Appendix F

The Fundamental Theorem of Algebra

In this appendix we provide a proof of the Fundamental Theorem of Algebra. This particular proof comes from the paper "The Fundamental Theorem of Algebra" by Frode Terkelsen in *The American Mathematical Monthly*, Vol. 83, No. 8, (Oct. 1976), p. 647. As is true with most proofs of this theorem, some complex analysis is required, and we will gloss over those points. While our proof will not be complete and rigorous, it is instructive to have some idea of how this important theorem is proved. For complete details, consult a text on complex analysis.

Theorem (The Fundamental Theorem of Algebra.)**.** *Every polynomial of degree 1 or greater in* $\mathbb{C}[x]$ *has a root in* $\mathbb{C}$.

Proof. Let $f(z) \in \mathbb{C}[z]$ be a non-constant polynomial. We know that $|f(z)| \to \infty$ as $|z| \to \infty$, and along with the continuity of the polynomial $f(z)$, this implies the existence of a complex number z_0 such that $|f(z_0)| \leq |f(z)|$ for all $z \in \mathbb{C}$. In other words, the function $|f(z)|$ attains a minimum value.* (Recall that the norm $|f(z)|$ of the complex number $f(z)$ is a nonnegative real number.) We can always translate f so that z_0 is at the origin (by considering the polynomial $f(z + z_0)$), so we will assume $z_0 = 0$ without loss of generality. Our job is then to show that $f(0) = 0$. We will proceed by contradiction and assume $f(0) \neq 0$.

First, we will rewrite $f(z)$ in a more useful form. By subtracting out the constant term of $f(z)$, we obtain a polynomial whose smallest degree term is n for some $n \geq 1$. Thus,

$$f(z) = a_0 + a_n z^n + z^{n+1} Q(z),$$

where $a_n \neq 0$ and $Q(z)$ is a polynomial. (As an illustration, consider the polynomial $f(z) = 2 + 3z^3 + 4z^4 + 5z^6$. Note that $n = 3$ and $Q(z) = 4 + 5z^2$ in this case.) For ease of notation, we let $a = a_0$ and $b = a_n$. Note that $f(0) \neq 0$ implies $a \neq 0$. We will now use the fact that every complex number has n^{th} roots. In particular, let ω be an n^{th} root of $-\frac{a}{b}$—that is $\omega^n = -\frac{a}{b}$. For each real number x, we have

$$\begin{aligned} f(x\omega) &= a + b(x\omega)^n + (x\omega)^{n+1} Q(x\omega) \\ &= a + bx^n \omega^n + (x\omega)^{n+1} Q(x\omega) \\ &= a + bx^n \left(-\frac{a}{b}\right) + (x\omega)^{n+1} Q(x\omega) \\ &= a(1 - x^n) + (x\omega)^{n+1} Q(x\omega). \end{aligned}$$

*The formal proof of this result requires some more sophisticated results from analysis and topology, but the idea is similar to why every even-degree polynomial $p(x)$ in $\mathbb{R}[x]$ attains a minimum value. For any such $p(x)$, the Extreme Value Theorem guarantees that $p(x)$ will attain a minimum value m_a on each closed interval of the form $[-a, a]$. But $p(x) \to \infty$ as $|x| \to \infty$, and so there must be some $a \in \mathbb{R}^+$ for which $p(x) > m_a$ for all $x < -a$ and all $x > a$. It follows that this m_a is the (global) minimum value of $p(x)$.

Suppose $Q(z) = q_0 + q_1 z + q_2 z^2 + \cdots + q_m z^m$. If $0 < t < 1$, then the triangle inequality shows that

$$|Q(t\omega)| \leq \sum_{k=0}^{m} |q_k (t\omega)^k| = \sum_{k=0}^{m} |t|^k |q_k \omega^k| < \sum_{k=0}^{m} |q_k \omega^k|.$$

Note that the quantity $\sum_{k=0}^{m} |q_k \omega^k|$ does not depend on t. Therefore, there exists $t \in \mathbb{R}$ with $0 < t < 1$ such that

$$t|\omega^{n+1} Q(t\,\omega)| < |a|.$$

Thus,

$$\begin{aligned} |f(t\omega)| &\leq |a|(1 - t^n) + t^n \left(t|\omega^{n+1} Q(t\omega)|\right) \\ &< |a|(1 - t^n) + t^n |a| \\ &= |a| \\ &= |f(0)|, \end{aligned}$$

which contradicts the fact that $|f(z)|$ attains its minimum value at 0. Therefore, it must be that $f(0) = 0$, and $f(z)$ has a root. ∎