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Gridline Graphs in Three and Higher Dimensions

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Introduction

In mathematics, a graph is a collection of vertices and edges represented by an image with vertices as points and edges as lines. While there are many different types of graphs in graph theory, we will focus on gridline graphs (also called *graphs of* $(0, 1)$ *matrices*, or rook's graphs).

Definition. *In two dimensions, a* ***gridline graph*** *is a graph whose vertices can be labeled by points in the plane in such a way that two vertices are adjacent, meaning they share an edge, if and only if they have one coordinate in common.*

In other words, in a two dimensional gridline graph, two vertices are not connected if they differ in both the x coordinate and the y coordinate. We can extend this idea to three and higher dimensions, as investigated by the GVSU Math REU in 2014 [2], by increasing the number of coordinates of each vertex and connecting them if they share at least one coordinate. Note that this definition is different than the generalization investigated by Peterson in [3] where an n-dimensional gridline graph is a graph which can be realized in n dimensions such that two vertices are adjacent if and only if they share $n-1$ coordinates. The motivation behind the definition we use is how the rook movement is defined in three and higher dimensions in [1]. Gridline graphs in two dimensions correspond to graphs representing how rooks attack in two dimensions. Correspondingly, our definition of gridline graphs in three and higher dimensions correspond to graphs representing how rooks attack

in three and higher dimensions. Peterson's definition of higher dimensional gridline graphs, on the other hand, assumes that rooks attack along lines in any dimension.

Gridline graphs in two dimensions can be characterized in terms of a minimum forbidden induced subgraph classification. An **induced subgraph** is a subset of the vertices of the original graph such that, if an edge connects two vertices in the original graph and those vertices are in the subgraph, then an edge will connect those vertices in the subgraph as well. A **forbidden induced subgraph** of a gridline graph is a minimal graph that is not an induced subgraph of a gridline graph. A **minimal forbidden induced graph** is a graph which is a forbidden induced graph none of whose subgraphs are forbidden.

It is known that in two dimensions, the diamond, claw, and odd hole with $n \geq 5$ are all forbidden induced subgraphs of gridline graphs [3]. Thus these graphs, and any graph with these as subgraphs, are not gridline graphs in two dimensions. Our goal in this project is to investigate minimal forbidden subgraphs in higher dimensions that generalize forbidden graphs in two and three dimensions.

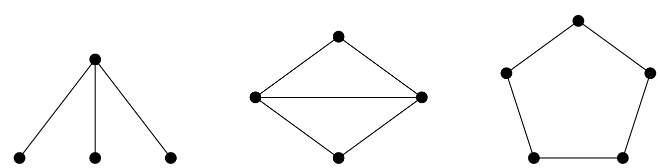

Figure 1: Forbidden gridline subgraphs in $n = 2$

Determining a Gridline Graph

Before exploring minimal forbidden induced subgraphs in higher dimensions, we first investigate how we determine if a graph is a gridline graph. We have two main methods to determine if a graph is a gridline graph: assigning coordinates and the coloring test.

Assigning Coordinates

To use the assigning coordinates method to determine if a graph is a gridline graph, each vertex has n coordinates in n dimensions. For example, a

graph in two dimensions has an x coordinate and a y coordinate. In three dimensions, vertex can be labeled with three coordinates, (x, y, z). We see that for a graph in n-dimensions, each vertex has n coordinates. We assign coordinates to the vertices where two vertices are adjacent if they share at least one coordinate. Note that two vertices with the same coordinate indicate that the two vertices are not distinct. This implies that when assigning coordinates, we cannot assign the same coordinate to two different vertices. It follows then that if two vertices do not share an edge, then they cannot have n identical coordinates. If all the vertices in the graph can have coordinates assigned in such a way that follows these rules, then the graph is a gridline graph.

As noted previously, in two dimensions the gridline graphs can be characterized as those that do not have the diamond, claw, or odd hole with $n \geq 5$ vertices as induced subgraphs. Let us consider a two dimensional forbidden subgraph in three dimensions to see if it is still forbidden. For example, suppose we have the diamond graph in three dimensions, as seen in Figure 2.

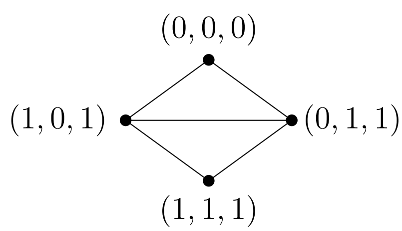

Figure 2: Diamond in $n = 3$

We will use the coordinate labeling approach to determine if this is a gridline graph. Without loss of generality, let us label the top vertex as $(0, 0, 0)$. Continuing clockwise, we see that the vertex to the right must have a zero as at least one of its coordinate since it shares an edge with vertex $(0, 0, 0)$. Thus, without loss of generality, suppose this vertex has coordinate $(0, 1, 1)$. To label the bottom vertex, we see that it must share at least one coordinate with the right-most vertex and none of the same coordinates as the very top middle vertex; thus labeling this as $(1, 1, 1)$ satisfies the connections. For the left-most vertex, we know that the vertex must have at least one of the same coordinates as $(0, 0, 0)$, $(1, 1, 1)$, and $(0, 0, 1)$; suppose $(1, 0, 0)$. Therefore, we

see that it is possible to label all the vertices of the diamond in such a way that follows our rules for labeling vertices, meaning this is a gridline graph in three dimensions. Similarly, the claw and the odd hole with $n \geq 5$ vertices can be shown to be gridline graphs in three dimensions.

Figure 3 shows an example of another gridline graph in three dimensions. Note that the vertices can be labeled in such a way that obeys the assigning coordinates method. This allows us to conclude that the graph is a gridline graph.

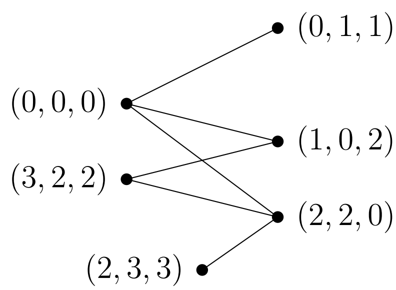

Figure 3: Gridline Graph in $n = 3$

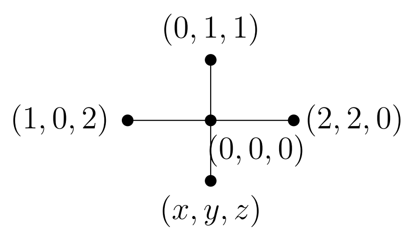

Figure 4: $K_{1,4}$ in $n = 3$

Figure 3 displays a gridline graph, as shown by the labeling. Now consider the graph $K_{1,4}$ shown in Figure 4. Let us see if this graph is a gridline graph in three dimensions. Without loss of generality, let us label the center hub as $(0, 0, 0)$. Then the very top middle vertex will have to share at least one coordinate with the center hub since these two vertices share an edge, suppose $(0, 1, 1)$. Similarly, the left-most vertex will need to have at least one of the same coordinates as the center hub vertex, but cannot share any of the same coordinates of the center top vertex since these two vertices do

not share an edge. Suppose we label the left-most vertex as $(1, 0, 2)$. Using a similar process, suppose the right-most vertex is assigned $(2, 2, 0)$. Note that the lower middle vertex must have at least one zero as one of the coordinates since this vertex shares an edge with the center hub. But note that the upper middle vertex has a zero as the first coordinate, the left-most vertex has a zero as the second coordinate, and the right-most vertex has a zero as the third coordinate, and since there are only three coordinates for each vertex, and since the lower middle coordinate does not share an edge with any vertex besides the middle hub, by the Pigeonhole Principle, the lower middle vertex would have to share a coordinate with another vertex besides the hub, which would disobey the assigning coordinates rule and thus $K_{1,4}$ is a forbidden subgraph in three dimensions. Using a similar proof, it is shown in [2] that in general, $K_{1,n}$ is a minimal forbidden subgraph of gridline graphs in n dimensions.

Figure 5 shows another example of a forbidden subgraph in three dimensions. Again, this graph can be generalized to a family of forbidden subgraphs in n dimensions of the form W_{2n+1} plus a pendant [2].

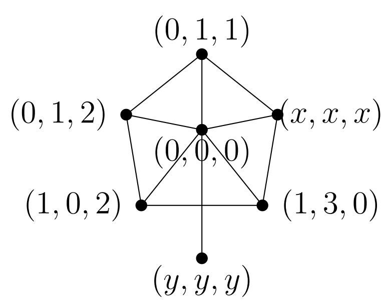

Figure 5: Wheel 5 plus Pendant in $n = 3$

Coloring Test

Another method to determine if a graph is a gridline graph is by using the coloring test. To determine if a graph is a gridline graph in n dimensions, we can use at most n colors to color the edges of the graphs. By assigning n colors to the n coordinates, we see that an edge is colored with the ith color if the two vertices incident with the edge share their ith coordinate. From the way we define gridline graphs, such a coloring then requires that at any

vertex, the edges colored in one color form a complete subgraph, i.e. a clique, including that vertex. It also follows that each edge can be colored with at most $n-1$ colors.

For example, consider the diamond in three dimensions, seen in Figure 2. Suppose we colored all the edges with the same first coordinate one color, suppose blue. Since vertices $(0,0,0)$ and $(0,1,1)$ share the same first coordinate, and since vertices $(1,0,1)$ and $(1,1,1)$ share the same first coordinate, we color the edges between these vertices blue. Next, we color the edges that connect vertices with the same second coordinate another color, suppose red. Then the edge between $(1,0,1)$ and $(0,0,0)$ and the edge between $(1,1,1)$ and $(0,1,1)$ are colored red. Lastly, we color the edges between vertices that share the same third coordinate, suppose green. Then the edge between $(1,0,1)$ and $(0,1,1)$, and the edge between $(1,0,1)$ and $(1,1,1)$, and the edge between $(0,1,1)$ and $(1,1,1)$ are colored green. Since we were able to color this three dimensional graph using three colors we are able to verify using the color test that the diamond is a gridline graph in three dimensions, as shown in Figure 6.

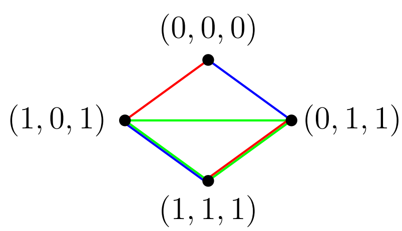

Figure 6: Diamond in $n=3$

We can apply the coloring test without assigning coordinates first. To do this, we must follow a few rules. As mentioned earlier, we can use at most n colors. Additionally, consider all edges that are the same color as a subgraph. Each colored subgraph must be a connected graph. Note that there can be multiple subgraphs of the same color, but each subgraph must be connected. Also, as shown in the example in Figure 6, an edge can be colored with more than one color. More precisely, a single edge can be colored with up to $n-1$ colors. Figure 7 shows how the graph from Figure 3 can be shown to be a gridline graph using the color test.

Assigning Coordinates in Higher Dimensions

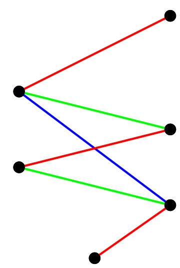

Figure 7: Colorable Gridline Graph in $n = 3$

Mireles and Volk [2] showed that $K_{3,3} + \mathit{edge}$, seen in Figure 8, is a forbidden induced subgraph in three dimensions, motivating us to consider if this generalized to $K_{n,n}+$ edge as a forbidden induced subgraph in $n > 3$ dimensions. The graph $K_{n,n}+$ edge is obtained by adding an edge between any two non-adjacent vertices in $K_{n,n}$.

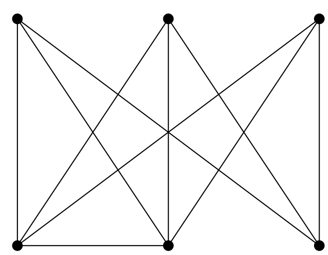

Figure 8: $K_{3,3}$+edge

Theorem 1. *For $n > 3$, $K_{n,n}+$ edge is a gridline graph in n dimensions.*

Proof. We will prove this by cases using the coordinate test. We will show that in each case, the vertices of $K_{n,n}+$ edge can be assigned n integer coordinates so that two vertices are adjacent if and only if at least one of their coordinates agree.

First we assume n is even. Then $n = 2k$, for some $k > 1$. Since we have $K_{n,n}+$ edge, we have $2n$ vertices. Since we have a bipartite graph with an extra edge, we have two sets of vertices, one set of vertices has no edges between them while the other set has exactly one edge between two vertices. Let

one set of vertices be $v_0, v_1, ...v_{n-1}$ and the other set $w_0, w_1, ...w_{n-1}$. Without loss of generality, assume the extra edge is between w_0 and w_1. We assign coordinates to v_i's where $0 \leq i \leq n-1$, so that $v_i = (i, i, \ldots, i)$ where i is repeated n times.

This leaves us to assign coordinates to the remaining vertices, w_i, $i = 0, \ldots, n-1$. Since each of these vertices connect to each v_i, to label each coordinate, we must use the integers from 0 to $n-1$ exactly once, creating a permutation of the numbers $0, \ldots, n-1$. Since we know w_0 and w_1 share an edge, these two vertices must share at least one coordinate, while satisfying the condition that each w_i connects to each v_i. Suppose w_0 and w_1 share $n-2$ coordinates, say the first $n-2$ coordinates without loss of generality. Then the last two coordinates of each of these vertices would be permuted. For example, let w_0 be assigned $(0, 1, 2, \ldots, n-3, n-2, n-1)$. Then w_1 be assigned $(0, 1, 2, \ldots, n-3, n-1, n-2)$. Since the last two coordinates are different, we know w_0 and w_1 are not the same vertex, and all integers from 0 to $n-1$ are included in their coordinates, assuring that both w_0 and w_1 connect to every v_i.

Since the coordinates of w_2 need to be a permutation of integers 0 through $(n-1)$, we can shift the permutation assigned to w_0 by two to the left to obtain another permutation that will have different entries from that of w_0. This permutation will also differ from the permutation assigned to w_1. To see this, note that the permutations assigned to w_0 and w_1 agree on the first $n-2$ entries. In the last two entries, w_0 has $n-2, n-1$, while w_1 has $n-1, n-2$, and w_2 has $0, 1$. We continue shifting the permutation by 1 each time when we move to the next vertex. Thus the coordinates assigned to w_i for $2 \leq i < n-1$ will be

$$(i \mod n, (i+1) \mod n, \ldots, (i+n-2) \mod n, (i+n-1) \mod n) .$$

In particular, the j-th coordinate of w_i is $(i + (j-1)) \mod n$.

For the j-th coordinate of w_{n-1} we use the following formula:

$$j\text{-th coordinates of } w_{n-1} = \begin{cases} (n-1) + (j-1) & \text{if } j \text{ is odd} \\ 1 + (j-1) & \text{if } j < n \text{ is even} \end{cases}$$

We now show that the coordinates of w_{n-1} do not overlap with coordinates of w_i for $i = 0, \ldots, n-2$. Recall that for $j \leq n-2$, the j-th coordinate of w_i is $(i + (j-1)) \mod n$, except for w_1 whose first $n-2$ coordinates are same as those of w_0. Therefore, for the discussion of the first $n-2$ coordinates, we remove w_1 from the w_i's we consider. For odd $j < n-1$, since $i \neq n-1$, $(i + (j-1)) \mod n \neq ((n-1) + (j-1)) \mod n$, which shows that for odd $j < n-1$, j-th coordinate of w_{n-1} is different than the j-th coordinate of w_i for all $i \neq n-1$. Similarly, for even $j < n$, since $i \neq 1$, $(i + (j-1)) \mod n \neq (1 + (j-1)) \mod n$, which shows that for even $j < n$, j-th coordinate of w_{n-1} is different than the j-th coordinate of w_i for all $i \neq n-1$.

Finally, we consider the last two coordinates. The next-to-last coordinate of w_i is $(i + (n-1) - 1) \mod n = i - 2$ for every i. Since the next-to-last coordinate of w_{n-1} is $((n-1) + (n-2)) \mod n = (n-1) - 2$, these coordinates are all different for every i. The last coordinate of w_i is $(i+n-1) \mod n$ for $i \neq 1$ and $n-2$ for $i = 1$. Since none of these entries are 0 and the last coordinate of w_{n-1}, the last coordinates are all different for w_i.

For the second case, we assume n is odd. Then $n = 2k+1$, for some $k > 1$. We proceed using the same methods as in the even case, until we assign all vertices except w_{n-1}. Before assigning w_{n-1}, for all w_j such that $2 \leq j \leq n-2$ we replace all coordinates that are 0 with a 1, and all coordinates that are 1 with a 0.

We find that the j-th coordinate of w_{n-1} can be found using the following formula.

$$j\text{-th coordinates of } w_{n-1} = \begin{cases} (n-1) & \text{if } j = 1 \\ (j-2) & \text{if } j < n \text{ is even} \\ j & \text{if } 1 < j < n \text{ is odd} \\ 1 & \text{if } j = n \end{cases}$$

Similar to the odd case, it can be shown that the coordinates of w_i are all permutations of the numbers $0 - (n-1)$ and coordinates of w_i for $i \geq 2$ are all different from each other and those of w_0, w_1. ■

Using the assigning coordinates algorithm explained in the proof above, Figure 9 shows the resulting coordinates for the $K_{n,n}$+ edge for $n = 6$ and $n = 7$, respectively.

w_0	0	1	2	3	4	5
w_1	0	1	2	3	5	4
w_2	2	3	4	5	0	1
w_3	3	4	5	0	1	2
w_4	4	5	0	1	2	3
w_5	5	2	1	4	3	0

w_0	0	1	2	3	4	5	6
w_1	0	1	2	3	4	6	5
w_2	2	3	4	5	6	1	0
w_3	3	4	5	6	1	0	2
w_4	4	5	6	1	0	2	3
w_5	5	6	1	0	2	3	4
w_6	6	0	3	2	5	4	1

Figure 9: Coordinates for $K_{n,n}+$ edge for $n = 6$ and $n = 7$

Further Exploration

In three dimensions, we know that the left graph shown in Figure 10 is a forbidden subgraph of a gridline graph. We explored possible generalizations of this graph into higher dimensions and found that in three dimensions, we have the center edge forming three triangles, and as we move into higher dimensions, we want this edge to form n triangles. Further, we need to have multiple vertices with degree $n + 1$ to force the coloring test to fail. Thus in four dimensions, we end up with the graph seen on the right in Figure 10.

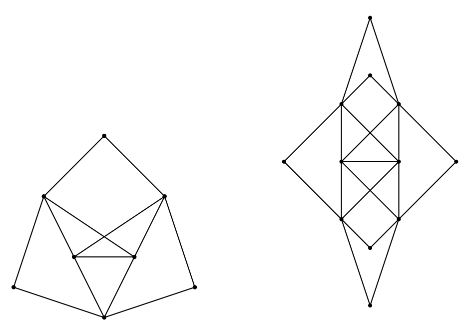

Figure 10: Forbidden subgraph in $n = 3$ and possible generalization in $n = 4$

This work served as an extension from the GVSU Math REU from 2014 where future research regarding the generalization of forbidden subgraphs in higher dimensions was suggested. Possibilities for extending this research include further generalizing two dimensional gridline graphs into higher dimensions and continuing the search to find forbidden induced subgraphs in n dimensions.

References

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[2] Josh Mireles and Adam Volk, “Gridline Graphs in Three and Higher Dimensions”, GVSU Math REU report, 2014.

[3] Dale Peterson, ”Gridline graphs: a review in two dimensions and an extension to higher dimensions”, *Discrete Applied Mathematics*, No. 126, 2003, pp. 223-239.