Automated Conjecture Making: Domination on Planar Graphs

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Conjectures in Mathematics

Mathematicians come up with conjectures. They either prove, disprove, or pose these conjectures to other mathematicians to attempt to answer. This is how a lot of mathematical research questions are generated.

- Conjectures are usually generated by humans.
  - They usually require some intuition, or
  - they require working through lots of examples to generate some intuition that may lead to a conjecture
- However computers can also generate conjectures!
  - This is helpful because computers can do repetitive tasks quickly.
  - Using computers helps automate the process of looking at lots of examples. As a result the software can generate conjectures that as humans we may not have thought of yet.
- We use a software package developed by Craig Larson and Nico VanCleemput for our investigation. It is what we refer to as Automated Conjecture Making.

What is a Graph?

A graph G

- is made up of dots called **vertices** and
- has lines between any two dots called **edges**
- is **planar** if it can be drawn so that the edges only intersect at the vertices. See Figure 1a and Figure 1b in this and next slide.

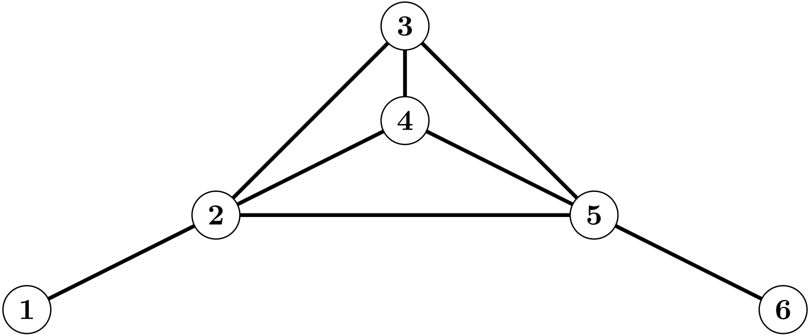

Figure: 1a Example of a Planar Graph

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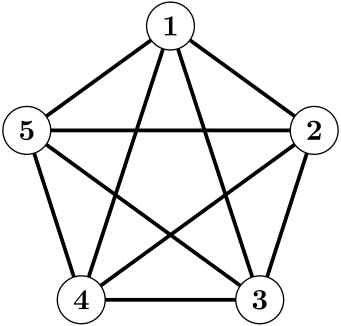

Figure: 1b Example of a Non-Planar Graph

Order, Size, and Max Degree of a Graph

For a graph G,

- its **order** is the number of vertices that it contains. The order of the graph in Figure 1a is 6.
- its **size** is the number of edges that it contains. The size of the graph in Figure 1a is 8.
- two vertices are **adjacent** provided that they have an edge drawn between them. In the graph in Figure 1a, we have vertex 2 and vertex 4 adjacent because we have a line or edge drawn between them.
- the **degree** of a vertex v in this graph is the number of vertices v is adjacent to. If we look at vertex 3 of the graph in Figure 1a, we see that it is adjacent to three other vertices. As a result it has degree 3.
- the **max degree** $\Delta(G)$ is the maximum degree over all the vertices in this graph. In Figure 1a, we can see that both vertex 2 and 5 have the largest degree of 4.

What is a Dominating Set on a Graph?

For a graph G,

- the **distance** between two vertices u and v is the shortest path between u and v.
- the **eccentricity** of a vertex v in a graph is the maximum distance possible starting at v. For example on the graph in Figure 1a, we find that if we start at vertex 2, the longest distance is from 2 to 6.
- the **radius** of the graph is the minimum eccentricity among all of its vertices. For the graph in Figure 1a, the radius is 2.
- the **diameter** of the graph is the maximum eccentricity among all of its vertices. For the graph in Figure 1a, the diameter is 3.

Isomorphisms and Invariants

Two graphs G and H

- are the "same" or **isomorphic** if we can move the vertices in one graph G around so that it looks like the other graph H.
- have special numerical properties called **invariants** that are the same for both.

Note

- Think of an invariant as a numerical property that both G and H have equal, in other words the values don't vary. As an example, two graphs being isomorphic means that they share for the same radius, or the same maximum degree.
- **Warning**, just because two graphs share any one specific invariant, does not mean that they are the same graph, only that they could be.
- The reason we care about invariants is because once we prove something about one specific type of graph, we can say the same thing about another "similar" or isomorphic graph.

Dominating Sets - Planar Graphs

A dominating set of a graph G

- is a set S where the vertices of G are either in S or are adjacent to at least one vertex in S.
- is not unique. This just means that a graph can have different dominating sets (See Figure 2a and 2b below and the next slide).

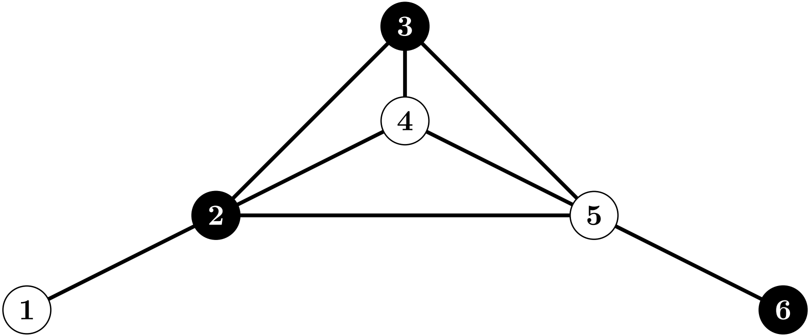

Figure: 2a Dominating Set $S_1 = \{2, 3, 6\}$

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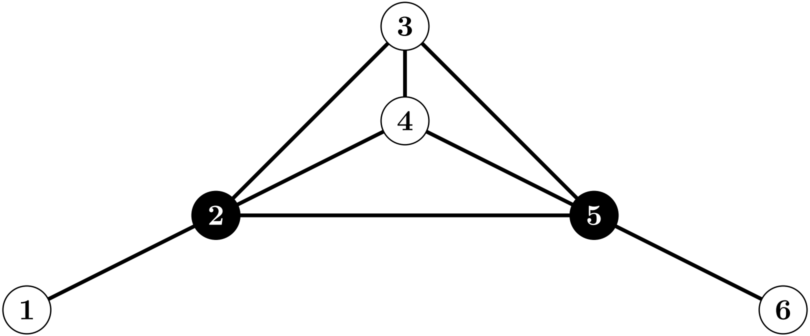

Figure: 2b Dominating Set $S_2 = \{2, 5\}$.

Domination Number

The domination number $\gamma(G)$ of a graph G

- is the smallest number of vertices of a dominating set S of the graph G.
- is only concerned with the smallest number of vertices. This means we can have multiple dominating sets with the smallest number of vertices.
- is the invariant we are most interested for this project.

Note

- If we look at Figure 2a, and 2b. We find that the graph in Figure 2a has 3 vertices in the dominating set, while the graph in Figure 2b has only 2 vertices in the dominating set.
- Try as we might, there is no way to select only one vertex to be a dominating set of the graphs in Figure 2a. So, the domination number for the graph in Figure 2a above is $\gamma(G) = 2$.

Running Conjecture - Input and Output

Sample Output

Here we have a sample output of conjectures generated by the software program that we are using. Specifically the software ran computational tests on the invariants of what we are calling planar graphs (see Figure 1a).

```
mainInvariant = basic_invariants.index(domination_number)
conj=conjecture(planar_graphs_5_8, basic_invariants, mainInvariant, precomputed=precomputedDictionary)
for c in conj:
    print c
```

```
    domination_number(x) <= 1/2*order(x)
    domination_number(x) <= radius(x)^2
    domination_number(x) <= 2*diameter(x) - 2
    domination_number(x) <= max_degree(x) + 1
    domination_number(x) <= -average_degree(x) + order(x)
    domination_number(x) <= -distinct_degrees(x) + order(x)
    domination_number(x) <= -max_degree(x) + order(x)
    domination_number(x) <= -radius(x) + size(x)
    domination_number(x) <= radius(x)^distinct_degrees(x)
    domination_number(x) <= diameter(x) + triangles_count(x)
    domination_number(x) <= maximum(triangles_count(x), diameter(x))
    domination_number(x) <= maximum(diameter(x), card_center(x))
    domination_number(x) <= maximum(diameter(x), card_periphery(x))
    domination_number(x) <= card_center(x) + card_periphery(x)
    domination_number(x) <= card_center(x) + distinct_degrees(x)
    domination_number(x) <= maximum(girth(x), max_degree(x))
    domination_number(x) <= floor(1/2*order(x))
```

Figure 3: Output conjectures on planar graphs.

Known Theory

A literature search was the first thing we did after getting a list of conjectures. Below are two examples of conjectures that turned out to be well-known theorems. They are the 1st and 7th theorems from Figure 3 above.

Theorem (Øystein Ore)

If a graph G of order n has no isolated vertices, then

$$\gamma(G) \leq \frac{n}{2}.$$

Theorem (Claude Berge)

For any graph G of order n,

$$\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G).$$

Finding a Counter Example

Conjecture

If G is a planar graph with order $n \geq 5$, then $\gamma(G) \leq$ diam$(G) + 1$.

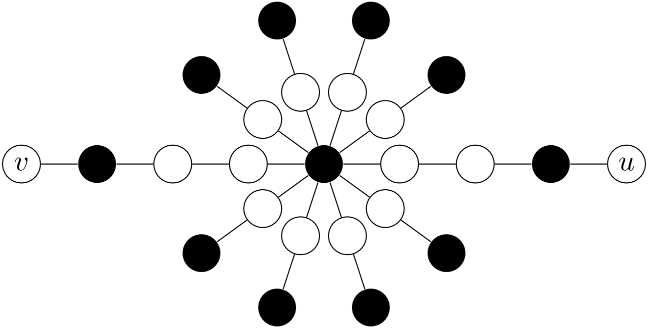

Figure 3: Counterexample Graph G; diam$(G) = 8$, $\gamma(G) = 11$.

Here we have an example where the conjecture is false because we can find a planar graph with order $n \geq 5$ such that the $\gamma(G) \geq \text{diam}(G) + 1$.

Proving Size-Radius Bound

Proving a conjecture is the next thing we try since we found nothing in literature and we could not find a counterexample. Here is an example of a conjecture that we proved, and converted to a Theorem statement!

Theorem

Let G be a planar graph of order $n \geq 5$, size m and radius r. Then

$$\gamma(G) \leq m - r.$$

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