# Doubly Chorded Cycles in Graphs

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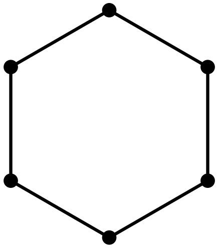
# Introduction

## Definition

A **cycle** is a set of points (**vertices**) connected in a cyclic fashion by **edges**.

Here is an example of a cycle on 6 vertices:

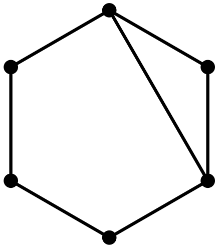


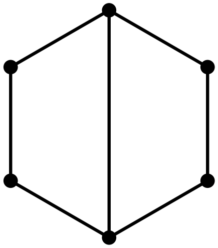
# Introduction

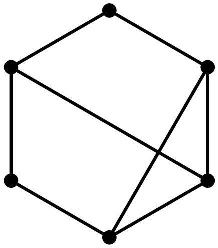
## Definition

A **chorded cycle** is a cycle with at least one additional edge.

It does not matter where the additional edge is situation in the graph. Also, there can be more than two additional edges, the definition only requires that there be at least one additional. Here are three examples of chorded cycles on 6 vertices:







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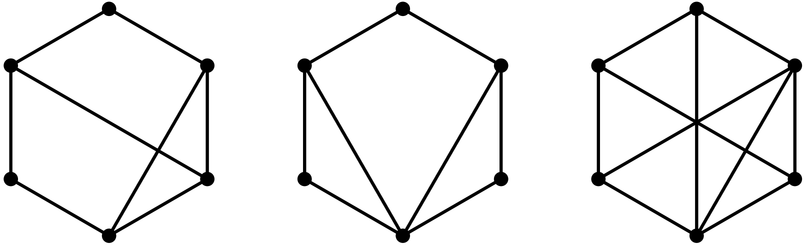


# Introduction

Our main constraint in our results will regard the minimum degree of a graph. Here are the definitions:

**Definition**

The **degree** of a vertex is the number of edges incident to that vertex (essentially the number of edges that are "coming out" of that vertex).

**Definition**

The **minimum degree**, $\delta(G)$, of a graph graph $G$ is the smallest degree over all vertices in $G$ (look at the degree of each vertex in the graph, then whatever the smallest was is the minimum degree of the graph).

# Introduction

Consider a **graph**(set of points and set of edges that connect two points) $G$.

It is not too difficult to see that if $|G| \geq 3$ (number of vertices in $G$ is at least 3) and $\delta(G) \geq 2$ (minimum degree of at least 2), then $G$ contains a cycle.

This intuitive and simple idea led to the following extension and result:

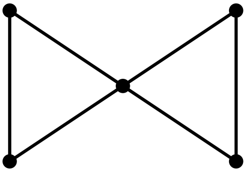
**Theorem (Corrádi - Hajnal, 1963)**

*If $|G| \geq 3k$ and $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.*

Where here $k$, and throughout the rest of these slides, is just some positive integer.

# Introduction

The meaning of disjoint in the previous result and the rest in this presentation is vertex disjoint. So if we state that a graph has disjoint cycles, those cycles do not share any common vertices. For example, the following graph clearly has to cycles:



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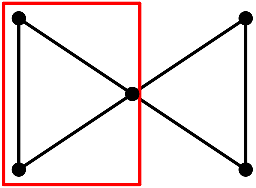

However, the graph only has one *disjoint* cycle. Since as soon as we choose one cycle (in red), we can no longer use the middle vertex to find a second cycle.

# Introduction

The previews result concluded cycles and so the next logical step is to consider chorded cycles, which has the following result:

**Theorem (Finkel, 2008)**

*If* $|G| \geq 4k$ *and* $\delta(G) \geq 3k$*, then* $G$ *contains* $k$ *disjoint chorded cycles.*

# Introduction

With a solution to chorded cycles, what happens with doubly chorded cycles? There are two main results in this area:

**Theorem (Hajnal - Szemerédi, 1970)**

*If $|G| = 4k$ and $\delta(G) \geq 3k$, then $G$ contains $k$ disjoint doubly chorded cycles.*

**Theorem (Gould - Hirohata - Horn, 2015)**

*If $|G| \geq 6k$ and $\delta(G) \geq 3k$, then $G$ contains $k$ disjoint doubly chorded cycles.*

**Our Goal**

Determine what minimum degree constraints will guarantee $k$ disjoint doubly chorded cycles between $4k$ and $6k$ vertices.

# Setup

We first extended Gould, Hirohata, and Horn's result by proving the following:

**Theorem**

*If $|G| \geq 5k$ and $\delta(G) \geq 3k$, then $G$ contains $k$ disjoint doubly chorded cycles.*

How do we actually do this?

What we do is let $G$ be an "edge-maximal" graph with $|G| \geq 5k$ and $\delta(G) \geq 3k$ that DOES NOT contain $k$ disjoint doubly chorded cycles. So if this graph does exist, it would contradict our statement that we wish to show. Our job then is to show that no such graph exists. If no such graph exists, there are no graphs that contradict our result and so therefore our statement is true. (This is a common Proof by Contradiction technique used in mathematics).

# Setup

So we have:

- $G$, an "edge-maximal" graph with $|G| \geq 5k$ and $\delta(G) \geq 3k$ that does not contain $k$ disjoint doubly chorded cycles
  $\Rightarrow$ The "edge maximality" allows us to conclude that $G$ contains $k-1$ disjoint doubly chorded cycles.

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(O1) $|\mathcal{C}|$ is minimum
(O2) subject to (O1), number of chords in $\mathcal{C}$ is maximum
(O3) subject to (O1) and (O2), length of the longest path in $R$ is maximum
(O4) subject to (O1), (O2), (O3), number of edges in $R$ is maximum

# Our Work

Using these conditions on $\mathcal{C}$, as well as our initial assumptions about $G$, we will be able to concluded that $G$ does not exist. We first determined how vertices in our remainder, $R$, interact with doubly chorded cycles in our collection, $\mathcal{C}$. We proved:

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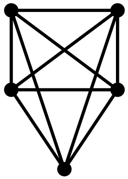
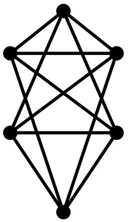
**Lemma (1)**

*For all $v \in R$ and $C \in \mathcal{C}$, $v$ can be adjacent to at most 4 vertices in $C$. When $v$ is adjacent to exactly 4, then $G$ along with $v$ (bottom vertex in pictures) is one of the following two strucutres:*





# Our Work

- Lemma (1) told us how vertices in $R$ interact with the collection $\mathcal{C}$ and actually told us a lot about the structure in $G$ and so is worth mentioning. It is a result we used again and again in proving later characteristics of $G$. Our next step was to determine what structure occurs within $R$.

# Our Work

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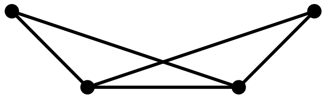
- Ultimately, we were able to prove that $R \cong K_{1,1,2}$, so it looks like



Note that $|R| = 4$. This is important.

# Our Work

Once we knew exactly what our remainder looked like, the bulk of our work was spent proving that all doubly chorded cycles in our collection were on at most 5 vertices. This is the key we needed to prove that this counter example graph does not exist.

Note that since the remainder had 4 vertices and we conclude that the graph has $k-1$ doubly chorded cycles in $\mathcal{C}$, all of which are on at most 5 vertices, then

$$|G| \leq 4 + 5(k-1) = 5k - 1.$$

However, we initially assumed that $|G| \geq 5k$. Hence we have a contradiction and have proven out result!

**Theorem**

*If $|G| \geq 5k$ and $\delta(G) \geq 3k$, then $G$ contains $k$ disjoint doubly chorded cycles.*

# Our Work

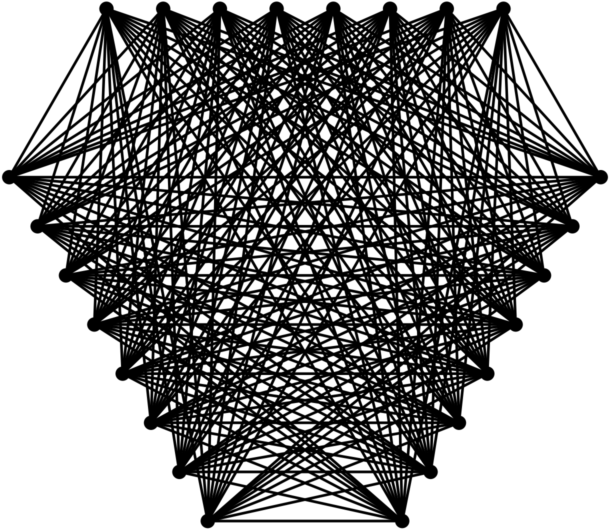
- What happens between $4k < |G| < 5k$?

Does a minimum degree of at least $3k$ still suffice to guarantee the existence of $k$ disjoint doubly chorded cycles?

Turns out a minimum degree of at least $3k$ is NOT enough. One actually needs about $\frac{10k}{3}$.

For those interested in why $\delta(G) \geq 3k$ is not enough, the following slide contains a counter example graph. This graph satisfies $4k < |G| < 5k$ as well as $\delta(G) \geq 3k$, however it DOES NOT contain $k$ disjoint doubly chorded cycles.

# (10/3)k vs 3k



Thank You!

I also want to thank Grand Valley State University which funded a Student Summer Scholars Program for me as well as my mentor Dr. Michael Santana.