NON-ATTACKING QUEEN AND ROOK PLACEMENTS

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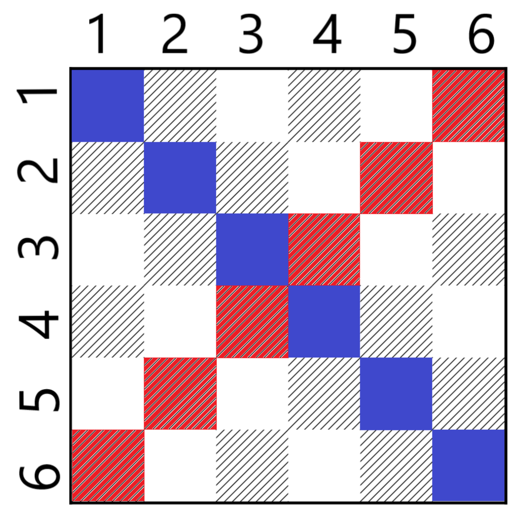
DEFINITIONS AND NOTATION

$P_n(q, r)$: The number of non-attacking placements, on an $n \times n$ board, of q queens and r rooks.

Valid Placement: A collection of pieces on a board in specific positions such that none of the pieces can attack each other.

Shadow: The cells used in a placement without regard to the specific pieces used.



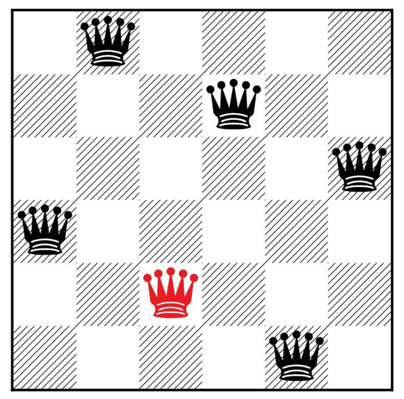
Blue Cells: Main Diagonal
Red Cells: Skew Diagonal

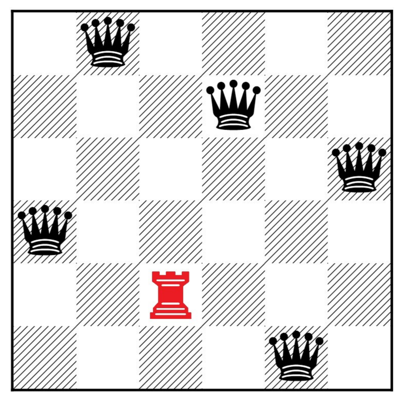
1 ROOK

$$P_n(n-1, 1) = n \cdot P_n(n, 0)$$

A shadow of $n-1$ queens and 1 rook = a shadow of n queens because if none of the queens can attack the rook, then a queen in that place could not attack the other queens.

We can get all the placements of $n-1$ queens and 1 rook by using n queens placements and replacing each of the queens with a rook one at a time.





LOWER BOUNDS

Table 1: Total Number of Valid Placements $P_n(n-r, r)$

Board Size	Number of Rooks Placed: 0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	0	0	2								
3	0	0	4	6							
4	2	8	20	24	24						
5	10	50	100	132	168	120					
6	4	24	120	432	996	1184	720				
7	40	280	992	2504	5288	8780	9668	5040			
8	92	736	3464	11416	28860	59472	92632	88488	40320		
9	352	3168	16048	58792	172992	416088	780488	1049940	894964	362880	
10	724	7240	46984	232264	900864	2710048	6206236	10611384	12951636	9944400	3628800

Let $L_n(q, r)$ be defined as the number of valid placements of q queens and r rooks which have the same shadow as a placement with more than q queens.

Theorem.

$$L_n(q, r) = \sum_{j=q+1}^{n} \binom{j}{j-q} \cdot (P_n(j, n-j) - L_n(j, n-j))$$

and $L_n(n, 0) = 0$.

Table 2: Lower Bounds on the Number of Valid Placements $L_n(n-r, r)$

Board Size	Number of Rooks Placed: 0	1	2	3	4	5	6	7	8	9	10
0	0										
1	0	1									
2	0	0	0								
3	0	0	0	4							
4	0	8	12	24	10						
5	0	50	100	100	114	96					
6	0	24	60	320	756	1080	520				
7	0	280	840	2160	4296	7400	8152	4424			
8	0	736	2576	10480	24440	49952	80336	81576	35064		
9	0	3168	12672	53200	148800	367352	696432	953952	820400	336856	
10	0	7240	32580	202112	766416	2428952	5637840	9722328	11974308	9441112	3398892

Table 3: Quotients between Lower Bounds and Actual Values $L_n(n-r, r)/P_n(n-r, r)$

Board Size	Number of Rooks Placed: 0	1	2	3	4	5	6	7	8	9	10	Averages
0	0											0
1	0	1										0.500
2	0	1	0									0.333
3	0	1	0	0.667								0.417
4	0	1	0.600	1	0.417							0.603
5	0	1	1	0.758	0.679	0.800						0.706
6	0	1	0.500	0.741	0.759	0.912	0.722					0.662
7	0	1	0.847	0.863	0.812	0.843	0.843	0.878				0.761
8	0	1	0.744	0.918	0.847	0.840	0.867	0.922	0.870			0.779
9	0	1	0.790	0.905	0.860	0.883	0.892	0.909	0.917	0.928		0.808
10	0	1	0.693	0.870	0.851	0.896	0.908	0.916	0.925	0.949	0.937	0.813

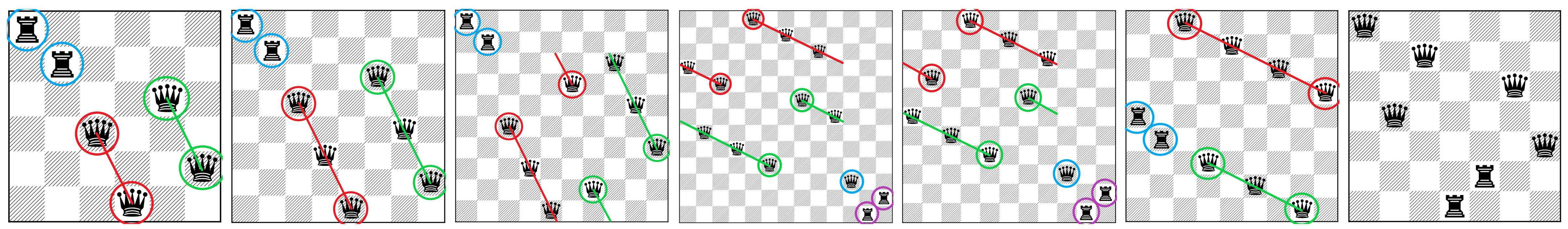
2 ROOK PROOF

$P_5(3, 2)$ was predicted exactly with our lower bound calculation. This was unexpected.

We have shown that that will never happen again for $n-2$ queens with 2 rooks.

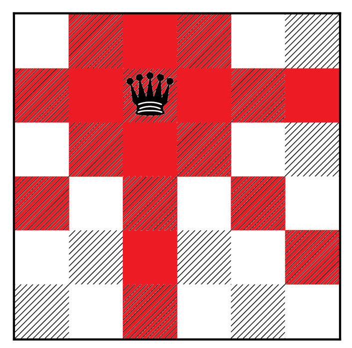
Placement algorithms for $n > 5$ to force 2 diagonal rooks is shown below.

These placements' shadows cannot match a shadow of an n queens placement so we know that there exists at least 1 placement which will not be counted by our lower bound.

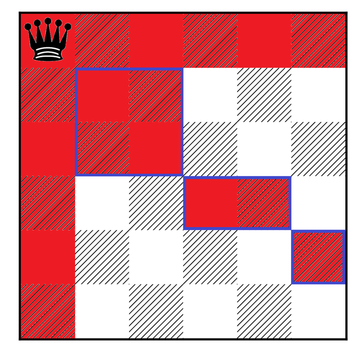


$n \pmod 6$: $\equiv 0$ $\equiv 2$ $\equiv 4$ $\equiv 1$ $\equiv 5$ $\equiv 3$ $n = 7$

1 QUEEN



Row and column swapping $\Longrightarrow$



All placements of a single queen will form the patterns of attacked squares highlighted above.

These correspond to the rook polynomials $(1 + 4x + 2x^2)$, $(1 + 2x)$, and $(1 + x)$.

The rook polynomial after a queen is placed will take the form: $(1 + 4x + 2x^2)^a(1 + 2x)^b(1 + x)^c$.

Rook polynomial coefficient on x^k: $r(k, a, b, c) = \sum_{p=0}^{a} \sum_{q=0}^{a-p} \sum_{s=0}^{b} 4^p \binom{a}{p} 2^q \binom{a-p}{q} 2^s \binom{b}{s} \binom{c}{k-p-2q-s}$

Total number of non-attacking placements: $P_n(1, n-1) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=0}^{n-1} (-1)^k \cdot (n-1-k)! \cdot r(k, a, b, c)$

REFERENCES

[1] Jordan Bell and Brett Stevens. A survey of known results and research areas for n-queens. *Discrete Mathematics*, 309(1):1–31, 2009.

[2] John G. Michaels and Kenneth H. Rosen. Arrangements with Forbidden Positions. In *Applications of Discrete Mathematics*, chapter 9, pages 158–173. McGraw-Hill College, 1991.

FUTURE RESEARCH

- The other perfect prediction
- 2 queens
- Connection to permutations

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