Section 10

Closed Sets in Metric Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What are boundary points, limit points, and isolated points of a set in a metric space? How are they related and how are they different?
- What does it mean for a set to be closed in a metric space?
- What important properties do closed sets have in relation to unions and intersections?
- How can we use closed sets to determine the continuity of a function?
- How are limit points related to sequences?
- How are boundary points related to sequences?
- What is the boundary of a set in a metric space?
- How are limit points and boundary points related to closed sets?
- What is the closure of a set in a metric space?
- How are closed sets related to sequences?

Introduction

Once we have defined open sets in metric spaces, it is natural to ask if there are closed sets. Recall that closed intervals are important in calculus because every continuous function on a closed interval attains an absolute maximum and absolute minimum value on that interval. If we have closed sets in metric spaces, we might consider if there is some result that is similar to this for continuous functions on closed sets. In this section we introduce the idea of closed sets in metric spaces and discover a few of their properties.

Every interval of the form $[a, b]$ in $\mathbb{R}$ is a closed set using the Euclidean metric. What distinguishes these closed intervals from the open intervals is that the open intervals do not contain either of their endpoints – this is what makes an open interval a neighborhood of each of its points. In general, what makes open sets open is that they do not contain their boundaries. If an open set doesn't contain its boundary, then its complement, by contrast, should contain its boundary. This leads to the definition of a closed set.

Definition 10.1. A subset C of a metric space X is **closed** if its complement $X \setminus C$ is open.

We said that open sets are open because they do not contain their boundary and closed sets are closed because they do contain their boundary. However, we did not define what we mean by boundary. The point a on the "boundary" of an open interval of the form $O = (a, b)$ in $\mathbb{R}$ with the Euclidean metric has the property that every open ball that contains a contains points in O and points not in O. This is what makes the point a lie on the boundary. We can also think of the point a as being at the very limit of the set O. This motivates the next definition.

Definition 10.2. Let X be a metric space, and let A be a subset of X. A **boundary point** of A is a point $x \in X$ such that every neighborhood of x contains a point in A and a point in $X \setminus A$.

For example, in $A = (0, 1)$ as a subset of $(\mathbb{R}, d_E)$, the number 0 is a boundary point of A because any open interval in $\mathbb{R}$ that contains 0 contains points in A and points not in A. Boundary points can arise in other ways. If $A = \{0, 1\}$ as a subset of $(\mathbb{R}, d_E)$, then 0 is again a boundary point because any open interval in $\mathbb{R}$ that contains 0 contains a point (0) in A and points not in A. However, 0 is the only point in A that is contained in any open interval that contains 0. In this case we call 0 an *isolated point* of A, and in the case of the set $A = (0, 1)$ we call 0 an *accumulation point* or a *limit point* of A (the use of the word "limit" here will become clear later).

Definition 10.3. Let X be a metric space, and let A be a subset of X.

(1) An **accumulation point** or **limit point** of A is a point $x \in X$ such that every neighborhood of x contains a point in A different from x.

(2) An **isolated point** of A is a point $a \in A$ such that there exists a neighborhood N of a in X with $N \cap A = \{a\}$.

You might wonder about the use of the term "limit point" and how limit points might be related to limits. As we will see later, limit points are limits of sequences, but the definition as we have given is one that will translate directly to topological spaces later.

Note that every boundary point is either an accumulation point or an isolated point. The proof is left as an exercise.

Preview Activity 10.1.

(1) For each of the given sets A, find all boundary points, limit points, and isolated points. Then determine if the set A is a closed set in the metric space (X, d). Explain your reasoning.

   (a) $X = \mathbb{R}$, $d = d_E$, the Euclidean metric, $A = [0, 0.5)$.

(b) $X = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$, $d = d_E$, the Euclidean metric, $A = (0, 0.5]$.

(c) $X = \{a, b, c, e\}$, d is the discrete metric defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

and $A = \{a, b\}$.

(2) Label each of the following statements as either true or false. If true, provide a convincing argument. If false, provide a specific counterexample.

(a) Every limit point is a boundary point.

(b) Every boundary point is a limit point.

(c) Every limit point is an isolated point

(d) Every isolated point is a limit point.

(e) Every boundary point is an isolated point.

(f) Every isolated point is a boundary point.

Closed Sets in Metric Spaces

Recall that Definition 10.1 defines a closed set in a metric space to be a set whose complement is open. We have seen that both $\emptyset$ and X are open subsets of X. We now ask the same question, this time in terms of closed sets.

Activity 10.1. Let X be a metric space.

(a) Is $\emptyset$ closed in X? Explain.

(b) Is X closed in X? Explain.

Note that a subset of a metric space can be both open and closed. We call such sets *clopen* (for closed-open). When we discussed open sets, we saw that an arbitrary union of open sets is open, but that an arbitrary intersection of open sets may not be open (although a finite intersection of open sets is open). Since closed sets are complements of open sets, we should expect a similar result for closed sets.

Activity 10.2. Let $X = \mathbb{R}$ with the Euclidean metric. Let $A_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ for each $n \in \mathbb{Z}^+$, $n \geq 2$.

(a) What is $\bigcup_{n \geq 2} A_n$? A proof is not necessary.

(b) Is $\bigcup_{n \geq 2} A_n$ closed in $\mathbb{R}$? Explain.

Activity 10.2 shows that an arbitrary union of closed sets is not necessarily closed. However, the following theorem tells us what we can say about unions and intersections of closed sets. The results should not be surprising given the relationship between open and closed sets.

Theorem 10.4. *Let X be a metric space.*

(1) Any intersection of closed sets in X is a closed set in X.

(2) Any finite union of closed sets in X is a closed set in X.

Proof. Let X be a metric space. To prove part 1, assume that $\{C_\alpha\}$ is a collection of closed sets in X for α in some indexing set I. DeMoivre's Theorem shows that

$$X \setminus \bigcap_{\alpha \in I} C_\alpha = \bigcup_{\alpha \in I} X \setminus C_\alpha.$$

The latter is an arbitrary union of open sets and so it an open set. By definition, then, $\bigcap_{\alpha \in I} C_\alpha$ is a closed set.

For part 2, assume that C_1, C_2, $\ldots$, C_n are closed sets in X for some $n \in \mathbb{Z}^+$. To show that $C = \bigcup_{k=1}^{n} C_k$ is a closed set, we will show that $X \setminus C$ is an open set. Now

$$X \setminus \bigcup_{i=1}^{n} C_i = \bigcap_{i=1}^{n} X \setminus C_i$$

is a finite intersection of open sets, and so is an open set. Therefore, $\bigcup_{i=1}^{n} C_i$ is a closed set. ■

Continuity and Closed Sets

Recall that we showed that a function f from a metric space (X, d_X) to a metric space (Y, d_Y) is continuous if and only if $f^{-1}(O)$ is open for every open set O in Y. We might conjecture that a similar result holds for closed sets. Since closed sets are complements of open sets, to make this connection we will want to know how $X \setminus f^{-1}(B)$ is related to $f^{-1}(Y \setminus B)$ for $B \subset Y$.

Activity 10.3. Let f be a function f from a metric space (X, d_X) to a metric space (Y, d_Y), and let B be a subset of Y.

(a) Let $x \in X \setminus f^{-1}(B)$.

   i. What does this tell us about $f(x)$?
   ii. What can we conclude about the relationship between $X \setminus f^{-1}(B)$ and $f^{-1}(Y \setminus B)$?

(b) Let $x \in f^{-1}(Y \setminus B)$.

   i. What does this tell us about $f(x)$?
   ii. What can we conclude about the relationship between $X \setminus f^{-1}(B)$ and $f^{-1}(Y \setminus B)$?

(c) What is the relationship between $X \setminus f^{-1}(B)$ and $f^{-1}(Y \setminus B)$?

Now we can consider the issue of continuity and closed sets.

Activity 10.4. Let f be a function from a metric space (X, d_X) to a metric space (Y, d_Y).

(a) Assume that f is continuous and that C is a closed set in Y. How does the result of Activity 10.3 tell us that $f^{-1}(C)$ is closed in X?

(b) Now assume that $f^{-1}(C)$ is closed in X whenever C is closed in Y. How does the result of Activity 10.3 tell us that f is a continuous function?

The result of Activity 10.4 is summarized in the following theorem.

Theorem 10.5. *Let f be a function from a metric space (X, d_X) to a metric space (Y, d_Y). Then f is continuous if and only if $f^{-1}(C)$ is closed in X whenever C is a closed set in Y.*

Limit Points, Boundary Points, Isolated Points, and Sequences

Recall that a limit point of a subset A of a metric space X is a point $x \in X$ such that every neighborhood of x contains a point in A different from x. You might wonder about the use of the word "limit" in the definition of limit point. The next activity should make this clear.

Activity 10.5. Let X be a metric space, let A be a subset of X, and let x be a limit point of A.

(a) Let $n \in \mathbb{Z}^+$. Explain why $B\left(x, \frac{1}{n}\right)$ must contain a point a_n in A different from x.

(b) What is $\lim a_n$? Why?

The result of Activity 10.5 is summarized in the following theorem.

Theorem 10.6. *Let X be a metric space, let A be a subset of X, and let x be a limit point of A. Then there is a sequence (a_n) in A that converges to x.*

Of course, the constant sequence (a) always converges to the point a, so every point in a set A is the limit of a sequence. With limit points there is a non-constant sequence that converges to the point. We might ask what we can say about a point $a \in A$ if the only sequences in A that converges to $a \in A$ are the eventually constant sequences (a). (By an eventually constant sequence (a_n), we mean that there is a positive integer K such that for $k \geq K$, we have $a_k = a$ for some element a.) That is the subject of our next activity.

Activity 10.6. Let (X, d) be a metric space, and let A be a subset of X.

(a) Let a be an isolated point of A. Prove that the only sequences in A that converge to a are the eventually constant sequences (a).

(b) Prove that if the only sequences in A that converges to a are the eventually constant sequences (a), then a is an isolated point of A.

Activity 10.6 shows us how we can understand isolated points in terms of sequences. We can do something similar with boundary points. Boundary points are points that are, in some sense, situated "between" a set and its complement. We will make this idea of "between" more concrete soon.

An argument just like the one in Activity 10.5 gives us the following result about boundary points. The proof is left for Exercise (7).

Theorem 10.7. *Let X be a metric space, let A be a subset of X, and let b be a boundary point of A. Then there are sequences (x_n) in $X \setminus A$ and (a_n) in A that converge to b.*

Limit Points and Closed Sets

There is a connection between limit points and closed sets that we examine in this section. The open set $(1, 2)$ in $(\mathbb{R}, d_E)$ does not contain all of its limit points or any of its boundary points, while the closed set $[1, 2]$ contains all of its boundary and limit points. This is an important attribute of closed sets. Recall that for a limit point x of a subset A of a metric space X, every neighborhood of x contains a point in A different from x. We can make the neighborhoods as small as we like so, in a sense, the limit points of A that are not in A are the points in X that are arbitrarily close to the set A. We denote the set of limit points of A as A', and the limit points of a set can tell us if the set is closed.

Theorem 10.8. *Let C be a subset of a metric space X, and let C' be the set of limit points of C. Then C is closed if and only if $C' \subseteq C$.*

Proof. Let X be a metric space, and let C be a subset of X. First we assume that C is closed and show that C contains all of its limit points. Let $x \in X$ be a limit point of C. We proceed by contradiction and assume that $x \notin C$. Then $x \in X \setminus C$, which is an open set. This implies that there is an $\epsilon > 0$ so that $B(x, \epsilon) \subseteq X \setminus C$. But then this neighborhood $B(x, \epsilon)$ contains no points in C, which contradicts the fact that x is a limit point of C. We conclude that $x \in C$ and C contains all of its limit points.

The converse of the result we just proved is the subject of the next activity.

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Activity 10.7. Let C be a subset of a metric space X, and let C' be the set of limit points of C. In this activity we prove that C is closed if C contains all of its limit points. So assume $C' \subseteq C$.

(a) What do we need to do to show that C is closed?

(b) If we proceed by contradiction to prove that C is closed, we assume that C is not closed. What does this tell us about $X \setminus C$?

(c) What does the conclusion of part (b) tells us?

(d) How does the result of (c) contradict the assumption that C contains all of its limit points?

The Closure of a Set

We have seen that the interior of a set is the largest open subset of that set. There is a similar result for closed sets. For example, let $A = (0, 1)$ in $(\mathbb{R}, d_E)$. The set A is an open set, but if we union A with its limit points, we obtain the closed set $C = [0, 1]$. Moreover, The set $[0, 1]$ is the smallest closed set that contains A. This leads to the idea of the *closure* of a set.

Definition 10.9. The **closure** of a subset A of a metric space X is the set

$$\overline{A} = A \cup A'.$$

In other words, the closure of a set is the collection of the elements of the set and the limit points of the set – those points that are on the "edge" of the set. The importance of the closure of a set A is that the closure of A is the smallest closed set that contains A.

Theorem 10.10. *Let X be a metric space and A a subset of X. The closure of A is a closed set. Moreover, the closure of A is the smallest closed subset of X that contains A.*

Proof. Let X be a metric space and A a subset of X. To prove that $\overline{A}$ is a closed set, we will prove that $\overline{A}$ contains its limit points. Let $x \in \overline{A}'$. To show that $x \in \overline{A}$, we proceed by contradiction and assume that $x \notin \overline{A}$. This implies that $x \notin A$ and $x \notin A'$. Since $x \notin A'$, there exists a neighborhood N of x that contains no points of A other than x. But $A \subseteq \overline{A}$ and $x \notin \overline{A}$, so it follows that $N \cap A = \emptyset$. This implies that there is an open ball $B \subseteq N$ centered at x so that $B \cap A = \emptyset$. The fact that $x \in \overline{A}'$ means that $B \cap \overline{A}$ contains a point y in $\overline{A}$ different from x. Since $B \cap A = \emptyset$, we must have $y \in A'$. But this, and the fact that B is a neighborhood of y, means that B must contain a point of A different than y. But $B \cap A = \emptyset$, so we have reached a contradiction. We conclude that $x \in \overline{A}$ and $\overline{A}' \subseteq \overline{A}$. This shows that $\overline{A}$ is a closed set.

The proof that $\overline{A}$ is the smallest closed subset of X that contains A is left for the next activity. ■

Activity 10.8. Let (X, d) be a metric space, and let A be a subset of X.

(a) What will we have to show to prove that $\overline{A}$ is the smallest closed subset of X that contains A?

(b) Suppose that C is a closed subset of X that contains A. To show that $\overline{A} \subseteq C$, why is it enough to demonstrate that $A' \subseteq C$?

(c) If $x \in A'$, what can we say about x?

(d) Complete the proof that $\overline{A} \subseteq C$.

One consequence of Theorem 10.10 is the following.

Corollary 10.11. *A subset C of a metric space X is closed if and only if $C = \overline{C}$.*

We can also characterize closed sets as sets that contain their boundaries.

Definition 10.12. The **boundary** $\text{Bdry}(A)$ of a subset A of a metric space X is the set of all boundary points of A.

Theorem 10.13. *A subset C of a metric space X is closed if and only if C contains its boundary.*

The proof of Theorem 10.13 is left to Exercise (10).

Recall that a boundary point of a subset A of a metric space X is a point $x \in X$ such that every neighborhood of x contains a point in A and a point in $X \setminus A$. The boundary points are those that are somehow "between" a set and its complement. For example if $A = (0, 1]$ in $\mathbb{R}$, the boundary of A is the set $\{0, 1\}$. We also have that $\overline{A} = [0, 1]$, $\mathbb{R} \setminus A = (-\infty, 0] \cup (1, \infty)$, and $\overline{\mathbb{R} \setminus A} = (-\infty, 0] \cup [1, \infty)$. Notice that $\text{Bdry}(A) = \overline{A} \cap \overline{\mathbb{R} \setminus A}$. That this is always true is formalized in the next theorem.

Theorem 10.14. *Let X be a metric space and A a subset of X. Then*

$$\text{Bdry}(A) = \overline{A} \cap \overline{X \setminus A}.$$

Proof. Let X be a metric space and A a subset of X. To prove $\text{Bdry}(A) = \overline{A} \cap \overline{X \setminus A}$ we need to verify the containment in each direction. Let $x \in \text{Bdry}(A)$ and let N be a neighborhood of x. Then N contains a point in A and a point in $X \setminus A$. We consider the cases of $x \in A$ or $x \notin A$.

- Suppose $x \in A$. Then $x \in \overline{A}$. Also, $x \notin X \setminus A$, so N must contain a point in $X \setminus A$ different from x. That makes x a limit point of $X \setminus A$ and so $X \in \overline{A} \cap \overline{X \setminus A}$.
- Suppose $x \notin A$. Then $x \in X \setminus A \subseteq \overline{X \setminus A}$. Also, $x \notin A$, so N must contain a point in A different from x. That makes x a limit point of A and so $X \in \overline{A} \cap \overline{X \setminus A}$.

In either case we have $x \in \overline{A} \cap \overline{X \setminus A}$ and so $\text{Bdry}(A) \subseteq \overline{A} \cap \overline{X \setminus A}$.

For the reverse implication, refer to the next activity.

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Activity 10.9. Let X be a metric space and A a subset of X. In this activity we prove that

$$\overline{A} \cap \overline{X \setminus A} \subseteq \text{Bdry}(A).$$

Let $x \in \overline{A} \cap \overline{X \setminus A}$.

(a) What must be true about x, given that x is in the intersection of two sets?

(b) Let N be a neighborhood of x. As we did in the proof of Theorem 10.14, we consider the cases $x \in A$ and $x \notin A$.

   i. Suppose $x \in A$. Why must N contain a point in A and a point not in A? What does this tell us about x?

   ii. Suppose $x \notin A$. Why must N contain a point in A and a point not in A? What does this tell us about x?

   iii. What can we conclude from parts i. and ii.?

Closed Sets and Limits of Sequences

Suppose we consider a sequence (a_n) in a subset A of a metric space X that converges to a point x. Must it be the case that $x \in A$? We consider this question in the next activity.

Activity 10.10. Let $A = (0, 1)$ and $B = [0, 1]$ in $(\mathbb{R}, d_E)$. For each positive integer n, let $a_n = \frac{1}{n}$. Note that the sequence (a_n) is contained in both sets A and B.

(a) To what does the sequence (a_n) converge in $\mathbb{R}$?

(b) Is $\lim a_n$ in A?

(c) Is $\lim a_n \in B$?

(d) Name two significant differences between the sets A and B that account for the different responses in parts (b) and (c)? Respond using the terminology we have introduced in this section.

The result of Activity 10.10 is encapsulated in the next theorem.

Theorem 10.15. *A subset C of a metric space X is closed if and only if whenever (c_n) is a sequence in C that converges to a point $c \in X$, then $c \in C$.*

Proof. Let X be a metric space and C a subset of X. First assume that C is closed. Let (c_n) be a convergent sequence in C with $c = \lim c_n$. So either $c \in C$ or c is a limit point of C. Since C contains its limit points, either case gives us $c \in C$. So $\lim c_n \in C$.

The proof of the remaining implication is left to the next activity.

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Activity 10.11. Let X be a metric space and C a subset of X. In this activity we will prove that if any time a sequence (c_n) in C converges to a point $c \in X$, the point c is in C, then C is a closed set.

(a) List three different ways that we can show that a subset of a metric space is closed. Which one might be relevant in this situation to show that the set C is closed?

(b) Let c be a limit point of C. What does that tell us?

(c) Complete the proof that C is a closed set.

Summary

Important ideas that we discussed in this section include the following.

- Let X be a metric space and A a subset of X.

    i. A point $x \in X$ is a boundary point of A if every neighborhood of x contains a point in A and a point in $X \setminus A$.

    ii. A point x is a limit point of A if every neighborhood of x contains a point in A different from x.

    iii. A point $a \in A$ is an isolated point of A if there is a neighborhood N of a such that $N \cap A = \{a\}$.

  Boundary points and limit points don't need to be in the set A, whereas an isolated point of A must be in A. In $A = (0, 1) \cup \{2\}$ as a subset of $= (\mathbb{R}, d_E)$, 0 is a boundary point but not an isolated point while 2 is a boundary point but not a limit point. Also, 0.5 is a limit point but neither a boundary or isolated point. With A as subset of $\mathbb{R}$ with the discrete metric, every point of A is an isolated point but no point in $\mathbb{R}$ is a boundary point or a limit point of A. So even though every boundary point is either a limit point or an isolated point, the three concepts are different.

- A subset A of a metric space X is closed if $X \setminus A$ is an open set.
- Any intersection of closed sets is closed while finite unions of closed sets are closed.
- A function f from a metric space X to a metric space Y is continuous if $f^{-1}(C)$ is a closed set in X whenever C is a closed set in Y.
- Let X be a metric space, let A be a subset of X, and let x be a limit point of A. Then there is a sequence (a_n) in A that converges to x.
- Let X be a metric space, let A be a subset of X, and let x be a boundary point of A. Then there are sequences (x_n) in $X \setminus A$ and (a_n) in A that converge to x.
- The boundary of a subset A of a metric space X is the set of boundary points of A.
- A subset A of a metric space X is closed if and only if A contains all of its limit points. Similarly, A is closed if and only if A contains all of its boundary points.
- The set of all limit points of a subset A of a metric space X is denoted by A'. The closure of A is the set $\overline{A} = A \cup A'$. The closure of A is the smallest closed set in X that contains A.
- A subset A of a metric space X is closed if and only if $\lim a_n$ is in A whenever (a_n) is a convergent sequence in A.

Exercises

(1) Informal, but convincing, arguments suffice for this problem.

(a) Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ as a subset of $(\mathbb{R}^2, d_E)$. Note that D is the unit disk in the plane. Determine all of the interior points, boundary points, accumulation points, and isolated points of D. Give reasons for your conclusions. Is D an open set? Is D a closed set? Explain.

(b) Let $A = \mathbb{Q}$, the set of rational numbers, as a subset of $(\mathbb{R}, d_E)$. Determine all of the interior points, boundary points, accumulation points, and isolated points of A. Give reasons for your conclusions. Is A an open set? Is A a closed set? Explain.

(c) Let $A = \left\{\frac{1}{n} \mid \in \mathbb{Z}^+\right\}$ as a subset of $(\mathbb{R}, d_E)$. Determine all of the interior points, boundary points, accumulation points, and isolated points of A. Give reasons for your conclusions. Is A an open set? Is A a closed set? Explain.

(2) Let (X, d) be a metric space. Let $a \in X$, and let $r > 0$. We know that the open ball $B(a, r) = \{x \in X \mid d(a, x) < r\}$ is an open set. Let

$$B[a, r] = \{x \in X \mid d(a, x) \leq r\}.$$

Prove or disprove: $B[a, r]$ is a closed set in X.

(3) Let (X, d) be a metric space. We have seen that it is possible for a subset of X to be both open and closed. There is a characterization of sets that are both open and closed in terms of their boundaries. Find and prove such a characterization. (Your statement should have the form: A subset A of a metric space X is both open and closed if and only if the boundary of A is _______.)

(4) Let A be a subset of a metric space. Let A' be the set of limit points of A and A^i the set of isolated points of A. Prove the following.

  (a) $A \cup A^i = A \cup A'$

  (b) $A' \cap A^i = \emptyset$

  (c) $A \subseteq A' \cup A^i$

  (d) $x \in \overline{A}$ if and only if there is a sequence of points of A which converges to x

  (e) $\overline{A}$ is the intersection of all closed sets that contain A

  (f) $\text{Int}(A)$ is the union of all open sets contained in A

  (g) $\overline{A}$ is the disjoint union of $\text{Int}(A)$ and $\text{Bdry}(A)$

  (h) $\overline{X \setminus A} = X \setminus \text{Int}(A)$

  (i) $\text{Int}(X \setminus A) = X \setminus \overline{A}$

(5) Let (X, d) be a metric space and A a subset of X. Prove that a point $x \in X$ is a limit point of A if and only if every open ball centered at x contains a point in A different from x.

(6) Let A be a subset of a metric space. Let A' be the set of limit points of A and A^i the set of isolated points of A.

  (a) Prove that $A' \cap A^i = \emptyset$ and $A \subseteq A' \cup A^i$.

  (b) Prove that $x \in \overline{A}$ if and only if there is a sequence of points of A which converges to x.

  (c) Prove that if F is a closed set such that $A \subseteq F$, then $\overline{A} \subseteq F$. Then prove that $\overline{A}$ is the intersection of all such closed sets F and hence is closed.

(7) Prove Theorem 10.7 that if A is a subset of a metric space X and b is a boundary point of A, then there are sequences (x_n) in $X \setminus A$ and (a_n) in A that converge to b.

(8) Recall that the distance from a point x in a metric space X to a nonempty subset A of X is

$$d(a, X) = \inf\{d(x, a) \mid a \in A\}.$$

Prove that a subset C of a metric space X is closed if and only if whenever $x \in X$ and $d(x, C) = 0$, then $x \in C$.

(9) Let (X, d) be a metric space. In this exercise we show that some subsets of X, other than $\emptyset$ and X must be closed. Show that any finite subset of X is closed. (Hint: What are the limits points of a finite subset?)

(10) Prove that a subset C of a metric space X is closed if and only if C contains its boundary.

(11) Let (X, d_X) and (Y, d_Y) be metric space and let $f : X \to Y$ be a function.

(a) Prove that f is continuous if and only if $f^{-1}(\text{Int}(B)) \subseteq \text{Int}(f^{-1}(B))$ for any subset B of Y.

(b) Give an example where the containment, and not the equality, in (a) is the best we can do.

(c) Give an example to show that the equality in (a) can actually be achieved.

(12) Let (X, d) be a metric space and let A be a subset of X. Prove that every boundary point of A is either a limit point or an isolated point of A.

(13) Let (X, d) be a metric space, and let A and B be subsets of X.

(a) Is it the case that $\overline{A \cup B} = \overline{A} \cup \overline{B}$? If true, prove it. If false, show why and prove any containment that is true.

(b) Is it the case that $\overline{A \cap B} = \overline{A} \cap \overline{B}$? If true, prove it. If false, show why and prove any containment that is true.

(14) Recall that an infinite union of closed sets in a metric space may not be closed, and that an infinite intersection of open sets in a metric space may not be open. In this exercise we explore situations in which we can conclude that an infinite union of closed sets is closed and an infinite intersection of open sets is open. Let (X, d) be a metric space.

(a) We first establish a preliminary result. Let C be a closed subset of X and $x \in X$. Prove that if $x \notin C$, then $d(x, C) > 0$.

(b) Let $\{C_\alpha\}$ be a collection of closed subsets of X for α in some indexing set I with the property that given any $x \in X$, there exists an $\epsilon_x > 0$ such that $B(x, \epsilon_x)$ intersects at most finitely many of the sets C_α. Prove that $\bigcup_{\alpha \in I} C_\alpha$ is closed.

(c) Determine and prove an analogous statement for open sets in X.

(15) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) If x is a point in a metric space X, then the singleton set $\{x\}$ is closed.

(b) The only subsets of $\mathbb{R}$ that are both open and closed under the standard metric are $\emptyset$ and $\mathbb{R}$.

(c) If (X, d) is the metric space with $X = \{1, 3, 5\}$ and $d(x, y) = xy - 1 \pmod 8$, then the set $\{1, 3\}$ is both open and closed in X.

(d) If X is a metric space and $A \subseteq X$, then $\text{Int}(\overline{A}) = A$.

(e) The boundary of any subset of a metric space X is a closed set.

(f) If A is a subset of a metric space X, then $A \subseteq A' \cup A^i$ where A' is the set of limit points of A and A^i is the set of isolated points of A