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The "Lights Out!" Game on Threshold Graphs

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Abstract

The ‘Lights Out’ game was originally played on a 5×5 grid of buttons that started as either on or off. Pressing the buttons flips the state of the button pressed and the directly adjacent buttons, the goal being to have all of the lights out. We can play ‘Lights Out’ on a graph if we assign 0 or 1 to each of the vertices on the graph, analogous to on and off in the original game, and switch the labels of a vertex and its neighbors from 0 to 1 and vice versa when the vertex is toggled. This paper studies a modified version of the game, where instead of labels coming from integers modulo 2, they come from integers modulo k. To win the game, a player toggles the vertices so that eventually all vertices have label 0 at the same time. For certain graphs and certain starting configurations, the game is not winnable. We investigate under what conditions the game can be won when applied to threshold graphs.*

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1 Introduction

The 'Lights Out!' game, developed by Tiger Electronics in 1995, involves a 5×5 grid of buttons that have two states, on and off. In the original game, pressing a button flips the state of the button pressed and the buttons directly adjacent to it. The states of the buttons can be equally represented as having states of 0 for off and 1 for on. The goal is to turn off all lights starting from any configuration. This game has been generalized to arbitrary graphs, where vertices represent buttons and edges represent the adjacency between them [5]. The game was further generalized to be played with any number of states, k [3, 4].

The generalization goes as follows. Let G be a graph with an initial labeling $\pi : V(G) \to \mathbb{Z}_k$ for $k \geq 2 \in \mathbb{N}$. When a vertex is toggled, it and every adjacent vertex has it's label increased by 1 mod k. We define a labeling to be k-winnable when there exists a sequence of toggles to the vertices of a graph that reduces the label for all vertices to 0. It is possible to have a graph structure or number of colors that makes it possible that an initial labeling cannot be won. The case where all of the initial labelings of the vertices are k-winnable is referred to as always winnable, or AW. We say a graph is always winnable with k colors as k-AW.

The mathematical exploration of the 'Lights Out!' game on graphs has revealed connections to linear algebra, graph theory, and domination theory. Anderson and Feil used linear algebra over $\mathbb{Z}_2$ to classify winnable configurations on $n \times n$ grids [2]. Sutner demonstrated that winnability in 'Lights Out!' is equivalent to determining the reachability problem in finite cellular automata [5]. Amin and Slater developed the concept of parity domination for graphs labeled with 0 and 1. The parity dominating sets studied correspond to winning strategies in the 'Lights Out' game. They identified winnable configurations in specific graph classes such as paths, spider graphs, and caterpillar graphs [1].

Building on these foundations, Giffen and Parker and Arangala et al. independently extended the game to graphs with vertices having k states, represented by elements of $\mathbb{Z}_k$. Arangala et al. called the game "multi-state lights out" and Giffen/Parker referred to the game as "neighborhood lights out." They both established conditions for when many families of graphs are always-winnable (AW) in $\mathbb{Z}_k$, and connected these conditions to domination theory in multi-colored 'Lights Out' puzzles [3, 4].

This paper focuses on the 'Lights Out!' game on threshold graphs. We investigate the conditions under which the 'Lights Out!' game is k-AW on threshold graphs. We define a threshold graph as follows:

Definition 1 (Threshold Graphs)**.** *Let a bit string $B = (b_1, b_2, \ldots, b_n)$, where $b_i \in \mathbb{Z}_2$ for $1 \leq i \leq n$.*

- *A graph G is a threshold graph associated with B when $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $v_i v_j \in E(G)$ only if $b_i = 1$ and $1 \leq i < j \leq n$.*
- *Let $W = \{w_1, w_2, \ldots, w_m\} \subseteq V(G)$ be the set of vertices such that they correspond to the 1s in B, listed in the same order as their corresponding bits in the string. Let L_0 be the set of vertices before w_1. Let L_i be the set of vertices after w_i but before w_{i+1} where $1 \leq i \leq m$. Let $\ell_i = |L_i|$.*

Our main theorem states that a threshold graph is k-AW if $\gcd(\ell_i, k) = 1$ for all $1 \leq i < m$. The development of the solution leverages tools from linear algebra and number theory to establish these conditions.

To toggle a vertex a negative number of times, say -5 times, we toggle the vertex $k - 5$ times, and because the vertices operate modulo k, that expression reduces to -5. In this paper, we define the natural numbers to be the non-negative integers.

2 Lemmas

To support the main theorem, we need some lemmas. First, we show that L_0 does not affect the winnability of a graph G.

Lemma 1. *Let G be a threshold graph with associated bit string $B = (b_1, b_2, \ldots, b_n)$. Assume that $b_i = 0$ for $1 \leq i \leq p$. Define G' as a sub-graph of G induced by the vertices $\{v_{p+1}, v_{p+2}, \ldots, v_n\}$. Let $\pi : V(G) \to \mathbb{Z}_k$ be the initial labeling of G and π' be a restriction of π to G'. If and only if π' is a k-winnable labeling in G', π is a k-winnable labeling in G.*

Proof. Assume that π' is k-winnable on G'.

If π' is k-winnable on G', then there exists a sequence of toggles on the vertices in G' that reduce the label of every vertex to 0. This does not include $v_1, v_2, \ldots, v_p$, which still have an arbitrary label on each vertex, $\pi(v_i)$ for all $1 \leq i \leq p$ because they share no edges with the sub-graph G' and we did not toggle any of them. Now we toggle each v_i $-\pi(v_i)$ times, reducing all of their labels to 0. We can do this without considering the other vertices in $v_1, v_2, \ldots, v_p$ and G' because each vertex in $v_1, v_2, \ldots, v_p$ has degree 0 by construction. We have reduced every vertex to zero, thus π' is a k-winnable labeling in G' and π is a k-winnable labeling in G.

If π is k-winnable on G, then there exists a sequence of toggles on the vertices in G that reduce the label of every vertex to 0. Because there are no edges between the first p vertices and the vertices in G', the same sequence of toggles that reduces the label of every vertex in G to 0 (π) also reduces the label of every vertex in G' (π'). Thus, if π is k-winnable on G, then π' is k-winnable on G', concluding our proof. □

After that step, we can assume that $L_0 = \emptyset$ without a loss of generality. Next we further reduce the problem by ensuring that the vertices in any threshold graph associated with a 0 in the bit string can have their labels reduced to 0 without a loss of generality.

Lemma 2. *Assume G is a threshold graph with associated bit string B. Assume $L_0 = \emptyset$. Let $\tau : V(G) \to \mathbb{Z}_k$ be the initial labeling for G. Then, we can toggle $V(G)$ such that for every vertex, $v \in L_i$ for $1 \leq i \leq m-1$ associated with a 0 in B, its label is 0.*

Proof. If we toggle each $v \in L_i$ a total of $-\tau(v)$ times, their labels are reduced to 0. The vertices v share no edges between them, so toggling each of them does not affect the others. We have shown that all vertices in L_i can have their labels reduced to zero, concluding our proof. □

The previous lemma applies to every threshold graph, so from here we assume that the vertices associated with zeros have label zero the graphs in our following lemmas. Next we are going to toggle the vertices of a threshold graph in that state to produce a system of linear equations that determine when the game is k-winnable.

Lemma 3. *Let G be a threshold graph with associated bit string B, and suppose $\tau : V(G) \to \mathbb{Z}_k$. Let $\tau(v_i) = 0$ for every v_i in every L_i. Assume that $L_0 = \emptyset$. Let $|L_i| = \ell_i$. Recall that w_i is the vertex associated with the i-th 1 in the B. Then, τ is a k-winnable labeling if and only if the following system of linear equations has a solution.*

$$\begin{cases} \tau(w_1) &= x_1\left(\sum_{i=1}^{m-1}\ell_i - 1\right) + x_2\left(\sum_{i=2}^{m-1}\ell_i - 1\right) + \ldots + x_{m-1}(\ell_{m-1}-1) - x_m \\ \tau(w_2) &= x_1\left(\sum_{i=2}^{m-1}\ell_i - 1\right) + x_2\left(\sum_{i=2}^{m-1}\ell_i - 1\right) + \ldots + x_{m-1}(\ell_{m-1}-1) - x_m \\ \vdots & \qquad\qquad\qquad\qquad \vdots \\ \tau(w_{m-1}) &= x_1(\ell_{m-1}-1) + x_2(\ell_{m-1}-1) + \ldots + x_{m-1}(\ell_{m-1}-1) - x_m \\ \tau(w_m) &= -x_1 - x_2 - \ldots - x_{m-1} - x_m \end{cases}$$

Proof. Assume that w_i is toggled x_i times for all $1 \leq i \leq m$. Due to the construction of the graph, a few things occur:

1. Since every w_i is adjacent to each other, this toggling adds every x_i to each of them once. This leaves the label for w_i as $\tau(w_i) + \sum_{n=1}^{m} x_n$ for every i.
2. Every vertex v in L_i is adjacent to every w_r for all $1 \leq r \leq i$. This implies that after every w_r is pressed, the label for every vertex in L_i will be $0 + \sum_{j=1}^{i} x_j$.

Each vertex v in each L_i is adjacent to every w_r for all $1 \leq r \leq i$. The only way to turn the vertices in L_i off is to toggle them. The only adjacent vertices that we could toggle to turn the L_i's off are the w_i's, and as we toggled those vertices to get the labels onto the vertices in L_i, we cannot toggle the w_i's to turn off each vertex in L_i, as we have toggled all the w_i's as much as they are going to be toggled. There are ℓ_i vertices in L_i, so pressing all of the vertices in each L_i leaves the total label of w_r for $1 \leq r \leq m$ as:

$$\tau(w_r)+\sum_{i=1}^{m} x_i-\ell_r\left(\sum_{i=1}^{r} x_i\right)-\ell_{r+1}\left(\sum_{i=1}^{r+1} x_i\right)-\ell_{r+2}\left(\sum_{i=1}^{r+2} x_i\right)-\ldots-\ell_{m-1}\left(\sum_{i=1}^{m-1} x_i\right)$$

The only way to win the game is if all of these expressions are equal to zero, as then all of the vertices in the graph would have label 0. We use algebra to rearrange this to make the coefficients x:

$$\tau(w_r) = x_1 \left(\sum_{i=r}^{m-1} \ell_i - 1 \right) + x_2 \left(\sum_{i=r}^{m-1} \ell_i - 1 \right) + \ldots$$
$$\ldots + x_r \left(\sum_{i=r}^{m-1} \ell_i - 1 \right) + x_{r+1} \left(\sum_{i=r+1}^{m-1} \ell_i - 1 \right) + \ldots + x_{m-1}(\ell_{m-1} - 1) - x_m$$

When we look at the full system of equations we get this:

$$\begin{cases} \tau(w_1) & = x_1 \left(\sum_{i=1}^{m-1} \ell_i - 1 \right) + x_2 \left(\sum_{i=2}^{m-1} \ell_i - 1 \right) + \ldots + x_{m-1} (\ell_{m-1} - 1) - x_m \\ \tau(w_2) & = x_1 \left(\sum_{i=2}^{m-1} \ell_i - 1 \right) + x_2 \left(\sum_{i=2}^{m-1} \ell_i - 1 \right) + \ldots + x_{m-1} (\ell_{m-1} - 1) - x_m \\ \vdots & \qquad\qquad\qquad \vdots \\ \tau(w_{m-1}) & = x_1 (\ell_{m-1} - 1) + x_2 (\ell_{m-1} - 1) + \ldots + x_{m-1} (\ell_{m-1} - 1) - x_m \\ \tau(w_m) & = -x_1 - x_2 - \ldots - x_{m-1} - x_m \end{cases}$$

Thus, it is possible regardless of the initial labeling to toggle every vertex to be zero, if and only if this equation has a solution. □

Next, we need to find when this system of linear equations has a solution, and the way to do this simply is to turn this system into a matrix. A solution always exists if and only if the matrix is invertible, so we find when the determinant has a multiplicative inverse modulo k, which implies an inverse.

Lemma 4. *Let $\ell_i \in \mathbb{N}$ for all $1 \leq i < m$. Then, an $m \times m$ matrix of the form:*

$$\begin{bmatrix} \left(\sum_{i=1}^{m-1} \ell_i - 1 \right) & \left(\sum_{i=2}^{m-1} \ell_i - 1 \right) & \ldots & (\ell_{m-1} - 1) & -1 \\ \left(\sum_{i=2}^{m-1} \ell_i - 1 \right) & \left(\sum_{i=2}^{m-1} \ell_i - 1 \right) & \ddots & (\ell_{m-1} - 1) & -1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ (\ell_{m-1} - 1) & (\ell_{m-1} - 1) & \ldots & (\ell_{m-1} - 1) & -1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

is invertible if and only if $\gcd(\ell_i, k) = 1$ *for all* $1 \leq i < m$.

Proof. Consider this $m \times m$ matrix:

$$\begin{bmatrix} \left(\sum_{i=1}^{m-1} \ell_i - 1\right) & \left(\sum_{i=2}^{m-1} \ell_i - 1\right) & \dots & (\ell_{m-1} - 1) & -1 \\ \left(\sum_{i=2}^{m-1} \ell_i - 1\right) & \left(\sum_{i=2}^{m-1} \ell_i - 1\right) & \ddots & (\ell_{m-1} - 1) & -1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ (\ell_{m-1} - 1) & (\ell_{m-1} - 1) & \dots & (\ell_{m-1} - 1) & -1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

The matrix A can be row reduced to a lower triangular matrix by subtracting each row from all of the rows above it. Considering that as you go down the rows, a term gets removed, what remains is the term corresponding to that row before the diagonal, and after the diagonal it's 0. For the j-th row, for $1 \leq j \leq m$, the row reduction looks like this:

$$(R_j): \left(\sum_{i=j}^{m-1} \ell_i - 1\right) \quad \left(\sum_{i=j}^{m-1} \ell_i - 1\right) \quad \dots \quad \left(\sum_{i=j}^{m-1} \ell_i - 1\right) \quad \left(\sum_{i=j+1}^{m-1} \ell_i - 1\right) \quad \dots \quad (\ell_{m-1} - 1) \quad -1$$

$$(R_j - R_m): \left(\sum_{i=j}^{m-1} \ell_i\right) \quad \left(\sum_{i=j}^{m-1} \ell_i\right) \quad \dots \quad \left(\sum_{i=j}^{m-1} \ell_i\right) \quad \left(\sum_{i=j+1}^{m-1} \ell_i\right) \quad \dots \quad \ell_{m-1} \quad 0$$

$$(R_j - R_m - R_{m-1}): \left(\sum_{i=j}^{m-2} \ell_i\right) \quad \left(\sum_{i=j}^{m-2} \ell_i\right) \quad \dots \quad \left(\sum_{i=j}^{m-2} \ell_i\right) \quad \left(\sum_{i=j+1}^{m-2} \ell_i\right) \quad \dots \quad 0 \quad 0$$

$$\vdots$$

$$(R_j - R_m - \ldots - R_{j+1}): \ell_j \quad \ell_j \quad \dots \quad \ell_j \quad 0 \quad \dots \quad 0 \quad 0$$

Thus the first j components of the j-th row are non-zero and the other components are 0, doing this for all rows leaves the matrix as lower triangular:

$$A' = \begin{bmatrix} \ell_1 & 0 & \dots & 0 & 0 \\ \ell_2 & \ell_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ \ell_{m-1} & \ell_{m-1} & \dots & \ell_{m-1} & 0 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

These operations do not affect the invertibility of A, because row operations preserve the determinant. The condition for when A is invertible is when $\det(A)$ has a multiplicative inverse modulo k, because we are doing these operations in $\mathbb{Z}_k$. Finding the determinant of A' is straightforward, as it is the product of the diagonal entries.

$$\det(A') = \det(A) = -1 \prod_{i=1}^{m-1} \ell_i$$

This product only has a multiplicative inverse modulo k when each of the factors are co-prime with k, so we can express the condition that A is invertible as:

$$\gcd(\ell_i, k) = 1$$
$$\text{for } 1 \leq i < m \text{ and } \gcd(-1, k) = 1$$

Note that the statement $\gcd(-1, k) = 1$ is trivially true for all k, so we can ignore it. This concludes our proof. □

3 Main Theorem

Now we have all the tools we need in order to show when threshold graphs are k-AW.

Theorem. *Any threshold graph G, with associated bit string B is k-AW if and only if for all $1 \leq i < m$, $\gcd(\ell_i, k) = 1$, where ℓ_i is the number of 0s after the i-th 1 in B.*

Proof. Assume that G is k-AW. By definition this means that every initial labeling τ is k-winnable. Since τ is k-winnable, the system of linear equations in Lemma 3 has a solution. This system of equations has a solution when the matrix it creates is invertible, and in Lemma 4 we find that the matrix is invertible if $\gcd(\ell_i, k) = 1$ for all $1 \leq i < m$. Therefore, when $\gcd(\ell_i, k) = 1$ for all $1 \leq i < m$, any threshold graph is G is k-AW.

Assume $\gcd(\ell_i, k) = 1$ for all $1 \leq i < m$ and τ is an arbitrary initial labeling for G. If $\gcd(\ell_i, k) = 1$ for all $1 \leq i < m$, then Lemma 4 states the matrix from Lemma 4 is invertible. When the Lemma 4 matrix is invertible, the system of linear equations in Lemma 3 has a solution. As shown in Lemma 3, when the system of linear equations has a solution, the initial labeling τ is k-winnable. Since τ is an arbitrary initial labeling, any threshold graph G must be k-AW when $\gcd(\ell_i, k) = 1$ for all $1 \leq i < m$. This concludes the proof.

□

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