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Classification of seven-dimensional solvable Lie algebras with $A_{5,2}$ and $A_{5,3}$ nilradicals

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Classification of seven-dimensional solvable Lie algebras with $A_{5,2}$ and $A_{5,3}$ nilradicals

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Abstract

This paper provides a classification of seven-dimensional indecomposable solvable Lie algebras over $\mathbb{R}$ for which the nilradical is isomorphic to $A_{5,2}$ and $A_{5,3}$. We follow a technique that was first introduced by Mubarakzyanov.

1. Introduction

For the elementary theory of Lie algebras refer to [4, 6, 7]. It has to be understood that classifying solvable Lie algebras is a different exercise from studying the semisimple algebras. The problem of classifying all semisimple Lie algebras over the field of complex numbers was solved by Cartan in 1894 [1], and over the field of real numbers by Gantmacher in 1939 [2]. For solvable indecomposable Lie algebras the problem is much more difficult. The classification of solvable Lie algebras only exists for low dimensions and was performed by, amongst others, Mubarakzyanov for solvable Lie algebras of dimension $n \leq 5$ over the field of real and partially over the field of complex numbers in [11] and [12]. Mubarakzyanov's results are summarized in [17]. Mubarakzyanov also considered dimension six and classified solvable Lie algebras with a co-dimension one nilradical [13]. Shabanskaya and Thompson refined his results and found some missing cases in [19, 20]. Then Turkowski classified six-dimensional solvable Lie algebras with a co-dimension two nilradical in [21]. Nilpotent Lie algebras in dimension six were studied as far back as Umlauf [22], and later by Morozov [9].

It is probably impossible to classify solvable Lie algebras in general in arbitrary dimension. The first step in classifying solvable Lie algebras in a specific dimension is to find the possible nilradicals. A general theorem asserts that if $\mathfrak{g}$ is an $n-$dimensional solvable Lie algebra, the dimension of its nilradical $\mathfrak{nil}(\mathfrak{g})$ is at least $\frac{n}{2}$ [13]. So for $n = 7$, the possible dimensions of the nilradical are seven, six, five, or four. The seven-dimensional nilradicals, called the nilpotents, were studied by Seely over $\mathbb{R}$ [18] and by Gong over $\mathbb{C}$ [3]. The four-dimensional nilradical case was studied by Hindeleh and Thompson [5]. The six-dimensional nilradical case was studied by Parry [16]. The five-dimensional nilradical case is still an open problem. A complete classification consists of all possible five-dimensional nilpotent algebras, including the decomposable ones.

We note that Ndogmo and Winternitz outlined methods for classifying solvable Lie algebras with abelian nilradical for a general dimension in [14, 15]. Also, Le, Vu A, et al. [8] posted in arXiv methods for the classification of seven-dimensional Lie algebras with five-dimensional nilradical. They conclude with the number of possible sub-classes without explicitly finding them. This paper provides a complete list of the seven-dimensional solvable Lie algebras with a nilradical isomorphic to the second and third nilpotent algebra of dimension five, denoted by $A_{5,2}$ and $A_{5,3}$ respectively.

In section 2, we recall basic definitions and properties related to the classification of solvable Lie algebras. Then in section 3, we use Turkowski's method [21] for classifying solvable Lie algebras with abelian nilradical, that is also outlined by Ndogmo and Winternitz [14, 15]. Finally, we list the adjoint matrices corresponding to our algebras with trivial and one-dimensional centers in sections 3 and 4.

2. A METHOD TO OBTAIN THE SOLVABLE ALGEBRAS

2.1. General Concepts

A Lie algebra $\mathfrak{g}$ is *solvable* if its derived series DS terminates, i.e.

$$DS = \{\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \dots, \mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}_{k-1}] = 0\}$$

for some $k \geq 1$.

A Lie algebra $\mathfrak{g}$ is *nilpotent* if its central series CS terminates, i.e.

$$CS = \{\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \dots, \mathfrak{g}^{(k)} = [\mathfrak{g}, \mathfrak{g}^{(k-1)}] = 0\}$$

for some $k \geq 1$.

A solvable algebra $\mathfrak{g}$ has a decomposition of the form

$$\mathfrak{g} = \mathfrak{nil}(\mathfrak{g}) \oplus X,$$

satisfying

$$\begin{aligned} [\mathfrak{nil}(\mathfrak{g}), \mathfrak{nil}(\mathfrak{g})] &\subset \mathfrak{nil}(\mathfrak{g}), \\ [\mathfrak{nil}(\mathfrak{g}), X] &\subseteq \mathfrak{nil}(\mathfrak{g}), \\ [X, X] &\subset \mathfrak{nil}(\mathfrak{g}), \end{aligned} \tag{1}$$

where $\mathfrak{nil}(\mathfrak{g})$ denotes the nilradical of $\mathfrak{g}$, the vector space X is spanned by the remaining generators, and $\oplus$ denotes the direct sum of vector spaces.

An element n of $\mathfrak{g}$ is *nilpotent* if it satisfies

$$[\dots[[x, n], n] \dots n] = 0$$

for all $x \in \mathfrak{g}$ when the commutator is taken sufficiently many times.

A set of elements $\{x_1, \dots, x_k\}$ of $\mathfrak{g}$ is called *nilindependent* if no non-trivial linear combination of them is nilpotent.

For $x \in \mathfrak{g}$, the *adjoint transformation* of x is a linear transformation $ad_x : \mathfrak{g} \to \mathfrak{g}$ defined by

$$ad_x(y) = [x, y],$$

for all $y \in \mathfrak{g}$. In this paper, the restriction of ad_x to the nilradical of $\mathfrak{g}$ denoted $ad_x|_{\mathfrak{nil}(\mathfrak{g})}$ is realized by matrices $A \in gl(5, \mathbb{R})$. Notice that if n is a nilpotent element of $\mathfrak{g}$, then $ad_n|_{\mathfrak{nil}(\mathfrak{g})}$ is a nilpotent matrix.

A set of matrices in $gl(n, \mathbb{R})$ will be called *linearly nilindependent* if no non-trivial linear combination of them is nilpotent.

2.2. Basic Structural Theorems

We shall choose a basis for $\mathfrak{g} = \langle e_1, e_2, \dots, e_5, x_1, x_2\rangle$ where $e_i \in \mathfrak{nil}(\mathfrak{g})$, $x_\alpha \in X$, for $i = 1, \dots, 5$, and $\alpha = 1, 2$.

To classify the seven-dimensional solvable Lie algebras with five-dimensional nilradical, one must start with a five-dimensional nilpotent algebra that will form $\mathfrak{nil}(\mathfrak{g})$, and add $X = \langle x_1, x_2\rangle$ satisfying the properties in (1). The following are all the nilpotent Lie algebras up to isomorphism in dimension five: $\mathbb{R}^5$, $A_{3,1} \oplus \mathbb{R}^2$, $A_{4,1} \oplus \mathbb{R}$, and $A_{5,1} - A_{5,6}$, where $\mathbb{R}^n$ denotes the $n-$dimensional abelian algebra, and $A_{n,k}$ denotes the k^{th} algebra of dimension n from Patera's list [17]. The case where the nilradical is isomorphic to $\mathbb{R}^5$, called the abelian nilradical case, was classified in a recently accepted work by Bakeberg, Blaine and Hindeleh. The focus of this article is on the second and third indecomposable nilpotent algebras of dimension 5, namely the $A_{5,2}$ and $A_{5,3}$ case.

Since the nilradical is $A_{5,2}$ or $A_{5,3}$ and basis elements must satisfy the relations in (1), we have

$$\begin{pmatrix} [x_\alpha, e_1] \\ \vdots \\ [x_\alpha, e_5] \end{pmatrix} = \begin{pmatrix} e_1 & \dots & e_5 \end{pmatrix} A^\alpha \tag{2a}$$

$$[x_1, x_2] = R^i e_i \tag{2b}$$

where $A^\alpha = ad_{x_\alpha}|_{\mathfrak{nil}(\mathfrak{g})}$, $\alpha = 1, 2$ and $i = 1, \ldots, 5$ and we use the Einstein summation notation, as well as

$$[e_2, e_5] = e_1, \quad [e_3, e_5] = e_2, \quad [e_4, e_5] = e_3$$

and

$$[e_3, e_4] = e_2, \quad [e_3, e_5] = e_1, \quad [e_4, e_5] = e_3$$

for the structure equations of $A_{5,2}$ and $A_{5,3}$ respectively. The classification of our Lie algebras thus amounts to classification of the matrices A^α and the constants R^i.

By the Jacobi identity involving x_1, x_2, and an e_i,

$$[[x_1, x_2], e_i] + [[x_2, e_i], x_1] + [[e_i, x_1], x_2] = 0.$$

Thus

$$\begin{aligned} ad_{[x_1,x_2]}(e_i) &= [x_1, [x_2, e_i]] - [x_2, [x_1, e_i]] \\ &= ad_{x_1}([x_2, e_i]) - ad_{x_2}([x_1, e_1]) \\ &= ad_{x_1}(ad_{x_2}(e_i)) - ad_{x_2}(ad_{x_1}(e_i)) \\ &= [ad_{x_1}, ad_{x_2}](e_i). \end{aligned}$$

Hence $[ad_{x_1}, ad_{x_2}]$ is an inner derivation of the nilradical.

We perform a combination of changes of basis until we reach our desired form. For $i = 1, \ldots, 5$, and $\alpha = 1, 2$, the following changes of basis preserve the nilradical:

1. Absorbtion-type change of basis
$$\bar{x}_\alpha = x_\alpha + r^i_\alpha e_i \qquad r^i_\alpha \in \mathbb{R}.$$
2. A change of basis in X
$$\bar{x}_\alpha = G^\beta_\alpha x_\beta \qquad G \in GL(2, \mathbb{R}).$$
3. A change of basis in $\mathfrak{nil}(\mathfrak{g})$
$$\bar{e_i} = S^j_i e_j \qquad S \in GL(5, \mathbb{R}),$$
where $S = (S^j_i)$ is the automorphism that will change every A^α to a similar matrix $SA^\alpha S^{-1}$.

Thus our classification problem reduces to finding the derivations of the nilradical that are not nilpotent and that satisfy the conditions above.

3. CLASSES OF SOLVABLE ALGEBRAS FOR $A_{5,2}$

We will determine all real solvable algebras $N = A_{5,2} \oplus X$ such that the dim$X = 2$. The dimension of the center of $\mathfrak{g}$ is

$$\dim Z(\mathfrak{g}) \leq 2 \dim \mathfrak{nil}(\mathfrak{g}) - \dim \mathfrak{g} = 3$$

(see Ref. [10]). The algebras that possess a center of dimension at least two are decomposable into a direct sum of lower-dimensional algebras [10]. Therefore, in the following, the classification problem is solved for the cases $\dim Z(\mathfrak{g}) = 0, 1$.

First, twenty Jacobi identities involving (x_α, e_i, e_j) and with additional transformations give

$$A^1 = \begin{bmatrix} 3\,p55 + p44 & p12 & p13 & p14 & p15 \\ 0 & 2\,p55 + p44 & p12 & p13 & p25 \\ 0 & 0 & p55 + p44 & p12 & p35 \\ 0 & 0 & 0 & p44 & p45 \\ 0 & 0 & 0 & 0 & p55 \end{bmatrix}$$

,

$$A^2 = \begin{bmatrix} 3\,q55 + q44 & q12 & q13 & q14 & q15 \\ 0 & 2\,q55 + q44 & q12 & q13 & q25 \\ 0 & 0 & q55 + q44 & q12 & q35 \\ 0 & 0 & 0 & q44 & q45 \\ 0 & 0 & 0 & 0 & q55 \end{bmatrix}$$

Next, five Jacobi identities involving (x_1, x_2, e_i) give

$$\begin{cases} p12q25 + p13q35 + p14q45 + p44q15 - p15q44 - 2p15q55 - p25q12 - p35q13 - p45q14 + 2p55q15 & = R2 \\ p12q35 + p13q45 + p44q25 - p25q44 - p25q55 - p35q12 - p45q13 + p55q25 & = R3 \\ p12q45 + p44q35 - p35q44 - p45q12 & = R4 \\ p12q55 - p55q12 & = R5 \\ 2p13q55 - 2p55q13 & = 0 \\ 3p14q55 - 3p55q14 & = 0 \\ -p44q45 + p45q44 - p45q55 + p55q45 & = 0 \end{cases}$$

From the last equation we can separate out two distinct cases with sub cases

1. $q55 \neq 0$
   (a) $q44 \neq q55$
   (b) $q44 = q55$: from the conditions that the adjoint matrices cannot be nilpotent, $q45 = 0$ is the only allowed subcase.
2. $q55 = 0$
   (a) $p55 = 0$: from nilpotent conditions $p44 \neq 0$ and $q44 \neq 0$
   (b) $p55 \neq 0$. Then $q13 = 0$ and $q14 = 0$ and $q44 \neq 0$.

3.1. Case 1

From case 1a we have

$$A^1 = \begin{bmatrix} 3\,p55 + p44 & 0 & \frac{q13\,p55}{q55} & \frac{q14\,p55}{q55} & 0 \\ 0 & 2\,p55 + p44 & 0 & \frac{q13\,p55}{q55} & 0 \\ 0 & 0 & p55 + p44 & 0 & 0 \\ 0 & 0 & 0 & p44 & \frac{q45\,(p44 - p55)}{q44 - q55} \\ 0 & 0 & 0 & 0 & p55 \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} 3\,q55 + q44 & 0 & q13 & q14 & 0 \\ 0 & 2\,q55 + q44 & 0 & q13 & 0 \\ 0 & 0 & q55 + q44 & 0 & 0 \\ 0 & 0 & 0 & q44 & q45 \\ 0 & 0 & 0 & 0 & q55 \end{bmatrix}$$

with

$$[x_1, x_2] = -\frac{q14\;q45\;(p44\;q55 - p55\;q44)}{q55\;(q44 - q55)}\,e2 - \frac{q13\;q45\;(p44\;q55 - p55\;q44)}{q55\;(q44 - q55)}\,e3.$$

From case 1b we have

$$A^1 = \begin{bmatrix} 3\,p55 + p44 & 0 & \frac{q13\,p55}{q55} & \frac{q14\,p55}{q55} & 0 \\ 0 & 2\,p55 + p44 & 0 & \frac{q13\,p55}{q55} & 0 \\ 0 & 0 & p55 + p44 & 0 & 0 \\ 0 & 0 & 0 & p44 & p45 \\ 0 & 0 & 0 & 0 & p55 \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} 4\,q55 & 0 & q13 & q14 & 0 \\ 0 & 3\,q55 & 0 & q13 & 0 \\ 0 & 0 & 2\,q55 & 0 & 0 \\ 0 & 0 & 0 & q55 & 0 \\ 0 & 0 & 0 & 0 & q55 \end{bmatrix}$$

with

$$[x_1, x_2] = -(p45 \cdot q14) \cdot e2 - (p45 \cdot q13) \cdot e3.$$

3.2. Case 2

From case 2a we have a violation of the linear nilindependence of the adjoint matrices. Thus case 2a is invalid. From case 2b we have

$$A^1 = \begin{bmatrix} 3\,p55 + p44 & 0 & p13 & p14 & 0 \\ 0 & 2\,p55 + p44 & 0 & p13 & 0 \\ 0 & 0 & p55 + p44 & 0 & 0 \\ 0 & 0 & 0 & p44 & \frac{q45\,(p44 - p55)}{q44} \\ 0 & 0 & 0 & 0 & p55 \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} q44 & 0 & 0 & 0 & 0 \\ 0 & q44 & 0 & 0 & 0 \\ 0 & 0 & q44 & 0 & 0 \\ 0 & 0 & 0 & q44 & q45 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with

$$[x_1, x_2] = -(p14 \cdot q45) \cdot e2 - (p13 \cdot q45) \cdot e3.$$

4. CLASSES OF SOLVABLE ALGEBRAS FOR $A_{5,3}$

First, twenty Jacobi identities involving (x_α, e_i, e_j) with additional transformations give

$$A^1 = \begin{bmatrix} 2\,p55 + p44 & p54 & p34 & p14 & p15 \\ p45 & p55 + 2\,p44 & -p35 & p24 & p25 \\ 0 & 0 & p55 + p44 & p34 & p35 \\ 0 & 0 & 0 & p44 & p45 \\ 0 & 0 & 0 & p54 & p55 \end{bmatrix}$$

,

$$A^2 = \begin{bmatrix} 2\,q55 + q44 & q54 & q34 & q14 & q15 \\ q45 & q55 + 2\,q44 & -q35 & q24 & q25 \\ 0 & 0 & q55 + q44 & q34 & q35 \\ 0 & 0 & 0 & q44 & q45 \\ 0 & 0 & 0 & q54 & q55 \end{bmatrix}.$$

Next, five Jacobi identities involving (x_1, x_2, e_i) give

$$\begin{cases} -p14q45 - p24q44 - p24q55 + p34q35 + p44q24 - p54q25 + p25q54 - p35q34 + p45q14 + p55q24 & = R3 \\ p34q45 + p44q35 - p35q44 - p45q34 & = R4 \\ p34q55 + p54q35 - p35q54 - p55q34 & = R5 \\ p54q45 - p45q54 & = 0 \\ p44q54 - p54q44 + p54q55 - p55q54 & = 0 \\ -p44q45 + p45q44 - p45q55 + p55q45 & = 0 \\ 2p14q55 + p24q54 - p54q24 + p54q15 - p15q54 - 2p55q14 & = 0 \\ -p24q45 - 2p44q25 + p15q45 + 2p25q44 + p45q24 - p45q15 & = 0 \\ -2p14q45 - p24q44 - p24q55 + p44q24 - p44q15 - 2p54q25 + p15q44 + p15q55 + 2p25q54 + 2p45q14 + p55q24 - p55q15 & = 0 \end{cases}$$

Put $C = \begin{pmatrix} 2p55 + p44 & p54 \\ p45 & p55{+}2p44 \end{pmatrix}$ and $D = \begin{pmatrix} 2q55 + q44 & q54 \\ q45 & q55{+}2q44 \end{pmatrix}$. The above equations show these blocks commute. Thus there exists a change of basis which will change these blocks to their Jordan form C' and D'. There are then three cases:

1. $C' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $D' = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ with $\lambda_1 \neq \lambda_2$ and $\mu_1 \neq \mu_2$,

2. $C' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ and $D' = \begin{pmatrix} \mu & \mu_1 \\ 0 & \mu \end{pmatrix}$, and

3. the case of complex eigenvalues where $C' = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$ and $D' = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_1 \end{pmatrix}$ with $\lambda_1 \neq \lambda_2$ and $\mu_1 \neq \mu_2$.

Note that in each case, with appropriate changes of basis we obtain the relation $[x_1, x_2] = R^1 e_1 + R^2 e_2$ for arbitrary constants R^1 and R^2. Thus in characterizing each case, we give only the forms of A^1 and A^2.

4.1. Case 1

In the first case we find to preserve the linear nilindependence of the adjoint matrices, we have the additional restriction

$$\text{rank} \begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix} = 2.$$

We then have four viable subcases which emerge from the Jacobi conditions above:

1. $\lambda 2 = 2\lambda_1$
2. $\lambda_1 = 2\lambda_2$
3. $\lambda_1 = -\lambda_2$
4. None of conditions 1-3 are true.

In the case where none of conditions 1-3 are true, we have two possible forms of solution which satisfy the Jacobi identity. The first is $\mu_1 \neq 0$ and the second is that $\mu_1 = 0$. In both cases we have $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ or case four degenerates to some other case.

In subcase 1 we have

$$A^1 \begin{bmatrix} \lambda 1 & 0 & 0 & 0 & p15 - p24 \\ 0 & 2\,\lambda 1 & 0 & 0 & p25 \\ 0 & 0 & \lambda 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} 3\,\mu 1 & 0 & 0 & 3\,q14 & -\frac{p24\,\mu 1 + p24\,\mu 2 - p15\,\mu 1 - p15\,\mu 2}{\lambda 1} \\ 0 & 3\,\mu 2 & 0 & 0 & -\frac{p25\,(-2\,\mu 2 + \mu 1)}{\lambda 1} \\ 0 & 0 & \mu 1 + \mu 2 & 0 & 0 \\ 0 & 0 & 0 & 2\,\mu 2 - \mu 1 & 0 \\ 0 & 0 & 0 & 0 & -\mu 2 + 2\,\mu 1 \end{bmatrix}$$

.

In subcase 2 we have

$$A^1 = \begin{bmatrix} 2\,\lambda 1 & 0 & 0 & 2\,p14 & 2\,p15 - 2\,p24 \\ 0 & \lambda 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda 1 \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} 3\,\mu 1 & 0 & 0 & 2\,\frac{p14\,(2\,\mu 1 - \mu 2)}{\lambda 1} & -2\,\frac{p24\,\mu 1 + p24\,\mu 2 - p15\,\mu 1 - p15\,\mu 2}{\lambda 1} \\ 0 & 3\,\mu 2 & 0 & 0 & 3\,q25 \\ 0 & 0 & \mu 2 + \mu 1 & 0 & 0 \\ 0 & 0 & 0 & 2\,\mu 2 - \mu 1 & 0 \\ 0 & 0 & 0 & 0 & 2\,\mu 1 - \mu 2 \end{bmatrix}.$$

In subcase 3 we have

$$A^1 = \begin{bmatrix} \lambda 1 & 0 & 0 & p14 & 0 \\ 0 & -\lambda 1 & 0 & 0 & p25 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda 1 \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} 3\,\mu 1 & 0 & 0 & \frac{p14\,(-\mu 2 + 2\,\mu 1)}{\lambda 1} & 3\,q15 - 3\,q24 \\ 0 & 3\,\mu 2 & 0 & 0 & \frac{p25\,(-2\,\mu 2 + \mu 1)}{\lambda 1} \\ 0 & 0 & \mu 2 + \mu 1 & 0 & 0 \\ 0 & 0 & 0 & 2\,\mu 2 - \mu 1 & 0 \\ 0 & 0 & 0 & 0 & -\mu 2 + 2\,\mu 1 \end{bmatrix}.$$

In subcase 4 note again that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ or the case is degenerate. Then we have

$$A^1 = \begin{bmatrix} 3\,\lambda 1 & 0 & 0 & 3\,p14 & 0 \\ 0 & 3\,\lambda 2 & 0 & 3\,p24 - 3\,p15 & 3\,p25 \\ 0 & 0 & \lambda 2 + \lambda 1 & 0 & 0 \\ 0 & 0 & 0 & 2\,\lambda 2 - \lambda 1 & 0 \\ 0 & 0 & 0 & 0 & -\lambda 2 + 2\,\lambda 1 \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} 3\,\mu 1 & 0 & 0 & 3\,\frac{p14\,(-\mu 2+2\,\mu 1)}{-\lambda 2+2\,\lambda 1} & 0 \\ 0 & 3\,\mu 2 & 0 & 3\,\frac{p24\,\mu 1+p24\,\mu 2-p15\,\mu 1-p15\,\mu 2}{\lambda 2+\lambda 1} & 3\,\frac{p25\,(-2\,\mu 2+\mu 1)}{-2\,\lambda 2+\lambda 1} \\ 0 & 0 & \mu 2 + \mu 1 & 0 & 0 \\ 0 & 0 & 0 & 2\,\mu 2 - \mu 1 & 0 \\ 0 & 0 & 0 & 0 & -\mu 2 + 2\,\mu 1 \end{bmatrix}.$$

4.2. Case 2

In the second case there is no combinations of values λ, μ, and μ_1 such that the adjoint matrices remain linearly nilindependent. Thus case 2 is an invalid case.

4.3. Case 3

To satisfy the Jacobi identity in the third case we find that either $\mu_2 = 0$ or $\lambda_2 = 0$. However the cases are isomorphic to one another, thus we describe the case where $\mu_2 = 0$ without loss of generality. Similarly to the first case we must have

$$\operatorname{rank}\begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix} = 2.$$

Then we have $\lambda_2 \neq 0$ as well as $\mu_1 \neq 0$ to preserve the linear nilindependence of the adjoint matrices. We then find,

$$A^1 = \begin{bmatrix} 3\,\lambda 1 & -3\,\lambda 2 & 0 & 3\,p14 & \frac{2\,p14\,\lambda 1\,\mu 1-3\,p24\,\lambda 2\,\mu 1+3\,p15\,\lambda 2\,\mu 1-2\,q14\,\lambda 1^2}{2\lambda 2\,\mu 1} \\ -3\,\lambda 2 & 3\,\lambda 1 & 0 & \frac{2\,p14\,\lambda 1\,\mu 1+3\,p24\,\lambda 2\,\mu 1-3\,p15\,\lambda 2\,\mu 1-2\,q14\,\lambda 1^2}{2\lambda 2\,\mu 1} & \beta \\ 0 & 0 & 2\,\lambda 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda 1 & -3\,\lambda 2 \\ 0 & 0 & 0 & -3\,\lambda 2 & \lambda 1 \end{bmatrix}$$

and

$$A^2 = \begin{bmatrix} 3\,\mu 1 & 0 & 0 & 3\,q14 & 2\,\frac{p14\,\mu 1-q14\,\lambda 1}{\lambda 2} \\ 0 & 3\,\mu 1 & 0 & 0 & \frac{2\,p14\,\lambda 1\,\mu 1+3\,p24\,\lambda 2\,\mu 1-3\,p15\,\lambda 2\,\mu 1-2\,q14\,\lambda 1^2+9\,q14\,\lambda 2^2}{3\lambda 2^2} \\ 0 & 0 & 2\,\mu 1 & 0 & 0 \\ 0 & 0 & 0 & \mu 1 & 0 \\ 0 & 0 & 0 & 0 & \mu 1 \end{bmatrix}$$

where $\beta = \frac{2\,p14\,\lambda 1^2\mu 1-9\,p14\,\mu 1\,\lambda 2^2+3\,p24\,\lambda 1\,\lambda 2\,\mu 1-3\,p15\,\lambda 1\,\lambda 2\,\mu 1-2\,q14\,\lambda 1^3+18\,q14\,\lambda 1\,\lambda 2^2}{3\mu 1\,\lambda 2^2}$.

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