Preface

Mathematical Reasoning: Writing and Proof is designed to be a text for the first course in the college mathematics curriculum that introduces students to the processes of constructing and writing proofs and focuses on the formal development of mathematics. The primary goals of the text are to help students:

- Develop logical thinking skills and to develop the ability to think more abstractly in a proof oriented setting.

- Develop the ability to construct and write mathematical proofs using standard methods of mathematical proof including direct proofs, proof by contradiction, mathematical induction, case analysis, and counterexamples.

- Develop the ability to read and understand written mathematical proofs.

- Develop talents for creative thinking and problem solving.

- Improve their quality of communication in mathematics. This includes improving writing techniques, reading comprehension, and oral communication in mathematics.

- Better understand the nature of mathematics and its language.

Another important goal of this text is to provide students with material that will be needed for their further study of mathematics.

This type of course has now become a standard part of the mathematics major at many colleges and universities. It is often referred to as a "transition course" from the calculus sequence to the upper-level courses in the major. The transition is from the problem-solving orientation of calculus to the more abstract and theoretical upper-level courses. This is needed today because many students complete their study of calculus without seeing a formal proof or having constructed a proof of their own. This is in contrast to many upper-level mathematics courses, where

the emphasis is on the formal development of abstract mathematical ideas, and the expectations are that students will be able to read and understand proofs and be able to construct and write coherent, understandable mathematical proofs. Students should be able to use this text with a background of one semester of calculus.

Important Features of the Book

Following are some of the important features of this text that will help with the transition from calculus to upper-level mathematics courses.

1. **Emphasis on Writing in Mathematics**

   Issues dealing with writing mathematical exposition are addressed throughout the book. Guidelines for writing mathematical proofs are incorporated into the book. These guidelines are introduced as needed and begin in Section 1.2. Appendix A contains a summary of all the guidelines for writing mathematical proofs that are introduced throughout the text. In addition, every attempt has been made to ensure that every completed proof presented in this text is written according to these guidelines. This provides students with examples of well-written proofs.

   One of the motivating factors for writing this book was to develop a textbook for the course "Communicating in Mathematics" at Grand Valley State University. This course is part of the university's Supplemental Writing Skills Program, and there was no text that dealt with writing issues in mathematics that was suitable for this course. This is why some of the writing guidelines in the text deal with the use of LaTeX or a word processor that is capable of producing the appropriate mathematical symbols and equations. However, the writing guidelines can easily be implemented for courses where students do not have access to this type of word processing.

2. **Instruction in the Process of Constructing Proofs**

   One of the primary goals of this book is to develop students' abilities to construct mathematical proofs. Another goal is to develop their abilities to write the proof in a coherent manner that conveys an understanding of the proof to the reader. These are two distinct skills.

   Instruction on how to write proofs begins in Section 1.2 and is developed further in Chapter 3. In addition, Chapter 4 is devoted to developing students' abilities to construct proofs using mathematical induction.

Students are introduced to a method to organize their thought processes when attempting to construct a proof that uses a so-called know-show table. (See Section 1.2 and Section 3.1.) Students use this table to work backward from what it is they are trying to prove while at the same time working forward from the assumptions of the problem. The know-show tables are used quite extensively in Chapters 1 and 3. However, the explicit use of know-show tables is gradually reduced and these tables are rarely used in the later chapters. One reason for this is that these tables may work well when there appears to be only one way of proving a certain result. As the proofs become more complicated or other methods of proof (such as proofs using cases) are used, these know-show tables become less useful.

So the know-show tables are not to be considered an absolute necessity in using the text. However, they are useful for students beginning to learn how to construct and write proofs. They provide a convenient way for students to organize their work. More importantly, they introduce students to a way of thinking about a problem. Instead of immediately trying to write a complete proof, the know-show table forces students to stop, think, and ask questions such as

- Just exactly what is it that I am trying to prove?
- How can I prove this?
- What methods do I have that may allow me to prove this?
- What are the assumptions?
- How can I use these assumptions to prove the result?

Being able to ask these questions is a big step in constructing a proof. The next task is to answer the questions and to use those answers to construct a proof.

3. Emphasis on Active Learning

One of the underlying premises of this text is that the best way to learn and understand mathematics is to be actively involved in the learning process. However, it is unlikely that students will learn all the mathematics in a given course on their own. Students actively involved in learning mathematics need appropriate materials that will provide guidance and support in their learning of mathematics. There are several ways this text promotes active learning.

- With the exception of Sections 1.1 and 3.6, each section has exactly two preview activities. These preview activities should be completed by the students prior to the classroom discussion of the section. The purpose of the preview activities is to prepare students to participate in the classroom discussion of the section. Some preview activities will review prior mathematical work that is necessary for the new section. This prior work may contain material from previous mathematical courses or it may contain material covered earlier in this text. Other preview activities will introduce new concepts and definitions that will be used when that section is discussed in class.

- Several progress checks are included in each section. These are either short exercises or short activities designed to help the students determine if they are understanding the material as it is presented. Some progress checks are also intended to prepare the student for the next topic in the section. Answers to the progress checks are provided in Appendix B.

- Explorations and activities are included at the end of the exercises of each section. These activities can be done individually or in a collaborative learning setting, where students work in groups to brainstorm, make conjectures, test each others' ideas, reach consensus, and, it is hoped, develop sound mathematical arguments to support their work. These activities can also be assigned as homework in addition to the other exercises at the end of each section.

4. Other Important Features of the Book

- Several sections of the text include exercises called Evaluation of Proofs. (The first such exercise appears in Section 3.1.) For these exercises, there is a proposed proof of a proposition. However, the proposition may be true or may be false. If a proposition is false, the proposed proof is, of course, incorrect, and the student is asked to find the error in the proof and then provide a counterexample showing that the proposition is false. However, if the proposition is true, the proof may be incorrect or not well written. In keeping with the emphasis on writing, students are then asked to correct the proof and/or provide a well-written proof according to the guidelines established in the book.

- To assist students with studying the material in the text, there is a summary at the end of each chapter. The summaries usually list the important definitions introduced in the chapter and the important results

proven in the chapter. If appropriate, the summary also describes the important proof techniques discussed in the chapter.

- Answers or hints for several exercises are included in an appendix. This was done in response to suggestions from many students at Grand Valley and some students from other institutions who were using the book. In addition, those exercises with an answer or a hint in the appendix are preceded by a star (⋆).

Content and Organization

Mathematical content is needed as a vehicle for learning how to construct and write proofs. The mathematical content for this text is drawn primarily from elementary number theory, including congruence arithmetic; elementary set theory; functions, including injections, surjections, and the inverse of a function; relations and equivalence relations; further topics in number theory such as greatest common divisors and prime factorizations; and cardinality of sets, including countable and uncountable sets. This material was chosen because it can be used to illustrate a broad range of proof techniques and it is needed as a prerequisite for many upper-level mathematics courses.

The chapters in the text can roughly be divided into the following classes:

- Constructing and Writing Proofs: Chapters 1, 3, and 4

- Logic: Chapter 2

- Mathematical Content: Chapters 5, 6, 7, 8, and 9

The first chapter sets the stage for the rest of the book. It introduces students to the use of conditional statements in mathematics, begins instruction in the process of constructing a direct proof of a conditional statement, and introduces many of the writing guidelines that will be used throughout the rest of the book. This is not meant to be a thorough introduction to methods of proof. Before this is done, it is necessary to introduce the students to the parts of logic that are needed to aid in the construction of proofs. This is done in Chapter 2.

Students need to learn some logic and gain experience in the traditional language and proof methods used in mathematics. Since this is a text that deals with constructing and writing mathematical proofs, the logic that is presented in Chapter 2 is intended to aid in the construction of proofs. The goals are to provide

students with a thorough understanding of conditional statements, quantifiers, and logical equivalencies. Emphasis is placed on writing correct and useful negations of statements, especially those involving quantifiers. The logical equivalencies that are presented provide the logical basis for some of the standard proof techniques, such as proof by contrapositive, proof by contradiction, and proof using cases.

The standard methods for mathematical proofs are discussed in detail in Chapter 3. The mathematical content that is introduced to illustrate these proof methods includes some elementary number theory, including congruence arithmetic. These concepts are used consistently throughout the text as a way to demonstrate ideas in direct proof, proof by contrapositive, proof by contradiction, proof using cases, and proofs using mathematical induction. This gives students a strong introduction to important mathematical ideas while providing the instructor a consistent reference point and an example of how mathematical notation can greatly simplify a concept.

The three sections of Chapter 4 are devoted to proofs using mathematical induction. Again, the emphasis is not only on understanding mathematical induction but also on developing the ability to construct and write proofs that use mathematical induction.

The last five chapters are considered "mathematical content" chapters. Concepts of set theory are introduced in Chapter 5, and the methods of proof studied in Chapter 3 are used to prove results about sets and operations on sets. The idea of an "element-chasing proof" is also introduced in Section 5.2.

Chapter 6 provides a thorough study of functions. Functions are studied before relations in order to begin with the more specific notion with which students have some familiarity and move toward the more general notion of a relation. The concept of a function is reviewed but with attention paid to being precise with terminology and is then extended to the general definition of a function. Various proof techniques are employed in the study of injections, surjections, composition of functions, inverses of functions, and functions acting on sets.

Chapter 7 introduces the concepts of relations and equivalence relations. Section 7.4 is included to provide a link between the concept of an equivalence relation and the number theory that has been discussed throughout the text.

Chapter 8 continues the study of number theory. The highlights include problems dealing with greatest common divisors, prime numbers, the Fundamental Theorem of Arithmetic, and linear Diophantine equations.

Finally, Chapter 9 deals with further topics in set theory, focusing on cardinality, finite sets, countable sets, and uncountable sets.

Designing a Course

Most instructors who use this text will design a course specifically suited to their needs and the needs of their institution. However, a standard one-semester course in constructing and writing proofs could cover the first six chapters of the text and at least one of Chapter 7, Chapter 8, or Chapter 9. Please note that Sections 4.3, 5.5, 6.6, 7.4, and 8.3 can be considered optional sections. These are interesting sections that contain important material, but the content of these sections is not essential to study the material in the rest of the book.

Supplementary Materials for the Instructor

Instructors for a course may obtain pdf files that contain the solutions for the preview activities and the solutions for the exercises.

To obtain these materials, send an email message to the author at mathreasoning@gmail.com, and please include the name of your institution (school, college, or university), the course for which you are considering using the text, and a link to a website that can be used to verify your position at your institution.

Although not part of the textbook, there are now 107 online videos with about 14 hours of content that span the first seven chapters of this book. These videos are freely available online at Grand Valley's Department of Mathematics YouTube channel on this playlist:

$$\text{http://gvsu.edu/s/0ll}$$

These online videos were created and developed by Dr. Robert Talbert of Grand Valley State University.

There is also a website for the textbook. For this website, go to

$$\text{www.tedsundstrom.com}$$

and click on the TEXTBOOKS button in the upper right corner.

You may find some things there that could be of help to your students. For example, there currently is a link to study guides for most of the sections of this textbook. If there are things that you think would be good additions to the book or the web site, please feel free to send me a message at mathreasoning@gmail.com.