Section 11

Subspaces and Products of Metric Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a subspace of a metric space?
- How do we find the open and closed sets in a subspace of a metric space?
- What is a product of metric spaces and how do we make a product of metric spaces into a metric space?

Introduction

Let (X, d) be a metric space, and let A be a subset of X. We can make A into a metric space itself in a very straightforward manner. When we do so, we say that A is a *subspace* of X.

Preview Activity 11.1. Let (X, d) be a metric space, and let A be a subset of X. To make the subset A into a metric space, we need to define a metric on A. For us to consider A as a subspace of X, we want the metric on A to agree with the metric on X. So we define $d' : A \times A \to \mathbb{R}$ by

$$d'(a_1, a_2) = d(a_1, a_2)$$

for all $a_1, a_2 \in A$. Note that d and d' are different functions because they have different domains.

(1) Show that d' is a metric on A.

Since d' is a metric on A it follows that (A, d') is a metric space. The metric d' is the *restriction* of d to $A \times A$ and can also be denoted by d_A.

Definition 11.1. Let (X, d) be a metric space. A **subspace** of (X, d) is a subset A of X together with the metric d_A from $A \times A$ to $\mathbb{R}$ defined by

$$d_A(a_1, a_2) = d(a_1, a_2)$$

for all $a_1, a_2 \in A$.

We might wonder what, if any, properties of the space X are inherited by a subspace.

(2) Let $(X, d) = (\mathbb{R}, d_E)$ and let $A = [0, 1]$. Let $0 < a < 1$. Is the set $[0, a)$ open in X? Is the set $[0, a)$ open in A? Explain.

(3) Let $(X, d) = (\mathbb{R}, d_E)$ and let $A = \mathbb{Z}$. What are the open subsets of A? Explain.

(4) Let $(X, d) = (\mathbb{R}^2, d_E)$, let $A = \{(x, 0) \mid x \in \mathbb{R}\}$ (the x-axis in $\mathbb{R}^2$), and let $Z = \{(x, 0) \mid 0 < x < 1\}$. Note that $Z \subset A \subset X$ and we can consider Z as a subspace of A and X, and A as a subspace of X.

   (a) Explain why A is a closed subset of X.

   (b) Explain why Z is an open subset of A.

   (c) Is Z an open subset of X? Explain

Open and Closed Sets in Subspaces

We saw in our preview activity that a subspace does not necessarily inherit the properties of the larger space. For example, a subset of a subspace might be open in the subspace, but not open in the larger space. However, there is a connection between the open subsets in a subspace and the open sets in the larger space.

Theorem 11.2. *Let (X, d) be a metric space and A a subset of X. A subset O_A of A is open in A if and only if there is an open set O_X in X so that $O_A = O_X \cap A$.*

Proof. Let (X, d) be a metric space and A a subset of X. Let O_A be an open subset of A. So for each $a \in A$ there is a $\delta_a > 0$ so that $B_A(a, \delta_a) \subseteq O_A$, where $B_A(a, \delta_a)$ is the open ball in A centered at a of radius δ_a. Then, $O_A = \bigcup_{a \in O_A} B_A(a, \delta_a)$. Now let $B_X(a, \delta_a)$ be the open ball in X centered at a of radius δ_a, and let $O_X = \bigcup_{a \in O_A} B_X(a, \delta_a)$. Note that

$$B_A(a, \delta_a) = B_X(a, \delta_a) \cap A.$$

As a union of open balls in X, the set O_X is open in X. Now

$$O_X \cap A = \left(\bigcup_{a \in O_A} B_X(a, \delta_a) \right) \cap A = \bigcup_{a \in O_A} (B_X(a, \delta_a) \cap A) = \bigcup_{a \in O_A} B_A(a, \delta_a) = O_A.$$

So there is an open set O_X in X such that $O_A = O_X \cap A$.

For the reverse implication, see the following activity. ■

Activity 11.1. Let (X, d) be a metric space and A a subset of X. Suppose that $O_A = A \cap O_X$ for some set O_X that is open in X. In this activity we will prove that O_A is an open subset of A.

(a) Let $a \in O_A$. Explain why there must exist a $\delta > 0$ such that $B_X(a, \delta)$, the open ball in X of radius δ around a in X, is a subset of O_X.

(b) What would be a natural candidate for an open ball in A centered at a that is contained in A? Prove your conjecture.

(c) What conclusion can we draw?

We might now wonder about closed sets in a subspace. If X is a metric space and A is a subspace, then by definition a subset C_A of A is closed if and only if $C_A = A \setminus O_A$ for some set O_A that is open in A. The analogy of Theorem 11.2 is true for closed sets in subspaces.

Theorem 11.3. *Let (X, d) be a metric space and A a subset of X. A subset C_A of A is closed in A if and only if there is a closed set C_X in X so that $C_A = C_X \cap A$.*

The proof is left to Exercise (4).

Products of Metric Spaces

If we have two metric spaces (X, d_1) and (X_2, d_2), we might wonder if we can make the set $X_1 \times X_2$ into a metric space. A natural approach might be to define a function $d : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$ by

$$d((x, y), (u, v)) = d_1(x, u)d_2(y, v)$$

for (x, y) and (u, v) in $X_1 \times X_2$. However, this d does not define a metric. For example, if $x \in X_1$ and $y \neq v$ in X_2, then $d((x, y), (x, v)) = 0$ even though $(x, y) \neq (x, v)$. To make a metric, we can take a clue from the Euclidean metric on $\mathbb{R} \times \mathbb{R}$. On $\mathbb{R}$, the metric has the form $d_1(x, y) = |x - y|$, while on $\mathbb{R}^2$ the metric is

$$d_E((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{d_1(x_1, y_1)^2 + d_1(x_2, y_2)^2}.$$

So on (X, d_1) and (X_2, d_2) we could consider defining $d : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$ by

$$d((x_1, x_2), (y_1, y_2)) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}. \tag{11.1}$$

Activity 11.2. In this activity we verify some of the properties that make d as defined in (11.1) a metric. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be in $X_1 \times X_2$

(a) Explain why $d(x, y)$ is greater than or equal to 0.

(b) Explain why $d(x, y) = d(y, x)$.

(c) Explain why $d(x, y) = 0$ if and only if $x = y$.

Activity 11.2 provides three of the four items that are necessary to prove that d as defined in (11.1) is a metric. We verify the last property, the triangle inequality, now.

Let x and y be defined as in Activity 11.2, and let $z = (z_1, z_2)$ be in $X_1 \times X_2$. Then

$$\begin{aligned} d(x,z)^2 &= d_1(x_1,z_1)^2 + d_2(x_2,z_2)^2 \\ &\leq (d_1(x_1,y_1) + d_1(y_1,z_1))^2 + (d_2(x_2,y_2) + d_2(y_2,z_2))^2 \\ &= \left(d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2\right) + 2\left(d_1(x_1,y_1)d_1(y_1,z_1) + d_2(x_2,y_2)d_2(y_2,z_2)\right) \\ &\quad + \left(d_1(y_1,z_1)^2 + d_2(y_2,z_2)^2\right) \\ &= d(x,y)^2 + 2\left(d_1(x_1,y_1)d_1(y_1,z_1) + d_2(x_2,y_2)d_2(y_2,z_2)\right) + d(y,z)^2 \\ &\leq d(x,y)^2 + d(y,z)^2. \end{aligned}$$

Since all terms are non-negative we conclude that

$$\begin{aligned} d(x,z) \leq \sqrt{d(x,y)^2 + d(y,z)^2} &\leq \sqrt{d(x,y)^2 + 2d(x,y)d(y,z) + d(y,z)^2} \\ &= \sqrt{(d(x,y) + d(y,z))^2} \\ &= d(x,y) + d(y,z). \end{aligned}$$

We conclude that d as defined in (11.1) is a metric on $X_1 \times X_2$, and so we can make the product of any two metric spaces into a metric space.

In the next activity we consider products of open balls and open sets in products of metric spaces.

Activity 11.3. Let $X_1 = [1,2]$ and $X_2 = [3,4]$ as subspaces of $\mathbb{R}^2$ using the Euclidean metric.

(a) Explain in detail what the product space $X_1 \times X_2$ looks like.

(b) If B_1 is an open ball in X_1 and B_2 is an open ball in X_2, is $B_1 \times B_2$ an open ball in $X_1 \times X_2$? Explain.

(c) If B_1 is an open ball in X_1 and B_2 is an open ball in X_2, is $B_1 \times B_2$ an open set in $X_1 \times X_2$? Explain.

(d) Find, if possible, an open subset of $X_1 \times X_2$ that is not of the form $O_1 \times O_2$ where O_1 is open in X_1 and O_2 is open in X_2.

Activity 11.3 shows that open sets in a product are more complicated than just products of open sets in the factors. We will return to product later when we consider topological spaces.

We conclude with one final comment about products. We can make the Cartesian product of any number of metric spaces into a metric space with the same construction we used for the product of two spaces.

Definition 11.4. Let (X_i, d_i) be metric spaces for i from 1 to some positive integer n. The **product metric space** (X, d) is the Cartesian product

$$X = X_1 \times X_2 \times \cdots \times X_n = \prod_{i=1}^{n} X_i$$

with metric d defined by

$$d(x,y) = \sqrt{\sum_{i=1}^{n} d_i(x_i,y_i)^2}$$

when $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in X.

The metric d is called the *product metric* and the spaces (X_i, d_i) are called the *coordinate* or *factor* spaces of (X, d). The proof that d is a metric is essentially the same as in the $n = 2$ case, and is left to Exercise (6).

Summary

Important ideas that we discussed in this section include the following.

- A subset A of a metric space (X, d) is a metric space, called a subspace, by using the metric $d|_{A\times A}$ on A.
- If X is a metric space and A is a subspace of X, a subset O_A of A is open in A if and only if $O_A = X \cap O$ for some open set O in X. A subset C_A of A is closed in A if $C_A = A \cap O_A$ for open set O_A in A. Alternatively, a set C_A is closed in A if $C_A = A \cap C$ for some closed set C in X.
- Let (X_i, d_i) be metric spaces for i from 1 to some positive integer n. The product metric space (X, d) is the Cartesian product

  $$X = X_1 \times X_2 \times \cdots \times X_n = \prod_{i=1}^{n} X_i$$

  with metric d defined by

  $$d(x,y) = \sqrt{\sum_{i=1}^{n} d_i(x_i,y_i)^2}$$

  when $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in X.

Exercises

(1) Determine if the following sets S are open in the subspace A of the topological space $(\mathbb{R}, d_E)$.

  (a) $S = [1, 2)$ in $A = [1, 3]$

  (b) $S = \{1, 2\}$ in $A = \mathbb{Q}$

  (c) $S = \{1, 2\}$ in $A = \mathbb{Z}$

(2) Let O be an open set in a metric space (X, d). Show that a subset U of O is open in $(O, d|_O)$ if and only if U is open in (X, d).

(3) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a continuous function. If A is a subspace of X, must the restriction $f|_A$ of f to A mapping A to Y be continuous? Give a proof that the restriction is continuous, or an example to show that the restriction need not be continuous.

(4) Prove Theorem 11.3. That is, let (X, d) be a metric space and A a subset of X. Prove that a subset C_A of A is closed in A if and only if there is a closed set C_X in X so that $C_A = C_X \cap A$.

(5) Let (X, d_X) and (Y, d_Y) be metric spaces. Prove or disprove: the function $d : X \times Y \to \mathbb{R}$ defined by

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

is a metric on $X \times Y$.

(6) Let (X_i, d_i) be metric spaces for i from 1 to some positive integer n. Let $d : \prod_{i=1}^{n} X_i \to \mathbb{R}$ be defined

$$d(x, y) = \sqrt{\sum_{i=1}^{n} d_i(x_i, y_i)^2}$$

when $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in X. Show that d is a metric on $\prod_{i=1}^{n} X_i$.

(7) Let (x_n) be a non-decreasing sequence of real numbers that is bounded above. That is, $x_n \leq x_{n+1}$ for every n and there is a positive real number K such that $x_n \leq K$ for every n. Show that the sequence (x_n) converges.

(8) It is possible to consider infinite products as metric spaces. One important example is a Hilbert space H, which consists of all infinite sequences (x_n) where $x_n \in \mathbb{R}$ for every n and $\sum_{k=1}^{\infty} x_k^2$ is finite. Hilbert space has important applications in physics, particularly in quantum mechanics.

  (a) Give two distinct elements in H and one infinite sequence that is not in H. Explain your examples.

  (b) We define the norm of an element $x = (x_n)$ in H as

$$\|x\| = \sqrt{\sum_{k=1}^{\infty} x_k^2}.$$

  From this norm we can define a distance between elements $x = (x_n)$ and $y = (y_n)$ in H as follows:

$$d(x, y) = \|x - y\|,$$

  where $x - y = (x_n - y_n)$. Another way to write d is

$$d(x, y) = \sqrt{\sum_{k=1}^{\infty} (x_k - y_k)^2}.$$

  One potential problem with this function d is that we need to know that if x and y are in H, then $x - y \in H$. That is, show that if $\sum_{k=1}^{\infty} x_k^2$ and $\sum_{k=1}^{\infty} y_k^2$ are finite, then $\sum_{k=1}^{\infty} (x_k - y_k)^2$ is also finite. (Hint: Consider a finite sum and use Exercise 7.)

(c) Show that d is a metric on H.

(d) Let $E^m = \{(x_n) \in H \mid x_k = 0 \text{ for } k > m\}$. Let $f : E^m \to \mathbb{R}^m$ be defined by $f((x_n)_{n=1}^{\infty}) = (x_n)_{n=1}^{m}$. Show that f is a bijection such that $d((x_n),(y_n)) = d_E(f((x_n)), f((y_n)))$ for any elements (x_n), (y_n) in H. So E^m is essentially the same as $\mathbb{R}^m$ and so we can consider the space $\mathbb{R}^m$ as embedded in H as a subspace of H for every $m \in \mathbb{Z}^+$.

(9) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) If d is the discrete metric on a metric space X, then for any subspace A of X, the restriction of d to A is the discrete metric.

(b) If d is a metric on a space X that is not the discrete metric, and if A is a subset of X, then $d|_A$ cannot be the discrete metric.

(c) Let A be a subspace of a metric space (X, d). If a sequence (a_n) is in A and $\lim a_n = a$ for some $a \in A$, then $\lim a_n = a$ in X.

(d) Let A be a subspace of a metric space (X, d). If a sequence (a_n) is in A and $\lim a_n = a$ for some $a \in X$, then $\lim a_n = a$ in A.

(e) If (X, d_X) and (Y, d_Y) are metric spaces, then the function $d : X \times Y \to \mathbb{R}$ defined by
$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$
is a metric on $X \times Y$.

(f) If (X, d_X) and (Y, d_Y) are metric spaces, then the function $d : X \times Y \to \mathbb{R}$ defined by
$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) d_Y(y_1, y_2)$$
is a metric on $X \times Y$.