

## Chapter 9

# Finite and Infinite Sets

### 9.1 Finite Sets

#### Preview Activity 1 (Equivalent Sets, Part 1)

- Let  $A$  and  $B$  be sets and let  $f$  be a function from  $A$  to  $B$ . ( $f : A \rightarrow B$ ). Carefully complete each of the following using appropriate quantifiers: (If necessary, review the material in Section 6.3.)
  - The function  $f$  is an injection provided that . . . .
  - The function  $f$  is not an injection provided that . . . .
  - The function  $f$  is a surjection provided that . . . .
  - The function  $f$  is not a surjection provided that . . . .
  - The function  $f$  is a bijection provided that . . . .

**Definition.** Let  $A$  and  $B$  be sets. The set  $A$  is **equivalent** to the set  $B$  provided that there exists a bijection from the set  $A$  onto the set  $B$ . In this case, we write  $A \approx B$ .

When  $A \approx B$ , we also say that the set  $A$  is in **one-to-one correspondence** with the set  $B$  and that the set  $A$  has the same **cardinality** as the set  $B$ .

**Note:** When  $A$  is not equivalent to  $B$ , we write  $A \not\approx B$ .

- For each of the following, use the definition of equivalent sets to determine if the first set is equivalent to the second set.

- (a)  $A = \{1, 2, 3\}$  and  $B = \{a, b, c\}$
- (b)  $C = \{1, 2\}$  and  $B = \{a, b, c\}$
- (c)  $X = \{1, 2, 3, \dots, 10\}$  and  $Y = \{57, 58, 59, \dots, 66\}$
3. Let  $D^+$  be the set of all odd natural numbers. Prove that the function  $f: \mathbb{N} \rightarrow D^+$  defined by  $f(x) = 2x - 1$ , for all  $x \in \mathbb{N}$ , is a bijection and hence that  $\mathbb{N} \approx D^+$ .
4. Let  $\mathbb{R}^+$  be the set of all positive real numbers. Prove that the function  $g: \mathbb{R} \rightarrow \mathbb{R}^+$  defined by  $g(x) = e^x$ , for all  $x \in \mathbb{R}$  is a bijection and hence, that  $\mathbb{R} \approx \mathbb{R}^+$ .

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**Preview Activity 2 (Equivalent Sets, Part 2)**

1. Review Theorem 6.20 in Section 6.4, Theorem 6.26 in Section 6.5, and Exercise (9) in Section 6.5.
2. Prove each part of the following theorem.

**Theorem 9.1.** *Let  $A$ ,  $B$ , and  $C$  be sets.*

- (a) *For each set  $A$ ,  $A \approx A$ .*
- (b) *For all sets  $A$  and  $B$ , if  $A \approx B$ , then  $B \approx A$ .*
- (c) *For all sets  $A$ ,  $B$ , and  $C$ , if  $A \approx B$  and  $B \approx C$ , then  $A \approx C$ .*

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**Equivalent Sets**

In Preview Activity 1, we introduced the concept of equivalent sets. The motivation for this definition was to have a formal method for determining whether or not two sets “have the same number of elements.” This idea was described in terms of a one-to-one correspondence (a bijection) from one set onto the other set. This idea may seem simple for finite sets, but as we will see, this idea has surprising consequences when we deal with infinite sets. (We will soon provide precise definitions for finite and infinite sets.)

**Technical Note:** The three properties we proved in Theorem 9.1 in Preview Activity 2 are very similar to the concepts of reflexive, symmetric, and transitive relations. However, we do not consider equivalence of sets to be an equivalence



relation on a set  $U$  since an equivalence relation requires an underlying (universal) set  $U$ . In this case, our elements would be the sets  $A$ ,  $B$ , and  $C$ , and these would then have to be subsets of some universal set  $W$  (elements of the power set of  $W$ ). For equivalence of sets, we are not requiring that the sets  $A$ ,  $B$ , and  $C$  be subsets of the same universal set. So we do not use the term relation in regards to the equivalence of sets. However, if  $A$  and  $B$  are sets and  $A \approx B$ , then we often say that  $A$  and  $B$  are **equivalent sets**.

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### Progress Check 9.2 (Examples of Equivalent Sets)

We will use the definition of equivalent sets from Preview Activity 1 in all parts of this progress check. It is no longer sufficient to say that two sets are equivalent by simply saying that the two sets have the same number of elements.

1. Let  $A = \{1, 2, 3, \dots, 99, 100\}$  and let  $B = \{351, 352, 353, \dots, 449, 450\}$ . Define  $f: A \rightarrow B$  by  $f(x) = x + 350$ , for each  $x$  in  $A$ . Prove that  $f$  is a bijection from the set  $A$  to the set  $B$  and hence,  $A \approx B$ .
2. Let  $E$  be the set of all even integers and let  $D$  be the set of all odd integers. Prove that  $E \approx D$  by proving that  $F: E \rightarrow D$ , where  $F(x) = x + 1$ , for all  $x \in E$ , is a bijection.
3. Let  $(0, 1)$  be the open interval of real numbers between 0 and 1. Similarly, if  $b \in \mathbb{R}$  with  $b > 0$ , let  $(0, b)$  be the open interval of real numbers between 0 and  $b$ .

Prove that the function  $f: (0, 1) \rightarrow (0, b)$  by  $f(x) = bx$ , for all  $x \in (0, 1)$ , is a bijection and hence  $(0, 1) \approx (0, b)$ .

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In Part (3) of Progress Check 9.2, notice that if  $b > 1$ , then  $(0, 1)$  is a proper subset of  $(0, b)$  and  $(0, 1) \approx (0, b)$ .

Also, in Part (3) of Preview Activity 1, we proved that the set  $D$  of all odd natural numbers is equivalent to  $\mathbb{N}$ , and we know that  $D$  is a proper subset of  $\mathbb{N}$ .

These results may seem a bit strange, but they are logical consequences of the definition of equivalent sets. Although we have not defined the terms yet, we will see that one thing that will distinguish an infinite set from a finite set is that an infinite set can be equivalent to one of its proper subsets, whereas a finite set cannot be equivalent to one of its proper subsets.



## Finite Sets

In Section 5.1, we defined the **cardinality** of a finite set  $A$ , denoted by  $\text{card}(A)$ , to be the number of elements in the set  $A$ . Now that we know about functions and bijections, we can define this concept more formally and more rigorously. First, for each  $k \in \mathbb{N}$ , we define  $\mathbb{N}_k$  to be the set of all natural numbers between 1 and  $k$ , inclusive. That is,

$$\mathbb{N}_k = \{1, 2, \dots, k\}.$$

We will use the concept of **equivalent sets** introduced in Preview Activity 1 to define a finite set.

**Definition.** A set  $A$  is a **finite set** provided that  $A = \emptyset$  or there exists a natural number  $k$  such that  $A \approx \mathbb{N}_k$ . A set is an **infinite set** provided that it is not a finite set.

If  $A \approx \mathbb{N}_k$ , we say that the set  $A$  has **cardinality  $k$**  (or **cardinal number  $k$** ), and we write  $\text{card}(A) = k$ .

In addition, we say that the empty set has **cardinality 0** (or **cardinal number 0**), and we write  $\text{card}(\emptyset) = 0$ .

Notice that by this definition, the empty set is a finite set. In addition, for each  $k \in \mathbb{N}$ , the identity function on  $\mathbb{N}_k$  is a bijection and hence, by definition, the set  $\mathbb{N}_k$  is a finite set with cardinality  $k$ .

**Theorem 9.3.** *Any set equivalent to a finite nonempty set  $A$  is a finite set and has the same cardinality as  $A$ .*

**Proof.** Suppose that  $A$  is a finite nonempty set,  $B$  is a set, and  $A \approx B$ . Since  $A$  is a finite set, there exists a  $k \in \mathbb{N}$  such that  $A \approx \mathbb{N}_k$ . We also have assumed that  $A \approx B$  and so by part (b) of Theorem 9.1 (in Preview Activity 2), we can conclude that  $B \approx A$ . Since  $A \approx \mathbb{N}_k$ , we can use part (c) of Theorem 9.1 to conclude that  $B \approx \mathbb{N}_k$ . Thus,  $B$  is finite and has the same cardinality as  $A$ . ■

It may seem that we have done a lot of work to prove an “obvious” result in Theorem 9.3. The same may be true of the remaining results in this section, which give further results about finite sets. One of the goals is to make sure that the concept of cardinality for a finite set corresponds to our intuitive notion of the number of elements in the set. Another important goal is to lay the groundwork for a more rigorous and mathematical treatment of infinite sets than we have encountered before. Along the way, we will see the mathematical distinction between finite and infinite sets.



The following two lemmas will be used to prove the theorem that states that every subset of a finite set is finite.

**Lemma 9.4.** *If  $A$  is a finite set and  $x \notin A$ , then  $A \cup \{x\}$  is a finite set and  $\text{card}(A \cup \{x\}) = \text{card}(A) + 1$ .*

**Proof.** Let  $A$  be a finite set and assume  $\text{card}(A) = k$ , where  $k = 0$  or  $k \in \mathbb{N}$ . Assume  $x \notin A$ .

If  $A = \emptyset$ , then  $\text{card}(A) = 0$  and  $A \cup \{x\} = \{x\}$ , which is equivalent to  $\mathbb{N}_1$ . Thus,  $A \cup \{x\}$  is finite with cardinality 1, which equals  $\text{card}(A) + 1$ .

If  $A \neq \emptyset$ , then  $A \approx \mathbb{N}_k$ , for some  $k \in \mathbb{N}$ . This means that  $\text{card}(A) = k$ , and there exists a bijection  $f: A \rightarrow \mathbb{N}_k$ . We will now use this bijection to define a function  $g: A \cup \{x\} \rightarrow \mathbb{N}_{k+1}$  and then prove that the function  $g$  is a bijection. We define  $g: A \cup \{x\} \rightarrow \mathbb{N}_{k+1}$  as follows: For each  $t \in A \cup \{x\}$ ,

$$g(t) = \begin{cases} f(t) & \text{if } t \in A \\ k+1 & \text{if } t = x. \end{cases}$$

To prove that  $g$  is an injection, we let  $x_1, x_2 \in A \cup \{x\}$  and assume  $x_1 \neq x_2$ .

- If  $x_1, x_2 \in A$ , then since  $f$  is a bijection,  $f(x_1) \neq f(x_2)$ , and this implies that  $g(x_1) \neq g(x_2)$ .
- If  $x_1 = x$ , then since  $x_2 \neq x_1$ , we conclude that  $x_2 \neq x$  and hence  $x_2 \in A$ . So  $g(x_1) = k+1$ , and since  $f(x_2) \in \mathbb{N}_k$  and  $g(x_2) = f(x_2)$ , we can conclude that  $g(x_1) \neq g(x_2)$ .
- The case where  $x_2 = x$  is handled similarly to the previous case.

This proves that the function  $g$  is an injection. The proof that  $g$  is a surjection is Exercise (1). Since  $g$  is a bijection, we conclude that  $A \cup \{x\} \approx \mathbb{N}_{k+1}$ , and

$$\text{card}(A \cup \{x\}) = k + 1.$$

Since  $\text{card}(A) = k$ , we have proved that  $\text{card}(A \cup \{x\}) = \text{card}(A) + 1$ . ■

**Lemma 9.5.** *For each natural number  $m$ , if  $A \subseteq \mathbb{N}_m$ , then  $A$  is a finite set and  $\text{card}(A) \leq m$ .*

**Proof.** We will use a proof using induction on  $m$ . For each  $m \in \mathbb{N}$ , let  $P(m)$  be, "If  $A \subseteq \mathbb{N}_m$ , then  $A$  is finite and  $\text{card}(A) \leq m$ ."



We first prove that  $P(1)$  is true. If  $A \subseteq \mathbb{N}_1$ , then  $A = \emptyset$  or  $A = \{1\}$ , both of which are finite and have cardinality less than or equal to the cardinality of  $\mathbb{N}_1$ . This proves that  $P(1)$  is true.

For the inductive step, let  $k \in \mathbb{N}$  and assume that  $P(k)$  is true. That is, assume that if  $B \subseteq \mathbb{N}_k$ , then  $B$  is a finite set and  $\text{card}(B) \leq k$ . We need to prove that  $P(k+1)$  is true.

So assume that  $A$  is a subset of  $\mathbb{N}_{k+1}$ . Then  $A - \{k+1\}$  is a subset of  $\mathbb{N}_k$ . Since  $P(k)$  is true,  $A - \{k+1\}$  is a finite set and

$$\text{card}(A - \{k+1\}) \leq k.$$

There are two cases to consider: Either  $k+1 \in A$  or  $k+1 \notin A$ .

If  $k+1 \notin A$ , then  $A = A - \{k+1\}$ . Hence,  $A$  is finite and

$$\text{card}(A) \leq k < k+1.$$

If  $k+1 \in A$ , then  $A = (A - \{k+1\}) \cup \{k+1\}$ . Hence, by Lemma 9.4,  $A$  is a finite set and

$$\text{card}(A) = \text{card}(A - \{k+1\}) + 1.$$

Since  $\text{card}(A - \{k+1\}) \leq k$ , we can conclude that  $\text{card}(A) \leq k+1$ .

This means that we have proved the inductive step. Hence, by mathematical induction, for each  $m \in \mathbb{N}$ , if  $A \subseteq \mathbb{N}_m$ , then  $A$  is finite and  $\text{card}(A) \leq m$ . ■

The preceding two lemmas were proved to aid in the proof of the following theorem.

**Theorem 9.6.** *If  $S$  is a finite set and  $A$  is a subset of  $S$ , then  $A$  is a finite set and  $\text{card}(A) \leq \text{card}(S)$ .*

**Proof.** Let  $S$  be a finite set and assume that  $A$  is a subset of  $S$ . If  $A = \emptyset$ , then  $A$  is a finite set and  $\text{card}(A) \leq \text{card}(S)$ . So we assume that  $A \neq \emptyset$ .

Since  $S$  is finite, there exists a bijection  $f: S \rightarrow \mathbb{N}_k$  for some  $k \in \mathbb{N}$ . In this case,  $\text{card}(S) = k$ . We need to show that  $A$  is equivalent to a finite set. To do this, we define  $g: A \rightarrow f(A)$  by

$$g(x) = f(x) \text{ for each } x \in A.$$



Since  $f$  is an injection, we conclude that  $g$  is an injection. Now let  $y \in f(A)$ . Then there exists an  $a \in A$  such that  $f(a) = y$ . But by the definition of  $g$ , this means that  $g(a) = y$ , and hence  $g$  is a surjection. This proves that  $g$  is a bijection.

Hence, we have proved that  $A \approx f(A)$ . Since  $f(A)$  is a subset of  $\mathbb{N}_k$ , we use Lemma 9.5 to conclude that  $f(A)$  is finite and  $\text{card}(f(A)) \leq k$ . In addition, by Theorem 9.3,  $A$  is a finite set and  $\text{card}(A) = \text{card}(f(A))$ . This proves that  $A$  is a finite set and  $\text{card}(A) \leq \text{card}(S)$ . ■

Lemma 9.4 implies that adding one element to a finite set increases its cardinality by 1. It is also true that removing one element from a finite nonempty set reduces the cardinality by 1. The proof of Corollary 9.7 is Exercise (4).

**Corollary 9.7.** *If  $A$  is a finite set and  $x \in A$ , then  $A - \{x\}$  is a finite set and  $\text{card}(A - \{x\}) = \text{card}(A) - 1$ .*

The next corollary will be used in the next section to provide a mathematical distinction between finite and infinite sets.

**Corollary 9.8.** *A finite set is not equivalent to any of its proper subsets.*

**Proof.** Let  $B$  be a finite set and assume that  $A$  is a proper subset of  $B$ . Since  $A$  is a proper subset of  $B$ , there exists an element  $x$  in  $B - A$ . This means that  $A$  is a subset of  $B - \{x\}$ . Hence, by Theorem 9.6,

$$\text{card}(A) \leq \text{card}(B - \{x\}).$$

Also, by Corollary 9.7

$$\text{card}(B - \{x\}) = \text{card}(B) - 1.$$

Hence, we may conclude that  $\text{card}(A) \leq \text{card}(B) - 1$  and that

$$\text{card}(A) < \text{card}(B).$$

Theorem 9.3 implies that  $B \not\approx A$ . This proves that a finite set is not equivalent to any of its proper subsets. ■

## The Pigeonhole Principle

The last property of finite sets that we will consider in this section is often called the **Pigeonhole Principle**. The “pigeonhole” version of this property says, “If  $m$  pigeons go into  $r$  pigeonholes and  $m > r$ , then at least one pigeonhole has more than one pigeon.”

In this situation, we can think of the set of pigeons as being equivalent to a set  $P$  with cardinality  $m$  and the set of pigeonholes as being equivalent to a set  $H$  with cardinality  $r$ . We can then define a function  $f: P \rightarrow H$  that maps each pigeon to its pigeonhole. The Pigeonhole Principle states that this function is not an injection. (It is not one-to-one since there are at least two pigeons “mapped” to the same pigeonhole.)

**Theorem 9.9 (The Pigeonhole Principle).** *Let  $A$  and  $B$  be finite sets. If  $\text{card}(A) > \text{card}(B)$ , then any function  $f: A \rightarrow B$  is not an injection.*

**Proof.** Let  $A$  and  $B$  be finite sets. We will prove the contrapositive of the theorem, which is, if there exists a function  $f: A \rightarrow B$  that is an injection, then  $\text{card}(A) \leq \text{card}(B)$ .

So assume that  $f: A \rightarrow B$  is an injection. As in Theorem 9.6, we define a function  $g: A \rightarrow f(A)$  by

$$g(x) = f(x) \text{ for each } x \in A.$$

As we saw in Theorem 9.6, the function  $g$  is a bijection. But then  $A \approx f(A)$  and  $f(A) \subseteq B$ . Hence,

$$\text{card}(A) = \text{card}(f(A)) \text{ and } \text{card}(f(A)) \leq \text{card}(B).$$

Hence,  $\text{card}(A) \leq \text{card}(B)$ , and this proves the contrapositive. Hence, if  $\text{card}(A) > \text{card}(B)$ , then any function  $f: A \rightarrow B$  is not an injection. ■

The Pigeonhole Principle has many applications in the branch of mathematics called “combinatorics.” Some of these will be explored in the exercises.

## Exercises 9.1

1. Prove that the function  $g: A \cup \{x\} \rightarrow \mathbb{N}_{k+1}$  in Lemma 9.4 is a surjection.





\* 2. Let  $A$  be a subset of some universal set  $U$ . Prove that if  $x \in U$ , then  $A \times \{x\} \approx A$ .

\* 3. Let  $E^+$  be the set of all even natural numbers. Prove that  $\mathbb{N} \approx E^+$ .

\* 4. Prove Corollary 9.7.

If  $A$  is a finite set and  $x \in A$ , then  $A - \{x\}$  is a finite set and  $\text{card}(A - \{x\}) = \text{card}(A) - 1$ .

**Hint:** One approach is to use the fact that  $A = (A - \{x\}) \cup \{x\}$ .

5. Let  $A$  and  $B$  be sets. Prove that

\* (a) If  $A$  is a finite set, then  $A \cap B$  is a finite set.

\* (b) If  $A \cup B$  is a finite set, then  $A$  and  $B$  are finite sets.

(c) If  $A \cap B$  is an infinite set, then  $A$  is an infinite set.

(d) If  $A$  is an infinite set or  $B$  is an infinite set, then  $A \cup B$  is an infinite set.

6. There are over 7 million people living in New York City. It is also known that the maximum number of hairs on a human head is less than 200,000. Use the Pigeonhole Principle to prove that there are at least two people in the city of New York with the same number of hairs on their heads.

7. Prove the following propositions:

\* (a) If  $A$ ,  $B$ ,  $C$ , and  $D$  are sets with  $A \approx B$  and  $C \approx D$ , then  $A \times C \approx B \times D$ .

(b) If  $A$ ,  $B$ ,  $C$ , and  $D$  are sets with  $A \approx B$  and  $C \approx D$  and if  $A$  and  $C$  are disjoint and  $B$  and  $D$  are disjoint, then  $A \cup C \approx B \cup D$ .

**Hint:** Since  $A \approx B$  and  $C \approx D$ , there exist bijections  $f : A \rightarrow B$  and  $g : C \rightarrow D$ . To prove that  $A \times C \approx B \times D$ , prove that  $h : A \times C \rightarrow B \times D$  is a bijection, where  $h(a, c) = (f(a), g(c))$ , for all  $(a, c) \in A \times C$ .

8. Let  $A = \{a, b, c\}$ .

\* (a) Construct a function  $f : \mathbb{N}_5 \rightarrow A$  such that  $f$  is a surjection.

(b) Use the function  $f$  to construct a function  $g : A \rightarrow \mathbb{N}_5$  so that  $f \circ g = I_A$ , where  $I_A$  is the identity function on the set  $A$ . Is the function  $g$  an injection? Explain.



9. This exercise is a generalization of Exercise (8). Let  $m$  be a natural number, let  $A$  be a set, and assume that  $f : \mathbb{N}_m \rightarrow A$  is a surjection. Define  $g : A \rightarrow \mathbb{N}_m$  as follows:

For each  $x \in A$ ,  $g(x) = j$ , where  $j$  is the least natural number in  $f^{-1}(\{x\})$ .

Prove that  $f \circ g = I_A$ , where  $I_A$  is the identity function on the set  $A$ , and prove that  $g$  is an injection.

10. Let  $B$  be a finite, nonempty set and assume that  $f : B \rightarrow A$  is a surjection. Prove that there exists a function  $h : A \rightarrow B$  such that  $f \circ h = I_A$  and  $h$  is an injection.

**Hint:** Since  $B$  is finite, there exists a natural number  $m$  such that  $\mathbb{N}_m \approx B$ . This means there exists a bijection  $k : \mathbb{N}_m \rightarrow B$ . Now let  $h = k \circ g$ , where  $g$  is the function constructed in Exercise (9).

### Explorations and Activities

11. **Using the Pigeonhole Principle.** For this activity, we will consider subsets of  $\mathbb{N}_{30}$  that contain eight elements.

- (a) One such set is  $A = \{3, 5, 11, 17, 21, 24, 26, 29\}$ . Notice that

$$\begin{aligned} \{3, 21, 24, 26\} \subseteq A & \quad \text{and} & \quad 3 + 21 + 24 + 26 = 74 \\ \{3, 5, 11, 26, 29\} \subseteq A & \quad \text{and} & \quad 3 + 5 + 11 + 26 + 29 = 74. \end{aligned}$$

Use this information to find two disjoint subsets of  $A$  whose elements have the same sum.

- (b) Let  $B = \{3, 6, 9, 12, 15, 18, 21, 24\}$ . Find two disjoint subsets of  $B$  whose elements have the same sum. **Note:** By convention, if  $T = \{a\}$ , where  $a \in \mathbb{N}$ , then the sum of the elements in  $T$  is equal to  $a$ .
- (c) Now let  $C$  be any subset of  $\mathbb{N}_{30}$  that contains eight elements.
- i. How many subsets does  $C$  have?
  - ii. The sum of the elements of the empty set is 0. What is the maximum sum for any subset of  $\mathbb{N}_{30}$  that contains eight elements? Let  $M$  be this maximum sum.
  - iii. Now define a function  $f : \mathcal{P}(C) \rightarrow \mathbb{N}_M$  so that for each  $X \in \mathcal{P}(C)$ ,  $f(X)$  is equal to the sum of the elements in  $X$ . Use the Pigeonhole Principle to prove that there exist two subsets of  $C$  whose elements have the same sum.



- (d) If the two subsets in part (11(c)iii) are not disjoint, use the idea presented in part (11a) to prove that there exist two disjoint subsets of  $C$  whose elements have the same sum.
- (e) Let  $S$  be a subset of  $\mathbb{N}_{99}$  that contains 10 elements. Use the Pigeon-hole Principle to prove that there exist two disjoint subsets of  $S$  whose elements have the same sum.

## 9.2 Countable Sets

### Preview Activity 1 (Introduction to Infinite Sets)

In Section 9.1, we defined a **finite set** to be the empty set or a set  $A$  such that  $A \approx \mathbb{N}_k$  for some natural number  $k$ . We also defined an **infinite set** to be a set that is not finite, but the question now is, “How do we know if a set is infinite?” One way to determine if a set is an infinite set is to use Corollary 9.8, which states that a finite set is not equivalent to any of its subsets. We can write this as a conditional statement as follows:

If  $A$  is a finite set, then  $A$  is not equivalent to any of its proper subsets.

or more formally as

For each set  $A$ , if  $A$  is a finite set, then for each proper subset  $B$  of  $A$ ,  $A \not\approx B$ .

1. Write the contrapositive of the preceding conditional statement. Then explain how this statement can be used to determine if a set is infinite.
2. Let  $D^+$  be the set of all odd natural numbers. In Preview Activity 1 from Section 9.1, we proved that  $\mathbb{N} \approx D^+$ .
  - (a) Use this to explain carefully why  $\mathbb{N}$  is an infinite set.
  - (b) Is  $D^+$  a finite set or an infinite set? Explain carefully how you know.
3. Let  $b$  be a positive real number. Let  $(0, 1)$  and  $(0, b)$  be the open intervals from 0 to 1 and 0 to  $b$ , respectively. In Part (3) of Progress Check 9.2 (on page 454), we proved that  $(0, 1) \approx (0, b)$ .
  - (a) Use a value for  $b$  where  $0 < b < 1$  to explain why  $(0, 1)$  is an infinite set.
  - (b) Use a value for  $b$  where  $b > 1$  to explain why  $(0, b)$  is an infinite set.



**Preview Activity 2 (A Function from  $\mathbb{N}$  to  $\mathbb{Z}$ )**

In this activity, we will define and explore a function  $f : \mathbb{N} \rightarrow \mathbb{Z}$ . We will start by defining  $f(n)$  for the first few natural numbers  $n$ .

$$\begin{array}{ll} f(1) = 0 & \\ f(2) = 1 & f(3) = -1 \\ f(4) = 2 & f(5) = -2 \\ f(6) = 3 & f(7) = -3 \end{array}$$

Notice that if we list the outputs of  $f$  in the order  $f(1), f(2), f(3), \dots$ , we create the following list of integers:  $0, 1, -1, 2, -2, 3, -3, \dots$ . We can also illustrate the outputs of this function with the following diagram:

1	2	3	4	5	6	7	8	9	10	...
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	...
0	1	-1	2	-2	3	-3	4	-4	5	...

Figure 9.1: A Function from  $\mathbb{N}$  to  $\mathbb{Z}$

1. If the pattern suggested by the function values we have defined continues, what are  $f(11)$  and  $f(12)$ ? What is  $f(n)$  for  $n$  from 13 to 16?
2. If the pattern of outputs continues, does the function  $f$  appear to be an injection? Does  $f$  appear to be a surjection? (Formal proofs are not required.)

We will now attempt to determine a formula for  $f(n)$ , where  $n \in \mathbb{N}$ . We will actually determine two formulas: one for when  $n$  is even and one for when  $n$  is odd.

3. Look at the pattern of the values of  $f(n)$  when  $n$  is even. What appears to be a formula for  $f(n)$  when  $n$  is even?
4. Look at the pattern of the values of  $f(n)$  when  $n$  is odd. What appears to be a formula for  $f(n)$  when  $n$  is odd?
5. Use the work in Part (3) and Part (4) to complete the following: Define  $f : \mathbb{N} \rightarrow \mathbb{Z}$ , where

$$f(n) = \begin{cases} ?? & \text{if } n \text{ is even} \\ ?? & \text{if } n \text{ is odd.} \end{cases}$$



6. Use the formula in Part (5) to
- Calculate  $f(1)$  through  $f(10)$ . Are these results consistent with the pattern exhibited at the start of this activity?
  - Calculate  $f(1000)$  and  $f(1001)$ .
  - Determine the value of  $n$  so that  $f(n) = 1000$ .

---

In this section, we will describe several infinite sets and define the cardinal number for so-called countable sets. Most of our examples will be subsets of some of our standard number systems such as  $\mathbb{N}$ ,  $\mathbb{Z}$ , and  $\mathbb{Q}$ .

### Infinite Sets

In Preview Activity 1, we saw how to use Corollary 9.8 to prove that a set is infinite. This corollary implies that if  $A$  is a finite set, then  $A$  is not equivalent to any of its proper subsets. By writing the contrapositive of this conditional statement, we can restate Corollary 9.8 in the following form:

**Corollary 9.8.** *If a set  $A$  is equivalent to one of its proper subsets, then  $A$  is infinite.*

In Preview Activity 1, we used Corollary 9.8 to prove that

- The set of natural numbers,  $\mathbb{N}$ , is an infinite set.
- The open interval  $(0, 1)$  is an infinite set.

Although Corollary 9.8 provides one way to prove that a set is infinite, it is sometimes more convenient to use a proof by contradiction to prove that a set is infinite. The idea is to use results from Section 9.1 about finite sets to help obtain a contradiction. This is illustrated in the next theorem.

**Theorem 9.10.** *Let  $A$  and  $B$  be sets.*

1. *If  $A$  is infinite and  $A \approx B$ , then  $B$  is infinite.*
2. *If  $A$  is infinite and  $A \subseteq B$ , then  $B$  is infinite.*

**Proof.** We will prove part (1). The proof of part (2) is exercise (3) on page 473.



To prove part (1), we use a proof by contradiction and assume that  $A$  is an infinite set,  $A \approx B$ , and  $B$  is not infinite. That is,  $B$  is a finite set. Since  $A \approx B$  and  $B$  is finite, Theorem 9.3 on page 455 implies that  $A$  is a finite set. This is a contradiction to the assumption that  $A$  is infinite. We have therefore proved that if  $A$  is infinite and  $A \approx B$ , then  $B$  is infinite. ■

---

### Progress Check 9.11 (Examples of Infinite Sets)

1. In Preview Activity 1, we used Corollary 9.8 to prove that  $\mathbb{N}$  is an infinite set. Now use this and Theorem 9.10 to explain why our standard number systems ( $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$ ) are infinite sets. Also, explain why the set of all positive rational numbers,  $\mathbb{Q}^+$ , and the set of all positive real numbers,  $\mathbb{R}^+$ , are infinite sets.
  2. Let  $D^+$  be the set of all odd natural numbers. In Part (2) of Preview Activity 1, we proved that  $D^+ \approx \mathbb{N}$ . Use Theorem 9.10 to explain why  $D^+$  is an infinite set.
  3. Prove that the set  $E^+$  of all even natural numbers is an infinite set.
- 

### Countably Infinite Sets

In Section 9.1, we used the set  $\mathbb{N}_k$  as the standard set with cardinality  $k$  in the sense that a set is finite if and only if it is equivalent to  $\mathbb{N}_k$ . In a similar manner, we will use some infinite sets as standard sets for certain infinite cardinal numbers. The first set we will use is  $\mathbb{N}$ .

We will formally define what it means to say the elements of a set can be “counted” using the natural numbers. The elements of a finite set can be “counted” by defining a bijection (one-to-one correspondence) between the set and  $\mathbb{N}_k$  for some natural number  $k$ . We will be able to “count” the elements of an infinite set if we can define a one-to-one correspondence between the set and  $\mathbb{N}$ .

**Definition.** The **cardinality of  $\mathbb{N}$**  is denoted by  $\aleph_0$ . The symbol  $\aleph$  is the first letter of the Hebrew alphabet, **aleph**. The subscript 0 is often read as “naught” (or sometimes as “zero” or “null”). So we write

$$\text{card}(\mathbb{N}) = \aleph_0$$

and say that the cardinality of  $\mathbb{N}$  is “aleph naught.”



**Definition.** A set  $A$  is **countably infinite** provided that  $A \approx \mathbb{N}$ . In this case, we write

$$\text{card}(A) = \aleph_0.$$

A set that is countably infinite is sometimes called a **denumerable** set. A set is **countable** provided that it is finite or countably infinite. An infinite set that is not countably infinite is called an **uncountable set**.

### Progress Check 9.12 (Examples of Countably Infinite Sets)

1. In Preview Activity 1 from Section 9.1, we proved that  $\mathbb{N} \approx D^+$ , where  $D^+$  is the set of all odd natural numbers. Explain why  $\text{card}(D^+) = \aleph_0$ .
2. Use a result from Progress Check 9.11 to explain why  $\text{card}(E^+) = \aleph_0$ .
3. At this point, if we wish to prove a set  $S$  is countably infinite, we must find a bijection between the set  $S$  and some set that is known to be countably infinite.

Let  $S$  be the set of all natural numbers that are perfect squares. Define a function

$$f: S \rightarrow \mathbb{N}$$

that can be used to prove that  $S \approx \mathbb{N}$  and, hence, that  $\text{card}(S) = \aleph_0$ .

The fact that the set of integers is a countably infinite set is important enough to be called a theorem. The function we will use to establish that  $\mathbb{N} \approx \mathbb{Z}$  was explored in Preview Activity 2.

**Theorem 9.13.** *The set  $\mathbb{Z}$  of integers is countably infinite, and so  $\text{card}(\mathbb{Z}) = \aleph_0$ .*

**Proof.** To prove that  $\mathbb{N} \approx \mathbb{Z}$ , we will use the following function:  $f: \mathbb{N} \rightarrow \mathbb{Z}$ , where

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{1-n}{2} & \text{if } n \text{ is odd.} \end{cases}$$

From our work in Preview Activity 2, it appears that if  $n$  is an even natural number, then  $f(n) > 0$ , and if  $n$  is an odd natural number, then  $f(n) \leq 0$ . So it seems reasonable to use cases to prove that  $f$  is a surjection and that  $f$  is an injection. To prove that  $f$  is a surjection, we let  $y \in \mathbb{Z}$ .



- If  $y > 0$ , then  $2y \in \mathbb{N}$ , and

$$f(2y) = \frac{2y}{2} = y.$$

- If  $y \leq 0$ , then  $-2y \geq 0$  and  $1 - 2y$  is an odd natural number. Hence,

$$f(1 - 2y) = \frac{1 - (1 - 2y)}{2} = \frac{2y}{2} = y.$$

These two cases prove that if  $y \in \mathbb{Z}$ , then there exists an  $n \in \mathbb{N}$  such that  $f(n) = y$ . Hence,  $f$  is a surjection.

To prove that  $f$  is an injection, we let  $m, n \in \mathbb{N}$  and assume that  $f(m) = f(n)$ . First note that if one of  $m$  and  $n$  is odd and the other is even, then one of  $f(m)$  and  $f(n)$  is positive and the other is less than or equal to 0. So if  $f(m) = f(n)$ , then both  $m$  and  $n$  must be even or both  $m$  and  $n$  must be odd.

- If both  $m$  and  $n$  are even, then

$$f(m) = f(n) \text{ implies that } \frac{m}{2} = \frac{n}{2}$$

and hence that  $m = n$ .

- If both  $m$  and  $n$  are odd, then

$$f(m) = f(n) \text{ implies that } \frac{1 - m}{2} = \frac{1 - n}{2}.$$

From this, we conclude that  $1 - m = 1 - n$  and hence that  $m = n$ . This proves that if  $f(m) = f(n)$ , then  $m = n$  and hence that  $f$  is an injection.

Since  $f$  is both a surjection and an injection, we see that  $f$  is a bijection and, therefore,  $\mathbb{N} \approx \mathbb{Z}$ . Hence,  $\mathbb{Z}$  is countably infinite and  $\text{card}(\mathbb{Z}) = \aleph_0$ . ■

The result in Theorem 9.13 can seem a bit surprising. It exhibits one of the distinctions between finite and infinite sets. If we add elements to a finite set, we will increase its size in the sense that the new set will have a greater cardinality than the old set. However, with infinite sets, we can add elements and the new set may still have the same cardinality as the original set. For example, there is a one-to-one correspondence between the elements of the sets  $\mathbb{N}$  and  $\mathbb{Z}$ . We say that these sets have the same cardinality.

Following is a summary of some of the main examples dealing with the cardinality of sets that we have explored.





- The sets  $\mathbb{N}_k$ , where  $k \in \mathbb{N}$ , are examples of sets that are countable and finite.
- The sets  $\mathbb{N}$ ,  $\mathbb{Z}$ , the set of all odd natural numbers, and the set of all even natural numbers are examples of sets that are countable and countably infinite.
- We have not yet proved that any set is uncountable.

### The Set of Positive Rational Numbers

If we expect to find an uncountable set in our usual number systems, the rational numbers might be the place to start looking. One of the main differences between the set of rational numbers and the integers is that given any integer  $m$ , there is a next integer, namely  $m + 1$ . This is not true for the set of rational numbers. We know that  $\mathbb{Q}$  is closed under division (by nonzero rational numbers) and we will see that this property implies that given any two rational numbers, we can also find a rational number between them. In fact, between any two rational numbers, we can find infinitely many rational numbers. It is this property that may lead us to believe that there are “more” rational numbers than there are integers.

The basic idea will be to “go half way” between two rational numbers. For example, if we use  $a = \frac{1}{3}$  and  $b = \frac{1}{2}$ , we can use

$$\frac{a+b}{2} = \frac{1}{2} \left( \frac{1}{3} + \frac{1}{2} \right) = \frac{5}{12}$$

as a rational number between  $a$  and  $b$ . We can then repeat this process to find a rational number between  $\frac{5}{12}$  and  $\frac{1}{2}$ .

So we will now let  $a$  and  $b$  be any two rational numbers with  $a < b$  and let  $c_1 = \frac{a+b}{2}$ . We then see that

$$\begin{aligned} c_1 - a &= \frac{a+b}{2} - a & b - c_1 &= b - \frac{a+b}{2} \\ &= \frac{a+b}{2} - \frac{2a}{2} & &= \frac{2b}{2} - \frac{a+b}{2} \\ &= \frac{b-a}{2} & &= \frac{b-a}{2} \end{aligned}$$

Since  $b > a$ , we see that  $b - a > 0$  and so the previous equations show that  $c_1 - a > 0$  and  $b - c_1 > 0$ . We can then conclude that  $a < c_1 < b$ .



We can now repeat this process by using  $c_2 = \frac{c_1 + b}{2}$  and proving that  $c_1 < c_2 < b$ . In fact, for each natural number, we can define

$$c_{k+1} = \frac{c_k + b}{2}$$

and obtain the result that  $a < c_1 < c_2 < \dots < c_n < \dots < b$  and this proves that the set  $\{c_k \mid k \in \mathbb{N}\}$  is a countably infinite set where each element is a rational number between  $a$  and  $b$ . (A formal proof can be completed using mathematical induction.) See Exercise (14).

This result is true no matter how close together  $a$  and  $b$  are. For example, we can now conclude that there are infinitely many rational numbers between 0 and  $\frac{1}{10000}$ . This might suggest that the set  $\mathbb{Q}$  of rational numbers is uncountable. Surprisingly, this is not the case. We start with a proof that the set of positive rational numbers is countable.

**Theorem 9.14.** *The set of positive rational numbers is countably infinite.*

**Proof.** We can write all the positive rational numbers in a two-dimensional array as shown in Figure 9.2. The top row in Figure 9.2 represents the numerator of the rational number, and the left column represents the denominator. We follow the arrows in Figure 9.2 to define  $f: \mathbb{N} \rightarrow \mathbb{Q}^+$ . The idea is to start in the upper left corner of the table and move to successive diagonals as follows:

- We start with all fractions in which the sum of the numerator and denominator is 2 (only  $\frac{1}{1}$ ). So  $f(1) = \frac{1}{1}$ .
- We next use those fractions in which the sum of the numerator and denominator is 3. So  $f(2) = \frac{2}{1}$  and  $f(3) = \frac{1}{2}$ .
- We next use those fractions in which the sum of the numerator and denominator is 4. So  $f(4) = \frac{1}{3}$ ,  $f(5) = \frac{3}{1}$ . We skipped  $\frac{2}{2}$  since  $\frac{2}{2} = \frac{1}{1}$ . In this way, we will ensure that the function  $f$  is a one-to-one function.

We now continue with successive diagonals omitting fractions that are not in lowest terms. This process guarantees that the function  $f$  will be an injection and a surjection. Therefore,  $\mathbb{N} \approx \mathbb{Q}^+$  and  $\text{card}(\mathbb{Q}^+) = \aleph_0$ . ■

**Note:** For another proof of Theorem 9.14, see exercise (15) on page 475.



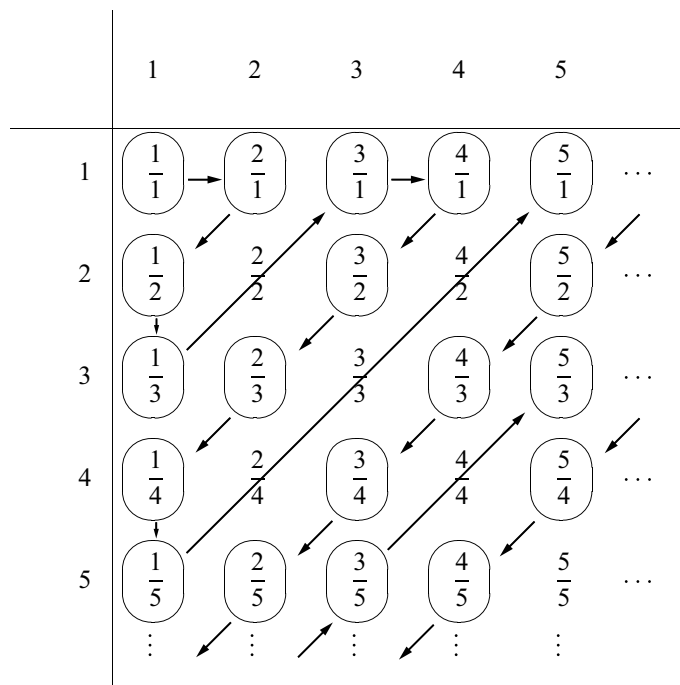


Figure 9.2: Counting the Positive Rational Numbers

Since  $\mathbb{Q}^+$  is countable, it seems reasonable to expect that  $\mathbb{Q}$  is countable. We will explore this soon. On the other hand, at this point, it may also seem reasonable to ask,

“Are there any uncountable sets?”

The answer to this question is yes, but we will wait until the next section to prove that certain sets are uncountable. We still have a few more issues to deal with concerning countable sets.

### Countably Infinite Sets

**Theorem 9.15.** *If  $A$  is a countably infinite set, then  $A \cup \{x\}$  is a countably infinite set.*

**Proof.** Let  $A$  be a countably infinite set. Then there exists a bijection  $f: \mathbb{N} \rightarrow A$ . Since  $x$  is either in  $A$  or not in  $A$ , we can consider two cases.



If  $x \in A$ , then  $A \cup \{x\} = A$  and  $A \cup \{x\}$  is countably infinite.

If  $x \notin A$ , define  $g: \mathbb{N} \rightarrow A \cup \{x\}$  by

$$g(n) = \begin{cases} x & \text{if } n = 1 \\ f(n-1) & \text{if } n > 1. \end{cases}$$

The proof that the function  $g$  is a bijection is Exercise (4). Since  $g$  is a bijection, we have proved that  $A \cup \{x\} \approx \mathbb{N}$  and hence,  $A \cup \{x\}$  is countably infinite. ■

**Theorem 9.16.** *If  $A$  is a countably infinite set and  $B$  is a finite set, then  $A \cup B$  is a countably infinite set.*

**Proof.** Exercise (5) on page 473. ■

Theorem 9.16 says that if we add a finite number of elements to a countably infinite set, the resulting set is still countably infinite. In other words, the cardinality of the new set is the same as the cardinality of the original set. Finite sets behave very differently in the sense that if we add elements to a finite set, we will change the cardinality. What may even be more surprising is the result in Theorem 9.17 that states that the union of two countably infinite (disjoint) sets is countably infinite. The proof of this result is similar to the proof that the integers are countably infinite (Theorem 9.13). In fact, if  $A = \{a_1, a_2, a_3, \dots\}$  and  $B = \{b_1, b_2, b_3, \dots\}$ , then we can use the following diagram to help define a bijection from  $\mathbb{N}$  to  $A \cup B$ .

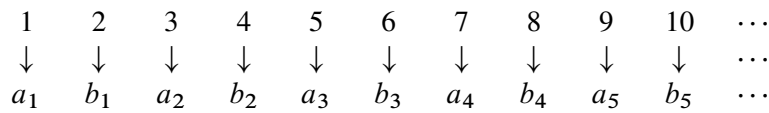


Figure 9.3: A Function from  $\mathbb{N}$  to  $A \cup B$

**Theorem 9.17.** *If  $A$  and  $B$  are disjoint countably infinite sets, then  $A \cup B$  is a countably infinite set.*

**Proof.** Let  $A$  and  $B$  be countably infinite sets and let  $f: \mathbb{N} \rightarrow A$  and  $g: \mathbb{N} \rightarrow B$  be bijections. Define  $h: \mathbb{N} \rightarrow A \cup B$  by

$$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$



It is left as Exercise (6) on page 474 to prove that the function  $h$  is a bijection. ■

Since we can write the set of rational numbers  $\mathbb{Q}$  as the union of the set of nonnegative rational numbers and the set of negative rational numbers, we can use the results in Theorem 9.14, Theorem 9.15, and Theorem 9.17 to prove the following theorem.

**Theorem 9.18.** *The set  $\mathbb{Q}$  of all rational numbers is countably infinite.*

**Proof.** Exercise (7) on page 474. ■

In Section 9.1, we proved that any subset of a finite set is finite (Theorem 9.6). A similar result should be expected for countable sets. We first prove that every subset of  $\mathbb{N}$  is countable. For an infinite subset  $B$  of  $\mathbb{N}$ , the idea of the proof is to define a function  $g: \mathbb{N} \rightarrow B$  by removing the elements from  $B$  from smallest to the next smallest to the next smallest, and so on. We do this by defining the function  $g$  recursively as follows:

- Let  $g(1)$  be the smallest natural number in  $B$ .
- Remove  $g(1)$  from  $B$  and let  $g(2)$  be the smallest natural number in  $B - \{g(1)\}$ .
- Remove  $g(2)$  and let  $g(3)$  be the smallest natural number in  $B - \{g(1), g(2)\}$ .
- We continue this process. The formal recursive definition of  $g: \mathbb{N} \rightarrow B$  is included in the proof of Theorem 9.19.

**Theorem 9.19.** *Every subset of the natural numbers is countable.*

**Proof.** Let  $B$  be a subset of  $\mathbb{N}$ . If  $B$  is finite, then  $B$  is countable. So we next assume that  $B$  is infinite. We will next give a recursive definition of a function  $g: \mathbb{N} \rightarrow B$  and then prove that  $g$  is a bijection.

- Let  $g(1)$  be the smallest natural number in  $B$ .
- For each  $n \in \mathbb{N}$ , the set  $B - \{g(1), g(2), \dots, g(n)\}$  is not empty since  $B$  is infinite. Define  $g(n + 1)$  to be the smallest natural number in  $B - \{g(1), g(2), \dots, g(n)\}$ .

The proof that the function  $g$  is a bijection is Exercise (11) on page 474. ■



**Corollary 9.20.** *Every subset of a countable set is countable.*

**Proof.** Exercise (12) on page 475. ■

## Exercises 9.2

- \* 1. State whether each of the following is true or false.
- (a) If a set  $A$  is countably infinite, then  $A$  is infinite.
  - (b) If a set  $A$  is countably infinite, then  $A$  is countable.
  - (c) If a set  $A$  is uncountable, then  $A$  is not countably infinite.
  - (d) If  $A \approx \mathbb{N}_k$  for some  $k \in \mathbb{N}$ , then  $A$  is not countable.
2. Prove that each of the following sets is countably infinite.
- (a) The set  $F^+$  of all natural numbers that are multiples of 5
  - (b) The set  $F$  of all integers that are multiples of 5
  - (c)  $\left\{ \frac{1}{2^k} \mid k \in \mathbb{N} \right\}$
  - (d)  $\{n \in \mathbb{Z} \mid n \geq -10\}$
  - \* (e)  $\mathbb{N} - \{4, 5, 6\}$
  - \* (f)  $\{m \in \mathbb{Z} \mid m \equiv 2 \pmod{3}\}$
3. Prove part (2) of Theorem 9.10.
- Let  $A$  and  $B$  be sets. If  $A$  is infinite and  $A \subseteq B$ , then  $B$  is infinite.
4. Complete the proof of Theorem 9.15 by proving the following:
- Let  $A$  be a countably infinite set and  $x \notin A$ . If  $f: \mathbb{N} \rightarrow A$  is a bijection, then  $g$  is a bijection, where  $g: \mathbb{N} \rightarrow A \cup \{x\}$  by
- $$g(n) = \begin{cases} x & \text{if } n = 1 \\ f(n-1) & \text{if } n > 1. \end{cases}$$
- \* 5. Prove Theorem 9.16.
- If  $A$  is a countably infinite set and  $B$  is a finite set, then  $A \cup B$  is a countably infinite set.
- Hint:** Let  $\text{card}(B) = n$  and use a proof by induction on  $n$ . Theorem 9.15 is the basis step.

- \* 6. Complete the proof of Theorem 9.17 by proving the following:

Let  $A$  and  $B$  be disjoint countably infinite sets and let  $f: \mathbb{N} \rightarrow A$  and  $g: \mathbb{N} \rightarrow B$  be bijections. Define  $h: \mathbb{N} \rightarrow A \cup B$  by

$$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd} \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$

Then the function  $h$  is a bijection.

- \* 7. Prove Theorem 9.18.

The set  $\mathbb{Q}$  of all rational numbers is countable.

**Hint:** Use Theorem 9.15 and Theorem 9.17.

- \* 8. Prove that if  $A$  is countably infinite and  $B$  is finite, then  $A - B$  is countably infinite.

9. Define  $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  as follows: For each  $(m, n) \in \mathbb{N} \times \mathbb{N}$ ,

$$f(m, n) = 2^{m-1}(2n - 1).$$

- (a) Prove that  $f$  is an injection. **Hint:** If  $f(m, n) = f(s, t)$ , there are three cases to consider:  $m > s$ ,  $m < s$ , and  $m = s$ . Use laws of exponents to prove that the first two cases lead to a contradiction.
- (b) Prove that  $f$  is a surjection. **Hint:** You may use the fact that if  $y \in \mathbb{N}$ , then  $y = 2^k x$ , where  $x$  is an odd natural number and  $k$  is a non-negative integer. This is actually a consequence of the Fundamental Theorem of Arithmetic, Theorem 8.15. [See Exercise (13) in Section 8.2.]
- (c) Prove that  $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$  and hence that  $\text{card}(\mathbb{N} \times \mathbb{N}) = \aleph_0$ .
10. Use Exercise (9) to prove that if  $A$  and  $B$  are countably infinite sets, then  $A \times B$  is a countably infinite set.
11. Complete the proof of Theorem 9.19 by proving that the function  $g$  defined in the proof is a bijection from  $\mathbb{N}$  to  $B$ .

**Hint:** To prove that  $g$  is an injection, it might be easier to prove that for all  $r, s \in \mathbb{N}$ , if  $r \neq s$ , then  $g(r) \neq g(s)$ . To do this, we may assume that  $r < s$  since one of the two numbers must be less than the other. Then notice that  $g(r) \in \{g(1), g(2), \dots, g(s-1)\}$ .



To prove that  $g$  is a surjection, let  $b \in B$  and notice that for some  $k \in \mathbb{N}$ , there will be  $k$  natural numbers in  $B$  that are less than  $b$ .

12. Prove Corollary 9.20, which states that every subset of a countable set is countable.

**Hint:** Let  $S$  be a countable set and assume that  $A \subseteq S$ . There are two cases:  $A$  is finite or  $A$  is infinite. If  $A$  is infinite, let  $f: S \rightarrow \mathbb{N}$  be a bijection and define  $g: A \rightarrow f(A)$  by  $g(x) = f(x)$ , for each  $x \in A$ .

13. Use Corollary 9.20 to prove that the set of all rational numbers between 0 and 1 is countably infinite.

14. Let  $a, b \in \mathbb{Q}$  with  $a < b$ . On page 468, we proved that  $c = \frac{a+b}{2}$  is a rational number and that  $a < c < b$ , which proves that there is a rational number between any two (unequal) rational numbers.

(a) Now let  $c_1 = \frac{a+b}{2}$ , and define  $c_2 = \frac{c_1+b}{2}$ . Prove that  $c_1 < c_2 < b$  and hence, that  $a < c_1 < c_2 < b$ .

(b) For each  $k \in \mathbb{N}$ , define

$$c_{k+1} = \frac{c_k + b}{2}.$$

Prove that for each  $k \in \mathbb{N}$ ,  $a < c_k < c_{k+1} < b$ . Use this to explain why the set  $\{c_k \mid k \in \mathbb{N}\}$  is an infinite set where each element is a rational number between  $a$  and  $b$ .

## Explorations and Activities

15. **Another Proof that  $\mathbb{Q}^+$  Is Countable.** For this activity, it may be helpful to use the Fundamental Theorem of Arithmetic (see Theorem 8.15 on page 432). Let  $\mathbb{Q}^+$  be the set of positive rational numbers. Every positive rational number has a unique representation as a fraction  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime natural numbers. We will now define a function  $f: \mathbb{Q}^+ \rightarrow \mathbb{N}$  as follows:

If  $x \in \mathbb{Q}^+$  and  $x = \frac{m}{n}$ , where  $m, n \in \mathbb{N}$ ,  $n \neq 1$  and  $\gcd(m, n) = 1$ , we write

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}, \quad \text{and} \\ n = q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s},$$





where  $p_1, p_2, \dots, p_r$  are distinct prime numbers,  $q_1, q_2, \dots, q_s$  are distinct prime numbers, and  $\alpha_1, \alpha_2, \dots, \alpha_r$  and  $\beta_1, \beta_2, \dots, \beta_s$  are natural numbers. We also write  $1 = 2^0$  when  $m = 1$ . We then define

$$f(x) = p_1^{2\alpha_1} p_2^{2\alpha_2} \dots p_r^{2\alpha_r} q_1^{2\beta_1-1} q_2^{2\beta_2-1} \dots q_s^{2\beta_s-1}.$$

If  $x = \frac{m}{1}$ , then we define  $f(x) = p_1^{2\alpha_1} p_2^{2\alpha_2} \dots p_r^{2\alpha_r} = m^2$ .

- (a) Determine  $f\left(\frac{2}{3}\right)$ ,  $f\left(\frac{5}{6}\right)$ ,  $f(6)$ ,  $f\left(\frac{12}{25}\right)$ ,  $f\left(\frac{375}{392}\right)$ , and  $f\left(\frac{2^3 \cdot 11^3}{3 \cdot 5^4}\right)$ .
- (b) If possible, find  $x \in \mathbb{Q}^+$  such that  $f(x) = 100$ .
- (c) If possible, find  $x \in \mathbb{Q}^+$  such that  $f(x) = 12$ .
- (d) If possible, find  $x \in \mathbb{Q}^+$  such that  $f(x) = 2^8 \cdot 3^5 \cdot 13 \cdot 17^2$ .
- (e) Prove that the function  $f$  is an injection.
- (f) Prove that the function  $f$  is a surjection.
- (g) What has been proved?

### 9.3 Uncountable Sets

#### Preview Activity 1 (The Game of Dodge Ball)

(From *The Heart of Mathematics: An Invitation to Effective Thinking* by Edward B. Burger and Michael Starbird, Key Publishing Company, ©2000 by Edward B. Burger and Michael Starbird.)

Dodge Ball is a game for two players. It is played on a game board such as the one shown in Figure 9.4. Player One has a 6 by 6 array to complete and Player Two has a 1 by 6 row to complete. Each player has six turns as described next.

- Player One begins by filling in the first horizontal row of his or her table with a sequence of six X's and O's, one in each square in the first row.
- Then Player Two places either an X or an O in the first box of his or her row. At this point, Player One has completed the first row and Player Two has filled in the first box of his or her row with one letter.



Player One's Array						
1						
2						
3						
4						
5						
6						

Player Two's Row					
1	2	3	4	5	6

Figure 9.4: Game Board for Dodge Ball

- The game continues with Player One completing a row with six letters (X's and O's), one in each box of the next row followed by Player Two writing one letter (an X or an O) in the next box of his or her row. The game is completed when Player One has completed all six rows and Player Two has completed all six boxes in his or her row.

### Winning the Game

- Player One wins if any horizontal row in the 6 by 6 array is identical to the row that Player Two created. (Player One matches Player Two.)
- Player Two wins if Player Two's row of six letters is different than each of the six rows produced by Player One. (Player Two "dodges" Player One.)



There is a winning strategy for one of the two players. This means that there is a plan by which one of the two players will always win. Which player has a winning strategy? Carefully describe this winning strategy.

### Applying the Winning Strategy to Lists of Real Numbers

Following is a list of real numbers between 0 and 1. Each real number is written as a decimal number.

$$\begin{array}{ll} a_1 = 0.1234567890 & a_6 = 0.0103492222 \\ a_2 = 0.3216400000 & a_7 = 0.0011223344 \\ a_3 = 0.4321593333 & a_8 = 0.7077700022 \\ a_4 = 0.9120930092 & a_9 = 0.2100000000 \\ a_5 = 0.0000234102 & a_{10} = 0.9870008943 \end{array}$$

Use a method similar to the winning strategy in the game of Dodge Ball to write a real number (in decimal form) between 0 and 1 that is not in this list of 10 numbers.

1. Do you think your method could be used for any list of 10 real numbers between 0 and 1 if the goal is to write a real number between 0 and 1 that is not in the list?
2. Do you think this method could be extended to a list of 20 different real numbers? To a list of 50 different real numbers?
3. Do you think this method could be extended to a countably infinite list of real numbers?

---

### Preview Activity 2 (Functions from a Set to Its Power Set)

Let  $A$  be a set. In Section 5.1, we defined the **power set**  $\mathcal{P}(A)$  of  $A$  to be the set of all subsets of  $A$ . This means that

$$X \in \mathcal{P}(A) \text{ if and only if } X \subseteq A.$$

Theorem 5.5 in Section 5.1 states that if a set  $A$  has  $n$  elements, then  $A$  has  $2^n$  subsets or that  $\mathcal{P}(A)$  has  $2^n$  elements. Using our current notation for cardinality, this means that

$$\text{if } \text{card}(A) = n, \text{ then } \text{card}(\mathcal{P}(A)) = 2^n.$$



(The proof of this theorem was Exercise (17) on page 229.)

We are now going to define and explore some functions from a set  $A$  to its power set  $\mathcal{P}(A)$ . This means that the input of the function will be an element of  $A$  and the output of the function will be a subset of  $A$ .

1. Let  $A = \{1, 2, 3, 4\}$ . Define  $f: A \rightarrow \mathcal{P}(A)$  by

$$\begin{aligned} f(1) &= \{1, 2, 3\} & f(3) &= \{1, 4\} \\ f(2) &= \{1, 3, 4\} & f(4) &= \{2, 4\}. \end{aligned}$$

- (a) Is  $1 \in f(1)$ ? Is  $2 \in f(2)$ ? Is  $3 \in f(3)$ ? Is  $4 \in f(4)$ ?  
 (b) Determine  $S = \{x \in A \mid x \notin f(x)\}$ .  
 (c) Notice that  $S \in \mathcal{P}(A)$ . Does there exist an element  $t$  in  $A$  such that  $f(t) = S$ ? That is, is  $S \in \text{range}(f)$ ?
2. Let  $A = \{1, 2, 3, 4\}$ . Define  $f: A \rightarrow \mathcal{P}(A)$  by

$$f(x) = A - \{x\} \text{ for each } x \in A.$$

- (a) Determine  $f(1)$ . Is  $1 \in f(1)$ ?      (c) Determine  $f(3)$ . Is  $3 \in f(3)$ ?  
 (b) Determine  $f(2)$ . Is  $2 \in f(2)$ ?      (d) Determine  $f(4)$ . Is  $4 \in f(4)$ ?  
 (e) Determine  $S = \{x \in A \mid x \notin f(x)\}$ .  
 (f) Notice that  $S \in \mathcal{P}(A)$ . Does there exist an element  $t$  in  $A$  such that  $f(t) = S$ ? That is, is  $S \in \text{range}(f)$ ?
3. Define  $f: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$  by

$$f(n) = \mathbb{N} - \{n^2, n^2 - 2n\}, \text{ for each } n \in \mathbb{N}.$$

- (a) Determine  $f(1)$ ,  $f(2)$ ,  $f(3)$ , and  $f(4)$ . In each of these cases, determine if  $k \in f(k)$ .  
 (b) Prove that if  $n > 3$ , then  $n \in f(n)$ . **Hint:** Prove that if  $n > 3$ , then  $n^2 > n$  and  $n^2 - 2n > n$ .  
 (c) Determine  $S = \{x \in \mathbb{N} \mid x \notin f(x)\}$ .  
 (d) Notice that  $S \in \mathcal{P}(\mathbb{N})$ . Does there exist an element  $t$  in  $\mathbb{N}$  such that  $f(t) = S$ ? That is, is  $S \in \text{range}(f)$ ?



We have seen examples of sets that are countably infinite, but we have not yet seen an example of an infinite set that is uncountable. We will do so in this section. The first example of an uncountable set will be the open interval of real numbers  $(0, 1)$ . The proof that this interval is uncountable uses a method similar to the winning strategy for Player Two in the game of Dodge Ball from Preview Activity 1. Before considering the proof, we need to state an important result about decimal expressions for real numbers.

### Decimal Expressions for Real Numbers

In its decimal form, any real number  $a$  in the interval  $(0, 1)$  can be written as  $a = 0.a_1a_2a_3a_4 \dots$ , where each  $a_i$  is an integer with  $0 \leq a_i \leq 9$ . For example,

$$\frac{5}{12} = 0.416666\dots$$

We often abbreviate this as  $\frac{5}{12} = 0.41\overline{6}$  to indicate that the 6 is repeated. We can also repeat a block of digits. For example,  $\frac{5}{26} = 0.19\overline{230769}$  to indicate that the block 230769 repeats. That is,

$$\frac{5}{26} = 0.19230769230769230769\dots$$

There is only one situation in which a real number can be represented as a decimal in more than one way. A decimal that ends with an infinite string of 9's is equal to one that ends with an infinite string of 0's. For example,  $0.319999\dots$  represents the same real number as  $0.320000\dots$ . Geometric series can be used to prove that a decimal that ends with an infinite string of 9's is equal to one that ends with an infinite string of 0's, but we will not do so here.

**Definition.** A decimal representation of a real number  $a$  is in **normalized form** provided that there is no natural number  $k$  such that for all natural numbers  $n$  with  $n > k$ ,  $a_n = 9$ . That is, the decimal representation of  $a$  is in normalized form if and only if it does not end with an infinite string of 9's.

One reason the normalized form is important is the following theorem (which will not be proved here).

**Theorem 9.21.** *Two decimal numbers in normalized form are equal if and only if they have identical digits in each decimal position.*



### Uncountable Subsets of $\mathbb{R}$

In the proof that follows, we will use only the normalized form for the decimal representation of a real number in the interval  $(0, 1)$ .

**Theorem 9.22.** *The open interval  $(0, 1)$  is an uncountable set.*

**Proof.** Since the interval  $(0, 1)$  contains the infinite subset  $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$ , we can use Theorem 9.10, to conclude that  $(0, 1)$  is an infinite set. So  $(0, 1)$  is either countably infinite or uncountable. We will prove that  $(0, 1)$  is uncountable by proving that any function from  $\mathbb{N}$  to  $(0, 1)$  is not a surjection, and hence, there is no bijection from  $\mathbb{N}$  to  $(0, 1)$ .

So suppose that  $f: \mathbb{N} \rightarrow (0, 1)$  is a function. We will show that  $f$  is not a surjection by showing that there exists an element in  $(0, 1)$  that cannot be in the range of  $f$ . Writing the images of the elements of  $\mathbb{N}$  in normalized form, we can write

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}a_{14}a_{15} \dots \\ f(2) &= 0.a_{21}a_{22}a_{23}a_{24}a_{25} \dots \\ f(3) &= 0.a_{31}a_{32}a_{33}a_{34}a_{35} \dots \\ f(4) &= 0.a_{41}a_{42}a_{43}a_{44}a_{45} \dots \\ f(5) &= 0.a_{51}a_{52}a_{53}a_{54}a_{55} \dots \\ &\vdots \\ f(n) &= 0.a_{n1}a_{n2}a_{n3}a_{n4}a_{n5} \dots \\ &\vdots \end{aligned}$$

Notice the use of the double subscripts. The number  $a_{ij}$  is the  $j^{\text{th}}$  digit to the right of the decimal point in the normalized decimal representation of  $f(i)$ .

We will now construct a real number  $b = 0.b_1b_2b_3b_4b_5 \dots$  in  $(0, 1)$  and in normalized form that is not in this list.

**Note:** The idea is to start in the upper left corner and move down the diagonal in a manner similar to the winning strategy for Player Two in the game in Preview Activity 1. At each step, we choose a digit that is not equal to the diagonal digit.

Start with  $a_{11}$  in  $f(1)$ . We want to choose  $b_1$  so that  $b_1 \neq 0$ ,  $b_1 \neq a_{11}$ , and  $b_1 \neq 9$ . (To ensure that we end up with a decimal that is in normalized form, we make sure that each digit is not equal to 9.) We then repeat this process with  $a_{22}$ ,



$a_{33}, a_{44}, a_{55}$ , and so on. So we let  $b$  be the real number  $b = 0.b_1b_2b_3b_4b_5\dots$ , where for each  $k \in \mathbb{N}$

$$b_k = \begin{cases} 3 & \text{if } a_{kk} \neq 3 \\ 5 & \text{if } a_{kk} = 3. \end{cases}$$

(The choice of 3 and 5 is arbitrary. Other choices of distinct digits will also work.)

Now for each  $n \in \mathbb{N}$ ,  $b \neq f(n)$  since  $b$  and  $f(n)$  are in normalized form and  $b$  and  $f(n)$  differ in the  $n$ th decimal place. Therefore,  $f$  is not a surjection. This proves that any function from  $\mathbb{N}$  to  $(0, 1)$  is not a surjection and hence, there is no bijection from  $\mathbb{N}$  to  $(0, 1)$ . Therefore,  $(0, 1)$  is not countably infinite and hence must be an uncountable set. ■

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**Progress Check 9.23 (Dodge Ball and Cantor's Diagonal Argument)**

The proof of Theorem 9.22 is often referred to as **Cantor's diagonal argument**. It is named after the mathematician Georg Cantor, who first published the proof in 1874. Explain the connection between the winning strategy for Player Two in Dodge Ball (see Preview Activity 1) and the proof of Theorem 9.22 using Cantor's diagonal argument.

---

The open interval  $(0, 1)$  is our first example of an uncountable set. The cardinal number of  $(0, 1)$  is defined to be  $c$ , which stands for **the cardinal number of the continuum**. So the two infinite cardinal numbers we have seen are  $\aleph_0$  for countably infinite sets and  $c$ .

**Definition.** A set  $A$  is said to have **cardinality  $c$**  provided that  $A$  is equivalent to  $(0, 1)$ . In this case, we write  $\text{card}(A) = c$  and say that the cardinal number of  $A$  is  $c$ .

The proof of Theorem 9.24 is included in Progress Check 9.25.

**Theorem 9.24.** *Let  $a$  and  $b$  be real numbers with  $a < b$ . The open interval  $(a, b)$  is uncountable and has cardinality  $c$ .*

**Progress Check 9.25 (Proof of Theorem 9.24)**

1. In Part (3) of Progress Check 9.2, we proved that if  $b \in \mathbb{R}$  and  $b > 0$ , then the open interval  $(0, 1)$  is equivalent to the open interval  $(0, b)$ . Now let  $a$  and  $b$  be real numbers with  $a < b$ . Find a function

$$f: (0, 1) \rightarrow (a, b)$$



that is a bijection and conclude that the open interval  $(0, 1) \approx (a, b)$ .

**Hint:** Find a linear function that passes through the points  $(0, a)$  and  $(1, b)$ . Use this to define the function  $f$ . Make sure you prove that this function  $f$  is a bijection.

- Let  $a, b, c, d$  be real numbers with  $a < b$  and  $c < d$ . Prove that the open interval  $(a, b)$  is equivalent to the open interval  $(c, d)$ .

---

**Theorem 9.26.** *The set of real numbers  $\mathbb{R}$  is uncountable and has cardinality  $\mathfrak{c}$ .*

**Proof.** Let  $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$  be defined by  $f(x) = \tan x$ , for each  $x \in \mathbb{R}$ . The function  $f$  is a bijection and, hence,  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \approx \mathbb{R}$ . So by Theorem 9.24,  $\mathbb{R}$  is uncountable and has cardinality  $\mathfrak{c}$ . ■

### Cantor's Theorem

We have now seen two different infinite cardinal numbers,  $\aleph_0$  and  $\mathfrak{c}$ . It can seem surprising that there is more than one infinite cardinal number. A reasonable question at this point is, “Are there any other infinite cardinal numbers?” The astonishing answer is that there are, and in fact, there are infinitely many different infinite cardinal numbers. The basis for this fact is the following theorem, which states that a set is not equivalent to its power set. The proof is due to Georg Cantor (1845–1918), and the idea for this proof was explored in Preview Activity 2. The basic idea of the proof is to prove that any function from a set  $A$  to its power set cannot be a surjection.

**Theorem 9.27 (Cantor's Theorem).** *For every set  $A$ ,  $A$  and  $\mathcal{P}(A)$  do not have the same cardinality.*

**Proof.** Let  $A$  be a set. If  $A = \emptyset$ , then  $\mathcal{P}(A) = \{\emptyset\}$ , which has cardinality 1. Therefore,  $\emptyset$  and  $\mathcal{P}(\emptyset)$  do not have the same cardinality.

Now suppose that  $A \neq \emptyset$ , and let  $f: A \rightarrow \mathcal{P}(A)$ . We will show that  $f$  cannot be a surjection, and hence there is no bijection from  $A$  to  $\mathcal{P}(A)$ . This will prove that  $A$  is not equivalent to  $\mathcal{P}(A)$ . Define

$$S = \{x \in A \mid x \notin f(x)\}.$$

Assume that there exists a  $t$  in  $A$  such that  $f(t) = S$ . Now, either  $t \in S$  or  $t \notin S$ .





- If  $t \in S$ , then  $t \in \{x \in A \mid x \notin f(x)\}$ . By the definition of  $S$ , this means that  $t \notin f(t)$ . However,  $f(t) = S$  and so we conclude that  $t \notin S$ . But now we have  $t \in S$  and  $t \notin S$ . This is a contradiction.
- If  $t \notin S$ , then  $t \notin \{x \in A \mid x \notin f(x)\}$ . By the definition of  $S$ , this means that  $t \in f(t)$ . However,  $f(t) = S$  and so we conclude that  $t \in S$ . But now we have  $t \notin S$  and  $t \in S$ . This is a contradiction.

So in both cases we have arrived at a contradiction. This means that there does not exist a  $t$  in  $A$  such that  $f(t) = S$ . Therefore, any function from  $A$  to  $\mathcal{P}(A)$  is not a surjection and hence not a bijection. Hence,  $A$  and  $\mathcal{P}(A)$  do not have the same cardinality. ■

**Corollary 9.28.**  $\mathcal{P}(\mathbb{N})$  is an infinite set that is not countably infinite.

*Proof.* Since  $\mathcal{P}(\mathbb{N})$  contains the infinite subset  $\{\{1\}, \{2\}, \{3\}, \dots\}$ , we can use Theorem 9.10, to conclude that  $\mathcal{P}(\mathbb{N})$  is an infinite set. By Cantor's Theorem (Theorem 9.27),  $\mathbb{N}$  and  $\mathcal{P}(\mathbb{N})$  do not have the same cardinality. Therefore,  $\mathcal{P}(\mathbb{N})$  is not countable and hence is an uncountable set. ■

### Some Final Comments about Uncountable Sets

1. We have now seen that any open interval of real numbers is uncountable and has cardinality  $\mathfrak{c}$ . In addition,  $\mathbb{R}$  is uncountable and has cardinality  $\mathfrak{c}$ . Now, Corollary 9.28 tells us that  $\mathcal{P}(\mathbb{N})$  is uncountable. A question that can be asked is,

“Does  $\mathcal{P}(\mathbb{N})$  have the same cardinality as  $\mathbb{R}$ ?”

The answer is yes, although we are not in a position to prove it yet. A proof of this fact uses the following theorem, which is known as the Cantor-Schröder-Bernstein Theorem.

**Theorem 9.29 (Cantor-Schröder-Bernstein).** *Let  $A$  and  $B$  be sets. If there exist injections  $f: A \rightarrow B$  and  $g: B \rightarrow A$ , then  $A \approx B$ .*

In the statement of this theorem, notice that it is not required that the function  $g$  be the inverse of the function  $f$ . We will not prove the Cantor-Schröder-Bernstein Theorem here. The following items will show some uses of this important theorem.



2. The Cantor-Schröder-Bernstein Theorem can also be used to prove that the closed interval  $[0, 1]$  is equivalent to the open interval  $(0, 1)$ . See Exercise (6) on page 486.
3. Another question that was posed earlier is,

“Are there other infinite cardinal numbers other than  $\aleph_0$  and  $c$ ?”

Again, the answer is yes, and the basis for this is Cantor’s Theorem (Theorem 9.27). We can start with  $\text{card}(\mathbb{N}) = \aleph_0$ . We then define the following infinite cardinal numbers:

$$\begin{array}{ll} \text{card}(\mathcal{P}(\mathbb{N})) = \alpha_1. & \text{card}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))) = \alpha_3. \\ \text{card}(\mathcal{P}(\mathcal{P}(\mathbb{N}))) = \alpha_2. & \vdots \end{array}$$

Cantor’s Theorem tells us that these are all different cardinal numbers, and so we are just using the lowercase Greek letter  $\alpha$  (alpha) to help give names to these cardinal numbers. In fact, although we will not define it here, there is a way to “order” these cardinal numbers in such a way that

$$\aleph_0 < \alpha_1 < \alpha_2 < \alpha_3 < \dots$$

Keep in mind, however, that even though these are different cardinal numbers, Cantor’s Theorem does not tell us that these are the only cardinal numbers.

4. In Comment (1), we indicated that  $\mathcal{P}(\mathbb{N})$  and  $\mathbb{R}$  have the same cardinality. Combining this with the notation in Comment (3), this means that

$$\alpha_1 = c.$$

However, this does not necessarily mean that  $c$  is the “next largest” cardinal number after  $\aleph_0$ . A reasonable question is, “Is there an infinite set with cardinality between  $\aleph_0$  and  $c$ ?” Rewording this in terms of the real number line, the question is, “On the real number line, is there an infinite set of points that is not equivalent to the entire line and also not equivalent to the set of natural numbers?” This question was asked by Cantor, but he was unable to find any such set. He conjectured that no such set exists. That is, he conjectured that  $c$  is really the next cardinal number after  $\aleph_0$ . This conjecture has come to be known as the **Continuum Hypothesis**. Stated somewhat more formally, the Continuum Hypothesis is



There is no set  $X$  such that  $\aleph_0 < \text{card}(X) < \mathfrak{c}$ .

The question of whether the Continuum Hypothesis is true or false is one of the most famous problems in modern mathematics.

Through the combined work of Kurt Gödel in the 1930s and Paul Cohen in 1963, it has been proved that the Continuum Hypothesis cannot be proved or disproved from the standard axioms of set theory. This means that either the Continuum Hypothesis or its negation can be added to the standard axioms of set theory without creating a contradiction.

### Exercises 9.3

1. Use an appropriate bijection to prove that each of the following sets has cardinality  $\mathfrak{c}$ .

\* (a)  $(0, \infty)$

(c)  $\mathbb{R} - \{0\}$

\* (b)  $(a, \infty)$ , for any  $a \in \mathbb{R}$

(d)  $\mathbb{R} - \{a\}$ , for any  $a \in \mathbb{R}$

- \* 2. Is the set of irrational numbers countable or uncountable? Prove that your answer is correct.
- \* 3. Prove that if  $A$  is uncountable and  $A \subseteq B$ , then  $B$  is uncountable.
- \* 4. Do two uncountable sets always have the same cardinality? Justify your conclusion.
5. Let  $C$  be the set of all infinite sequences, each of whose entries is the digit 0 or the digit 1. For example,

$$(1, 0, 1, 0, 1, 0, 1, 0, \dots) \in C;$$

$$(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, \dots) \in C;$$

$$(2, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, \dots) \notin C.$$

Is the set  $C$  a countable set or an uncountable set? Justify your conclusion.

6. The goal of this exercise is to use the Cantor-Schröder-Bernstein Theorem to prove that the cardinality of the closed interval  $[0, 1]$  is  $\mathfrak{c}$ .

(a) Find an injection  $f: (0, 1) \rightarrow [0, 1]$ .



- (b) Find an injection  $h: [0, 1] \rightarrow (-1, 2)$ .
- (c) Use the fact that  $(-1, 2) \approx (0, 1)$  to prove that there exists an injection  $g: [0, 1] \rightarrow (0, 1)$ . (It is only necessary to prove that the injection  $g$  exists. It is not necessary to determine a specific formula for  $g(x)$ .)
- Note:** Instead of doing Part (b) as stated, another approach is to find an injection  $k: [0, 1] \rightarrow (0, 1)$ . Then, it is possible to skip Part (c) and go directly to Part (d).
- (d) Use the Cantor-Schröder-Bernstein Theorem to conclude that  $[0, 1] \approx (0, 1)$  and hence that the cardinality of  $[0, 1]$  is  $c$ .
7. In Exercise (6), we proved that the closed interval  $[0, 1]$  is uncountable and has cardinality  $c$ . Now let  $a, b \in \mathbb{R}$  with  $a < b$ . Prove that  $[a, b] \approx [0, 1]$  and hence that  $[a, b]$  is uncountable and has cardinality  $c$ .
8. Is the set of all finite subsets of  $\mathbb{N}$  countable or uncountable? Let  $F$  be the set of all finite subsets of  $\mathbb{N}$ . Determine the cardinality of the set  $F$ .

Consider defining a function  $f: F \rightarrow \mathbb{N}$  that produces the following.

- If  $A = \{1, 2, 6\}$ , then  $f(A) = 2^1 3^2 5^6$ .
- If  $B = \{3, 6\}$ , then  $f(B) = 2^3 3^6$ .
- If  $C = \{m_1, m_2, m_3, m_4\}$  with  $m_1 < m_2 < m_3 < m_4$ , then  $f(C) = 2^{m_1} 3^{m_2} 5^{m_3} 7^{m_4}$ .

It might be helpful to use the Fundamental Theorem of Arithmetic on page 432 and to denote the set of all primes as  $P = \{p_1, p_2, p_3, p_4, \dots\}$  with  $p_1 < p_2 < p_3 < p_4 \dots$ . Using the sets  $A$ ,  $B$ , and  $C$  defined above, we would then write

$$f(A) = p_1^1 p_2^2 p_3^6, \quad f(B) = p_1^3 p_2^6, \quad \text{and} \quad f(C) = p_1^{m_1} p_2^{m_2} p_3^{m_3} p_4^{m_4}.$$

9. In Exercise (2), we showed that the set of irrational numbers is uncountable. However, we still do not know the cardinality of the set of irrational numbers. Notice that we can use  $\mathbb{Q}^c$  to stand for the set of irrational numbers.

- (a) Construct a function  $f: \mathbb{Q}^c \rightarrow \mathbb{R}$  that is an injection.

We know that any real number  $a$  can be represented in decimal form as follows:

$$a = A.a_1 a_2 a_3 a_4 \cdots a_n \cdots,$$



where  $A$  is an integer and the decimal part  $(0.a_1a_2a_3a_4\cdots)$  is in normalized form. (See page 480.) We also know that the real number  $a$  is an irrational number if and only if  $a$  has an infinite non-repeating decimal expansion. We now associate with  $a$  the real number

$$A.a_10a_211a_3000a_41111a_500000a_6111111\cdots. \quad (1)$$

Notice that to construct the real number in (1), we started with the decimal expansion of  $a$ , inserted a 0 to the right of the first digit after the decimal point, inserted two 1's to the right of the second digit to the right of the decimal point, inserted three 0's to the right of the third digit to the right of the decimal point, and so on.

- (b) Explain why the real number in (1) is an irrational number.
- (c) Use these ideas to construct a function  $g: \mathbb{R} \rightarrow \mathbb{Q}^c$  that is an injection.
- (d) What can we now conclude by using the Cantor-Schröder-Bernstein Theorem?

- 10.** Let  $J$  be the unit open interval. That is,  $J = \{x \in \mathbb{R} \mid 0 < x < 1\}$  and let  $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid 0 < x < 1 \text{ and } 0 < y < 1\}$ . We call  $S$  the unit open square. We will now define a function  $f$  from  $S$  to  $J$ . Let  $(a, b) \in S$  and write the decimal expansions of  $a$  and  $b$  in normalized form as

$$\begin{aligned} a &= 0.a_1a_2a_3a_4\cdots a_n\cdots \\ b &= 0.b_1b_2b_3b_4\cdots b_n\cdots \end{aligned}$$

We then define  $f(a, b) = 0.a_1b_1a_2b_2a_3b_3a_4b_4\cdots a_nb_n\cdots$ .

- (a) Determine the values of  $f(0.3, 0.625)$ ,  $f\left(\frac{1}{3}, \frac{1}{4}\right)$ , and  $f\left(\frac{1}{6}, \frac{5}{6}\right)$ .
- (b) If possible, find  $(x, y) \in S$  such that  $f(x, y) = 0.2345$ .
- (c) If possible, find  $(x, y) \in S$  such that  $f(x, y) = \frac{1}{3}$ .
- (d) If possible, find  $(x, y) \in S$  such that  $f(x, y) = \frac{1}{2}$ .
- (e) Explain why the function  $f: S \rightarrow J$  is an injection but is not a surjection.
- (f) Use the Cantor-Schröder-Bernstein Theorem to prove that the cardinality of the unit open square  $S$  is equal to  $c$ . If this result seems surprising, you are in good company. In a letter written in 1877 to the mathematician Richard Dedekind describing this result that he had discovered, Georg Cantor wrote, "I see it but I do not believe it."



### Explorations and Activities

**11. The Closed Interval [0, 1].** In Exercise (6), the Cantor-Schröder-Bernstein Theorem was used to prove that the closed interval  $[0, 1]$  has cardinality  $\mathfrak{c}$ . This may seem a bit unsatisfactory since we have not proved the Cantor-Schröder-Bernstein Theorem. In this activity, we will prove that  $\text{card}([0, 1]) = \mathfrak{c}$  by using appropriate bijections.

(a) Let  $f: [0, 1] \rightarrow [0, 1)$  by

$$f(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise.} \end{cases}$$

- i. Determine  $f(0)$ ,  $f(1)$ ,  $f\left(\frac{1}{2}\right)$ ,  $f\left(\frac{1}{3}\right)$ ,  $f\left(\frac{1}{4}\right)$ , and  $f\left(\frac{1}{5}\right)$ .
- ii. Sketch a graph of the function  $f$ . **Hint:** Start with the graph of  $y = x$  for  $0 \leq x \leq 1$ . Remove the point  $(1, 1)$  and replace it with the point  $\left(1, \frac{1}{2}\right)$ . Next, remove the point  $\left(\frac{1}{2}, \frac{1}{2}\right)$  and replace it with the point  $\left(\frac{1}{2}, \frac{1}{3}\right)$ . Continue this process of removing points on the graph of  $y = x$  and replacing them with the points determined from the information in Part (11(a)i). Stop after repeating this four or five times so that pattern of this process becomes apparent.
- iii. Explain why the function  $f$  is a bijection.
- iv. Prove that  $[0, 1] \approx [0, 1)$ .

(b) Let  $g: [0, 1) \rightarrow (0, 1)$  by

$$g(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise.} \end{cases}$$

- i. Follow the procedure suggested in Part (11a) to sketch a graph of  $g$ .
  - ii. Explain why the function  $g$  is a bijection.
  - iii. Prove that  $[0, 1) \approx (0, 1)$ .
- (c) Prove that  $[0, 1]$  and  $(0, 1)$  are both uncountable and have cardinality  $\mathfrak{c}$ .

## 9.4 Chapter 9 Summary

### Important Definitions

- Equivalent sets, page 452
- Sets with the same cardinality, page 452
- Finite set, page 455
- Infinite set, page 455
- Cardinality of a finite set, page 455
- Cardinality of  $\mathbb{N}$ , page 465
- $\aleph_0$ , page 465
- Countably infinite set, page 466
- Denumerable set, page 466
- Uncountable set, page 466

### Important Theorems and Results about Finite and Infinite Sets

- **Theorem 9.3.** Any set equivalent to a finite nonempty set  $A$  is a finite set and has the same cardinality as  $A$ .
- **Theorem 9.6.** If  $S$  is a finite set and  $A$  is a subset of  $S$ , then  $A$  is finite and  $\text{card}(A) \leq \text{card}(S)$ .
- **Corollary 9.8.** A finite set is not equivalent to any of its proper subsets.
- **Theorem 9.9 [The Pigeonhole Principle].** Let  $A$  and  $B$  be finite sets. If  $\text{card}(A) > \text{card}(B)$ , then any function  $f: A \rightarrow B$  is not an injection.
- **Theorem 9.10.** Let  $A$  and  $B$  be sets.
  1. If  $A$  is infinite and  $A \approx B$ , then  $B$  is infinite.
  2. If  $A$  is infinite and  $A \subseteq B$ , then  $B$  is infinite.
- **Theorem 9.13.** The set  $\mathbb{Z}$  of integers is countably infinite, and so  $\text{card}(\mathbb{Z}) = \aleph_0$ .
- **Theorem 9.14.** The set of positive rational numbers is countably infinite.
- **Theorem 9.16.** If  $A$  is a countably infinite set and  $B$  is a finite set, then  $A \cup B$  is a countably infinite set.
- **Theorem 9.17.** If  $A$  and  $B$  are disjoint countably infinite sets, then  $A \cup B$  is a countably infinite set.



- **Theorem 9.18.** *The set  $\mathbb{Q}$  of all rational numbers is countably infinite.*
  - **Theorem 9.19.** *Every subset of the natural numbers is countable.*
  - **Corollary 9.20.** *Every subset of a countable set is countable.*
  - **Theorem 9.22.** *The open interval  $(0, 1)$  is an uncountable set.*
  - **Theorem 9.24.** *Let  $a$  and  $b$  be real numbers with  $a < b$ . The open interval  $(a, b)$  is uncountable and has cardinality  $c$ .*
  - **Theorem 9.26.** *The set of real numbers  $\mathbb{R}$  is uncountable and has cardinality  $c$ .*
  - **Theorem 9.27 [Cantor's Theorem].** *For every set  $A$ ,  $A$  and  $\mathcal{P}(A)$  do not have the same cardinality.*
  - **Corollary 9.28.**  *$\mathcal{P}(\mathbb{N})$  is an infinite set that is not countably infinite.*
  - **Theorem 9.29 [Cantor-Schröder-Bernstein].** *Let  $A$  and  $B$  be sets. If there exist injections  $f_1 : A \rightarrow B$  and  $f_2 : B \rightarrow A$ , then  $A \approx B$ .*
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