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The Fractal Geometry of Grand Rapids

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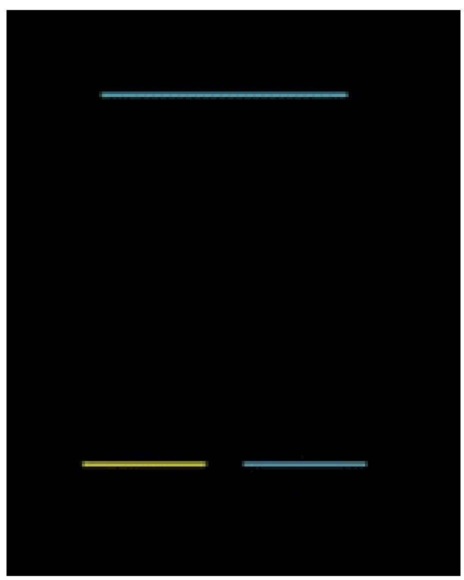
Abstract

The fractal nature of cities' geometry has been widely studied. By analyzing a city's fractal pattern, we can obtain it's fractal dimension, or measure of how complex it is. We will create models on how Grand Rapids' complexity changed since its inception, and use those models to predict what the complexity is like today.

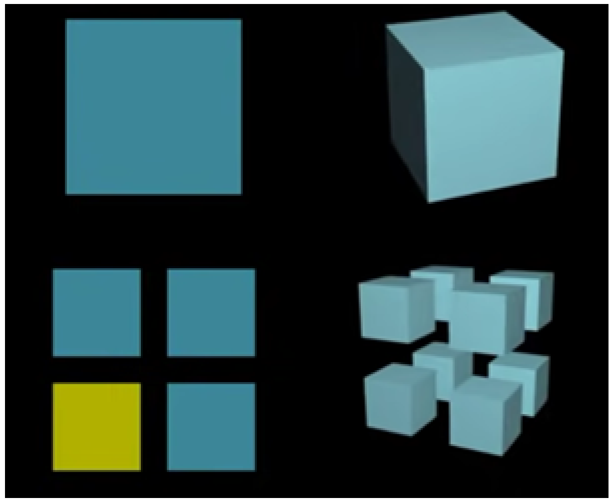
1 Introduction

Benoit Mandelbrot, the mathematician who coined fractals, once said that fractals are "The art of roughness." Before he had named the geometric shapes, he wrote a paper titled "How long is the coastline of Britain?" [Man67] In it, he argued the perimeter of Britain could not be measured. This was because as the measurements of length became finer, more detail was found, and length increased with no limit. Thus, Mandelbrot worked to find a way to describe this phenomena.
The paper is notable because it is a stepping stone that reveals this concept of self similarity, as well as showing how to think of fractals, and also how to apply them to geography.

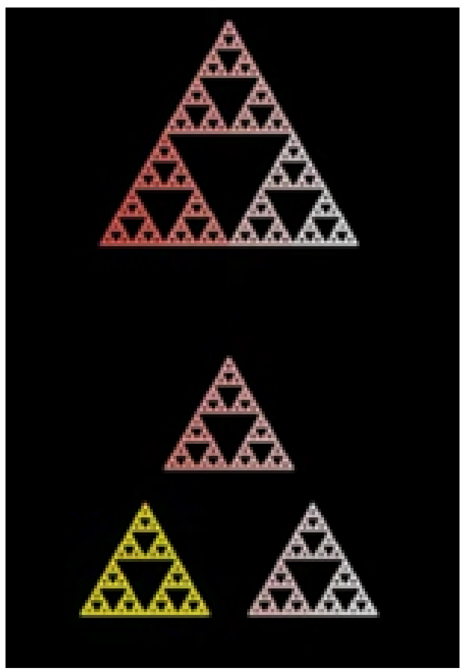
To understand fractal dimension, we must understand the concept of self-similarity. Consider the line below:



A line is one-dimensional, and if we scale the line by a linear scaling factor of $\frac{1}{2}$, then the line will have $\frac{1}{2}$, or $(\frac{1}{2})^1$ the length. Next consider a square and cube:



A square is 2-dimensional, and when scaled by $\frac{1}{2}$, it will have $\frac{1}{4}$, or $(\frac{1}{2})^2$ area. Similarly, a cube is 3-dimensional and when scaled by $\frac{1}{2}$, it will have $\frac{1}{8}$, or $(\frac{1}{2})^3$ volume. In every case above, the new mass is equivalent to the scaling factor raised to the power of the dimension, or $(\frac{1}{2})^D$, where D is the shape's dimension. We'll now apply this concept to the Sierpinski Triangle:



From the image, we can see that when the Sierpinski Triangle is scaled by $\frac{1}{2}$, it will have $\frac{1}{3}$ of its mass. Thus to obtain its dimension, we find:

$$\begin{aligned}
\frac{1}{2^D} &= \frac{1}{3} \\
log(\frac{1}{2^D}) &= log(\frac{1}{3}) \\
D \cdot log(\frac{1}{2}) &= log(\frac{1}{3}) \\
D &= \frac{log(\frac{1}{3})}{log(\frac{1}{2})} \\
&\approx 1.585
\end{aligned}$$

Thus, in general the fractal dimension of a shape, D, is $D = \frac{log(B)}{\log(c)}$, where B is the new mass, and c is the linear scaling factor. Now that we know how fractal dimension is found, we can use computer software to find it for us. Generally, programs use the box-counting method. This involves overlaying a shape with a grid, measuring how many boxes the shape touches, then scaling the shape and measuring the touching boxes again [GS17]. An example of this is included below with Britain:

On the left is the initial picture, and the yellow boxes are touched by the shape of Britain. Then on the right, the shape of Britain is scaled up and the yellow boxes are counted again. Thus, the fractal dimension can be obtained with $D = \frac{log(B)}{\log(c)}$. Additionally, the process is repeated at different scales for the best estimate.

2 Gathering Data

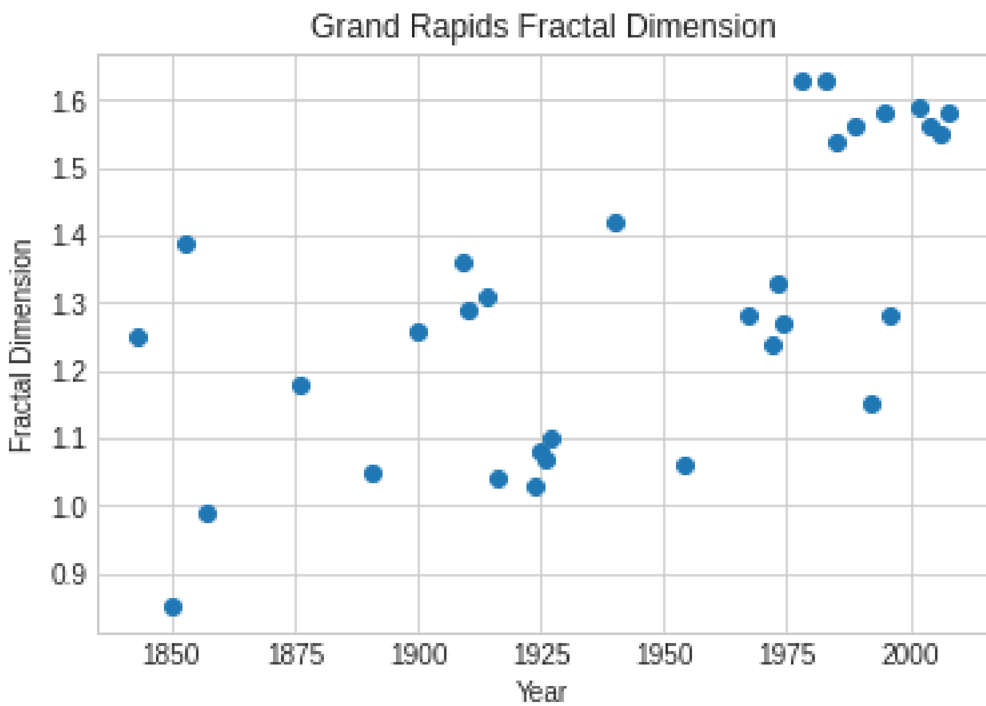
2.1 Obtaining Maps

To measure how Grand Rapids' fractal dimension has changed, we will collect maps from many years and from a number of sources. The majority of maps came from the archive at the Grand Rapids Public Library. Their maps range in years from 1843 to 2008. After taking pictures of those maps, online sources were used. Online archives include Google Maps, oldmapsonline.org, the Grand Rapids Public Library's online collections, and the Grand Rapids Public Museum's historical collections of maps.

2.2 Fractal Dimension Estimator

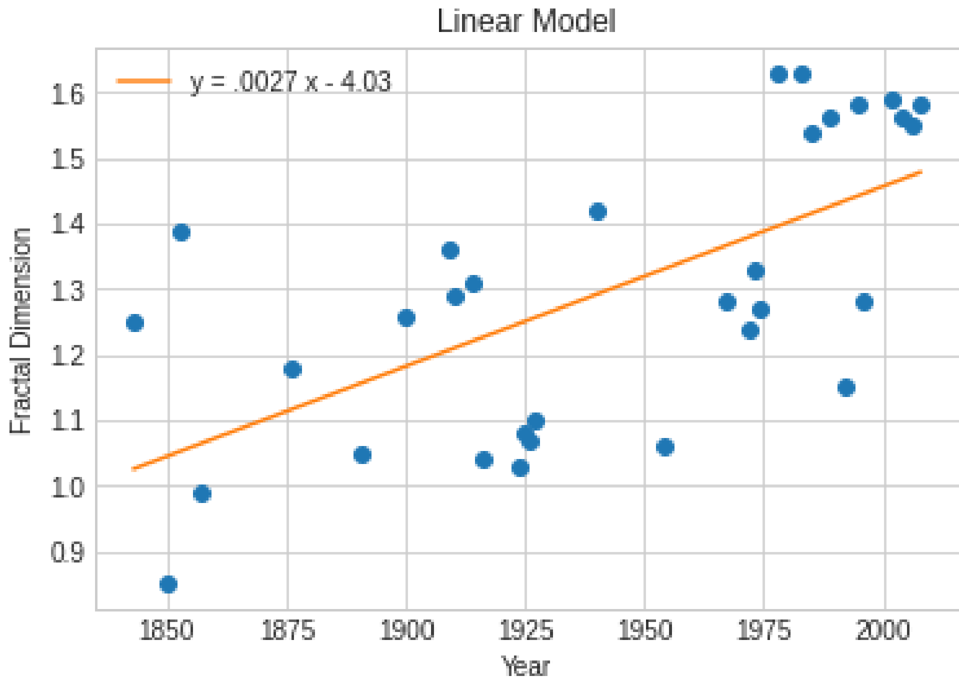
After we have obtained the necessary maps, we can start using software to find their fractal dimension. The program we will use is called Fractal Dimension Estimator. In the program, we upload our photos, and can then focus on the black pixels in the image, as we want to focus on the streets, and not any unnecessary data in the pictures. Once we have selected that, Fractal Dimension Estimator will give us it's fractal dimension.

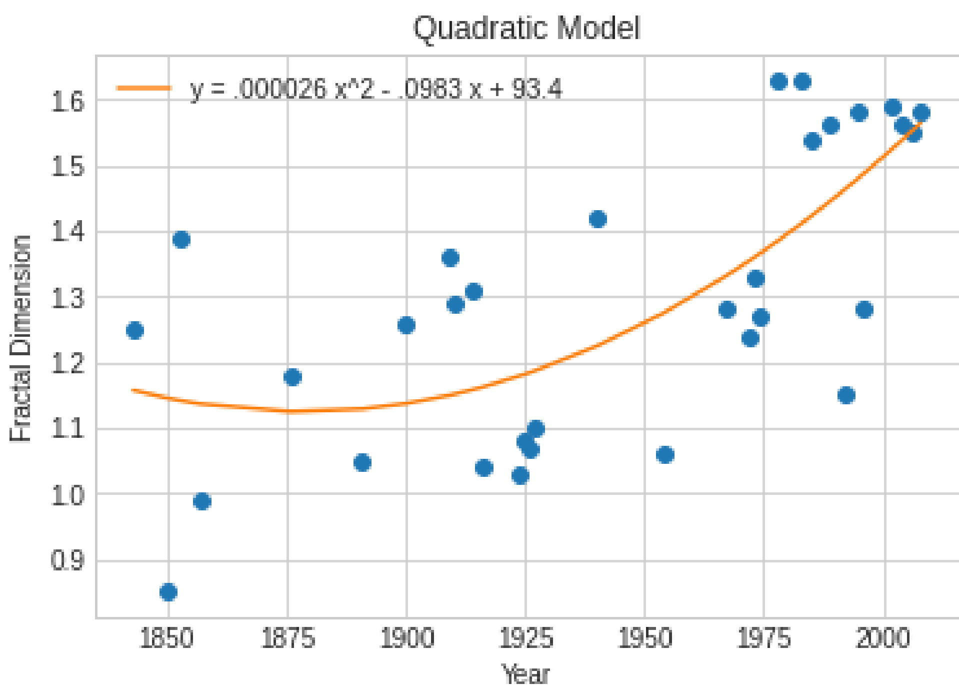
Once we have our data, we can plot it to obtain

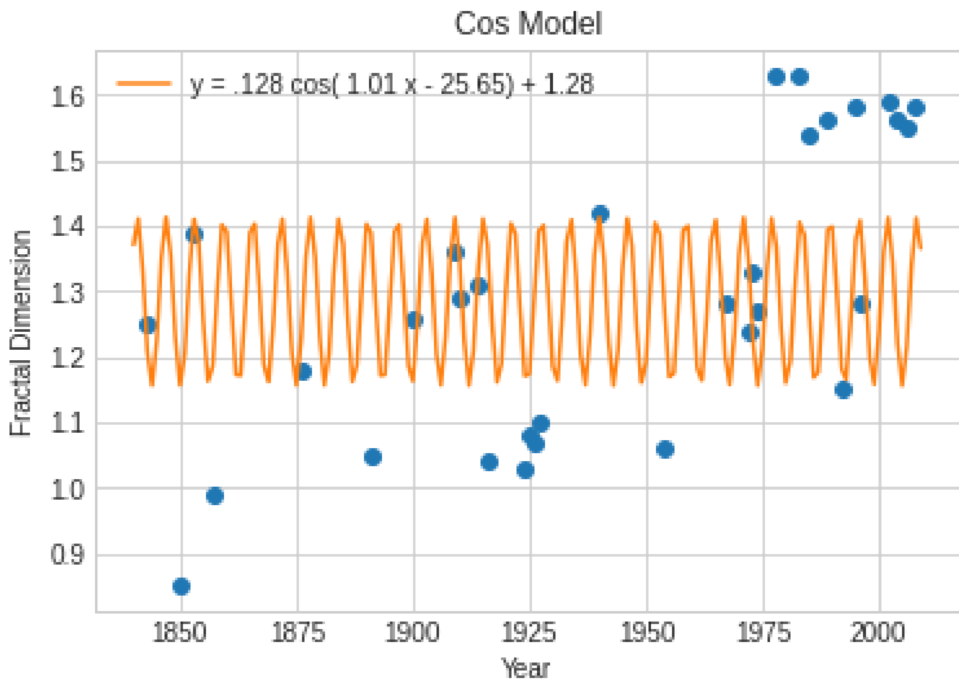


3 The Models

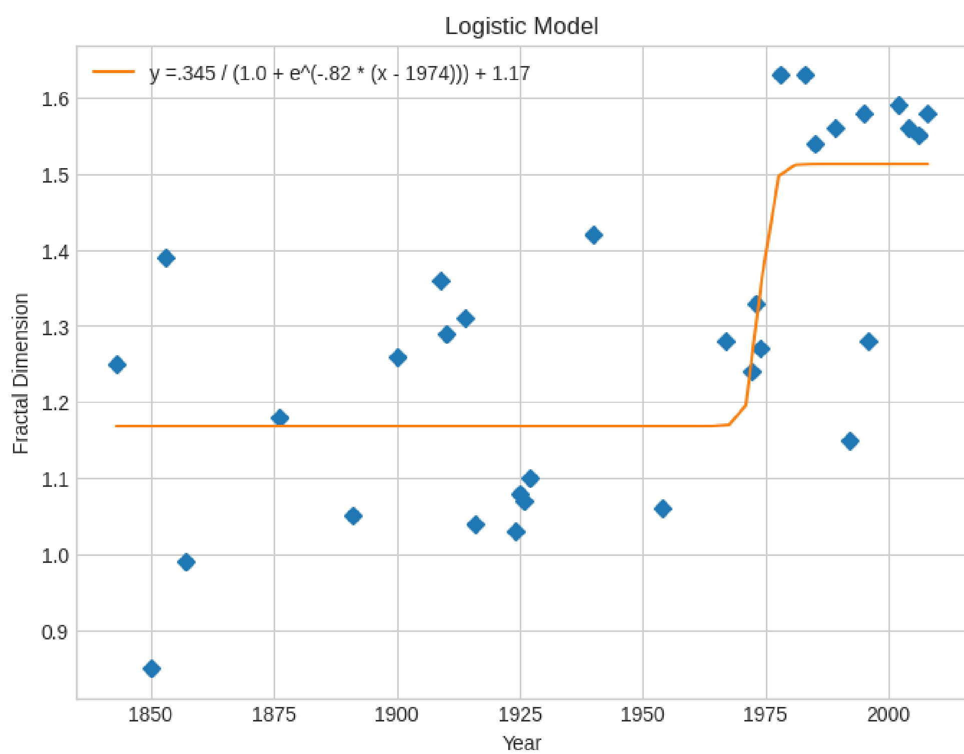
Now that we have our data, we can fit our models to it. Using SciPy in Python, we will create linear, quadratic, cosine, and logarithmic functions as seen below:







Cos Model
y = .128 cos(1.01 x - 25.65) + 1.28
Fractal Dimension
Year
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
1850
1875
1900
1925
1950
1975
2000



Logistic Model
y =.345 / (1.0 + e^(-.82 * (x - 1974))) + 1.17
Fractal Dimension
Year
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
1850
1875
1900
1925
1950
1975
2000

4 Conclusions

We will have two criteria to judge these models. The first will be root mean square error and R^2 values:

Model	RMSE	R^2
Linear	0.165	0.413
Quadratic	0.154	0.490
Cos	0.193	0.200
Logarithmic	0.145	0.547

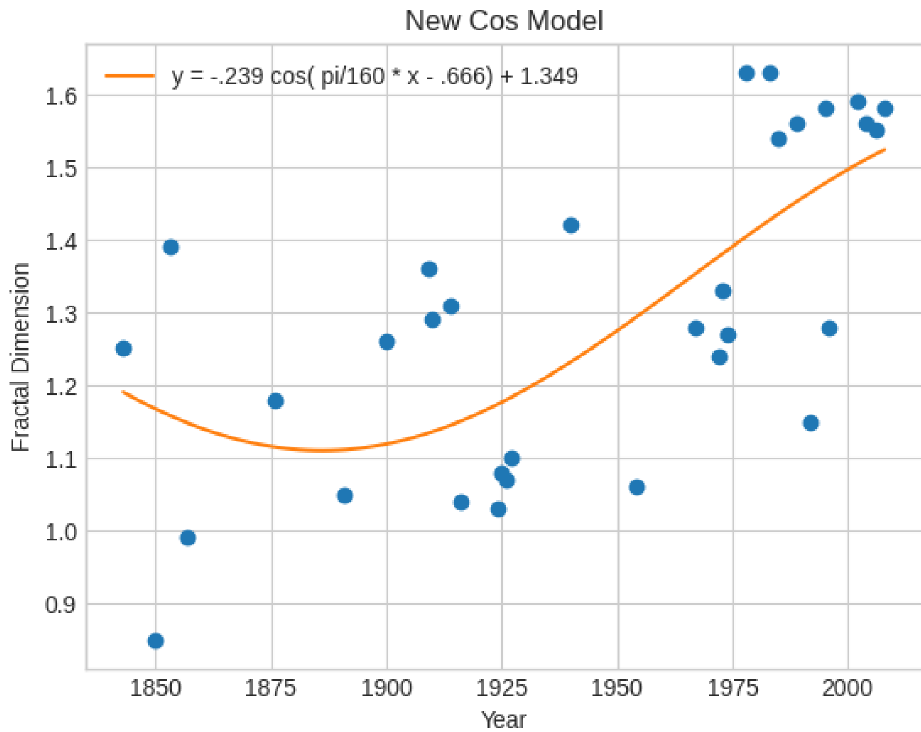
Table 1: RMSE and R^2 of Models

From these we can see that the Logarithmic model had the lowest error, and the highest Coefficient of Determination. These metrics show that the logistic model is the best at fitting the data. This matches up with research on Shenzen, China, where logarithmic models were found to be the best at matching data [MC20]. Conversely, the Cosine model is the worst at fitting the data.
The next way we'll test the models is with their predictive power. We will see how good our models predict the fractal dimension in 2022. Since 2022 was not part of our data set used to make the models, it should be a good choice to test this. In 2022, the fractal dimension of Grand Rapids was 1.69; below are the predictions and errors for each model:

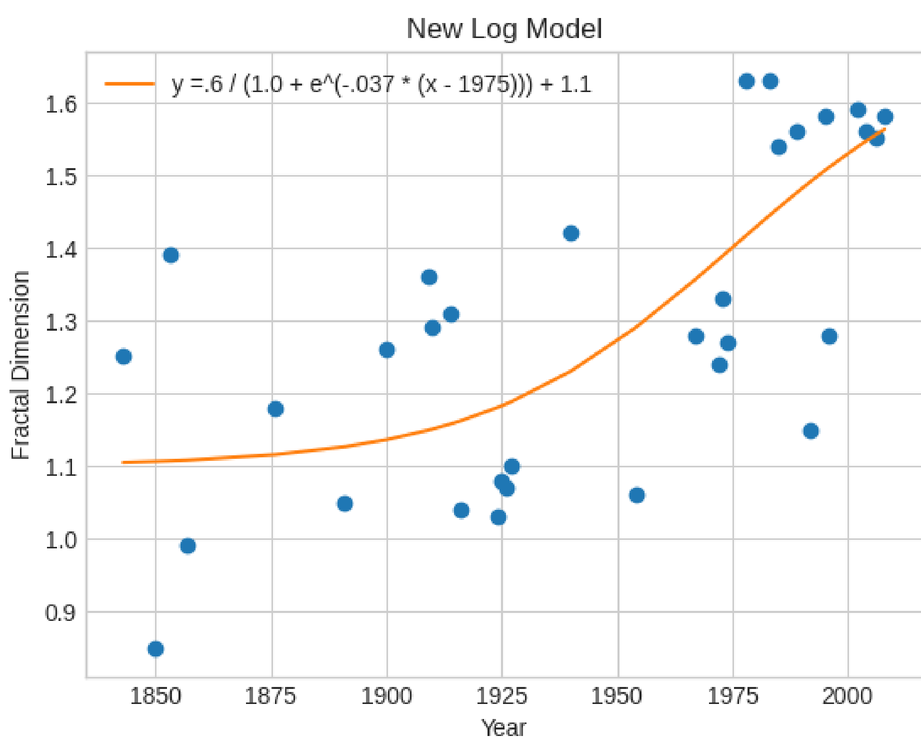
Model	Prediction	Error
Linear	1.517	0.173
Quadratic	1.665	0.025
Cos	1.293	0.397
Logarithmic	1.515	0.175

Table 2: Future Estimations with the Models

Thus, we find that the quadratic model has the lowest error and is therefore the best at predicting future values. While this may make the quadratic appear as the best model, consider that the maximum fractal dimension that Grand Rapids could have is 2. This is because it takes in data from a map, which is a plane. However, since the linear and quadratic can increase infinitely, the viability of these models falters. The cosine model also had a large amount of error and wasn't a good fit, so next we will control some of the coefficients more to make the model fit as we please. By forcing the period of the cos wave to increase, we obtain:

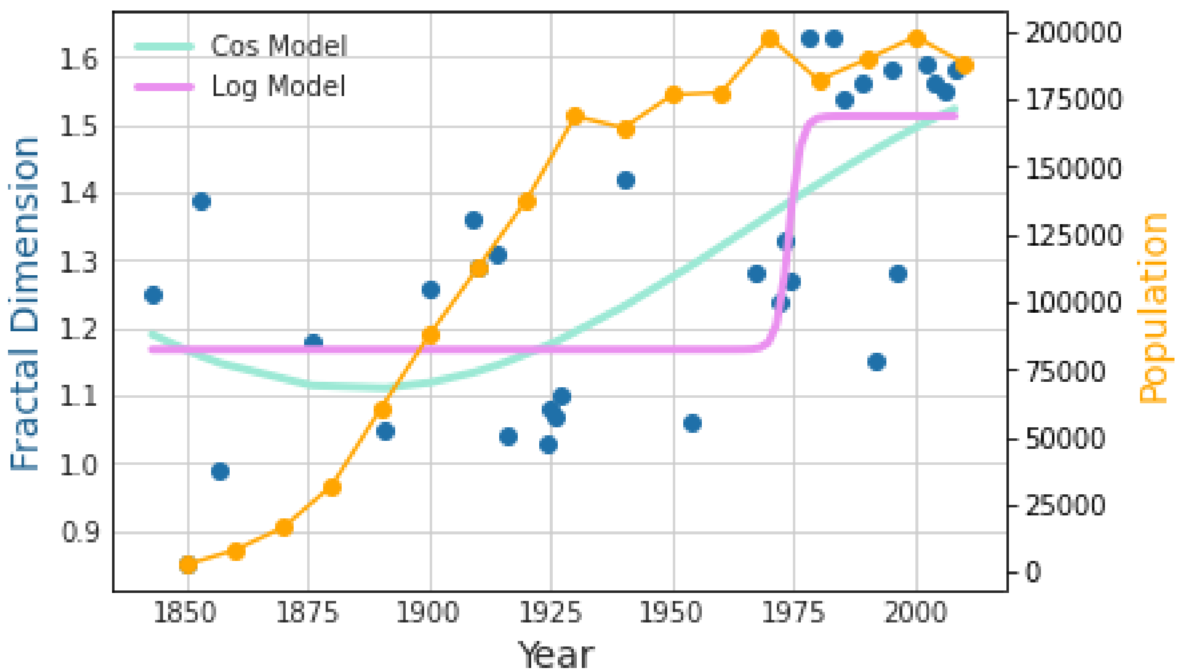


This new cos model is similar to the quadratic model, and its RMSE is 0.156 and R^2 is 0.474. Thus the changes we made to the model have more accurate. Its prediction is also more accurate; it predicts the 2022 fractal dimension is 1.561, which is off by 0.129. Let's make similar changes to our logarithmic model now:



This Log model has higher RMSE and lower R^2, at 0.155 and 0.484 respectively, but it is the best model at predicting the current fractal dimension, with a predicted value of 1.609. These two examples show that our models in Python aren't set in stone, and have some respects where they could be improved.

Finally, we'll consider how the population of Grand Rapids affects its fractal dimension. Using data from population.us [Pop], we can plot the population along with our models:



From this graph we observe that fractal dimension and population have both increased over time, and are likely correlated. This gives explanation to potential reasons why fractal dimension has increased, though more study should be conducted and other factors, such as GDP, should be considered.

5 Acknowledgements

Lastly, I would like to thank Dr. Lora Bailey for being my advisor for this project. Her support and guidance has been invaluable, and without her, this project would not have been possible.

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