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Fault-Free Tilings

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Abstract

Given a $p \times q$ rectangular board (height p and width q), we may fill in the area with 2×1 tiles. We say the board is *tileable* if the board can be filled with non-overlapping tiles leaving no open space. Not all boards are tileable, for example a 5×5 board is not tileable because it has an odd area. A *fault-free* tiling exists on the board if every line through the board parallel to a side goes through a tile. Previous work on this subject has been completed by Ron Graham in "Fault-Free Tilings of Rectangles" and Emily Montelius in "Fault-Free Tileability of Rectangles, Cylinders, Tori, and Möbius Strips with Dominoes". Here, we fill in the details of Graham's proof of exactly which boards have fault-free tilings and give an efficient way to create such tilings.

1 Introduction

We begin by imagining a rectangular board with a given height and width, which can be filled by tiles. The area of the board must be fully covered by the tiles in a way that there is no overlap and all the tiles are within the board. If this can be done, we say a board is **tileable**. We will look at tilings of rectangular boards with 2×1 tilings, which gives us the following definition.

Definition 1.1. For $p, q \in \mathbb{N}$ a **tiling** of a $p \times q$ rectangle is a placement of 2×1 tiles so that no tiles overlap and the board is entirely covered. We say a board is *tileable* if such a tiling exists.

Not all boards will have such a tiling. For example, boards with odd area are not tileable. Figure 1 shows a tiling of a 7×7 board. Since a 7×7 board has an area of 49, we cannot use 2×1 tiles to fill it, as these tiles consist of an area of 2.

Definition 1.2. A tiling is **fault-free** if every line through the board parallel to a side goes through a tile.

A fault-free tiling of a 6×5 board is seen in Figure 2. Any of the 5 horizontal lines through the interior of the board cuts through a tile and the same is true of any of the 4 vertical lines. In Figure 3, we see a tiling that is not fault-free. There are two horizontal lines that do not cut through a tile, we call this a *fault-line*. There are also 4 vertical fault-lines. Because of this error, we say that the tiling given in Figure 1 is not a fault-free tiling. In [2], Graham provides a complete characterization of the existence of fault-free tilings of $p \times q$ boards with 2×1 tiles. We include here that characterization.

Theorem 1.1. *[2] A fault-free tiling of a $p \times q$ rectangle with 2×1 tiles exists if and only if*

1. *pq is divisible by 2*

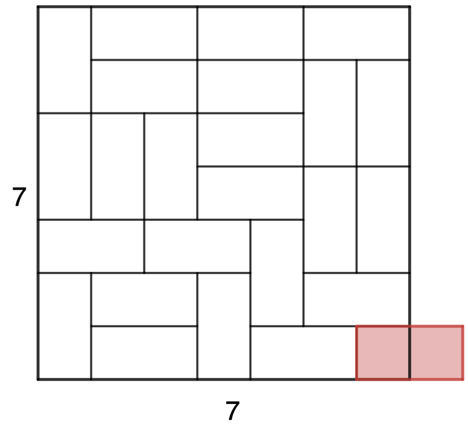

Figure 1: A 7×7 board can not be tiled with 2×1 tiles.

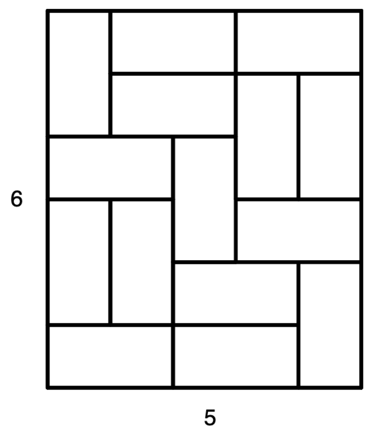

Figure 2: A fault-free tiling of a 6×5 rectangle.

2. *$p \geq 5$ and $q \geq 5$, and*
3. *$(p,q) \neq (6,6)$.*

Graham provides a complete proof that the listed conditions are necessary, but does not provide a complete proof of their sufficiency. He states "One way this can be provide is by starting with small fault-free tilings, such as 5-by-8, 6-by-8 and our earlier 5-by-6 and building up larger fault-free tilings from these." In this paper we discuss a way to do this efficiently.

Strategy Application To describe our strategy for building up larger fault-free tilings from smaller ones, we first need some definitions.

Definition 1.3. A **fissure break** is a set of connected line segments going from one side of a board to the opposite side that do not intersect any tiles.

We may find a fissure break as a line starting on an edge of a board and continuing until it hits a tile, then moving left, right, up, or down by a length of one, then continuing in the same direction until we reach the end of the board. This can be seen in Figure 4.

To be able to describe fissure breaks (see Definition 1.3), we must be able to locate and label coordinates. We use Cartesian coordinates with the upper left corner being the point $(0,0)$. For example, Figure 5

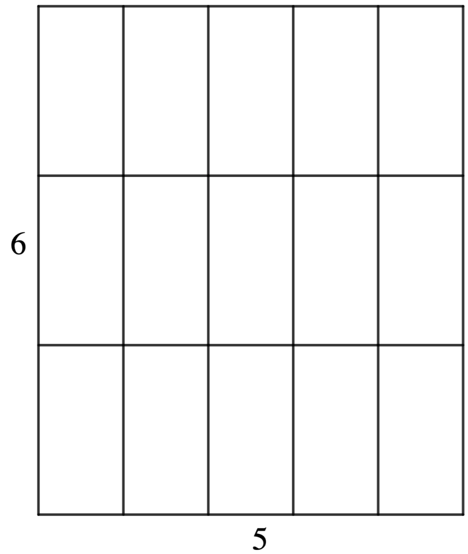

Figure 3: A non-fault-free tiling of a 6×5 rectangle.

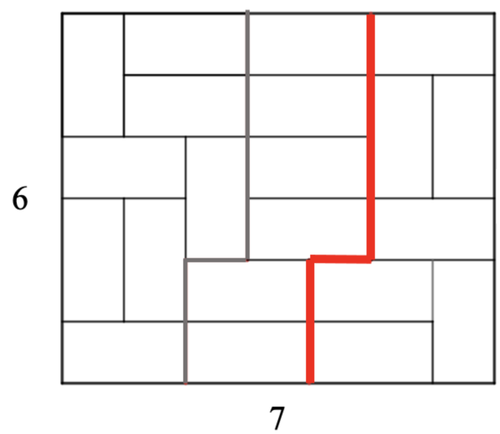

Figure 4: A fault-free tiling of a 6×7 board, with a fissure line highlighted in red.

shows a labeled 6 by 5 board. The bottom left corner of this is denoted $(0,6)$. We describe the fissure break in Figure 4 has a fissure break highlighted in red that would be denoted $(5,0)$—$(5,4)$—$(4,4)$—$(4,6)$.

The primary way to add additional area to a fault-free tiling is to insert additional tiles after breaking along the fissure break. By doing this we preserve the same fissure breaks, and so the same strategy can be repeated. The tiles will be inserted in a way to break potentially introduced fault-lines.

2 Sufficient Conditions

This section will conclude with a complete proof that the conditions on p and q given in Theorem 1.1 are sufficient. We begin with some lemmas to describe how to extend boards. Since pq must even for a board to be tileable one of p or q must be even. We assume, without loss of generality, that p is even.

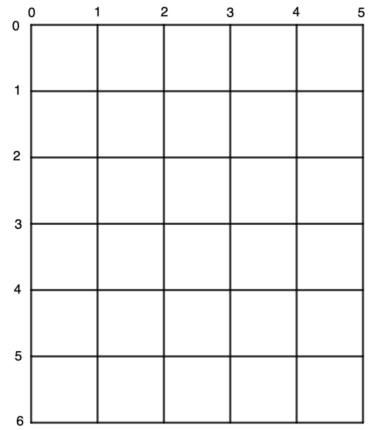

Figure 5: A grid representation of a 6×5 board.

2.1 Extending Boards

We will show that we can add any even number of rows and any even or odd number of columns to a $p \times q$ board. To start, we show that any integer at least 2 can be written as the sum of a non-negative multiple of 3 and a non-negative multiple of 2.

Lemma 2.1. *Given any $y \in \mathbb{N}$ with $y \geq 2$, there exist non-negative integers m and n such that*

$$y = 3m + 2n.$$

Proof. We will assume that y is a natural number such that $y \geq 2$, and $m,\ n \in \mathbb{Z}_{\geq 0}$. We will prove that $y = 3m + 2n$ using a proof by cases based on whether y is even or odd.

Case 1 If y is even and a positive integer, then we know that $\frac{y}{2}$ will always yield a positive integer. We can then set $m = 0$ and $n = \frac{y}{2}$, which then gives us the following

$$y = 3(0) + 2\left(\frac{y}{2}\right)$$

We have now proven that our lemma holds if y is even.

Case 2 Suppose y is odd. Since y is an odd integer and $y \geq 2$, we know $y - 3$ is a non-negative even integer. Let $m = 1$ and $n = \frac{y-3}{2}$. Note n is a non-negative integer. Then

$$y = 3(1) + 2\left(\frac{y-3}{2}\right).$$

Since we have shown the lemma to be true in the case where y is even and the case where y is odd, we have proven the lemma. $\square$

In the Lemma 2.2 we will show that we can add an even number of rows to a 6×5 board.

Lemma 2.2. *For all $k \in \mathbb{N}$, $(6 + 2k) \times 5$ has a fault-free tiling with the fissure break*

$$(0,2) - (3,2) - (3,3) - (5,3).$$

Proof. We will assume $k \in \mathbb{N}$, and we tile a 6×5 board as seen in Figure 2. We will show that $(6+2k) \times 5$ has a valid fault-free tiling using induction on k.

Basis Step (Base Case)

When $k = 1$, we consider a

$$(6+2(1)) \times 5 = 8 \times 5$$

board. Using the tiling in Figure 2 we see the fissure break

$$(0,2) - (3,2) - (3,3) - (5,3).$$

We break along this fissure break and insert 5 vertically upright tiles in a row (see Figure 6). We claim this is a fault-free tiling. Since the tiling of the 6×5 board is fault-free, we only need to consider the 2 potential fault lines introduced at heights 3 and 4. As seen in Figure 6, these fault lines are broken by the inserted tiles. Moreover, the same fissure break still appears in the new tiling.

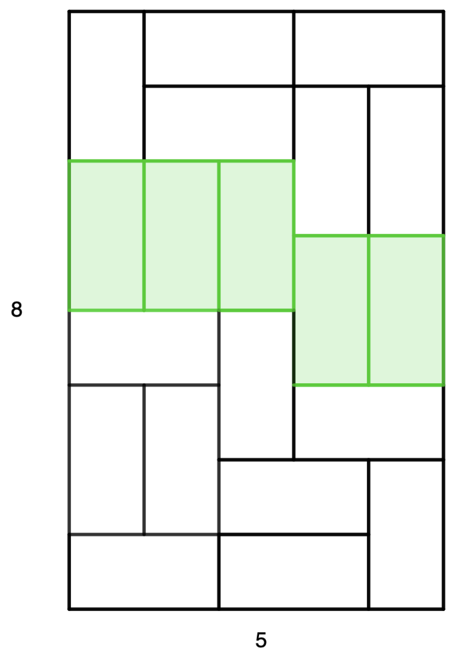

Figure 6: A fault-free tiling of a 8×5 board, with inserted tiles highlighted in green.

Inductive Step

Suppose there exists a tiling of $(6+2k) \times 5$ with the given fissure break. We will show there's a tiling of $6 + 2(k+1) \times 5$ that has the same fissure break.

By the inductive hypothesis the same fissure break exists, and so we repeat what we did in the base case. Break along this fissure break and insert 5 vertical tiles. We know that is a fault-free tiling of $(6 + 2k + 2) \times 5$ because the two potential fault lines introduced at heights 3 and 4 are broken by the inserted tiles. Moreover, we have the same fissure break given in our theorem statement.

By the Principle of Mathematical Induction, our proof is complete. $\square$

In the Lemma 2.3 we will show that we can add 2 or more columns to a 6×5 board.

Lemma 2.3. *For all $m, \ell \in \mathbb{N} \cup \{0\}$, a $6 \times (5+2m+3\ell)$ board has a fault-free tiling with the fissure break*

$$(5+2m+3\ell-2,0)-(5+2m+3\ell-2,4)-(5+2m+3\ell-3,4)-(5+2m+3\ell-3,6).$$

Proof. We will assume that $m, \ell \in \mathbb{N} \cup \{0\}$. We will prove that $6 \times (5+2m+3\ell)$ is fault-free tileable by a direct proof with cases.

Case 1 (Increasing the width by multiples of 2)
First note that in Figure 2, we have the given fissure break in a tiling of a 6×5 board. We may use this fissure break to add additional tiles to the board. We'll add width $2m$ by breaking along the fissure break and insert horizontal tiles (see Figure 7). When we add one set of horizontal tiles introduce two possible fault-lines, both of which are broken by our break (see Figure 7). Also, the same fissure break exists, so we can continue to add and break fault lines until we have added width $2m$.

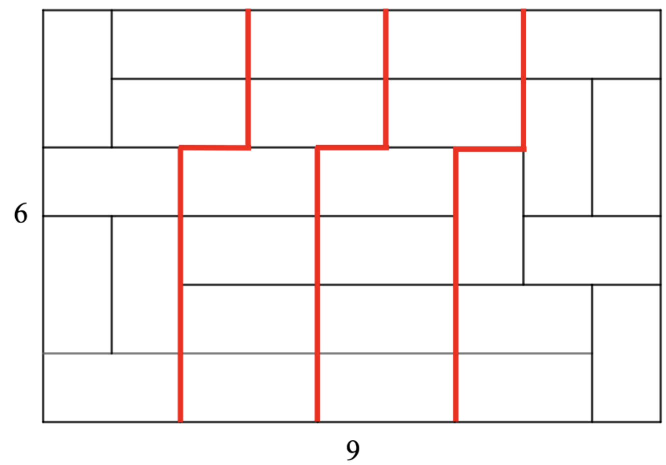

Figure 7: Fault-free tiling of a 6×9 board, with fissure breaks highlighted in red.

Case 2 (Increasing the width by multiples of 3)
Now we will add the 3ℓ. Note that we have maintained the same fissure break. Let's start by adding 3. We break along the given fissure break and add tiles as seen in Figure 8. When we make room for width 3 we are introducing 3 potential fault lines. At the top we add one vertical tile on the left and two horizontal tiles. This breaks the rightmost of the potential fault lines introduced. In the middle height 3 block we insert the same pattern which breaks the middle potential fault line introduced. In the bottom height 3 block we put the same pattern but mirrored. This breaks the potential fault line on the left. This breaks all the potential fault lines, and keeps the same fissure break. We repeat this to add width 3ℓ.

□

2.2 Tiling a $p \times q$ board

As discussed in the introduction, Graham has Theorem 1.1 in [2], but the proof is left to the reader. In Theorem 2.4 we will prove the conditions given are sufficient.

Theorem 2.4. *[2] Let $p, q \in \mathbb{N}$. If*

1. *pq is divisible by 2*

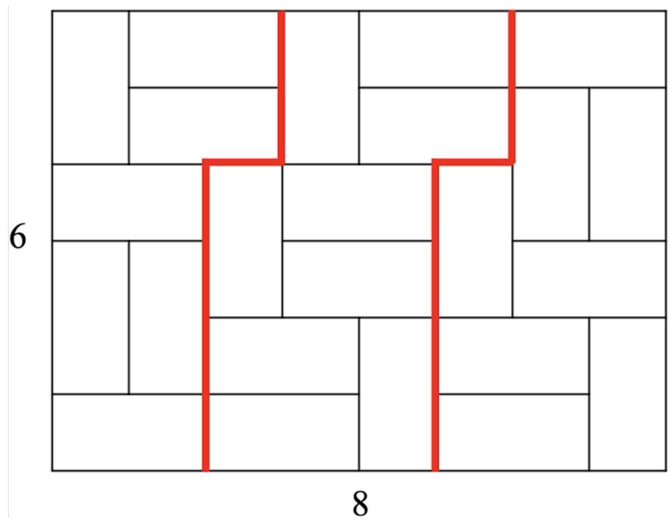

Figure 8: Fault-free tiling of a 6×8 board, with fissure breaks highlighted in red.

2. $p \geq 5$ and $q \geq 5$, and

3. $(p,q) \neq (6,6)$

then a fault-free tiling of a $p \times q$ rectangle with 2×1 tiles exists.

Proof. We will assume p and q are integers such that pq is divisible by 2, $p \geq 5$ and $5 \geq q$, and $(p,q) \neq (6,6)$. Without loss of generality, assume p is even. We will prove that for any such p and q, a fault-free tiling of a $p \times q$ board exists. We use Lemmas 2.1, 2.2, and 2.3.

First, write $p = 6 + 2n$ for some $n \in \mathbb{N} \cup \{0\}$ and $q = 5 + 2m + 3\ell$ for some $m, \ell \in \mathbb{N} \cup \{0\}$. This can be done for any even integer $p \geq 6$ and any integer $q \geq 5$ for $q \neq 6$ by Lemma 2.1. Start with the tiling of a 6×5 board given in Figure **??**.

By Lemma 2.2, we know that we can add any even number of height and break all potential fault lines. So we add height $2n$ This gives a fault-free tiling of a $(6+2n) \times 5$ board. Note that this board will have the fault-line described in Lemma 2.3. So, we insert tiles to increase the width by $2m+3\ell$. The inserted tiles break any newly introduced potential fault lines. With all fault lines broken, we can say that the board will have a fault-free tiling.

In the case where $q = 6$ we know $p \neq 6$. Using the technique above we can construct a fault-free tiling of a $6 \times p$ board and rotate it to have a tiling of $p \times 6$ board. Thus, the conditions are given are sufficient to have a fault-free tiling. □

3 Future Work

In [2] there is also a theorem about when a $p \times q$ board has a fault-free tiling with $a \times b$ tiles. Other work has been completed by Emily Montelius in "Fault-Free Tileability of Rectangles, Cylinders, Tori, and Möbius Strips with Dominoes" [3] where she extends these tilings to other shapes. Additionally in [1] Alabi and Dresden count the number of fault-free tilings of $3 \times n$ rectangles using squares and dominoes. In [2] Graham suggests looking at 3 or more dimensions, or requiring each fault-line to be broken by at least 2 (or more) tiles. These questions appear to have not been looked at yet!

References

[1] Oluwatobi Jemima Alabi and Greg Dresden. “Fault-free tilings of the $3 \times n$ rectangle with squares and dominos”. In: *J. Integer Seq.* 24.1 (2021), Art. 21.1.2, 11.

[2] R.L. Graham. *Fault-free Tilings of Rectangles*. 1981. URL: `https://mathweb.ucsd.edu/~ronspubs/81_01_fault_free_tilings.pdf`.

[3] Emily Montelius. *Fault-Free Tileability of Rectangles, Cylinders, Tori, and Möbius Strips with Dominoes*. 2019. DOI: `10.48550/ARXIV.1912.04445`. URL: `https://arxiv.org/abs/1912.04445`.