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Mathematical Modeling in Finance

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Abstract

Financial tools play an integral role in the day-to-day lives of individuals and businesses. Many of these tools use pre-defined formulas to calculate items such as loan payments, interest and capital structure components. These tools do not usually provide the flexibility needed when new parameters are introduced. By utilizing mathematical modeling, these standard formulas can be derived and even improved to provide the needed flexibility.

Modeling in Finance

Financial tools dictate the monetary flow between individuals and businesses. However, many of these tools use very basic, standard formulas and are limited to certain financial decisions. As a student at the Seidman College of Business, I felt I could use my knowledge of these basic tools to devise more appropriate measures that can include financial decisions beyond the standard scope. I’ve chosen to focus on improving financial tools in two areas: annuities and optimal capital structure. In order to gain a better understanding of the intricate mathematics behind the new models I planned to create, I’ve had the pleasure of collaborating with Professor Firas Hindeleh of the GVSU Mathematics Department.

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Chapter 1 – Annuities

An annuity is a series of equal payments made at equal intervals where the final balance is \$0. An annuity is a blanket term that includes any financial tool that makes equal payments at equal intervals, such as car loan payments.

For example, this table represents a three-year annuity; interest is added, and a payment is made once per year. There are two types of annuities: an ordinary annuity and an annuity due. An ordinary annuity is when payments are made at the end of a period, after interest is calculated. An annuity due is when payments are made at the beginning of a period, before interest is calculated. During the modeling process, I used ordinary annuities and later followed the same process for annuity dues.

Year	*Beg. Bal.*	*Interest*	*Distrib.*	*End. Bal.*
1	\$10,000.00	\$500.00	\$3,672.09	\$6,827.91
2	\$6,827.91	\$341.40	\$3,672.09	\$3,497.22
3	\$3,497.22	\$174.86	\$3,672.09	\$0.00

Throughout each example to follow, a set of base variables will be used. "P_0" will represent the initial amount invested, called the principal. "r" will represent the annual interest rate given by a creditor, "k" will represent the number of times interest compounds per year, "T" will represent the duration of the annuity, "d" will represent the distribution dollar amount, and "A" will represent the growth factor for interest.

- P_0 - Principal Amount
- r – Annual Interest Rate
- k – Number of Compounding Periods per Year
- T – Duration (Time)
- d – Payment Amount
- $A = \left(1 + \frac{r}{k}\right)$ – Growth Factor

Section 1.1: Basic Model

We created a basic model of an ordinary annuity using only the base variables. The first step we needed to take was to figure out how to get from one balance to the next; from P_0 to P_1, P_1 to P_2 and so on, which can be represented by $P_n = AP_{n-1} - d$.

The next step was to restate P_{n-1} in terms of the known variable P_0, which can be represented by $P_{n-1} = A^{n-1} * P_0 - d * \frac{1-A^{n-1}}{1-A}$. This was achieved using the concept of a finite geometric sum. Now that we have a formula to find the balance of a period in terms of known variables, we can plug a term in for "n" that gives us the final balance – which we know is $0 – to find the formula for the previously unknown "d", represented by $d = \frac{P_0\left(\frac{r}{k}\right)}{\left(1-\left(1+\frac{r}{k}\right)^{-tk}\right)}$.

The final step is to come up with the formula for total interest. Since we know that $P_{n-1}\left(\frac{r}{k}\right)$ gives us the interest for P_n, we can state the formula for total interest as $Interest_{Total} = \sum_{n=1}^{T} P_{n-1}\left(\frac{r}{k}\right)$. Since we already have P_{n-1} stated in terms of P_0, we can restate the formula for total interest as $Interest_{Total} = P_0 \sum_{n=1}^{t}(A^{n-1}) - \frac{d}{1-A}\sum_{n=1}^{t}(1 - A^{n-1})$. After finding the sums of each term, the simplified formula for total interest can be stated as $I_{Total} = (1 - A^T)\left(\frac{d}{A-1} - P_0\right) + Td$.

Section 1.2: Modified Ordinary Annuity – 1st Scenario

After testing the first model until I was certain it was correct, I began creating a second model. This model was to be used if payments were made less often than interest is calculated. For example, if interest is calculated per month for a car loan, but payments are made every two months, then this model would be used instead. In order to include this new change in the model, I created two new variables; "w", which represents the number of periods between payments, and "v", which represents the total number of payments. Upon rigorous testing of this model, it was determined that "w" must be an integer greater than or equal to 1, and that $k * T = w * v$.

Displayed below is the model that was built for the decision criteria. The top left input table displays the base variables and their corresponding values; the principal is $10,000 , the annual interest rate is 5%, the duration of the annuity is 10 years, interest compounds once per year or “annually”, and the growth factor “A” is 1.05 .

Po	$10,000.00
r	5.00%
T	10.00
k	1.00
A	1.05

Year	*Beg. Bal.*	*Interest*	*Distrib.*	*End. Bal.*
1	$10,000.00	$500.00	$1,295.05	$9,204.95
2	$9,204.95	$460.25	$1,295.05	$8,370.16
3	$8,370.16	$418.51	$1,295.05	$7,493.62
4	$7,493.62	$374.68	$1,295.05	$6,573.25
5	$6,573.25	$328.66	$1,295.05	$5,606.87
6	$5,606.87	$280.34	$1,295.05	$4,592.17
7	$4,592.17	$229.61	$1,295.05	$3,526.73
8	$3,526.73	$176.34	$1,295.05	$2,408.02
9	$2,408.02	$120.40	$1,295.05	$1,233.38
10	$1,233.38	$61.67	$1,295.05	$0.00
			w	1.00
			v	10.00
			d	$1,295.05

Year	*Beg. Bal.*	*Interest*	*Distrib.*	*End. Bal.*
1	$10,000.00	$500.00		$10,500.00
2	$10,500.00	$525.00	$2,654.84	$8,370.16
3	$8,370.16	$418.51		$8,788.66
4	$8,788.66	$439.43	$2,654.84	$6,573.25
5	$6,573.25	$328.66		$6,901.92
6	$6,901.92	$345.10	$2,654.84	$4,592.17
7	$4,592.17	$229.61		$4,821.78
8	$4,821.78	$241.09	$2,654.84	$2,408.02
9	$2,408.02	$120.40		$2,528.42
10	$2,528.42	$126.42	$2,654.84	$0.00
			w	2.00
			v	5.00
			d	$2,654.84

Year	*Beg. Bal.*	*Interest*	*Distrib.*	*End. Bal.*
1	$10,000.00	$500.00		$10,500.00
2	$10,500.00	$525.00		$11,025.00
3	$11,025.00	$551.25	$4,435.25	$7,141.00
4	$7,141.00	$357.05		$7,498.04
5	$7,498.04	$374.90		$7,872.95
6	$7,872.95	$393.65	$4,435.25	$3,831.34
7	$3,831.34	$191.57		$4,022.91
8	$4,022.91	$201.15		$4,224.05
9	$4,224.05	$211.20	$4,435.25	$0.00
			w	3.00
			v	3.00
			d	$4,435.25

The three data tables represent three different scenarios; the first is when payments are made every year. The second is when payments are made every two years, and the third is when payments are made every three years. This “width” between payments is “w”, and “v” is the total number of payments. As the number of payments decreases, the payment amount “d” increases.

This model requires similar steps to those of the basic model. The first step is to figure out how to get from once balance to the next, however, instead of finding the balance of each year, we will only find the ending balances for the years that make payments. Using this example, the balance after the first payment can be modeled with the formula $P_3 = P_w = P_0A^3 - d$. The balance after the second payment will be $P_6 = P_{2w} = P_3A^3 - d$. The balance for the final payment, which we can call P_{vw} since $v = 3$, will be $P_9 = P_{3w} = P_{vw} = P_6A^3 - d$. The next step in the process is to restate P_{vw} in terms of P_0, since it is known. The final balance can be represented as $P_{vw} = P_0A^{vw} - d\left(\frac{1-A^{vw}}{1-A^w}\right)$. Since we know that the final balance of a loan will be

Year	*Beg. Bal.*	*Interest*	*Distrib.*	*End. Bal.*
1	$10,000.00	$500.00		$10,500.00
2	$10,500.00	$525.00		$11,025.00
3	$11,025.00	$551.25	$4,435.25	$7,141.00
4	$7,141.00	$357.05		$7,498.04
5	$7,498.04	$374.90		$7,872.95
6	$7,872.95	$393.65	$4,435.25	$3,831.34
7	$3,831.34	$191.57		$4,022.91
8	$4,022.91	$201.15		$4,224.05
9	$4,224.05	$211.20	$4,435.25	$0.00
			w	3.00
			v	3.00
			d	$4,435.25

\$0, we can set this formula equal to zero in order to determine the formula for the unknown "d", which can be represented as $d = P_0 A^{kT} \left(\frac{1-A^w}{1-A^{kT}}\right)$.

The final step is to derive the formula for the total interest paid. Since we know that P_0 plus interest minus the payment equals P_w, we can determine that the interest for each period is the balance minus the preceding balance plus the payment. We can represent the summation of the interest from each period using a telescoping sum.

$$(P_w - P_0 + d) + (P_{2w} - P_w + d) + (P_{3w} - P_{2w} + d) \dots + \left(P_{vw} - P_{(v-1)w} + d\right)$$

The first "P" term of each period is cancelled out by the second "P" term of the following period. This leaves us with $P_{vw} - P_0 + vd$. We also know that the final balance P_{vw} must be 0, so the final expression is $vd - P_0$. The formula for total interest can be achieved by re-substituting "d" and can be represented as $I_{Total} = P_0 \left(vA^{kT} \left(\frac{A^w - 1}{A^{kT} - 1}\right) - 1\right)$.

Section 1.3: Modified Ordinary Annuity – 2nd Scenario

Now that I was able to calculate the total interest paid if payments were less frequent than interest accruals, my goal was to model a situation when payments are more frequent than interest accruals. For this scenario, "w" now represents the number of periods between interest accruals instead of payments and must be an integer greater than 1. A new variable is also introduced, "u", which represents the total number of interest accruals and $u = k * T = w * v$.

Displayed is a table that represents this scenario. The principal is 10,000, the annual interest rate is 5%, interest compounds four times per year or “quarterly”, the duration of the annuity is three years or 36 months, and the growth factor “A” is 1.0125. Payments are made every month, but interest accrues quarterly, which is every three months. This three-month period is “w”, and the total amount of interest accruals is “u”.

Po	$10,000.00
r	5.00%
k	4.00
T	3.00
d	$298.37
w	3.00
v	4.00
A	1.0125
u	12.00

Month	*Beg. Bal.*	*Interest*	*Payment*	*End. Bal.*
1	$10,000.00		$298.37	$9,701.63
2	$9,701.63		$298.37	$9,403.25
3	$9,403.25	$117.54	$298.37	$9,222.42
4	$9,222.42		$298.37	$8,924.04
5	$8,924.04		$298.37	$8,625.67
6	$8,625.67	$107.82	$298.37	$8,435.11
7	$8,435.11		$298.37	$8,136.74
8	$8,136.74		$298.37	$7,838.36
9	$7,838.36	$97.98	$298.37	$7,637.97
10	$7,637.97		$298.37	$7,339.60
11	$7,339.60		$298.37	$7,041.22
12	$7,041.22	$88.02	$298.37	$6,830.86
13	$6,830.86		$298.37	$6,532.49
14	$6,532.49		$298.37	$6,234.11
15	$6,234.11	$77.93	$298.37	$6,013.66
16	$6,013.66		$298.37	$5,715.29
17	$5,715.29		$298.37	$5,416.91
18	$5,416.91	$67.71	$298.37	$5,186.25
19	$5,186.25		$298.37	$4,887.88
20	$4,887.88		$298.37	$4,589.50
21	$4,589.50	$57.37	$298.37	$4,348.50
22	$4,348.50		$298.37	$4,050.12
23	$4,050.12		$298.37	$3,751.75
24	$3,751.75	$46.90	$298.37	$3,500.27
25	$3,500.27		$298.37	$3,201.90
26	$3,201.90		$298.37	$2,903.52
27	$2,903.52	$36.29	$298.37	$2,641.44
28	$2,641.44		$298.37	$2,343.07
29	$2,343.07		$298.37	$2,044.69
30	$2,044.69	$25.56	$298.37	$1,771.87
31	$1,771.87		$298.37	$1,473.50
32	$1,473.50		$298.37	$1,175.13
33	$1,175.13	$14.69	$298.37	$891.44
34	$891.44		$298.37	$593.07
35	$593.07		$298.37	$294.69
36	$294.69	$3.68	$298.37	$0.00

Using this example, the balance after the first payment is $P_3 = P_w = A(P_0 - 2d) - d$, since two payments are made, interest is calculated, then another payment is made. The process of subtracting two payments, multiplying by the growth factor, then subtracting another payment continues until the last period, P_{uw} .

The next step in the process is to state P_{uw} in terms of P_0 , since it is known. The final balance can be represented as $P_{uw} = A^u P_0 - d(A(w-1)+1)\left(\frac{1-A^u}{1-A}\right)$. Now that we have the formula for the final balance, we can set this formula equal to zero in order to determine the formula for the unknown “d”, which can be represented as $d = \frac{P_0 A^{kT-1}(1-A)}{(w-1)(1-A^{kT})}$.

The final step is the exact same as the previous model, with P_{uw} replacing P_{vw} as the final balance.

$$(P_w - P_0 + d) + (P_{2w} - P_w + d) + (P_{3w} - P_{2w} + d) \ldots + \left(P_{uw} - P_{(u-1)w} + d\right)$$

The telescoping sum leaves us with $P_{uw} - P_0 + uwd$, though we know that P_{uw} equals 0. Once "d" has been re-substituted and "u" is substitued with "k*T", the formula for total interest for this scenario can be represented as $I_{Total} = P_0\left(\frac{wkTA^{kT-1}(1-A)}{(w-1)(1-A^{kT})} - 1\right)$.

Section 1.4: Modified Annuity Due under 1st and 2nd Scenario

So far, we have looked at two different scenarios; the first when the number of interest accruals is greater than or equal to the number of payments, and the second is when the number of interest accruals is less than the number of payments. The calculations have dealt with ordinary annuities, which make payments at the end of a period. When modeling annuity dues, which make payments at the beginning of a period, the steps taken throughout the process are identical. The resulting formulas for total interest are $I_{Total} = P_0\left(vA^{kT-1}\left(\frac{A^w-1}{A^{kT}-1}\right) - 1\right)$ under the 1st scenario and $I_{Total} = P_0\left(\frac{kTA^{kT-1}(1-A)}{(1-A^{kT})} - 1\right)$ under the 2nd scenario for annuity dues. There is very little variation between the resulting formulas for ordinary annuities and annuity dues.

Section 1.5: Integration of Ordinary Annuity

The final model we created was for the integration of an ordinary annuity, which assumes that interest compounds continuously. For this model, interest is always compounding rather than by the month or year. It is also important to note that "t" represents time and is variable and "T" represents the duration of the annuity in periods and is constant.

For the first step, we determined the differential equation using the separation of variables, $\frac{dP}{dt} = r * P(t) - d$, which represents the change from one balance to the next. Next, we integrated the differential equation, resulting in $P(t) = \frac{1}{r}(C * e^{rt} + d)$. Once integrated, we

needed to determine “C” by plugging in 0 for “t”. We know that $P(0) = P_0$, so we plug in P_0 for $P(0)$ and find that $C = rP_0 - d$. Now that C is known, we can plug it back in for C of the integrated formula. The final balance can be represented as $P(T) = \frac{1}{r}\left((rP_0 - d)e^{rt} + d\right)$.

Now that we have the formula for the final balance, we can set this formula equal to zero in order to determine the formula for the unknown “d”, which can be represented as $d = \frac{rP_0e^{rT}}{(e^{rT}-1)}$. Re-substituting “d” back into the formula for the final balance, the integral of a continuously compounding ordinary annuity can be represented as $P(t) = \frac{P_0e^{rT}\left(e^{r(t-T)}-1\right)}{(1-e^{rT})}$. This formula will return the balance of year “t”.

Section 1.6: Conclusion

Given the principal amount, the annual interest rate, the number of compounding periods per year, and the duration of the annuity, these formulas will return the balances after each payment and the total interest paid under each scenario. These tools can be useful in situations when an election is made to make payments more frequent – 1st Scenario – or to make payments less frequent – 2nd Scenario – than those of a traditional annuity.

Chapter 2 – Capital Structure

An important part of financial planning for a firm is determining how much external funding is required and where it should come from. For businesses, external funding can come from two sources. The first source is through the issuance of debt, which can be bought by individuals, other firms, or governments. The second source is through the issuance of equity, which can be bought by legal entites such as individuals or corporations in the form of stocks. Each of these sources has its own cost – the cost of debt is represented by the interest rate at which the unpaid balance accrues, and the cost of equity is represented by the influences of the market on the firm. By balancing the amount of funding raised through debt with the amount raised through equity, a firm can minimize its overall cost of external funding, called the cost of capital.

Section 2.1: Weighted Average Cost of Capital

Capital raised for a firm comes from the issuance of debt or equity, meaning that the amount of debt and the amount of equity can be represented as percentages of the total amount of capital raised. We call these the weight of debt and the weight of equity, represented as w_D and w_E. Since these are the only two sources of capital, $w_D + w_E = 100\%$ or 1. w_D and w_E have perfect collinearity, which means that when one weighting changes, the other changes in an indirect proportion. In addition, when a firm raises capital, a cost is associated with that issuance, represented as a percentage. The cost of issuance, however, also depends on the amount of capital received from a particular source prior. In other words, the cost of issuing debt depends on how much debt a firm has already issued, and the cost of issuing equity depends on how much equity has already been issued. This means that the cost of debt, r_D, is a function of the weight of debt, and that the cost of equity, r_E, is a function of the weight of equity. In order to

determine the overall cost of capital, the weight of debt must be multiplied by the cost of debt, then added to the weight of equity multiplied by the cost of equity. This is called the weighted average cost of capital, and is represented as $WACC = w_D * r_D * (1 - T) + w_E * r_E$ where T is the tax rate for the firm and will be discussed in the following section.

Section 2.2: Cost of Debt

When determining the cost of debt, the assumption is that the relationship between w_D and r_D is linear, and can be represented as $r_D = f(w_D) = aw_D + b$. It is important to note that this is considered the "pre-tax" cost of debt. This is because when interest is paid on outstanding debt, it is deducted as an expense on the income statement, meaning that the cost of debt is diluted by a tax effect. The premise of $r_D * (1 - T)$ is to adjust for this tax effect, and is called the "after-tax" cost of debt.

Section 2.3: Cost of Equity

The cost of equity is a product of two market influences and the firm's expected rate of return. The first market influence is the levered beta of a firm, represented as β_L, which represents the volatility of the firm in comparison to the market with the assumption that the firm has outstanding debt. A levered beta of 1.00 means the company is as volatile as the market. A levered beta of less than 1.00 means the firm is less volatile, and a levered beta of more than 1.00 means the firm is more volatile. The second market influence is the risk-free rate of return on investments. This rate is typically comparitive to the rate of a 10-year U.S. Treasury Note, and is represented as R_f. The expected rate of return, represented as E_R, is the rate at which a firm expects to profit from investments. The risk-free rate and the expected rate of return comprise the market risk premium, represented as $\left(E_R - R_f\right)$, which is the excess return as firm will receive

over the risk-free rate. All of these componenets together make up the cost of equity, which can be represented as $r_E = \beta_L(E_R - R_f) + R_f$.

Section 2.4: Adjustments to Cost of Debt and Cost of Equity

Since we know that a firm aims to minimize its weighted average cost of capital and that the cost of capital is a function of w_D and w_E, we need to figure out the optimal weight of debt and equity. For this case, we are determining the optimal $WACC$, which is the future $WACC$, and can be represented as $WACC = w_{D_1} * r_D * (1-T) + w_{E_1} * r_E$ where w_{D_1} is the optimal weight of debt and w_{E_1} is the optimal weight of equity. We can achieve this by using the known current weight of debt w_{D_0}, current weight of equity w_{E_0}, current levered beta β_{L_0}, risk-free rate R_f, expected rate of return E_R, and tax rate T to determine the unknown optimal weight of debt w_{D_1}, optimal weight of equity w_{E_1}, and $WACC$.

Because the optimal $WACC$ uses future measurements, we must first adjust the cost of debt and the cost of equity to reflect the future costs that utilize the optimal weightings. Since r_D is represented as $r_D = aw_D + b$, we must restate it as $r_D = aw_{D_1} + b$. The cost of equity must also be adjusted, but entails a more complicated process. Using the current levered beta, the cost of equity is stated as $r_E = \beta_{L_0}(E_R - R_f) + R_f$, but we must figure out what the future levered beta will be since it is a function of the changing percentages of w_D and w_E. To do this, we must utilize the Hamada Equation, which is $\beta_L = \beta_U\left(1 + (1-T)\frac{w_D}{w_E}\right)$. First, we must solve for β_U – the unlevered beta of the firm with the assumption that the firm has no debt – using the current levered beta and current capital weightings. This returns the formula $\beta_U = \frac{\beta_{L_0}}{\left(1+(1-T)\frac{w_{D_0}}{w_{E_0}}\right)}$. Next is to restate the Hamada Equation in terms of the future levered beta and future (optimal) capital

weightings, represented as $\beta_{L_1} = \beta_U \left(1 + (1-T)\frac{w_{D_1}}{w_{E_1}}\right)$. Since β_U is now solved for, we can substitute it into the formula for β_{L_1}, now represented as $\beta_{L_1} = \beta_{L_0} \frac{\left(1+(1-T)\frac{w_{D_1}}{w_{E_1}}\right)}{\left(1+(1-T)\frac{w_{D_0}}{w_{E_0}}\right)}$. Finally, we can substitute β_{L_1} into the formula for r_E, now represented as $r_E = \beta_{L_0} \frac{\left(1+(1-T)\frac{w_{D_1}}{w_{E_1}}\right)}{\left(1+(1-T)\frac{w_{D_0}}{w_{E_0}}\right)}(E_R - R_f) + R_f$.

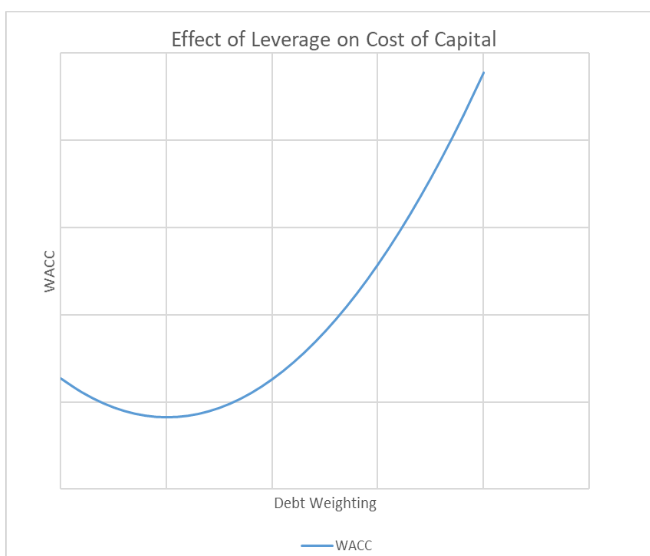
Section 2.5: Optimal Weight of Debt and Weighted Average Cost of Capital

Now that we all components of $WACC$ stated in terms of known variables, we can state it as $WACC = w_{D_1}(aw_{D_1} + b)(1-T) + w_{E_1}\left(\beta_{L_0} \frac{\left(1+(1-T)\frac{w_{D_1}}{w_{E_1}}\right)}{\left(1+(1-T)\frac{w_{D_0}}{w_{E_0}}\right)} \overbrace{(E_R - R_f)}^{Market\ Risk\ Premium} + R_f\right)$. Since we also know $w_D + w_E = 1$, we can state w_E in terms of w_D as $w_E = 1 - w_D$. Now we have

$$WACC = w_{D_1}(aw_{D_1} + b)(1-T) + (1 - w_{D_1})\left(\beta_{L_0} \frac{\left(1+(1-T)\frac{w_{D_1}}{1-w_{D_1}}\right)}{\left(1+(1-T)\frac{w_{D_0}}{1-w_{D_0}}\right)} \overbrace{(E_R - R_f)}^{Market\ Risk\ Premium} + R_f\right)$$

where $WACC$ is a function of w_{D_1}, the only unknown variable.

We want to find the minimum value possible for $WACC$ and we know that this is a quadratic function, represented by the $w_{D_1}(aw_{D_1} + b)(1-T)$ term, so we know that we are looking for the vertex of a concave up parabola, represented by the adjacent graph. To find the x-value of the vertex, we can use the formula $x = -\frac{B}{2A}$ where $x = w_{D_1}$ from the function in the form of $WACC = Aw_{D_1}{}^2 + Bw_{D_1} + C$.



Before we continue, we must restate the formula for $WACC$ to follow this form and determine the equations for A, B, and C.

To determine these equations, we will need to find the first and second derivate of the formula for $WACC$. Once derived, we can plug in 0 for w_{D_1} of the original formula, the first derivatice, and second derivative. The second derivative returns $A = \frac{1}{2}f''(0) = a(1-T)$. The first derivate returns $B = f'(0) = b\,(1-T) - \frac{\beta_{L_0}T(1-w_{D_0})}{(1-Tw_{D_0})}(E_R - R_f) - R_f$, and the original formula for $WACC$ returns $C = f(0) = \beta_{L_0}\frac{(1-w_{D_0})}{(1-Tw_{D_0})}(E_R - R_f) + R_f$. Now that we have the coefficients for the quadtratic function, we can substitute A and B into $x = -\frac{B}{2A}$. w_{D_1} can now be represented as $w_{D_1} = \frac{\frac{\beta_{L_0}T(1-w_{D_0})}{(1-Tw_{D_0})}(E_R-R_f)+R_f-b(1-T)}{2a(1-T)}$. This formula represents the x-value of the vertex, so we must now combine the coefficients with the formula for w_{D_1} to restate the formula for $WACC$, which can be represented as $minWACC = \frac{\left(\frac{\beta_{L_0}T(1-w_{D_0})}{(1-w_{D_0}T)}(E_R-R_f)+R_f-b(1-T)\right)^2}{4a(1-T)} +$

$$\frac{(bw_{D_0}T^2+T(w_{D_0}(\beta_{L_0}(E_R-R_f)+R-b)+\beta_{L_0}(E_R-R_f)+b)-R_f+b)\left(\frac{\beta_{L_0}T(1-w_{D_0})}{(1-w_{D_0}T)}(E_R-R_f)+R_f-b(1-T)\right)}{2a(1-w_{D_0}T)(1-T)} +$$

$$\frac{\beta_{L_0}(1-w_{D_0})(E_R-R_f)+R_f(1-w_{D_0}T)}{1-w_{D_0}T}.$$

Section 2.6: Conclusion

Given the known variables of the current weight of debt, the current levered beta, the expected rate of return, the risk-free rate of return, the applicable tax rate, and the linear regression of the cost of debt, these formulas will return the optimal weight of debt and the

optimal weighted average cost of capital. This tool can be effective for firms that have this important information readily available and wish to improve their capital structure.

Chapter 3 – Effect of $WACC$ on Total Firm Value

Value of operations is a financial measure used to apply a dollar amount to the overall value of a firm. Value of operations, or V_{op}, uses a method of estimating and discounting future free cash flows to the present value of those cash flows. The total present value of these cash flows represents the firm's value of operations, which is then added to any non-operating assets to comprise the total firm value. The rate at which the cash flows are discounted depends on the historical growth rate of the firm's free cash flows and the firm's $WACC$, as determined in the previous chapter.

Section 3.1: Free Cash Flows

A free cash flow, or FCF, is the cash left over after expenses have been paid that is available for use in the following period. An important component of FCF is the direct cost of issuing equity, called the flotation cost. The flotation cost is the percentage of capital raised that is paid to a brokerage firm contracted to issue the shares of stock. It is important to note that flotation costs are only inccured when equity is issued; an increase in the weight of equity or a decrease in the weight of debt incurs flotation costs.

There are two methods of determining the free cash flow for a period: the direct method and the indirect method. The direct method tracks the cash receipts and payments directly from the statement of cash flows. The indirect method begins with the net income and adds or subtracts the changes in asset and liability account balances to estimate the free cash flow. For this model, we will use the indirect method.

Section 3.2: Indirect Method of FCF

The indirect method finds the changes in certain balance sheet accounts and adds or subtracts them from net income, then adds back non-cash expenses such as depreciation and

amortization to determine FCF. Flotation costs are then subtracted if equity has been issued this period, which is determined by multiplying the market value of the firm, represented as the stock price times the number of oustanting shares plus the outstanding debt, by the flotation percentage "f" charged by the brokerage multiplied by the change in debt. The flotation cost can be represented as $Float = \begin{cases} f(P_0 CS_0 + D)(w_{D_0} - w_{D_1}), & w_{D_0} - w_{D_1} > 0 \\ 0, & w_{D_0} - w_{D_1} \leq 0 \end{cases}$, where the flotation cost is 0 when the change in the weight of debt is positive (no equity is issued).

The FCF for a period can be represented as $FCF_0 = NI_0 + (Am_0 + Dp_0) + U_0 - |AR_0 - AR_{-1}| - |AP_0 - AP_{-1}| + |Inv_0 - Inv_{-1}| - Float$. Amortization, depreciation, and unusual expenses are added to net income, the absolute value of the change in accounts receivable from the previous year to the current year is subtracted, the absolute value of the change in accounts payable is subtracted, the absolute value of the change in inventory dollar amount is added, then the flotation costs are subtracted if equity has been issued.

Section 3.3: Growth Rate

The next step to determine the value of operations is to find the historical average growth rate of free cash flows. The number of years of historical data chosen is up to professional judgement, so it will be represented by the variable "n". The historical average geometric growth rate can be represented as $g = \sqrt[n-1]{\frac{FCF_0}{FCF_{-n}}} - 1$, where the quotient of the current free cash flow and the cash flow of the first historical year chosen is rooted by $n - 1$, which represents the number of growth rates averaged.

Section 3.4: Value of Operations

Now that the current free cash flow and the historical growth rate have been determined, the value of operations can be calculated, represented as $V_{op} = \frac{FCF_0(1+g)}{WACC-g}$. This represents the

value generated through operations. As shown by the formula, the value of operations is inversely related to the difference of the $WACC$ and the growth rate, further reinforcing the idea of minimizing a firm's $WACC$, although it is now apparent that the true goal is to reduce $WACC$ to be as close to the growth rate as possible without going beyond.

Section 3.5: Total Firm Value and Intrinsic Stock Price

The total firm value is determined by adding any non-operating assets, such as short-term investments, to the value of operations. This is represented as $V_{total} = V_{op} + NOA_0$. Once the total firm value is determined, the firm's outstanding debt is subtracted from V_{total} to generate the intrinsic equity, represented as $E = V_{total} - D$. The final step in the process is to calculate the intrinsic stock price, which represents the value of a stock as determined by the value created during operations. The intrinsic stock price is the quotient of the intrinsic equity and the number of shares outstanding, and can be represented as $ISP = \frac{E}{CS_0}$.

Section 3.6: Conclusion

Given the weighted average cost of capital and the current and historical financial data, a firm can determine the value it has created through operations, its total firm value, and its intrinsic stock price. These referents can be used to benchmark certain performance goals to maximize the value of the firm. These referents can also be used in situations where the firm is to be acquired by another firm to determine the maximum price of acquisition.

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