Section 1

Sets

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a set?
- What is a subset of a set?
- What is the union of two sets? How do we define the union of an arbitrary collection of sets?
- What is the intersection of two sets? How do we define the intersection of an arbitrary collection of sets?
- What is the complement of a set?
- What is the Cartesian product of sets?

Introduction

At its most basic level topology deals with sets and how we can deform sets into other sets. So to start our study of topology, we begin with sets. Much of this material should be familiar, but some might be new. The first issue for us to settle on is as precise a definition of "set" as possible.

Preview Activity 1.1. Suppose we try to define a set to be a collection of elements. So, by definition, the elements are the objects contained in the set. We use the symbol $\in$ to denote that an object is an element of a set. So $\notin$ means an object is not in the set – if x is an object in a set S we write $x \in S$, and if x is not an object in a set S we write $x \notin S$. We write sets using set brackets. For example, the set $\{a, b, c\}$ is the set whose elements are the symbols a, b, and c. We can also include in the set notation conditions on elements of the set. For example, $\{x \in \mathbb{R} : x > 0\}$ is the set of positive real numbers. We typically use capital letters to denote sets. Some familiar examples of sets are $\mathbb{R}$, the set of real numbers; $\mathbb{Q}$, the set of rational numbers; and $\mathbb{Z}$, the set of integers. Sets can also contain sets as elements. For example, the power set of a set S is the set of subsets of S. So the power set of $S = \{1, 2\}$ is the set $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. (We will define subsets and the empty set later in this activity).

(1) Consider the following "set" S, which is a collection of objects:

$$S = \{A \text{ is a set} \mid A \notin A\}.$$

That is, S is the collection of sets that do not have themselves as elements.

Given any object x, either $x \in S$ or $x \notin S$.

(a) Is S an element of S? Explain.

(b) Is it the case that $S \notin S$? Explain.

(c) Based on your responses to parts (a) and (b), explain why our current concept of a set as a collection of elements is not a good one.

(2) Assume that we have a working definition of a set. In this part of the activity we define a subset of a set. The notation we will use is $A \subset S$ if A is a subset of S that is not equal to S, and $A \subseteq S$ if A is a subset of S that could be the entire set S. We will also say that A is *contained* in S if A is a subset of S, and call the relation $A \subset S$ (or $A \subseteq S$) a *containment*.

(a) How should we define a subset of a set? Give a specific example of a set and two examples of subsets of that set.

(b) If A is a set, is A a subset of A? Explain.

(c) What is the empty set $\emptyset$? If A is a set, is $\emptyset$ a subset of A? explain.

The Basic Idea of Topology

If you like geometry, you will probably like topology. Geometry is the study of objects with certain attributes (e.g., shape and size), while topology is more general than geometry. In topology, we aren't concerned about the attributes (shape and size) of an object, only about those characteristics that don't change when we transform the object in different ways (any way that doesn't involve tearing or poking holes the object). There are lots of really interesting theorems in topology – for example, the Hairy Ball Theorem which states that if you have a ball with hair all over it (think of a tribble from Star Trek – if that isn't too old of a reference), it is impossible to comb the hairs continuously and have all the hairs lay flat. Some hair must be sticking straight up!

Activity 1.1.

(a) Take a pipe cleaner, a rubber band, or a piece of string and make a square from it. You are allowed to change the square by moving parts of the square without breaking it or lifting it off the surface it is on. To which of the following shapes can you transform your square? Explain.

(i.) a circle (ii.) the letter S (iii.) a five point star ☆ (iv.) the letter D

(b) Now take some play-doh (if you don't have any play-doh, just use your imagination). Use the play-doh (or your imagination) to determine which of the following shape can be transformed into others without breaking or making holes.

(i.) a filled sphere (ii.) a doughnut (iii.) a bowl (iv.) a coffee mug with handle

This idea of transforming one set into another as we explored in Activity 1.1 is formally done with functions. As we progress through this subject, we will need to have more rigorous definitions of functions and sets. We begin with sets and discuss functions in Section 2.

Intervals

We will begin with one of the most basic type of set we will encounter – intervals. The open intervals will be important as they will form a basis for the standard topology on $\mathbb{R}$. We are likely familiar with intervals from algebra and calculus, sets like $(0, 1)$, $[2, 3)$. To really understand intervals, we will need a rigorous definition.

Definition 1.1. A subset I of $\mathbb{R}$ is an **interval** if for all a, b, and c in $\mathbb{R}$ (allowing for a or b to be $\pm\infty$) with $a < c < b$, if a and b are in I, then c is in I.

With this definition, the set of all real numbers x satisfying $0 < x < 1$ is an interval that we denote by $(0, 1)$ (it is important to understand the context – we also use the notation $(0, 1)$ to denote an ordered pair). The general notation we use for intervals is the following:

- $(a, b) = \{t \in \mathbb{R} \mid a < t < b\}$ (a or b could be $\pm\infty$)
- $[a, b) = \{t \in \mathbb{R} \mid a \leq t < b\}$ (b could be $\pm\infty$)
- $(a, b] = \{t \in \mathbb{R} \mid a < t \leq b\}$ (a could be $\pm\infty$)
- $[a, b] = \{t \in \mathbb{R} \mid a \leq t \leq b\}$.

In this notation, $\mathbb{R} = (-\infty, \infty)$. Intervals of the form (a, b) are called *open* intervals, intervals of the form $[a, b]$ are called *closed* intervals, and intervals of the form $[a, b)$ or $(a, b]$ are *half-open* (or *half-closed*) intervals. The reason for this terminology should become more clear as we introduce open and closed sets later.

Note that nothing in the definition indicates that we must have $a < b$ in the interval notation. This implies that $(1, 1)$ is an interval. Since there are no real numbers larger than 1 and less than 1, $\emptyset = (1, 1)$ is an interval. We could also have an interval of the form $[a, a]$ where a is any real number. This means that $\{a\} = [a, a]$, and any single point set is an interval. The intervals $\emptyset$ and $[a, a]$ for any real number a are called *degenerate* intervals.

One last note about intervals. Some require that a be less than b in the definition of an interval, with the result that there are no degenerate intervals. This is a matter of debate that we won't get into. In almost all of our work, we will consider only non-degenerate intervals so this won't be an issue for us.

Unions, Intersections, and Complements of Sets

In mathematics, the collection of points that make up a string or a blob of play-doh as in Activity 1.1 is represented as a set. Topology is then the study of these sets and what properties of the sets don't change when transformations are applied to the sets. To study topology we will need a solid understanding of sets and different operations on sets.

What we saw in Preview Activity 1.1 is what is called a *paradox*. Our original attempt to define a set led to an impossible situation since both $S \in S$ and $S \notin S$ lead to contradictions. This paradox is called *Russell's paradox* after Bertrand Russell, although it was apparently known before Russell. The moral of the story is that we need to be careful when making definitions. A set might seem like a simple object, and in our experience usually is, but formally defining a set can be problematic. As a result, we won't state a formal definition, but rather take a set to be a collection of objects that doesn't lead to a paradox. The objects are called the elements of the set. (In axiomatic set theory, a set is taken to be an undefined primitive – much as a point is undefined in Euclidean geometry.)

In order to effectively work with sets, we need to have an understanding what it means for two sets to be equal.

Activity 1.2.

(a) What should it mean for two sets to be equal? If A and B are sets, how do we prove that $A = B$? (This is question that requires discussion, which is different than a question that only asks for a computation or an example. Activities throughout this text will ask both types of questions.)

(b) Let $A = \{x \in \mathbb{R} \mid x < 2\}$ and $B = \{x \in \mathbb{R} \mid x - 1 < 1\}$. Is $A = B$? If yes, prove your answer. If no, prove any containment that you can.

(c) Let $A = \{n \in \mathbb{Z} \mid 2 \text{ divides } n\}$ and $B = \{n \in \mathbb{Z} \mid 4 \text{ divides } (n-2)\}$. Is $A = B$? If yes, prove your answer. If no, prove any containment that you can.

(d) Let $A = \{n \in \mathbb{Z} \mid n \text{ is odd}\}$ and $B = \{n \in \mathbb{Z} \mid 4 \text{ divides } (n-1) \text{ or } 4 \text{ divides } (n-3)\}$. Is $A = B$? If yes, prove your answer. If no, prove any containment that you can.

Once we have the notion of a set, we can build new sets from existing ones. For example, we define the union, intersection, set difference, and complement of a set as follows.

- The **union** of sets A and B is the set $A \cup B$ defined as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

- The **intersection** of sets A and B is the set $A \cap B$ defined as

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

- Let A and B be sets. The **set difference** $A \setminus B$ is the set

$$A \setminus B = \{a \in A \mid a \notin B\}.$$

- Let A be a subset of a set U. The **complement** of A in U is the set

$$U \setminus A = \{x \in U \mid x \notin A\}.$$

  The complement of a set A in a set U is also denoted by $C_U(A)$, $C(A)$ (if the set U is understood), A^c, or even $U - A$.

We can visualize these sets using Venn diagrams. A Venn diagram is a depiction of sets using geometric figures. For example, if U is a set containing all other sets of interest (we call U the *universal* set), we can represent U as a large container (say a rectangle) with subsets A and B as smaller containers (say circles), and shade the elements in a given set. The Venn diagrams in Figure 1.1 depict the sets A, B, $A \cup B$, $A \cap B$, A^c, and B^c.

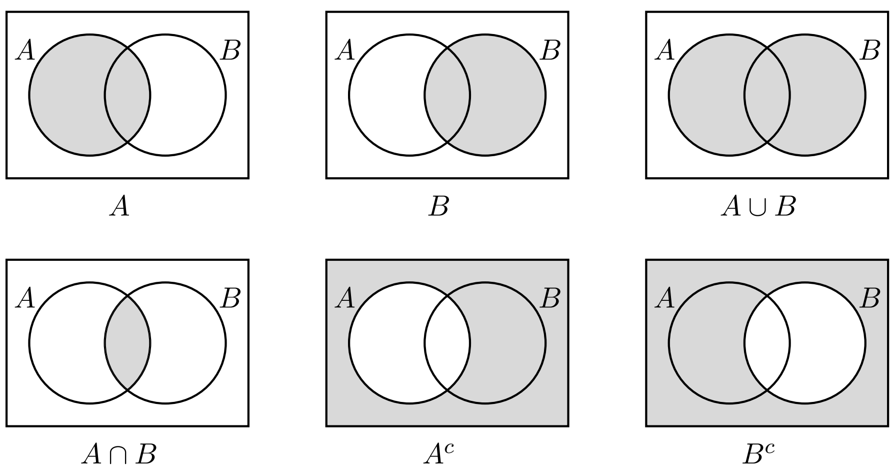

Figure 1.1: Vennn diagrams

As we have discussed, to prove that two sets X and Y are equal we prove that each is a subset of the other. The next example provides another illustration of the idea.

Example 1.2. Let A, B, and C be sets. We will prove that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

To prove this set equality we must prove that $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$ and $(A \cap B) \setminus (A \cap C) \subseteq A \cap (B \setminus C)$. We start with $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$.

To prove that $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$, we need to demonstrate that every element in $A \cap (B \setminus C)$ is also in $(A \cap B) \setminus (A \cap C)$. To do this, we select an arbitrary element in $A \cap (B \setminus C)$ and show that this element is in $(A \cap B) \setminus (A \cap C)$. Let $x \in A \cap (B \setminus C)$. Then $x \in A$ and $x \in B \setminus C$. The fact that $x \in B \setminus C$ implies that $x \in B$ but $x \notin C$. Therefore, $x \in A$ and $x \in B$, but $x \notin C$. This implies that $x \in A$ and $x \in B$, but $x \in A$ and $x \notin C$. So $x \in A$ and $x \in B$, but $x \notin A \cap C$. We conclude that $x \in (A \cap B) \setminus (A \cap C)$. This proves that $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$.

For the reverse containment, we let $y \in (A \cap B) \setminus (A \cap C)$. So $y \in A \cap B$ but $y \notin A \cap C$. Since $y \in A \cap B$, we know that $y \in A$ and $y \in B$. The fact that $y \notin A \cap C$ means that $y \notin C$. So $y \in A$, $y \in B$, and $y \notin C$. Thus, $y \in A$ and $y \in B \setminus C$. We conclude that $y \in A \cap (B \setminus C)$, which shows that $(A \cap B) \setminus (A \cap C) \subseteq A \cap (B \setminus C)$. The two containments, $A \cap (B \setminus C) \subseteq (A \cap B) \setminus (A \cap C)$ and $(A \cap B) \setminus (A \cap C) \subseteq A \cap (B \setminus C)$ demonstrate that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

We will use the ideas in Activity 1.2 and Example 1.2 to prove set equalities throughout this text. The next activity will provide some additional practice.

Activity 1.3. In this activity we work with unions, intersections, and complements of sets. Let A and B be sets.

(a) Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8, 10\}$, with $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

i. Determine the elements in $A \cup B$ and $A \cap B$. What are the elements in $(A \cup B)^c$ and $(A \cap B)^c$?

ii. Determine the elements in $A^c \cup B^c$ and $A^c \cap B^c$.

(b) Let A and B be arbitrary subsets of a universal set U. There are connections between A, B and their complements, unions, and intersections.

i. Use Venn diagrams to draw $(A \cup B)^c$ and $(A \cap B)^c$.

ii. Use the Venn diagrams and the result of (a) to find and prove a relationship between A^c, B^c and $(A \cup B)^c$.

iii. Use the Venn diagrams and the result of (a) to find and prove a relationship between A^c, B^c and $(A \cap B)^c$.

In Activity 1.3 we worked with the union and intersection of two sets. There is no reason to restrict these definitions to only two sets, as the next activity illustrates.

Activity 1.4. To define an infinite collection of sets we often use what is called an *indexing set*. An indexing set allows us to consider a collection of objects that are in one-to-one correspondence with a set like the positive integers, or even the real numbers. When using an indexing set, we generally make a statement such as "let $\{A_\alpha\}$ for $\alpha \in I$ be a collection of sets indexed by some set I". The collection $\{A_\alpha\}_{\alpha \in I}$ is called an *indexed family of sets*.

(a) The set I could be finite. As an example, let $A_n = \{1, 2, 3, \ldots n\}$ for n in the set $I = \{1, 2, 3, \ldots, 10\}$.

i. What is A_5? What is A_8?

ii. How many sets are in the indexed family $\{A_n\}_{n \in I}$?

(b) The indexing set can be infinite. For example, let $A_\alpha = [0, |\alpha|)$ for α in the set $\mathbb{R}$ (where $[a, b)$ is the interval consisting of the real numbers x such that $a \leq x < b$). In this case, what is A_5? What is A_π? What is $A_{-\frac{2}{3}}$?

(c) We have defined the union and intersection of two sets. The same idea can be extended to define the union and intersection of an indexed collection of sets.

i. Recall that if A and B are sets, the intersection $A \cap B$ is the set $\{x \mid x \in A \text{ and } x \in B\}$. How can we extend this definition from two sets to any collection of sets? In other words, how do we **define**

$$\bigcap_{\alpha \in I} A_\alpha?$$

In the example in (b), what set is $\bigcap_{\alpha \in \mathbb{R}} A_\alpha$?

ii. Recall that if A and B are sets, the union $A \cup B$ is the set $\{x \mid x \in A \text{ or } x \in B\}$. How can we extend this definition from two sets to any collection of sets? In other words, how do we **define**

$$\bigcup_{\alpha \in I} A_\alpha?$$

In the example in (b), what set is $\bigcup_{\alpha \in \mathbb{R}} A_\alpha$?

These properties $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$ that we learned about in Activity 1.3 are called DeMorgan's Laws. These laws apply to any union or intersection of sets, finite or infinite. The proofs are left Exercise (4).

Theorem 1.3 (DeMorgan's Laws). *Let $\{A_\alpha\}$ is a collection of sets indexed by a set I in some universal set U. Then*

(1) $$\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} A_\alpha^c$$

(2) $$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$$

Activity 1.5.

(a) Verify DeMorgan's Laws in the specific case of $A_\alpha = \{1, 2, 3, \ldots \alpha\}$ in $U = \mathbb{Z}$, where α is any element of the indexing set $I = \mathbb{Z}^+$.

(b) Why should the complement of a union be an intersection and why should the complement of an intersection be a union? (Hint: Consider the definitions of unions and intersections.)

Cartesian Products of Sets

The final operation on sets that we discuss is the *Cartesian product* (or *cross product*). This is an operation that we have seen before. When we draw the graph of a line $y = mx + b$ in the plane, we plot the points $(x, mx + b)$. These points are ordered pairs of real numbers. We can extend this idea to any sets.

Definition 1.4. Let A and B are sets. The **Cartesian product** of A and B is the set

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

In other words, the Cartesian product of A and B is the set of ordered pairs (a, b) with a coming from A and b coming from B. Note that the order is important.

Activity 1.6.

(a) List all of the elements in $\{\text{red}, \text{blue}\} \times \{\text{car}, \text{truck}, \text{van}\}$.

(b) If A has m elements and B has n elements, how many elements does the set $A \times B$ have? Explain.

There is no reason to restrict ourselves to a Cartesian product of just two sets. This is an idea that we have encountered before. The Cartesian product $\mathbb{R} \times \mathbb{R}$ is the standard real plane that we denote as $\mathbb{R}^2$ and the Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the three-dimensional real space denoted as $\mathbb{R}^3$. If we have an indexed collection $\{X_i\}$ of sets, with i running through the set of positive integers, then we can define the Cartesian product of the sets X_i as the set of infinite sequences $(x_1, x_2, \ldots, x_n, \ldots)$, where $x_i \in X_i$ for each $i \in \mathbb{Z}^+$. We denote this cartesian product as

$$\Pi_{i \in \mathbb{Z}^+} X_i = \Pi_{i=1}^{\infty} X_i.$$

The capital pi (Π) is used to represent a product an an analog of the capital sigma (Σ) that is used to represent a sum. We will study sequences in more detail later.

To conclude this section we summarize some properties of sets. Many of these properties can be extended to arbitrary collections of sets. Most of the proofs are straightforward. The associative and distributive laws are left for Exercise (3).

Theorem 1.5. *Let A, B, and C be subsets of a universal set U.*

Properties of the Empty Set.

i. $A \cap \emptyset = \emptyset$

ii. $A \cup \emptyset = A$

iii. $A - \emptyset = A$

iv. $\emptyset^c = U$

Properties of the Universal Set.

i. $A \cap U = A$

ii. $A \cup U = U$

iii. $A - U = \emptyset$

iv. $U^c = \emptyset$

Idempotent Laws.

i. $A \cap A = A$

ii. $A \cup A = A$

Commutative Laws.

i. $A \cap B = B \cap A$

ii. $A \cup B = B \cup A$

Associative Laws.

i. $(A \cap B) \cap C = A \cap (B \cap C)$

ii. $(A \cup B) \cup C = A \cup (B \cup C)$

Distributive Laws.

i. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

ii. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Basic Properties.

i. $(A^c)^c = A$

ii. $A - B = A \cap B^c$

Subsets and Complements.

$A \subseteq B$ *if and only if* $B^c \subseteq A^c$

Summary

Important ideas that we discussed in this section include the following.

- We can consider a set to be a well-defined collection of elements.
- A subset of a set is any collection of elements from that set. That is, a subset S of a set X is a set with the property that if $s \in S$, then $s \in X$.
- If X and Y are sets, then the union $X \cup Y$ is the set

$$X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}.$$

  The union of an arbitrary collection $\{X_\alpha\}$ of sets for α in some indexing set I is the set

$$\bigcup_{\alpha \in I} X_\alpha = \{z \mid z \in X_\beta \text{ for some } \beta \in I\}.$$

- If X and Y are sets, then the intersection $X \cap Y$ is the set

$$X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}.$$

  The intersection of an arbitrary collection $\{X_\alpha\}$ of sets for α in some indexing set I is the set

$$\bigcap_{\alpha \in I} X_\alpha = \{z \mid z \in X_\beta \text{ for all } \beta \in I\}.$$

- If X is a set and A is a subset of X, then the complement of A in X is the set

$$A^c = \{x \in X \mid x \notin A\}.$$

- If $\{X_i\}$ is a collection of sets with i in some indexing set I, where I is finite or I is the set of positive integers, the Cartesian product $\Pi_{i \in I} X_i$ of the sets X_i as the set of all ordered tuples of the form (x_i) where $i \in I$.

Exercises

(1) Let A, B, and C be subsets of a set X. Express each of the following sets in mathematical notation using the symbols $\cup$, $\cap$, and $\setminus$.

(a) The elements of X that belong to A and B, but not C.

(b) The elements of X that belong to C and either A or B.

(c) The elements of X that belong to A but not to both B and C.

(d) The elements of X that belong to none of the sets A, B, and C.

(e) The elements of X that fail to belong to at least two of the sets A, B, and C.

(f) The elements of X that fail to belong to at most one of the sets A, B, and C.

(2) Let $X \subset Y \subset Z$. Prove or disprove.

(a) $C_Y(X) \subset C_Z(X)$.

(b) $Z \setminus (Y \setminus X) = X \cup (Z \setminus Y)$.

(3) Let A and B be subsets of a universal set U. Prove the associative and distributive laws. That is, prove each of the following.

(a) $(A \cap B) \cap C = A \cap (B \cap C)$

(b) $(A \cup B) \cup C = A \cup (B \cup C)$

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(4) Prove DeMorgan's Laws. That is, let $\{A_\alpha\}$ be a collection of sets indexed by a set I in some universal set U. Prove that

(a) $\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} A_\alpha^c$

(b) $\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$

(5) What familiar set is $\emptyset \times A$ for any set A? Explain.

(6) If A is a set, the power set of A, denoted 2^A is the collection of all subsets of A.

(a) List the elements of $2^{\{1,2\}}$.

(b) If A is a set with three elements, how many elements are in 2^A?

(c) If A is a set with n elements, make a conjecture about the number of elements in 2^A. Prove your conjecture?

(7) If A is a set, the power set of A, denoted 2^A is the collection of all subsets of A. (See Exercise (6).) Critique each of the following statements. Doe the statement make sense or not? If not, explain why and then correct the statement to something that is true (and non-trivial).

(a) If A is a set, then $A \in 2^A$.

(b) If A is a set, then $A \subset 2^A$.

(c) If A is a set, then $\{A\} \subset 2^A$.

(d) If A is a set, then $\emptyset \in 2^A$.

(e) If A is a set, then $\emptyset \subset 2^A$.

(f) If A and B are sets and $A \subseteq B$, then $2^A \subseteq 2^B$.

(8) Let A and B be sets, both of which have at least two distinct members. Prove that there is a subset $W \subset A \times B$ that is not the Cartesian product of a subset of A with a subset of B. [Thus, not every subset of a Cartesian product is the Cartesian product of a pair of subsets.]

(9) Let I be the set of real numbers that are greater than 0. For each $x \in I$, let A_x be the open interval $(0, x)$. Prove that $\bigcap_{x \in I} A_x = \emptyset$, $\bigcup_{x \in I} A_x = I$. For each $x \in I$, let B_x be the closed interval $[0, x]$. Prove that $\bigcap_{x \in I} B_x = \{0\}$, $\bigcup_{x \in I} B_x = I \cup \{0\}$.

(10) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

As an example of a true statement, consider the statement

Let A, B, and C be sets such that $A \cap B = A \cap C$ and $A \cap B \neq \emptyset$. Then $B \cap C \neq \emptyset$.

We can justify the truth of this statement with a short argument. Since $A \cap B \neq \emptyset$, there is an element $x \in A \cap B$. Then $x \in B$. Since $A \cap B = A \cap C$, we also must have $x \in A \cap C$, which implies that $x \in C$. Thus, $x \in B \cap C$ and $B \cap C \neq \emptyset$.

As an example of a false statement, consider the statement

Let A, B, and C be sets such that $A \cap B = A \cap C$. Then $B = C$.

We can show that this statement is false by providing a counterexample. For example, let $A = \{0, 1\}$, $B = \{1\}$, and $C = \{1, 2\}$. Then $A \cap B = \{1\} = A \cap C$, but $B \neq C$.

(a) If A, B, and C are sets and $A \subseteq B$ and $A \subseteq C$, then $A \subseteq (B \cap C)$.

(b) If A, B, and C are sets and $A \subseteq C$ and $B \subseteq C$, then $(A \cup B) \subseteq C$.

(c) If A and B are subsets of a set X and $A \subseteq B$, then $(X \setminus A) \subseteq (X \setminus B)$.

(d) If A and B are subsets of a set X and $A \subseteq B$, then $(X \setminus B) \subseteq (X \setminus A)$.

(e) If A and B are sets, then $(A \cup B) \setminus B = A$.

(f) If A and B are sets, then $A \setminus (A \setminus B) = B$.

(g) If A, B, and C are sets, then $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

(h) If A and C are subsets of a set X, then $(A \setminus C) = A \cap (X \setminus C)$.

(i) There are no elements of the set $\{\emptyset\}$.

(j) There are two distinct objects that belong to the set $\{\emptyset, \{\emptyset\}\}$.