Appendix G

Complex Roots of Unity

Focus Questions

By the end of this investigation, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the investigation.

- How do we represent complex numbers using a polar or trigonometric form?
- What is a complex root of unity?
- What is de Moivre's theorem and what does it tell us about powers of complex numbers?
- How do we find all complex nth roots of unity?
- How can we find complex nth roots of an arbitrary complex number?

Preview Activity G.1. Consider the problem of determining all solutions to the polynomial equation $x^3 - 1 = 0$. We know that $x = \sqrt[3]{1} = 1$ is one solution, but the Fundamental Theorem of Algebra tells us that there are two more solutions. That is, there are three complex numbers that satisfy the equation $x^3 - 1 = 0$. These solutions are called the *third roots of unity* (where 1 is the unity).

(a) Let $z_0 = \cos\left(\frac{0\pi}{3}\right) + i\sin\left(\frac{0\pi}{3}\right)$. Evaluate the trigonometric functions at $\frac{0\pi}{3}$ and calculate *exact* numeric values for the real and imaginary parts of z_0. Then calculate z_0^2 and z_0^3.

(b) Let $z_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$. Evaluate the trigonometric functions at $\frac{2\pi}{3}$ and calculate *exact* numeric values for the real and imaginary parts of z_1. Then calculate z_1^2 and z_1^3.

(c) Let $z_2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$. Evaluate the trigonometric functions at $\frac{4\pi}{3}$ and calculate *exact* numeric values for the real and imaginary parts of z_2. Then calculate z_2^2 and z_2^3.

(d) How are z_0, z_1, and z_2 related to the polynomial $x^3 - 1$?

In this investigation we will expand on Preview Activity G.1 to explicitly determine all complex roots of unity and see how these roots of unity help us solve polynomial equations.

The Trigonometric form of a Complex Number

Multiplication of complex numbers is a pretty straightforward algebraic process, but the algebra doesn't provide much visual insight into a complex product. One idea that helps us visualize complex multiplication and makes the determination of complex roots of unity possible is the trigonometric (or polar) form of a complex number. This trigonometric form connects the algebra of complex numbers to trigonometry in a very useful way.

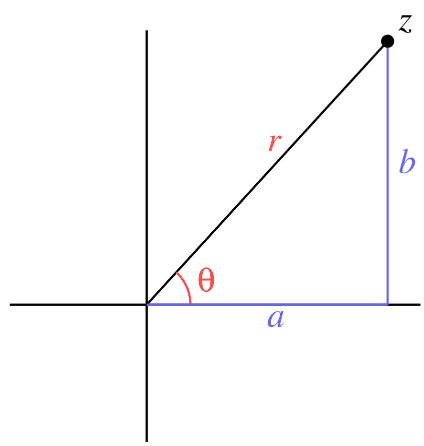

Figure G.1
Trigonometric form of a complex number.

If $z = a + bi$ is a complex number, then we can plot z in the plane as shown in Figure G.1. In this situation, we will let r be the magnitude of z (that is, the distance from z to the origin) and θ the angle z makes with the positive real axis as shown in Figure G.1.

We can use trigonometry to see that

$$a = r\cos(\theta) \text{ and } b = r\sin(\theta).$$

We can then write z in trigonometric form as

$$z = r(\cos(\theta) + i\sin(\theta)) \tag{G.1}$$

and call (G.1) the trigonometric (or polar) form (or representation) of the complex number z. (The word *polar* here comes from the fact that this process can be viewed as occurring with polar coordinates.) The angle θ is called the *argument* of the complex number z and the real number r is the *modulus* or *norm* of z. To find the polar representation of a complex number $z = a + bi$, we first notice that

$$r = |z| = \sqrt{a^2 + b^2}.$$

To find θ, we have to consider cases.

- If $z = 0 = 0 + 0i$, then $r = 0$ and θ can have any real value.

- If $z \neq 0$, then $\tan(\theta) = \frac{b}{a}$ if $a \neq 0$ and θ is determined by the quadrant in which z lies.
- If $z \neq 0$ and $a = 0$ (so $b \neq 0$), then
  - $\theta = \frac{\pi}{2}$ if $b > 0$
  - $\theta = -\frac{\pi}{2}$ if $b < 0$.

Activity G.2.

(a) Find the polar form of the complex numbers $w = 4 + 4\sqrt{3}i$ and $z = 1 - i$.

(b) Find $a, b \in \mathbb{R}$ so that $a + bi = 3\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$.

Products of Complex Numbers in Polar Form

There is an important product formula for complex numbers that the polar form provides. We illustrate with an example.

Example G.3. Let $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z = \sqrt{3} + i$. Now $|w| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ and the argument of w satisfies $\tan(\theta) = -\sqrt{3}$. Since w is in the second quadrant, we see that $\theta = \frac{2\pi}{3}$, so the polar form of w is

$$w = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right).$$

Also, $|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$ and the argument of z satisfies $\tan(\theta) = \frac{1}{\sqrt{3}}$. Since z is in the first quadrant, we know that $\theta = \frac{\pi}{6}$ and the polar form of z is

$$z = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right].$$

Computing wz directly gives

$$\begin{aligned} wz &= \left(\sqrt{3} + i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= -\sqrt{3} + i. \end{aligned}$$

Now $|wz| = 2$ and the argument of zw satisfies $\tan(\theta) = -\frac{1}{\sqrt{3}}$. Since wz is in quadrant II, we see that $\theta = \frac{5\pi}{6}$ and the polar form of wz is

$$wz = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right].$$

Notice that $|wz| = |w|\,|z|$ and that the argument of zw is $\frac{2\pi}{3} + \frac{\pi}{6}$ or the sum of the arguments of w and z.

In general, we have the following important result about the product of two complex numbers.

Theorem G.4. *Let $w = r(\cos(\alpha) + i\sin(\alpha))$ and $z = s(\cos(\beta) + i\sin(\beta))$ be complex numbers in polar form. Then the polar form of the complex product wz is given by*

$$wz = rs\left(\cos(\alpha + \beta) + i\sin(\alpha + \beta)\right).$$

The verification is left for Exercise (2).

An illustration of this is given in Figure G.2. Theorem G.4 tells us that to multiply two complex numbers together, we add their arguments and multiply their magnitudes.

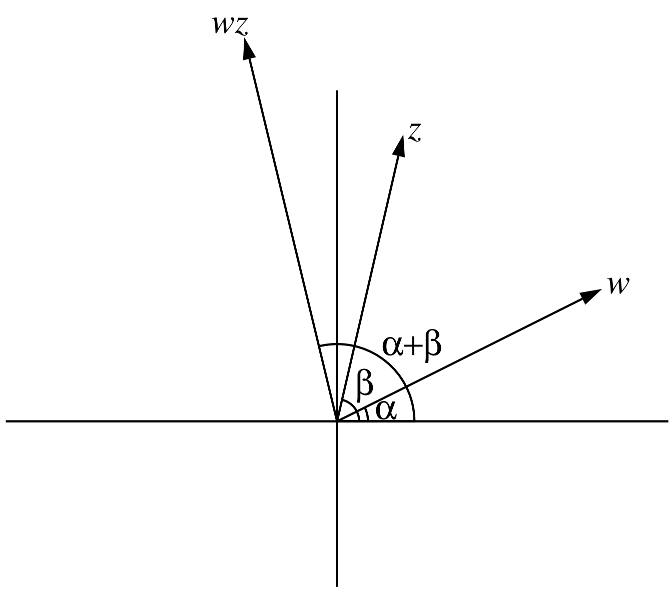

Figure G.2
A geometric interpretation of complex multiplication.

Activity G.5. Let $w = 3\left[\cos\left(\frac{5\pi}{3}\right) + \sin\left(\frac{5\pi}{3}\right)\right]$ and $z = 2\left[\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right)\right]$.

(a) What is $|wz|$?

(b) What is the argument of wz?

(c) In which quadrant is wz? Explain.

(d) Find the polar form of wz.

An alternate representation for the polar form of a complex number is the *exponential form* of a complex number. If $w = r(\cos(\alpha) + i\sin(\alpha))$ and $z = s(\cos(\beta) + i\sin(\beta))$, then the polar product

$$wz = rs\left(\cos(\alpha + \beta) + i\sin(\alpha + \beta)\right)$$

allows us to multiply the magnitudes and add the arguments. Exponential functions have the same properties. That is, if $u = ae^x$ and $v = be^y$, then

$$uv = abe^{x+y}.$$

This similarity provides a shorthand way of representing the polar form of a complex number. We write $e^{i\theta}$ to mean

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

(this equation is called *Euler's formula*). Then the polar form $r(\cos(\theta) + i\sin(\theta))$ of a complex number can be written as $re^{i\theta}$:

$$re^{i\theta} = r(\cos(\theta) + i\sin(\theta)).$$

An interesting consequence is the elegant and unexpected relationship between four of the most famous mathematical constants (0, 1, e, and π). If we substitute π for θ in $e^{i\theta}$ we see that

$$e^{i\pi} = -1 \quad \text{or} \quad e^{i\pi} + 1 = 0.$$

This last equation is called *Euler's identity*.

Theorem G.4 gives us a quick way to compute powers of a complex number, and hence to find roots of complex numbers. If $z = r(\cos(\theta) + i\sin(\theta))$, then Theorem G.4 shows us that

$$\begin{aligned} z^2 &= (r)(r)\left(\cos(\theta+\theta) + i\sin(\theta+\theta)\right) \\ &= r^2\left(\cos(2\theta) + i\sin(2\theta)\right). \end{aligned}$$

We can continue this process to obtain the following theorem.

Theorem G.6 (de Moivre's Theorem)**.** *Let* $z = r(\cos(\theta) + i\sin(\theta))$ *be a complex number and* n *any integer. Then*

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)). \tag{G.2}$$

The proof is left for Exercise (5). In exponential form we have

$$\left(re^{i\theta}\right)^n = r^n e^{in\theta}.$$

Note that de Moivre's Theorem works for negative integer powers as well as positive integer powers.

Activity G.7. Use de Moivre's Theorem to find z^{10} where $z = 1 - i$.

Roots of Unity

Now we return to the problem of solving the polynomial equation $x^3 = 1$. If we draw the graph of $y = x^3 - 1$ we see that the graph intersects the x-axis at only one point, so there is only one real solution to $x^3 = 1$. That means the other two solutions must be complex. The key to finding the two complex third roots of unity is de Moivre's Theorem. Suppose

$$z = r\left[\cos(\theta) + i\sin(\theta)\right]$$

is a solution to $x^3 = 1$. Then

$$1 = z^3 = r^3(\cos(3\theta) + i\sin(3\theta)).$$

This implies that $r = 1$ (or $r = -1$, but we can incorporate the latter case into our choice of angle). We then have

$$1 = \cos(3\theta) + i\sin(3\theta). \tag{G.3}$$

Equation (G.3) has solutions when $\cos(3\theta) = 1$ and $\sin(3\theta) = 0$. This will occur when $3\theta = 2\pi k$, or $\theta = \frac{2\pi k}{3}$, where k is any integer. The distinct integer multiples of $\frac{2\pi k}{3}$ on the unit circle occur when $k = 0$ and $\theta = 0$, $k = 1$ and $\theta = \frac{2\pi}{3}$, and $k = 2$ with $\theta = \frac{4\pi}{3}$. In other words, the solutions to $x^3 = 1$ should be

$$\begin{aligned}
z_0 &= \cos(0) + i\sin(0) = 1\\
z_1 &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i\\
z_2 &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.
\end{aligned}$$

We already know that $z_0^3 = 1^3 = 1$, so z_0 actually is a solution to $x^3 = 1$. To check that z_1 and z_2 are also solutions to $x^3 = 1$, we apply de Moivre's Theorem:

$$\begin{aligned}
z_1^3 &= \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]^3\\
&= \cos\left(3\frac{2\pi}{3}\right) + i\sin\left(3\frac{2\pi}{3}\right)\\
&= \cos(2\pi) + i\sin(2\pi)\\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
z_2^3 &= \left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right]^3\\
&= \cos\left(3\frac{4\pi}{3}\right) + i\sin\left(3\frac{4\pi}{3}\right)\\
&= \cos(4\pi) + i\sin(4\pi)\\
&= 1.
\end{aligned}$$

Thus, $z_1^3 = 1$ and $z_2^3 = 1$ and we have found three solutions to the equation $x^3 = 1$. Since a cubic can have only three solutions, we have found them all.

In general, the solutions to $x^n = 1$ for a positive integer n are called the nth roots of unity.

Definition G.8. If n is a positive integer, the nth **roots of unity** are the n solutions to the equation $z^n = 1$.

The argument above can be extended to show that the nth roots of unity are given by

$$e^{\frac{2k\pi i}{n}} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

for k from 0 to $n - 1$. To verify this statement, note that

$$\left(e^{\frac{2k\pi i}{n}}\right)^n = e^{2k\pi i} = \cos(2k\pi) + i\sin(2k\pi) = 1.$$

The complex numbers $e^{\frac{2k\pi i}{n}}$ are distinct for k from 0 to $n - 1$, so we have the n different complex roots of 1.

Activity G.9.

(a) Find all solutions to $x^4 = 1$. That is, find the fourth roots of unity.

(b) Find all sixth roots of unity.

Activity G.9 illustrates something interesting. There are four complex roots of 1, given by 1, ω, ω^2, and ω^3, where

$$\omega = e^{\frac{2\pi i}{4}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i.$$

Now $\omega^2 = -1$, and $1^2 = 1$ and $(-1)^2 = 1$, while ω^2 and $\left(\omega^3\right)^2$ are not equal to 1. So 1 and -1 are also square roots of 1. We say that an nth root of unity z is a *primitive* nth root of unity if z is not a root of unity for some smaller power.

Definition G.10. Let n be a positive integer. The complex number z is a **primitive** nth root of unity if $z^n = 1$ but $z^k \neq 1$ for any k with $1 \leq k < n$.

The complex number $e^{\frac{2\pi i}{n}}$ is always a primitive nth root of unity, but this is not the only primitive nth root of unity. We determine all of the primitive nth roots of unity in the concluding activities.

Roots of Complex Numbers

The final topic we will consider in this investigation is extending this process for finding roots of unity to calculate roots of other complex numbers. To find the solutions to the equation $x^n = a+bi$, where n is a positive integer and $a + bi$ is a complex number with trigonometric form

$$a + bi = r\left[\cos(\theta) + i\sin(\theta)\right],$$

with $r > 0$, we assume that $z = s\left[\cos(\alpha) + i\sin(\alpha)\right]$ with $s > 0$ is a solution to $x^n = a + bi$. Then

$$\begin{aligned} a + bi &= z^n \\ &= \left(s\left[\cos(\alpha) + i\sin(\alpha)\right]\right)^n \\ &= s^n\left[\cos(n\alpha) + i\sin(n\alpha)\right] \end{aligned}$$

and so

$$s^n = r \quad \text{and} \quad \cos(\theta) + i\sin(\theta) = \cos(n\alpha) + i\sin(n\alpha).$$

Therefore,

$$s^n = r \quad \text{and} \quad n\alpha = \theta + 2\pi k$$

where k is any integer. This give us

$$s = \sqrt[n]{r} \quad \text{and} \quad \alpha = \frac{\theta + 2\pi k}{n}.$$

We will get n different solutions for $k = 0, 1, 2, \ldots, n-1$, and these will be all of the solutions. These solutions are called the nth roots of the complex number $a + bi$. We summarize the results in the next theorem.

Theorem G.11. *Let $n \in \mathbb{Z}^+$. The nth roots of the complex number $r\left[\cos(\theta) + i\sin(\theta)\right]$ are given by*

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, (n-1)$.

In other words, if r is an nth root of a complex number z, then all of the nth roots of z have the form $r\omega^k$ for $0 \leq k \leq n-1$, where ω is a primitive nth root of unity.

Example G.12. We solve the equation

$$x^4 = -8 + 8\sqrt{3}i.$$

Note that we can write the right hand side of this equation in trigonometric form as

$$-8 + 8\sqrt{3}i = 16\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right).$$

The fourth roots of $-8 + 8\sqrt{3}i$ are then

$$\begin{aligned}
x_0 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3} + 2\pi(0)}{4}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi(0)}{4}\right)\right] \\
&= 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right] \\
&= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \\
&= \sqrt{3} + i,
\end{aligned}$$

$$\begin{aligned}
x_1 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3} + 2\pi(1)}{4}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi(1)}{4}\right)\right] \\
&= 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] \\
&= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= -1 + \sqrt{3}i,
\end{aligned}$$

$$\begin{aligned}
x_2 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3} + 2\pi(2)}{4}\right) + i\sin\left(\frac{\frac{2\pi}{3} + 2\pi(2)}{4}\right)\right] \\
&= 2\left[\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right] \\
&= 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) \\
&= -\sqrt{3} - i,
\end{aligned}$$

and

$$\begin{aligned} x_3 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3}+2\pi(3)}{4}\right)+i\sin\left(\frac{\frac{2\pi}{3}+2\pi(3)}{4}\right)\right] \\ &= 2\left[\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)\right] \\ &= 2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) \\ &= 1-\sqrt{3}i. \end{aligned}$$

Activity G.13. Find all fourth roots of -256, that is find all solutions to $x^4 = -256$.

Concluding Activities

Activity G.14. Let ω be a primitive sixth root of unity.

(a) Determine the powers of ω^2. What does this say about ω^2?

(b) Determine the powers of ω^3. What does this say about ω^3?

(c) Parts (a) and (b) illustrate the following theorem.

Theorem G.15. *Let F be a field that contains a primitive nth root of unity. Then F contains a primitive kth root of unity for every positive divisor k of n.*

Complete the following steps to prove this theorem.

i. Let ω be a primitive nth root of unity in F, and let k be a positive divisor of n. Let $d = \frac{n}{k}$. Let $\rho = \omega^d$. Show that $\rho^k = 1$.

ii. We have shown that ρ is a kth root of unity. To prove that ρ is a primitive kth root of unity, we must demonstrate that $\rho^m \neq 1$ for $1 \leq m < k$. Let m be an integer with $1 \leq m < k$. Show that $\rho^m \neq 1$.

Activity G.16. Let n be a positive integer. The complex number $\omega = e^{\frac{2\pi i}{n}}$ is always a primitive nth root of unity, but this is not the only primitive nth root of unity. We know that the nth roots of unity are ω^k for $0 \leq k \leq n-1$. In this exercise we determine exactly which of these roots are primitive nth roots of unity.

(a) Consider the case with $n = 6$. Determine which powers ω^k are primitive sixth roots of unity. What is the relationship between k and 6 if ω^k is a primitive sixth root of unity?

(b) Repeat part (a) with $n = 8$.

(c) Now generalize parts (a) and (b) and determine in general which powers of ω are primitive nth roots of unity.

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Exercises

(1) Let $w = 2 - 5i$ and $z = -4 + 7i$.

(a) Compute $w + z$

(b) Compute wz

(c) Find the polar forms of w and z.

(d) Use the polar forms to compute wz. Compare to your earlier result.

(2) Prove Theorem G.4 that if $w = r(\cos(\alpha) + i\sin(\alpha))$ and $z = s(\cos(\beta) + i\sin(\beta))$, then

$$wz = rs\left(\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right).$$

(3) Find the cube roots of $a = -2 + 2i$.

(4) In this problem we will find the fifth roots of unity, or the solutions to the equation $z^5 - 1 = 0$. This is much more complicated than finding the fourth or sixth roots of unity due to the difficulty of finding $\cos\left(\frac{\pi}{5}\right)$ and $\sin\left(\frac{\pi}{5}\right)$. Assume the identity

$$\sin(5\alpha) = 5\sin(\alpha) - 20\sin^3(\alpha) + 16\sin^5(\alpha). \tag{G.4}$$

WARNING: This is a very messy problem!

(a) Use (G.4) to find an equation for $\sin\left(\frac{\pi}{5}\right)$.

(b) Let $w = \sin\left(\frac{\pi}{5}\right)$. Show that the value of $\sin\left(\frac{\pi}{5}\right)$ is a solution to

$$16w^4 - 20w^2 + 5 = 0.$$

(c) Show that $\sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{10-2\sqrt{5}}}{4}$.

(d) Find the exact values of the fifth roots of unity. Hint: Let $\varphi = \frac{1+\sqrt{5}}{2}$. Show that $\sin\left(\frac{2\pi}{5}\right) = \frac{1}{2}\sqrt{2+\varphi}$ and $\cos\left(\frac{2\pi}{5}\right) = \left(\frac{1}{2}\right)(\varphi - 1)$. Note that $\varphi^2 = \varphi + 1$. Express all fifth roots of unity in terms of φ. Note also that computing inverses in $\mathbb{C}$ can be easier than computing products.

(e) Verify (G.4) for any angle α. You may use the following well-known trigonometric identities that are valid for any angles α and β:

$$\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha) \tag{G.5}$$
$$\sin(2\alpha) = 2\cos(\alpha)\sin(\alpha) \tag{G.6}$$
$$\cos(2\alpha) = 1 - 2\sin^2(\alpha) \tag{G.7}$$
$$\cos^2(\alpha) + \sin^2(\alpha) = 1. \tag{G.8}$$

(5) Prove de Moivre's Theorem that if $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number and n any integer, then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)). \tag{G.9}$$