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Fractal Geometry and its Correlation to the Efficiency of Biological Structures

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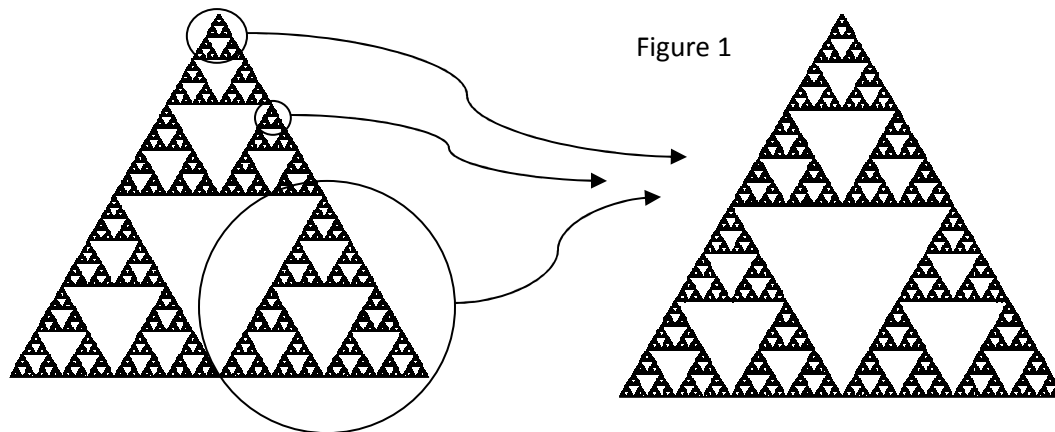
Abstract

Fractal geometry is a branch of mathematics that deals with, on a basic level, repeating geometric patterns that maintain the same level of complexity for any scale used to observe them. By observing the many facets of fractal geometry, including fractal dimension and points within fractal sets, we can draw comparisons to real-world phenomena. Fractal geometry appears in nature and biological systems where efficiency is needed, such as the surface area of the brain or lungs, or the branching patterns of leaves on a tree. This report examines the fractal geometry that exists within these biological systems, and how it relates to their overall output and efficiency. We will be gathering our information from print and online sources, from both mathematical and biological perspectives. From this project, we hope to gain a better understanding of the many ways mathematics permeates our universe, and how these correlations help to explain the seemingly infinite complexity of life.

Introduction

How can we use simple geometry to describe something we see every day? Take, for instance, a cloud. We can't, by any means, say that any cloud is perfectly spherical or ellipsoidal; we also can't begin to use shapes with sharp edges like triangles or squares to describe the wisps of vapor that make up a cloud's shape. The task of describing some things that occur in nature – things that have seemingly indescribable complexity, like a cloud or a snowflake – is accomplished with a particular kind of geometry that fits these unique needs of precision and flexibility. This is called fractal geometry.

The word “fractal” didn't even exist until a mathematician by the name of Benoit B. Mandelbrot came up with it in 1975. The root of the word comes from the Latin *fractus*, which is used in English words *fracture* and *fraction*. The best way to describe a fractal is to consider its complexity; fractals are shapes that maintain the same complexity no matter how much you “zoom in”, or narrow your focus. A good example of this is the Sierpinski Triangle, shown below in Figure 1. As you can see, we can zoom in on any piece of the triangle and end up with the Sierpinski Triangle once again! The level of complexity of the shape is maintained no matter how small a piece is that we look at.



One way to think about constructing fractals is using iterations. For the Sierpinski Triangle, we start with simply an equilateral triangle, which will act as the outline of the entire fractal. Then our next iteration involves connecting the midpoints of each side of our triangle, which forms three interior triangles. We then repeat this step with the three new triangles, and this is another iteration. Just repeat the iterations *ad infinitum*, and the full Sierpinski Triangle is formed. This concept of self-similarity and recursive patterns is a huge basis for thinking with fractals.

Let's put some numbers and calculations to all this fractal talk. Think about typical Euclidean dimensions, i.e. a point has dimension 0, a line 1, a grid 2, and a cube 3. How many dimensions does the Sierpinski Triangle have? It certainly isn't 3-dimensional, but it doesn't completely fill a 2-D space (look at the empty upside-down triangle in the middle!), so we can't say it's 2-dimensional either. And it's definitely not one-dimensional, because we can trace more than one line, so what is the dimensional value of the shape? Similar problems arise when we consider something like a ball of yarn. James Gleick, author of a book on chaos, describes a similar situation with a ball of twine, saying, "twine turns to three-dimensional columns, the columns resolve themselves into one-dimensional fibers, the solid material dissolves into zero-dimensional points."¹ The solution to this problem is to think in terms of a dimension between integer values: a fractional dimension, or fractal dimension.

We can use this idea of fractal dimension to delve deeper into the finer concepts of fractals. For example, the fractal dimension of the Sierpinski Triangle above is approximately 1.585. This tells us that

¹ Gleick, James. "Chaos." Viking Penguin Inc. 1987. p.97.

the shape doesn't completely fill the two-dimensional space it encloses, but it does a better job than a line. In general, the closer the fractal dimension is to an integer value, the closer that fractal is to filling that integer's dimension. The closer the fractal dimension is to being exactly between two integers, the generally more broken up and jagged the fractal looks. To demonstrate this, consider the Koch Snowflake, below in Figure 2. This fractal has a dimension of about 1.2619. So this tells us the Sierpinski Triangle is more space-filling than the Koch curve, and that the Koch curve looks more like a line than the Sierpinski Triangle.

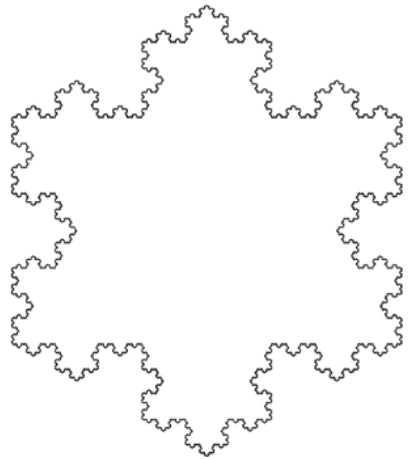


Figure 2

You might be thinking that we're pulling these fractal dimension values out of the air, but there are actually many ways to calculate fractal dimension. This is very advantageous because some situations give access to only certain types of data, like point values, or pictures on a screen. For the fractals above, we can use a method that involves observing how the fractals scale as we "zoom in," and how the shapes change as we proceed from iteration to iteration. We calculate the fractal dimension by looking at how many copies of a previous iteration exist in the next iteration after, and call this C . Then we look at how each of those copies scales down in size, take the reciprocal of that rate, and call this F . Then we take the log of these two values, take their ratio, and that's the fractal dimension.

But equations can be much easier to read than directions. For this method, the equation for the fractal dimension D is

$$D = \frac{\log(C)}{\log(F)}.$$

We can tell that for the Sierpinski Triangle, there are three new copies of the previous iteration in the next iteration after (start with the outline triangle, then the next iteration divides it into three triangles), and each scales down by a factor of one half after each iteration (side lengths of the triangles are cut in half at

each step), and the reciprocal of this scale factor is 2. We can now compute that $\frac{\log(3)}{\log(2)} \approx 1.585$. We can use this method for any fractal whose iterations scale at a constant rate that we can observe. Other methods involve measuring the total length of a curve and comparing that value to the length of the ruler used, which is the method Mandelbrot used to find the fractal dimension of the coast of Great Britain.² Another method, called the box-counting method, uses grid squares instead of ruler sticks. And there are other methods still.

Now we should not (and will not) focus too intently on fractals that exist purely in the minds of mathematicians. In fact, there are many diverse things that exist in nature that exhibit patterns of fractals. Granted, these structures aren't "perfect" fractals – they don't exhibit truly infinite levels of complexity, but we can definitely see the relationship. Just look at the head of a broccoflower (Figure 3), the veins on

your arm, the branches on a tree, or the paths of open space through a sponge. Some of these fractals are very opportunely found, because the things they are a part of require an increasing level of complexity, or need to completely or partially fill the space they take up. Consider a component of a cell that needs to maximize its surface area for diffusing nutrients across its membrane, but keep its volume to a minimum. Or perhaps we have a network of very thin (practically one-dimensional) veins that need to fill a 2-

Figure 3



dimensional space, like on the fin of a fish. It turns out that the closer these biological structures are to exhibiting fractal geometry, the more efficiently they can perform. The recursive nature of fractal geometry is embedded in many biological systems naturally by adaptation, and by examining this relationship between mathematics and biology we can get a better grasp of the role mathematics plays in the seemingly infinite complexity of life.

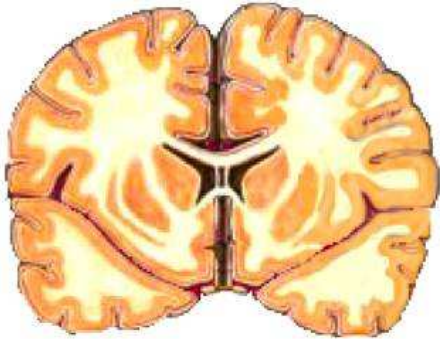
Some examples in Biology

Not surprisingly, a number of studies have already been completed concerning fractal geometry and its relevance to biological subfields. Somewhat astonishing is the relationship fractals have with Alzheimer's disease. To get an idea of where this relationship comes from, we have an image of the cross

² Mandelbrot, Benoit B. "Fractal Geometry of Nature." Macmillan 1983. p.29

section of the brain below in Figure 5. We can see that the brain is composed of tissues that fold in on themselves at the edges and in towards the center of the entire structure.

Figure 5



The brain obviously has some fractal-like characteristics when viewed from this angle. It isn't a perfect fractal by any means (there are scales at which spatial complexity terminates), but that doesn't take away from the fact that it has some fractal properties. In fact, the entire brain, when viewed as a three-dimensional object, shows fractal characteristics that correspond to a dimension between 2

and 3. The average for a healthy person lies at around 2.4, according to a 2010 study.³ The connection this has to Alzheimer's disease was discussed in a study published in 2012. The results allowed the researchers to conclude that "[Fractal dimension] relates to functional connectivity and is a reliable and reproducible measure of complexity. [Fractal dimension] decreases as [Alzheimer's disease] progresses."⁴ Here we see evidence that fractal geometry has a definite association with positive mental health, and that the dimension of the brain may be an indicator of the presence of Alzheimer's disease, and perhaps others.

Another example from biology that fractals play a key role in is mammalian lung function. The way lungs work is through branching; our lungs are filled with tubes that divide again and again until they reach microscopic size. At this level, the ends of these pathways lead to alveoli, which are essentially clusters of sacs for air to enter. These alveoli line the terminal ends of the branching bronchioles (tiny lung tubes), waiting to receive fresh air we breathe in. The alveoli are covered in diffusive membranes that essentially switch the carbon dioxide with the oxygen as we breathe in. We can see from this that the most important aspect of respiration efficiency isn't in how big our alveolar sacs are, it's in how much surfaces of alveolar membranes we can fit into the space of our lungs, since the main way we get oxygen from breathing is by diffusion across the alveolar membrane. An abundance of alveolar membrane surface area is preferable to an abundance of volume within individual alveoli. In this sense, our lungs should ideally be maximized for interior surface area with a given volume restraint. What better of a

³ King, R.D. et al, "Fractal Dimension analysis of the cortical ribbon in mild Alzheimer's Disease." *Neuroimage*, vol. 53, Issue 2, 1 November 2010. p. 475

⁴ Warsi, Mohammed A. "Fractal Dimension Compared to Resting State Network Analysis of BOLD MRI in Alzheimer's Disease." *Biological Psychiatry*, vol. 71, Issue 8, Supplement. pp 170s. April 2012, Elsevier Inc.

model to accomplish this than a fractal model? Below we can more easily see how this relationship arises. On the left of Figure 6 is a picture of a typical lung structure and on the right is a computer-generated fractal using recursedrawing.com.

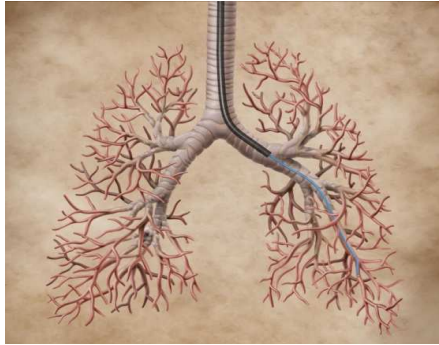
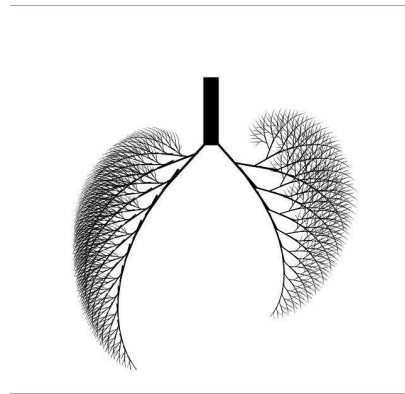


Figure 6



In fact, the fractal model has proved itself to be a great descriptor (if not the best at this point) of the airways of lungs. There have been multiple studies that have demonstrated and used this. Published in 1988, this paper⁵ examines many different methods for creating a fractal that fits just right with the boundary conditions imposed by the space of a lung. For example, we know the trachea is the singular tube that leads to the airway network of the lungs, so that was one constraint. Another obvious one was that the lung had a maximum volume limit. Some other variable conditions were the developmental environment, presence of diseases, and other growth limiting factors, like mutations, or even the difference of the angle at which the air-carrying tubes branch out. Another study found a way to better determine the presence of disease in lungs using fractal methods. The strength of the fractal approach is described towards the end of the paper, where the authors can conclude, "...bronchiolitis may indicate proximal disease, close to bronchi, and distal disease, which is also close to the alveolar ducts. This cannot be judged from 2D histological sections."⁶ It is becoming more and more evident that fractal geometry not only applies to these biological systems in the descriptive sense, but they also can give us insight into the condition of these systems.

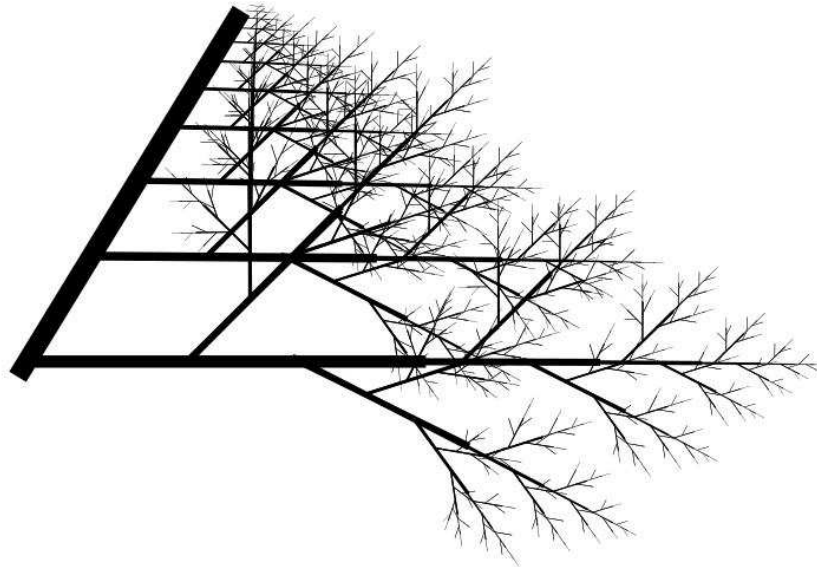
Mutation and Adaptation

⁵ Nelson, T. R. "Modeling of Lung Morphogenesis Using Fractal Geometries." IEEE Transactions on Medical Imaging, vol. 7, Issue 4, December 1988. p. 321-327.

⁶ Verbeken, E. K. "Classifying interstitial lung diseases in a fractal lung: a morphologist's view "anno Domini 2000"." European Respiratory Journal, vol. 18, Issue 32, Supplement, July 1 2001. pp. 111s.

Mutations occur inevitably in nature. They can lead to a vast number of different possible results, which can range from benign to tragic to inconsequential. In this section, we will use techniques of fractal geometry and dimension to analyze the results of a mutation that could easily take place, and compare this result to what we may have originally expected. First, we need to imagine the structure which will be undergoing the mutation. For this case, we'll analyze a branching network of neurons, possibly existing in an olfactory organ or a retina. The idea is that these branches will eventually terminate at some level of complexity, and that's where the neurons will be making contact with the membrane for sensory input. Figure 7 shows a computer-generated image of such a possible network, again created using recursivedrawing.com. Let's say this represents the neural network of a section of an animal's retina.

Figure 7



The pattern in this fractal is a little different from what we've been seeing. The scaling is non-uniform; that is, the branches don't all decrease in length and thickness at the same rate. An easy way to see this is by looking at the bottom. This (and the ones that branch off of it) isn't quite as long or thick as the middle branch. Since this is our standard model, our control, we can't really make any assumptions about it. All we can assume is that it accomplishes what it accomplishes at whatever its efficiency is. However, we can maybe get an idea of this efficiency by looking at the fractal dimension. Since scaling isn't uniform, we can't use

$$D = \frac{\log(C)}{\log(F)}$$

like before. This time, we have to use the box-counting method of finding fractal dimension. This involves overlaying the fractal with a grid, starting at first with a single square that covers the entire shape. Then the grid is subdivided so the squares inside have smaller lengths. The number of boxes needed to cover the fractal is compared to the side length of the boxes in this equation,

$$D = \lim_{\epsilon \rightarrow 0} \frac{\log(N_{\epsilon})}{\log\left(\frac{1}{\epsilon}\right)}$$

with N_{ϵ} being the number of boxes of side length ϵ needed to cover the fractal. Below in Table 1 the fractal dimension is approximated.

Table 1		
ϵ	N_{ϵ}	$\frac{\log(N_{\epsilon})}{\log\left(\frac{1}{\epsilon}\right)}$
1" = 1/6 units	16	1.547
1/2" = 1/12 units	50	1.574
1/3" = 1/18 units	94	1.572
1/4" = 1/24 units	150	1.577
1/6" = 1/36 units	328	1.617

So our data shows us that the structure has a fractal dimension of approximately 1.617.

Let's now introduce a mutation into the mix. Say the top branch's DNA coding has too many protein sequences and the branch ends up thickening in the process. Our new structure has changed slightly, and we can see that change below in Figure 8.

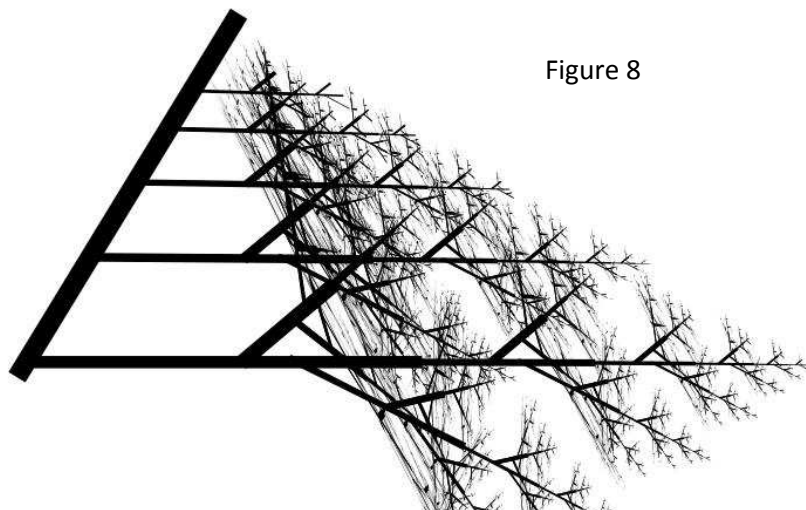


Figure 8

It looks like these nerve endings may be more concentrated, if we look to the middle section of the picture. But the question now is, does this structure do a better job at covering its section of area with neurons, or is the previous structure better? The process of actually counting the number of nerve endings would be nigh impossible, if not just repetitive, so to answer this we must look to fractal dimension. The fractal dimension of our new structure is estimated below in Table 2.

Table 2		
ϵ	N_ϵ	$\frac{\log(N_\epsilon)}{\log\left(\frac{1}{\epsilon}\right)}$
1" = 1/6 units	16	1.547
1/2" = 1/12 units	47	1.549
1/3" = 1/18 units	90	1.557
1/4" = 1/24 units	140	1.555
1/6" = 1/36 units	301	1.593

Our data shows us that the fractal dimension of our new structure is approximately 1.593. This, when compared with the fractal dimension of the previous, non-mutated structure, 1.617, tells us that the mutation is acting as a detriment to this animal's eyesight. Imagine if our mutation had yielded a higher fractal dimension – still less than 2, but more than 1.617. This could be achieved with additional branches or a different scale factor, perhaps, in the existing branches. This new mutation would increase the animal's eyesight, because even though the surface of the retina may not be completely covered with nerve endings, it would at least be more uniformly covered, yielding a higher-efficiency structure. This would allow the creature to see prey from further off, perhaps detect changes in motion more easily, or perhaps better night vision. Either way, the measurements of fractal dimension tell us unique things about these systems that would otherwise be unavailable.

Conclusions

The wonders of nature are deeply complex. Mathematics of many diverse subfields has been used all across history to explain and model both objects and phenomena that occur all around us at all

times. Fractal geometry is one of these subfields, and it provides us with unique tools that address infinite complexity in a way no other subfield has been able to emulate. Phenomena like scaling, self-repetition, and irregularity are all addressed in fractal geometry, all of which are embedded in genetic codes and growth patterns. By using the techniques of fractal geometry, and specifically fractal dimension, we can discern new things about the world around us, and better understand what goes into the seemingly infinite details of our universe.

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