and after toggling v_4

One variant of Lights Out game that has been studied is known as the neighborhood game.[4, 7] This variant follows of the rules of the standard game, but instead of dealing with labels of 1 or 0, representing on and off states, the labels can now come from $\mathbb{Z}_k$. In this version of the game, toggling a vertex $v \in V(G)$ has the effect of adding 1 to the label of that vertex that was toggled, as well adding 1 to the label of all of the vertices that are adjacent to it.

A paper by Giffen and Parker[4] laid out alot of the background research for defining this neighborhood variant of the game. Another paper, by Arangala, also defined a multi-state variant of the lights out game[1, 2]. The Giffen paper also investigated the winnability of the neighborhood game when played on paths, cycles and complete bipartite graphs[4].

This paper will be focused on different variant of Lights Out known as the group labeling version of the game. In this version, we use labels from an algebraic group, H. There has been previous work done on the winnability of the group game when dealing with labels coming from $H = \mathbb{Z}_k$[5]. In terms of how the game is played, we utilize the same basic idea as before, where toggling a vertex will affect the label of that vertex and its adjacent vertices, but the way that the labels are affected is a bit different. In the group labeling game, the labels are changed by adding whatever the label of the button that was pressed was. Figure 3 below shows how this game is played using labels from $\mathbb{Z}_4$

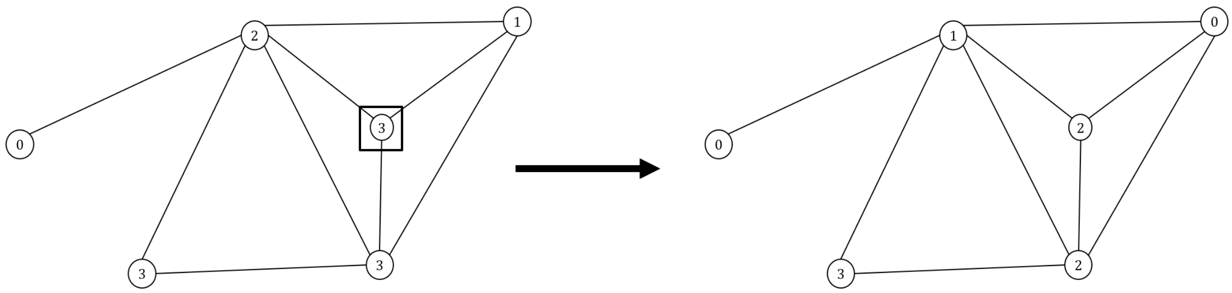

Figure 3: This figure shows how toggling in the group game works with $\mathbb{Z}_4$.We can see that toggling v_4 adds 3 to its label, which gives us a label of $3 + 3 = 6 mod(4) = 2$. Additionally, all of the vertices that are adjacent to the toggled vertex also have 3 added to their labels

Previously, there was been some work done looking into the winnability of this group game on path and cycle graphs [3], but there are still many classes of graphs for which the winnability conditions of the group game are still unknown.

The main research question of this paper is to investigate the winnability conditions of the group labeling Lights Out game on a class of graphs known as complete bipartite graphs. In section 2 of this paper, we will investigate the winnability of the game when using labels from Z_{2^k} on the complete bipartite graph $K_{m,n}$. More specifically, we will be proving a theorem that states the Z_{2^k} game is always winnable on $K_{m,n}$ if either m or n is even. We will also try to prove that the game is not always winnable when both m and n are odd and set up some lingering questions for further research.

2 Winnability of Group Labeling Game on Complete Bi-Partite Graphs

We want to investigate how the $\mathbb{Z}_k$ group game plays on complete bipartite graphs. More specifically, we want to know if and when the group game is can be won on complete bipartite graphs. We can start by defining a few terms that will help us throughout the paper.

Definition. A **complete bi-partite graph**, denoted as $K_{m,n}$, can be defined as a graph with two partitions, T and B, of the vertex set $V(K_{m,n})$. Part T having m vertices in its partition and part B having n vertices in its partition. Then the edge set of this graph can be defined as $E(G) = \{vw : v \in T, w \in B\}$ where v and w are vertices of the graph.

Definition. We will define a **vertex labeling** of a graph G as an assignment of numbers, or labels, to each $v \in V(G)$. In our case, a labeling can be defined as a function $\pi : V(G) \to \mathbb{Z}_k$

Furthermore, we can define a labeling as **winnable** if we are able to toggle the vertices of the graph in some way such that all of the vertex labels are equal to 0. The group labeling game on a graph G is considered **always winnable** if every possible labeling of that graph is winnable.

We want to look into the when the graph $K_{m,n}$ is always winnable. In a paper by Zadorozhnyy, it was shown that if we are investigating when the group game is always winnable using $\mathbb{Z}_k$, then we only have to look at the cases where k is a power of 2. This is expressed in the following theorem from that paper.[5]

Theorem 2.1. For the group labeling Lights Out game on a graph G with labels in $\mathbb{Z}_n$,

1.) If n is odd, then the only winnable labeling is the zero labeling.

2.) If $n = 2^k d$ where d is odd, then there is a one-to-one correspondence between the winnable labelings of G with labels in $\mathbb{Z}_n$ and winnable labelings of G with labels in $\mathbb{Z}_{2^n}$,

This means that for the rest of this paper, we will only be focused on the $\mathbb{Z}_{2^k}$ group game when looking at the winnability on complete bipartite graphs

To investigate the winnability, the general idea is that we will first show that we can get any initial labeling of a complete bipartite graph to labeling that is easier to work with, which we we will denote as a standard labeling, and then see what conditions are needed for the standard labeling to be always winnable. We will revisit the idea of a standard labeling further on, but, to provide some background context for the standard labeling we first need to investigate what happens when we focus on toggling just one of the vertices. Ideally, we want to explore what happens to some initial label on a vertex after pressing it a certain amount of times. The following lemma helps us answer this question.

Lemma 2.2. Assume we have complete bipartite graph $K_{m,n}$ and we pick a vertex v from the graph with an initial label denoted as ℓ. If we press the button q times, where $q \in \mathbb{N}$, repeatedly, we will end up with a label of $2^q(\ell)$

Proof. Assume we have a vertex v with an initial label denoted as ℓ. We will prove that pressing the button q times will turn that label to $2^q(\ell)$ using induction.

To begin, we will first look at the basis step with q=1 and prove that our lemma holds for this case. If we press a button once, we will have a label of $\ell+\ell = 2\ell = 2^1\ell$. Therefore, we have proved that pressing a vertex 1 time will turn that vertex's label to $2^1(\ell)$. Thus, our lemma is correct for $q = 1$.

For our induction step, we will assume that the lemma is correct for $q = n$. We will prove that the theorem also holds for $q = n+1$, that is, that pressing a vertex $n+1$ times will give a labeling of $2^{n+1}\ell$. If we press an arbitrary label of ℓ, n times, we will end up with a a label of $2^n(l)$ according to our induction hypothesis. If we press the button once more, we will add $2^n(l)$ to our vertex label , so we will have a label of $2^n(l) + 2^n(\ell)$. Since $2^n(l) + 2^n(\ell) = 2^{n+1}\ell$,we have proven the lemma holds for $q = n + 1$ □

As stated before, in terms of looking for when the group labeling game is always winnable, it only makes sense to think about the question using labels from $\mathbb{Z}_{2^k}$. Thus, we can apply Lemma 2.2 to a vertex using labels from $\mathbb{Z}_{2^k}$ to arrive at the following lemma.

Lemma 2.3. Assume we have complete bipartite graph $K_{m,n}$ and we pick a vertex v from the graph with an initial label denoted as ℓ. If we are dealing with labels from $\mathbb{Z}_{2^k}$, then pressing the vertex k times will give a labeling of 0.

Proof. According to Lemma 2.2, toggling the vertex v, q times will give a final label of $2^q(\ell)$. So if we let $q = k$, or in other words, if we toggle our vertex k times, then we will have a label of $2^k(\ell)$. Since $2^k = 0$ in $\mathbb{Z}_{2^k}$, our final label becomes $2^k(\ell) = 0(\ell) = 0$. This completes the proof. □

These two results tell us that if we toggle a single vertex enough times, precisely k times when dealing with labels from $\mathbb{Z}_{2^k}$ then we should be able to turn that vertex off. Since vertices in a complete bipartite graph are not connected to the other vertices in the same part, then this means that we should be able to use this result to be able to turn off all of the vertices in one of the parts of the graphs. This result will help us get our initial labeling to a standard labeling. The basic idea is that from Lemma 2.3, we will be able to turn off all of the lights in one of the parts, and then turn of all of the lights in the other part, since all of the vertices in the same part are not connected. Doing this will give us a labeling with one part of vertices with all the same label, and the other part of vertices all having a label of 0. As we will see later, this standard labeling will make investigating the winnability of the group game easier. For now, we can focus on the following lemmas that show how we can achieve a standard labeling.

Lemma 2.4. If we are playing the $\mathbb{Z}_{2^k}$ group game on the complete bipartite graph $K_{m,n}$, with two parts, T denoting the part with m vertices and B denoting the part with n vertices, then we can get to a standard labeling with all of the vertices in the part T having the same label, and with all of the vertices in part B having the label of 0.

Proof. Let's consider an initial labeling $\pi : V(G) \to \mathbb{Z}_k$ of the graph $K_{m,n}$ with labels coming from $\mathbb{Z}_{2^k}$ with k being some integer. Let each of the vertices in the part T have labels of $a_1, a_2, a_3...a_m$ and each of the vertices in part B having a labeling of $b_1, b_2, b_3...b_n$. If we start with the vertex labeled b_1,we can turn this button off by applying Lemma **??**. Each of our vertices from part T will now have some new label as a result of how many times we pushed our b_1 vertex to get it to be 0 .We can use this idea again on each of the vertices in part B until we have every label equal to 0. We can now apply Lemma 2.3 to turn off all of the vertices of part T. Since all of the part B labels started as 0 this time, and were all affected by the pressing of each of the buttons from part T in the same way, we know that they must all have the same label. Since we have one part of our complete bipartite graph with a label of 0 for all of the vertices, and the other part having the same label for all of the vertices in that part, we have proven that we can reach a a standard labeling from any initial labeling of $K_{m,n}$.

□

Now that we have shown that we can get any initial labeling to a standard labeling, we now have to show that this standard labeling is indeed easier to deal with, and will allow us to more easily investigate the winnability of the group game. The following lemmas explore the effect that toggling a vertex k times has on its adjacent vertices. We will later use this this to show that if we turn off all the vertices in the part of our standard labeling that have a label other than 0, we will get a new standard labeling, with all of the vertices that previously had a label of 0 getting an new label and all of the vertices that had a nonzero label changing to a label of 0.

Lemma 2.5. Assume are using labels from $\mathbb{Z}_{2^k}$ and we have the complete bipartite graph $K_{m,n}$, If we pick a vertex v from the graph with an initial label denoted as ℓ, then toggling the vertex k times will add a label of $2^k - 1\ell$ to all of the vertices adjacent to v.

Proof. If we apply Lemma 2.3, then we know that toggling a vertex k times will give that vertex a label of $2^k\ell$, which is equal to 0 using labels from $\mathbb{Z}_{2^k}$. Since that vertex started with a label of ℓ, we know that toggling that vertex k times had the effect of adding $(2^k - 1)\ell$ to the original label since $(2^k - 1)\ell + \ell = 2^k\ell = 0$. We know that toggling any vertex will have the effect of adding it's own label to itself, as well as adding it's label to all of the vertices that are adjacent to it. This means that since, after k toggles, we added $(2^k - 1)\ell$ to the label of our toggled vertex, we must also have added $(2^k - 1)\ell$ to all of the vertices that are adjacent to that vertex. This completes the proof. □

The following lemma is broken into two parts that show that if we turn off all of the vertices in the part of our standard labeling that has vertices with non-zero labels, we arrive at another standard labeling, but with different labels that will help us when trying to get to a winning labeling.

Lemma 2.6. 1. Assume that we have a standard labeling as in Lemma 2.4 of a graph $K_{m,n}$, with one part, B having n vertices with vertex labels of 0 and the opposite part, T having m vertices with vertex labels of c, and we are using labels from $\mathbb{Z}_{2^k}$.

Then we can toggle the vertices to get to a new labeling with all of the vertices in part T having a label of 0 and all of the vertices in part B having a label of $-mc$.

Proof. If we are dealing with labels from $\mathbb{Z}_{2^k}$, we know from Lemma 2.3 that pressing each of the vertices in part T k times will turn them to 0. By applying Lemma 2.5, we know that after toggling each vertex of part T k times, we will be adding$(2^k-1)c$ to the vertices of part B. There are m vertices in part T, so after turning off all of them, we will add m multiples of $(2^k-1)c$ to the vertices of B. In other words we will get a final label of $m(2^k-1)c$ for each vertex in part B. Since we are dealing with labels from $\mathbb{Z}_{2^k}$, $(2^k-1) = -1$ so $m(2^k-1)c = -mc$. This completes our proof for this part of the lemma. □

Once we get to this "new" standard labeling, we can use the same principles for the next part of the lemma to see what happens when we turn off all of the labeled vertices. This time, if we turn off all of the buttons in the labeled part, we will now get back to having labels on the vertices of the part that originally had the label of c, but now, we have a multiple of n and m which come from the amount of vertices in each of the parts.

2. Assume we are using labels from $\mathbb{Z}_{2^k}$. If we have a labeling for our $K_{m,n}$ graph with n vertices in part B being labeled with $-mc$ and m vertices in part T being labeled with 0, then we can get to a labeling with all of the vertices in part B being labeled with 0 and all of the vertices in part T having a label of nmc.

   Proof. If we press each of the part B vertices k times, then by applying Lemma 2.3, we know that the labels of these vertices will go to 0. Furthermore, by applying Lemma 2.5, after k toggles of each of the part B vertices, we will add $(2^k-1)(-mc)$ for each of the n vertices in part B. This means that the labels on the part T vertices after k toggles of each of the part B vertices will be $n(2^k-1)(-mc)$. Since $(2^k-1) = -1$ when using labels from $\mathbb{Z}_{2^k}$, our label becomes $n(-1)(-mc) = nmc$. This completes the proof.

   □

We saw before that carrying out Lemma 2.6 gets us to a labeling where we have a multiple of the number of vertices in each of the part multiplied by our original label. The following lemma shows that doing this process multiple times will just multiply the original label by another multiple of n and m each time.

Lemma 2.7. If we turn apply Lemma 2.6 q times on a $K_{m,n}$ graph with all of the vertices in one part having a label of c and all of the vertices in the other part having a label of 0, then we will end up with all of the vertices in one part having a label of $n^q m^q c$ and all of the vertices in the other part having a label of 0.

Proof. We will prove that carrying out Lemma 2.6 q times on a standard labeling will give a final labeling of with all of the vertices in one part having a label of $n^q m^q c$ and all of the vertices in the other part having a label of 0 using induction.

Basis case: q=1. If we apply Lemma 2.6 once to a standard labeling with one part having a label of c and the other part having a label 0, then we will get to a labeling where one part has a label of nmc and the other part has a label of 0. Since $nmc = n^1 m^1 c$, our lemma holds true for q=1.

Induction Step: Assume that our Induction hypothesis is true for q=w. We will prove that it is also true for q=w+1, or in other words, we will prove that applying Lemma 2.6 and $w + 1$ times to a standard labeling with one part having a label of c and the other part having a label 0, then we will get to a labeling where one part has a label of $n^{w+1} m^{w+1} c$ and the other part has a label of 0. From our Induction hypothesis, we know that applying Lemma 2.6 w times to a standard labeling will give a final labeling with one part of vertices having a label of $n^w m^w c$. If we know let this label be the label for a new standard labeling where all of the vertices in one part have a label of $c' = n^w m^w c$ and all of the vertices in the other part have a label of 0, then applying Lemma 2.6 again will give a labeling of $nmc' = nm(n^w m^w c) = n^{w+1} m^{w+1} c$. Thus, we have proved that applying Lemma 2.6 q times will give a labeling with all of the vertices in one part having a label of $n^q m^q c$ and all of the vertices in the other part having a label of 0

□

Pulling everything together, we can see that using these lemmas allow us to get a final labeling that is a multiple of n^q and m^q. This helps us when we consider that if either of m or n are even, then our final label will be a multiple of 2^q. Which means that if we have labels from $\mathbb{Z}_{2^k}$, all we have to do is carry out Lemma 2.6 k times to get a label of 0 for all of our vertices. This results in the following theorem.

Theorem 2.8. The group labeling "Lights Out" game is always winnable on the complete bipartite graph $K_{m,n}$ when m or n are even when using labels from $\mathbb{Z}_{2^k}$

Proof. From Lemma 2.4, we know that regardless of the initial labeling of the graph, we can get to a standard labeling with part T having labels of c and part B having labels of 0 for each of the vertices. From Lemma 2.7, we know that we can get from our standard labeling to a labeling with a label of $n^q m^q c$ for our vertices in the part T and a label of 0 for the vertices in part B . Let's assume that m is even. This means we can write m as $2w$ where w is some integer. Thus, our labeling after applying Lemma 2.7is $n^q (2w)^q c = n^q 2^q w^q c$. If we let q=k, then we get a labeling of $n^k 2^k w^k c$. Since 2^k is equal to 0 in Z_{2^k}, this labeling must also be 0, and since the labels in set B were already 0, all of the labels in our graph are now 0, and we have won the game. Thus, if either m or n is even, we know that the group labeling $\mathbb{Z}_{2^k}$ Lights Out game is always winnable. □

We have investigated the winnability of the group labeling $\mathbb{Z}_k$ game on the graph $K_{m,n}$ when either m or n are even, but we must also consider the case where m and n are both odd numbers. From the using the standard labeling method, it seemed that in the case where m and n are odd, that the game is not always winnable. However, this intuition is not sufficient enough proof to tell us that the game is not always winnable when m and n are odd since it would only tell us that we will not be able to always win the game following that specific method. Luckily, we can attack the problem from a different perspective. We can start by looking into the winnability of the standard neighborhood game on $K_{m,n}$ and seeing if there is a relationship between the winnability of the neighborhood game and the group game. Below is a result from [6] that will lead us in the direction of proving that the odd case is not always winnable when playing the $\mathbb{Z}_{2^k}$ on $K_{m,n}$.

Lemma 2.9. Let G be a graph with labels from $\mathbb{Z}_k$. The neighborhood Lights Out game on $K_{m,n}$ is always winnable in $\mathbb{Z}_k$ if and only if the $gcd(k, mn-1) = 1$.

We can use this result and apply it to the case where $k = 2$ and where m and n are both odd.

Lemma 2.10. The neighborhood Lights Out game is not always winnable on the complete bipartite graph $K_{m,n}$ when m and n are both odd when using labels from $\mathbb{Z}_2$

Proof. We will prove that the neighborhood Lights Out game is not always winnable when played on the graph $K_{m,n}$ when m and n are both odd using labels from $\mathbb{Z}_2$ by applying Lemma 2.9. If we have $k = 2$ and m and n both be odd numbers, then the Lemma states that the neighborhood game should always be winnable if the $gcd(2, mn-1)$ is equal to 1. If m and n are both odd numbers, $mn-1$ will be an even number, which means the $gcd(2, mn-1)$ will not be equal to one. This means that the $\mathbb{Z}_2$ game is not always winnable when m and n are odd. This completes the proof. □

Lemma 2.10 tells us that there exists at least one labeling that is not winnable for the $\mathbb{Z}_2$ neighborhood game when m and n are odd. We would like to be able to extend this fact to the $\mathbb{Z}_{2^k}$ group game somehow. We can do this by first showing that the group $\mathbb{Z}_2$ game is a more restrictive version of the $\mathbb{Z}_2$ game, so if there exists a labeling in the neighborhood game that is not winnable, then the same should be true for the $\mathbb{Z}_2$ group game. To provide some context for this idea, we note that the set of possible toggles in the group $\mathbb{Z}_2$ game is a subset of the set of possible toggles in the neighborhood $\mathbb{Z}_2$ game. This is because in the neighborhood $\mathbb{Z}_2$ game, we have two possible labels for each vertex, either 1 or 0. We are allowed to toggle each of the these vertices with the effect of switching the label of the toggled vertex and the label of vertices adjacent to that vertex. In the group $\mathbb{Z}_2$ game, we also only have two possible labels, 1 or 0. Since in the group game, toggling a vertex adds the label of the toggled vertex to itself and the label of its adjacent vertices, toggling a vertex labeled with 1 in the group game will have the same effect as toggling a vertex labeled with 1 in the neighborhood game. Toggling a vertex labeled with 0 in the group game will have no effect at all since it will be adding 0 to the label of itself and the labels of all of the adjacent vertices.

Lemma 2.11. Assume we have the graph $K_{m,n}$. If there exists a labeling that is not winnable in the neighborhood $\mathbb{Z}_2$ game, then that labeling is also not winnable in the group $\mathbb{Z}_2$ game.

Proof. We will prove that if there exists a labeling that is not winnable in the neighborhood $\mathbb{Z}_2$ game, then that labeling is also not winnable in the group $\mathbb{Z}_2$ game by using the fact that the set of possible toggles in the group $\mathbb{Z}_2$ game is a subset of the set of possible toggles in the neighborhood $\mathbb{Z}_2$ game. If there exists a labeling that is not winnable in the neighborhood game, then it means that there is no possible combination of toggles that would allow you to get all of the vertices of the graph to have a label of 0. Since the set of toggles in the group game is a subset of the set of toggles in the neighborhood game, this means that there is also no possible combination of toggles in the group game that would be able to get all of the vertices of the graph have a label of 0. This completes the proof □

We can use this lemma to prove the following lemma.

Lemma 2.12. Assume we have the graph $K_{m,n}$ with m and n being odd numbers. Then the $\mathbb{Z}_2$ group game is not always winnable.

Proof. To prove that the group $\mathbb{Z}_2$ game is not always winnable on $K_{m,n}$ when both m and n are odd numbers, we want to show that there exist some labeling that is not winnable. If we look at Lemma 2.11, it tells us that if there exists a labeling that is not winnable in the neighborhood $\mathbb{Z}_2$ game, then this labeling is also not winnable in the $\mathbb{Z}_2$ group game. From Lemma 2.10, we know that the neighborhood $\mathbb{Z}_2$ game is not always winnable when played on the graph $K_{m,n}$ when m and n are both odd numbers. This means that by Lemma 2.10, the $\mathbb{Z}_2$ group game is also not always winnable. This completes the proof. □

Now that we know that the $\mathbb{Z}_2$ group game is not always winnable we would like to extend this idea to proving that the $\mathbb{Z}_{2^k}$ group game is also not always winnable. We can do this by showing that there is some correspondence between the $\mathbb{Z}_{2^k}$ game and the $\mathbb{Z}_2$ game. For a labeling in the $\mathbb{Z}_2$ group game, there is also a corresponding labeling in the $\mathbb{Z}_{2^k}$ group game when we use labels of 0 and 2^{k-1} as the labels for the vertices that were labeled with 0 and 1, respectively, in the $\mathbb{Z}_2$ game. Furthermore, the winnability of these corresponding labelings should be the same. If we press a vertex labeled with 0 in either game, the label of all of the vertices of the graph will remain unaffected. If we press a vertex labeled 1 in the $\mathbb{Z}_2$ game, then it will flip the state of its label to 0 and add 1 to all of the labels of its adjacent vertices. If those adjacent vertices have a label of 0, then the label will go to 1. If the adjacent vertex label is already a 1,then that label will go to 0 after the toggle. Similarly, If we press a vertex labeled 2^{k-1} in the $\mathbb{Z}_{2^k}$ game, then it will flip the state of its label to 0 and add 2^{k-1} to all of the labels of its adjacent vertices. If those adjacent vertices have a label of 0, then the label will go to 2^{k-1}. If the adjacent vertex label is already 2^{k-1}, then that label will go to 0 after the toggle. This means that the toggles needed to win a labeling in the $\mathbb{Z}_2$ game are the same as the toggles need to win in the corresponding labeling in the $\mathbb{Z}_{2^k}$ game as long as each vertex labeled 1 in the $\mathbb{Z}_2$ labeling corresponds to a vertex labeled with 2^{k-1} in the $\mathbb{Z}_{2^k}$ labeling and vertices labeled with 0 are in the $\mathbb{Z}_2$ labeling are also labeled with 0 in the $\mathbb{Z}_{2^k}$ labeling. We will use this correspondence to prove the following theorem.

Theorem 2.13. Assume we have the graph $K_{m,n}$ where m and n are both odd, then if the $\mathbb{Z}_2$ group labeling game is not always winnable, then the group $\mathbb{Z}_{2^k}$ game is also not always winnable.

Proof. To prove that if the $\mathbb{Z}_2$ group labeling game is not always winnable, then the group $\mathbb{Z}_{2^k}$ game is also not always winnable, we will use the fact that if there is a combination of toggles in the $\mathbb{Z}_2$ game that is winnable, then that same set of toggles will win the $\mathbb{Z}_{2^k}$ game using labels of 0 and 2^{k-1} . We can do this if every time we would toggle a vertex labeled with 1 in the $\mathbb{Z}_2$ game, we make the same toggle on the corresponding vertex labeled with 2^{k-1} in the $\mathbb{Z}_{2^k}$ game. Since the combination of toggles needed to win are the same in each game, it means that if there exists a labeling that is not winnable in the $\mathbb{Z}_2$ game, then there must also be a corresponding labeling that is also not winnable in the $\mathbb{Z}_{2^k}$ game. From Lemma 2.12, we know that if m and n are both odd, that the $\mathbb{Z}_2$ group game on the graph $K_{m,n}$ is not always winnable, which means that there is at leas one labeling that is not winnable in the $\mathbb{Z}_2$ group game. Thus, because of the correspondence between the $\mathbb{Z}_2$ game and the $\mathbb{Z}_{2^k}$ game, we know that the $\mathbb{Z}_{2^k}$ game is also not always winnable. This completes the proof. □

3 Future Directions

After investigating the winnability of the group game on complete bipartite graphs, we have begun to explore using similar strategies on complete tripartite graphs and have arrived at the following conjecture.

Conjecture 3.1. The complete tripartite graph $K_{m,n,o}$ is always winnable when using labels from Z_{2^k} when at least two of m, n, o are even.

By getting the tripartite graph to a "standard" labeling, where all of the vertex labels of two of the parts are 0 and all of the vertex labels of the last part are all the same value, c, then it should be possible to prove that this conjecture is true and this method could also possibly be applied to any multipartite graph, but it remains to be seen how much of the complete bipartite proofs would generalize. Another future direction would be to explore the winnability of the group game when played with labels from different groups other than $\mathbb{Z}_k$, as the winnability of the game is highly dependant on the nature of the group and operation used for the labels. Finally, another direction would be to investigate the winnability of the $\mathbb{Z}_2^k$ group game on other families of graphs, such as a spider graph.

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