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Preston C. Hammer
Grand Valley State University

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The accelerating deterioration of our environments is now attracting more and more attention. Waste products pollute the air, the waters, the land, even the moon. Overpopulation is another polluting threat. Yet the attention given to these polluting activities is directed to the immediate rather than to the primary causes. One primary cause of the irresponsibility we demonstrate lies in the attitudes, habits, goals, and philosophies we have. This may be called MIND POLLUTION.

In the U.S.A., for example, much has been said about the detrimental effects of cigarette smoking. Yet despite the measures against the spread of this habit, the tobacco companies have managed, by innuendo and spurious advertising, to boost their sales. Every public cigarette-vending machine is a de-facto violation of the laws against selling cigarettes to minors. Yet we condone this illegal activity. Why? It's a matter of attitudes and habits. Mind pollution.

Again, for example, universities supposedly represent the best in thinking and teaching. Yet their goals have led to extremes of specialization and to disregard of the process of educating. The improvement of education in recent years has been almost negligible. At base, the cause is attitudes, philosophies, habits. Mind pollution.

In this essay, I concentrate on one important area of mind pollution, that of education in mathematics. Mathematics, in contrast to many other aspects of knowing, is basically simple; that is, I am aware of no area of mathematics that could not be learned during or before the secondary-school level, if it were conceded to be important enough. Other fields such as sociology, history, political science, biology, psychology, law, and medicine have aspects for which adulthood may be prerequisite. Mathematics is taught to every school child. Only the study of language competes with mathematics for the time it consumes in schools. For this reason, it is important that education in mathematics be optimal and effective, that our attitude toward it be correct.

I find that the neglect of mathematics education is closely related to the attitudes of professional mathematicians. I demonstrate here several instances of the failure of mathematicians to properly interpret the most important concepts of mathematics. I will also indicate some of the instances of abuses of terminology. My interpretation of the actions needed to improve mathematics education cannot be said to have been tested or even considered by mathematics educators. Nor can it be considered before minds are changed. The basic problem is to establish a cybernetic system which enables continuing progress in education at all levels. No rigid system of schooling can be adjustable enough to provide adequate educational opportunities.

Failure of Mathematics Education

A basic goal of mathematics education should be to assure a level of understanding mathematics—what it is, what it does, and what it fails to do. Ask any baccalaureate-degree holder with a major in mathematics what mathematics is. He will not be able to give a sensible answer, no matter how much time he is given. Ask a Ph.D. in mathematics the same question. He also will not know, or else he may reply with some nonsense statements like "mathematics is deductive science" or "mathematics is what mathematicians do."

Is it reasonable that not even a Ph.D. in mathematics will know what a theorem or a proof is when every school teacher should know? It is not! It might be said that mathematics is too difficult to be presented in a meaningful way. I claim that the emphasis on technical results has led to the anomaly that the understanding of mathematics has been left out of consideration.

Chemistry and biology are inherently at least as difficult as mathematics. Yet, a one-year course in either chemistry or biology will give the student a better cultural view of these subjects than he can achieve after years of study of mathematics. It is possible to do much better with mathematics than is now being done.

What is wrong with the present education in mathematics? The general ideas which relate mathematics to other human activities not only are not taught, THEY ARE NOT KNOWN! Not even by most mathematicians. There are no effective presentations of mathematics as a whole. Its structure is not outlined. Concepts which could have significance beyond mathematics are trivialized so as to be merely of mathematical interest, mostly to the mathematicians. Big ideas are reduced to the ashes of axiomatics. General principles and approaches which provide patterns for details are simply omitted. Some examples:

Functions

I first present a spectacular instance of the separation of mathematics from reality, in the consistent abuse of one of its most important concepts—"function." So important are functions deemed that some mathematicians have recommended that they be made a basis for all mathematics. Various "definitions" have been proposed, no one of which is particularly useful in dealing with functions. Certain textbooks suggest "transformation" and "map" or "mapping" as defining synonyms for "function."

Suddenly, one evening in March, 1970, after giving a lecture on computer science, I realized that a function corresponds to a VERB of a special kind. Thus y=f(x) can be diagrammed as X/F/Y or XFY, in which x is the nominative, f is the verb, and y is the object, direct or indirect, of the verb. The practice of writing a binary relation in the form xRy, "x is related to y," indicates the sentence diagram of binary relations. In speaking of functions, the various statements "x goes into y," "x becomes y," "x determines y," "x is labelled y," and "x is represented by y" show that the function is usually considered as a verb in the present tense.

I have tried this interpretation of function on a number of mathematicians. They have, without exception, agreed that it is superior, conceptually, to the previous interpretation. Now I ask: how have people for such a long period of time (nearly 200 years) who "knew" what verbs and functions are, not recognized this obvious comparison? It appears they have simply accepted the separation of mathematical concepts from those in other areas, in particular, language. We have been inhibited from recognizing patterns which should have been obvious; mathematics is split from interaction with other areas.

Since this example is so illuminating, I pursue it further. The present tense ordinarily used suggests other sentence forms. Consider "x will go into y (at a later time)"; "x went into y"; "x will have determined y"; "x ought to become y"; "probably x will go into y." You will have a variety of ways of thinking of functions, not all of which have mathematical models now.

For example, the imperative and conditional imperative modes of sentences correspond to the controlling devices of computing. A control function is one which tells other functions when to act and on what. The program formats DO, LET, GO TO, IF–THEN show the imperative and conditional imperative aspects of the control of computers.

The capability of computers to "learn" is based on implementation of the IF–THEN commands. I believe that an effective theory of computers can now be based on functions and relations, the relations being the state of the machine at a given cycle time and the functions carrying one relation into the next relation or state.

A computer, incidentally, does transforming—it is not a transformation; a mapper, not a mapping. The confusion of the object of the sentence with the verb is, most regretably, standard practice. A university is a (partial) transformer of students; it is not a transformation.

The lesson to be drawn from the above example is obvious. A most-important concept of mathematics, "Function," has never been well treated. Only by clearing away the misinterpretations by which issues have been befogged can we glimpse the deeper mysteries beneath.

Filters and Neighborhoods, another example

For years I have struggled with the concept of "neighborhood," trying to find out what it means basically—i.e., to the non-topologists. Finally, I arrived at the following interpretation. The neighborhoods of an objective are the conditions which MUST be met to achieve the objective. Thus neighborhoods PROTECT an objective from trivial attainment. For example, to achieve a B.S. degree in a university, the student must meet stated conditions which are the "neighborhoods" of the degree. If he meets all conditions, he has CONVERGED, i.e., he gets the degree. This conceptualization of "neighborhood" is simple, it can be explained to children, and it relates the topological concept to a much wider range of human activities.

Now, in the present state of education you will never see such a simple and non-technical discussion of neighborhood. If topology is made trivial by such interpretations, topology is trivial. The filling of minds with technical concepts without establishing their relationships to reality is a form of pollution. Topology would be much more useful if more people understood it in their terms.

I once read a technical definition of "filter" in a topology text. It was not satisfactory to me, so I asked several topologists "how do topology filters filter?" They did not know! I decided then to define filters myself by considering the filters I knew about—pipe filters, chemistry filters, air filters, and electronic filters. I then, in a few minutes, decided that a FILTER in a set is any device which passes or does not pass each element in a set. That is to say, a filter is a device which makes binary decision.

Now, I emphasize DEVICE since, in the examples I had selected, the filter was a mechanism for producing the decisions; it was not the result of its application. In a rather short time I had defined relational filters and learned how to interpret the neighborhood filters of topology. The collection of all neighborhoods of a point is called the neighborhood filter of the point in topology. Its filtering action is to pass all sets which are close to the point (the point is in the closure of each accepted set) and to reject all other sets. I ask, why do topologists use the term FILTER and divorce it from other filters. It is against the interest of good education to use terms in this way.

Now filters, as I defined them, become a unifying concept for mathematics and also have practical examples accessible everywhere. Equations are filters, inequalities are filters, an axiom system is a filter, definitions are filters, and so on. I feel most strongly that by early and repeated use of such a general concept, mathematics education can be made more enjoyable; the interrelationships among concepts and structures previously regarded as unrelated CAN BE SHOWN. Since I first defined my concept of filter in 1967, it is reasonable to ask: how many educational opportunities have been missed? Many, I am certain!

My conclusion is that if mathematics is not worth understanding, it occupies too much time. Much more work needs to be done on the sense of mathematics.

Information, Approximation, and Continuity

ACCESSIBILITY and INACCESSIBILITY are terms common to trade, science, education, governments, religions. Tradesmen seek access to markets, at the same time trying to prevent access to their processes and techniques. Scientists seek access to the mysteries of the universe and so do religions. Education is designed to provide access to certain forms of knowledge—at the same time making other kinds inaccessible. In one interpretation, mathematicians increase access to information by providing theorems, language, formulas, methods, and algorithms.

One of the most useful kinds of activity in reducing "real systems" to mathematical systems is variously labelled as ABSTRACTING, MODELLING, or APPROXIMATING. Thus the plane of geometry may be considered as abstraction or idealization of real surfaces.

The advantages of using the geometrical entity are numerous. First, it approximates adequately many surfaces and it is a reasonable replacement. Next, it is simpler than any real surface and it is amenable to manipulation. For its proper uses, the plane contains as LITTLE information or structure as possible. On the other hand, imagining the plane to be comprised of points leads to an enormous number of configurations which had no known counterparts in reality. Some of these configurations then serve as design elements, and from these man makes objects to approximate geometrical objects. Accordingly, there are not only mathematical models of real systems, there are also physical models of mathematical "objects."

It is an unfortunate aspect of mathematics education that many pupils do not well experience this relationship between systems. Yet it is critical that the interactions between mathematics and other activities be clearly understood.

Approximation theorists of modern vintage have confined the term "approximation" to a very small area, mainly in linear vector spaces endowed with norms. Yet the idea behind approximation has no need for such an esoteric limitation. After some years of considering the matter, I have come up with the following approach. The result of an approximating process is the substitution of one entity for another, with the intention that the former shall play the same role in some regards as the latter. For example, oleomargarine is an approximation to butter, when it is used as a substitute.

One grievous error in interpreting approximations is to allow only good approximations. In the example above, I may consider oleomargarine as an approximation to butter without making any statement concerning how good it is as an approximation. To some people this approximation is bad, to others it is good (they use oleomargarine), and to others, oleomargarine is superior to butter—they like it better. Who is right?

But read one, two, or one-hundred discussions of approximations in any mathematics text. Will you find any sensible discussion of the concept? I have yet to see one. This means that important concepts are being ignored simply because they have been severed from reality in order to be applied to very technical and narrow fields.

The fundamentally poor attitude involved is that of a disregard for simple reality that the young can understand, and this is the result of mind pollution.

Measures and Distances

The temptation to lay down axiom systems to define certain mathematical concepts is great. Since 1900, many mathematical systems have been formalized. The advantages of axiom systems are well advertised: they provide, in a sense, basic guidelines for a concept and thus provide a useful means of developing the underlying assumptions.

The disadvantages of axiomatizations are less well understood. The process of selecting an axiom system is a process of inductive rather than deductive reasoning, and it is therefore subject to the level of understanding which the producer of the axiom system possesses.

For example, the current definition of a "topological space" by axioms would scarecely have been accepted if the matter had been given careful thought.

The famous French mathematician Henri Lebesgue developed a generalization of the concepts of ARC LENGTH, AREA, and VOLUME. This generalization was very well adapted to many problems of analysis. Later studies led to an axiomatic presentation of Lebesgue's generalization. Unfortunately, the term "measure" is used in the sense of the axiomatization.

This use is a clear case of pollution. The "measure theorists" now have a definition of "measure" which they CANNOT use properly in their own theory! Thus, to every literate individual, an external or exterior measure must be a special kind of measure. But no! In measure theory, an exterior measure is not usually a "measure." Projection-measures, important in measure theory, are also not "measures" by definition!

More important, however, is that the axiomatic definition of measure excluded most of the important and known measures of mathematics. It is clear that the axiomatizers were not well-informed concerning the meanings of "measure."

How much better it would have been if that word had not been so ill treated. In this case, I think that it is important not to let a small and evidently uninformed sector of mathematicians dictate the use of an important big concept. Is it any wonder that students on whom this usage has been forced cannot grasp mathematics?

Incidentally, I have written a better definition of measure which does embrace most of the examples I know of. It is at a level which requires no technical background.

Distances are MEASURES of separation or inaccessibility. The first widely accepted axiomatization of distance was given by M. Frechet as a definition of a metric. Unfortunately, this definition again did not embrace the distances known in mathematics at the time it was published; however, in this case I can see the reasoning behind the definition. The metric system of units is based, in part, on the meter, a measure of distance. Moreover, the definition itself stimulated more thinking about geometric-type distances than before.

The damage here is not due to the term itself; it is that the assumption was made that distances were subsumed in metrics. My point is that virtually no mathematics teacher knows that there are DISTANCES and MEASURES all around which are not metrics. If he did know it, he could use the facts to excellent advantage.

Ideas Concerning Mathematics Education

So far, I have discussed a few concepts of recognized importance in mathematics. Let me now consider how the system of education falls short in a general way.

The order of difficulty of subjects in mathematics seems to be roughly as follows: algebra, combinatory geometry, arithmetic, infinitesimal geometry including analysis and topology. Actually arithmetic, the way it is taught, may be more difficult than analysis. I rate algebras as least difficult because the axiom systems for algebras are readily expressed in language. This is not to say that there are not unsolved problems and untouched branches of every area. Geometry is difficult because it has many concepts which cannot be verbalized, such as ANGLE, AREA, CURVE, and PLANE.

Whether or not arithmetic can be made less difficult, I am not certain. Teaching arithmetic well would seem to require concomitant instruction in the relevant algebras and, even at the best, it involves the difficult ideas of rational numbers in which there is available an infinite set of names for each number. The algorithms of arithmetic in themselves are a considerable feat of learning.

A child starts off in arithmetic with several functions of two variables (before he has experienced the perhaps-less-natural functions of one variable). He is compelled to be a machine, doing things for which computers are much more reliable. Boolean arithmetic is naturally easier, and the use of set algebra is one of the more hopeful aspects of early mathematics education.

Forms of geometry should appear early in education. In my estimate, every high-school graduate should have some idea of 4-dimensional geometry and of 3-dimensional projective geometry, if only to enable him to use space-time and to understand better the distortion of the world through his eyes.

The useful aspects of logics should unfold during the schooling. Probabilistic models can be used early. Functions, relations, and concepts like filters should be woven into education throughout. Concepts should be named more-or-less simultaneously with the appearance of examples.

The calculus as it is yet taught is an intellectual disgrace, despite the fact that it could serve as a carrier for many recent concepts. I would not necessarily favor putting calculus in grade nine or even later in high school, in its present form. However, some applications of infinitistic mathematics might well be learned in high school.

The major (and not precisely defined) objective I would suggest is that every individual, on receiving a high-school diploma, have some understanding of mathematics as a whole. This is a goal not achieved now in colleges, or even in graduate schools.

All along the way, the pupil should be made acquainted with the roles which mathematics plays and those it does not play. Students should have some experience with creating mathematical systems (actually easy to acquire). Mathematics should be related to other areas consistently; in particular to language, at the beginning.

A Cybernetic System for Mathematics Education

First, what are the prospects for reform? In the U.S.A., I believe the burden for change must rest in computer scientists rather than on those with extensive classical mathematical training. In Western Europe, the same role may be played by cybernetics and informatics. I see no indication that mathematicians will apply themselves to the task.

What steps, then, should be taken to get mathematics education revitalized? My basic tenet is that general concepts are comparatively simple to grasp—becoming a good specialist is difficult.

The first step is to search out the structures of mathematics and, when necessary, to provide better terminology. This task may be called meta-mathematical. I have taken some initial steps in this direction and have published a "Chart of Elemental Mathematics." *(See my ADVANCES IN MATHEMATICAL SYSTEMS THEORY, The Pennsylvania State University Press, 1967.)* This chart is crude, but revisions with the help of others should be of great help in getting areas of mathematics sorted out. This work is necessary anyway if a reasonable information-retrieval system for computer science is to be devised. It could have multiple applications.

A simultaneous effort needs to be made to classify, organize, or even generate the general concepts of mathematics and to relate them to other areas. How should this first part of the work get started? The answer is that there is required only a few, from 5 to 20, individuals to make significant progress. As these individuals start to produce reports, support from others will be forthcoming.

In a comparatively short while after the beginning of the initial effort, the second step must be taken. This step will involve writing a book to increase interest, especially of teachers. In this, the aims of the initial task force should be set forth and some of the current findings presented with great care.

Step three is the publication of a journal on the STRUCTURE AND LANGUAGE OF MATHEMATICAL SCIENCES. This journal will serve to publish projected standards before their submission. This will serve as a means of calling attention to the problems and of getting a wider base of support.

Step four will take the form of an international organization devoted to education, with national branches. If the early work is well-done, there will be, at this stage, a rather large number of supporters. The tasks will now be increased to include full schedules of education in the mathematical sciences through college. The basic idea now is to have materials prepared with support and criticism. Every means of assuring better learning both for pupils and teachers will be used. Achieving the status of being permitted to write a text will be considered a very unusual honor. In general, scholarly task forces will underwrite every venture in preparing materials and testing them.

Now, for the cybernetic system to work, feedback must be used quickly on all experiments, and means of gauging successes and failures must be devised. One blunder in the U.S.A. is the failure to prepare teachers in colleges to teach the so-called new mathematics. This error must not be repeated. Teachers should be prepared with the care which the responsibility of the work requires.

So far, I have mentioned only undergraduate and school levels. Obviously, most college professors are prepared in graduate schools. Again, the preparation of such instructors is critical. Two courses of action here may be open. One is to establish in existing universities a graduate program. Here outstanding computer-science departments in the U.S.A. are the best bet. The other course of action is to start institutes to provide the graduate education needed. The idea is to not water down the new approaches with the old ones.

My suggestions here do not require the present school system. The idea is to lay a sensible basis for education in mathematical sciences. This requires an initial study of the overall structure of mathematics and the sorting out of the semantics. If the current attitudes toward education cannot be altered in mathematics, then I see little prospect for substantially decreasing the pollution of minds.