Section 12

Topological Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a topology and what is a topological space?
- What important properties do open sets have in relation to unions and intersections?
- What is a basis for a topology? Why is a basis for a topology useful?
- What is a neighborhood in a topological space?
- What is an interior point and the interior of a set in a topological space?
- What is the connection between the interior of a set and open sets in a topological space?

Introduction

Many of the properties that we introduced in metric spaces (continuity, limit points, boundary) could be phrased in terms of the open sets in the space. With that in mind, we can broaden our concept of space by eliminating the metric and just defining the opens sets in the space. This produces what are called *topological spaces*.

Recall that the open sets in a metric space satisfied certain properties, including that the arbitrary union of open sets is open and any finite intersection of opens sets is open. We will now take these properties as our axioms in defining topological spaces.

Definition 12.1. Let X be a nonempty set. A set τ[1] of subsets of X is said to a **topology** on X if

(1) X and $\emptyset$ belong to τ,

[1]The symbol τ is the Greek lowercase letter tau.

(2) any union of sets in τ is a set in τ, and

(3) any finite intersection of sets in τ is a set in τ.

A *topological space* is then any set on which a topology is defined. If X is the space and τ a topology on X, we denote the topological space as (X, τ). The elements of τ are called the *open sets* in the topological space. When the topology is clear from the context, we simple refer to X as the topological space. Some examples are in order.

Preview Activity 12.1.

(1) Suppose $X = \{a, b, c\}$. Is the set $\tau = \{a, b\}$ a topology on X? Justify your response.

(2) Suppose $X = \{a, b, c, d\}$. Is the collection of subsets consisting of $\tau = \{\{a\}, \{b\}, \{a, b\}\}$ a topology on X? Justify your response. If not, what is the smallest collection of subsets of X that need to be added to τ to make τ a topology on X?

(3) Suppose $X = \{a, b, c, d\}$. Is the collection of subsets consisting of

$$\tau = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, X\}$$

a topology on X? Justify your response. If not, what is the smallest collection of subsets of X that need to be added to τ to make τ a topology on X?.

(4) Suppose $X = \{a, b, c, d\}$. Is the collection of subsets consisting of

$$\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, X\}$$

a topology on X? Justify your response. If not, what is the smallest collection of subsets of X that need to be added to τ to make τ a topology on X?

(5) Let F be the collection of finite subsets of $\mathbb{R}$. Let $\tau = \{\emptyset, \mathbb{R}\} \cup F$. First, list three members of F and three sets that are not in F. Next, is τ a topology on $\mathbb{R}$? Justify your response.

(6) Let $\tau = \{\emptyset, \mathbb{R}, \{0\}\}$. Is τ a topology on $\mathbb{R}$? Justify your response.

(7) Let X be a set and let $\tau = \{\emptyset, X\}$. Is τ a topology on X? Justify your response.

(8) Let X be a set and let τ be the collection of all subsets of X. Is τ a topology on X? Justify your response.

Examples of Topologies

In our preview activity we saw several examples of topologies. Suppose X is a nonempty set.

- The topology consisting of all subsets of X is called the *discrete topology*.
- The topology $\{\emptyset, X\}$ is the *indiscrete topology*.

- If (X, d) is a metric space, then the collection consisting of unions of all open balls is a topology called the *metric topology*. This result tells us that every metric space is topological space under the metric topology. We will see later than not every topological space is a metric space.

The discrete and indiscrete topologies are topologies that can be defined on any set and are often used to use to generate examples. Another topology that can be defined on any set is in the next activity.

Activity 12.1. Let X be any set and let τ_{FC} consist of the empty set along with all subsets O of X such that $X \setminus O$ is finite.

(a) Prove that τ_{FC} is a topology on X. (The topology τ_{FC} is called the *finite complement topology* or the *cofinite topology*.

(b) Explain why τ_{FC} is the discrete topology when X is finite.

Bases for Topologies

It can be difficult to completely describe the open sets in a topology. Instead, we can describe the topology using a collection of sets that generate the topology. For example, if (X, d) is a metric space then the collection of open sets in X forms a topology on X, called the *metric* topology. We also saw that in a metric space, every open set in X is a union of open balls. For that reason we called the collection of open balls a *basis* for the open sets in X. We can do the same thing in any topological space. As a non-trivial example, an interesting topology defined on the positive integers is due to S.W. Golomb. One can use this topology to prove that there are infinitely many primes. This topology also makes the positive integers into a connected Hausdorff space (more on these concepts later). The Golomb topology is defined as follows. If a and b are coprime integers in $\mathbb{Z}^+$ (that is, a and b have no common positive factors other than 1 so that the greatest common divisor of a and b is 1), let

$$B_{a,b} = \{a + bn \mid n \geq 0\}.$$

The collection of sets $B_{a,b}$ is a basis for the Golomb topology, and the topological space $(\mathbb{Z}^+, \tau)$ is called the *Golomb space*. It is an exercise in number theory to prove that the sets $B_{a,b}$ form a basis for a topology, so we will not go into the details.

Activity 12.2. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X)$. You may assume that τ is a topology on X. Explain why any nonempty open set in the topological space (X, τ) can be written in terms of arbitrary unions and finite intersections of $\{a\}$, $\{b\}$, and $\{c, d\}$.

Activity 12.2 shows that, just like the open balls in a metric space, a topology can have a collection of subsets whose unions make up all of the open sets in the topology. We do need to take a little care, though. A basis will generate the collection of open sets for a topology, so the basis sets we start with should themselves be open sets. In addition, every element in the topological space should be an element of one of the basis sets, and since the basis elements are to produce all of the open sets in the topology, every set in the topology (except the empty set) should be a union of sets in a basis. It also must be the case that we can ensure that any finite intersection of sets in the topology remains a set in the topology when we write the sets in the topology in terms of the sets in a basis. To make the last two conditions happen, we will see that it is enough to insist that for any point in the intersection of basis elements, there is another basis element in that intersection that contains the point. This is summarized in Theorem 12.2.

Theorem 12.2. *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X such that*

(1) For each $x \in X$, there is a set in $\mathcal{B}$ that contains x.

(2) If $x \in X$ is an element of $B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there is a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then the set τ that consists of the empty set and unions of elements of $\mathcal{B}$ is a topology on X.

Before we prove Theorem 12.2, we will need to know one fact about the set $\mathcal{B}$.

Activity 12.3. Let X be a set and $\mathcal{B}$ a collection of subsets of X such that

(1) For each $x \in X$, there is a set in $\mathcal{B}$ that contains x.

(2) If $x \in X$ is an element of $B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there is a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Let B_1, B_2, $\ldots$, B_n be in $\mathcal{B}$. Our goal in this activity is to extend property 2 and show that if $x \in \bigcap_{1 \leq k \leq n} B_k$, then there is a set $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq \bigcap_{1 \leq k \leq n} B_k$.

(a) Since the statement we want to prove depends on a positive integer n, we will use mathematical induction. Explain why the $n = 1$ and $n = 2$ cases are true.

(b) What is the inductive hypothesis and what do we want to prove in the inductive step?

(c) Use the inductive hypothesis and condition 2 to complete the proof of the following lemma.

Lemma 12.3. *Let X be a set and $\mathcal{B}$ a collection of subsets of X such that*

(1) For each $x \in X$, there is a set in $\mathcal{B}$ that contains x.

(2) If $x \in X$ is an element of $B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there is a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Let B_1, B_2, $\ldots$, B_n be in $\mathcal{B}$. If $x \in \bigcap_{1 \leq k \leq n} B_k$, then there is a set $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq \bigcap_{1 \leq k \leq n} B_k$.

Now we can prove Theorem 12.2.

Proof of Theorem 12.2. Let X be a topological space, and let $\mathcal{B}$ and τ satisfy the given conditions. By definition, $\emptyset \in \tau$. For each $x \in X$ there is a set $B_x \in \mathcal{B}$ such that $x \in B_x$. Then $X = \bigcup_{x \in X} B_x$, and $X \in \tau$. To complete our proof that τ is a topology on X, we need to demonstrate that τ is closed under arbitrary unions and finite intersections. We first consider unions. Let $\{U_\alpha\}$ be a collection of sets in τ for α in some indexing set I. By definition, each U_α is empty or is a union of elements of $\mathcal{B}$. So either $U = \bigcup_{\alpha \in I} U_\alpha$ is empty, or is a union of sets in $\mathcal{B}$. Thus, $U \in \tau$ and τ is closed under arbitrary unions.

Now we show that τ is closed under finite intersections. Let n be a positive integer and let $\{U_k\}$ a collection of sets in τ for $1 \leq k \leq n$. Let $U = U_1 \cap U_2 \cap \cdots \cap U_n$. If $U_k = \emptyset$ for any k, then $U = \emptyset$ is in τ. So suppose that $U_k \neq \emptyset$ for each k between 1 and n. Let $x \in U$. Then $x \in U_k$ for each k. For every m between 1 and n, the fact that U_m is a union of elements in $\mathcal{B}$ implies that there exists $B_m \subseteq U_m$ with $x \in B_m$. Thus, $x \in \bigcap_{1 \leq m \leq n} B_m$.

Lemma 12.3 shows that there is a set $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subseteq \bigcap_{1 \leq m \leq n} B_m \subseteq \bigcap_{1 \leq m \leq n} U_m = U$. Since x is an arbitrary element of U, we must have $U \subseteq \bigcup_{x \in U} B_x$. But each B_x is subset of U, so $\bigcup_{x \in U} B_x \subseteq U$. It follows that

$$U = \bigcup_{x \in U} B_x$$

and $U \in \tau$. Therefore, τ is a topology on X. ■

Any collection $\mathcal{B}$ of sets as given in Theorem 12.2 is given a special name.

Definition 12.4. Let X be a set. A set $\mathcal{B}$ is a **basis for a topology** (or just a **basis**) on X if

(1) For each $x \in X$, there is a set in $\mathcal{B}$ that contains x.

(2) If $x \in X$ is an element of $B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there is a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

The elements of a basis $\mathcal{B}$ are called *basis elements* or the *basic open sets*. A basis for a topology on a set X defines a topology on X as shown in Theorem 12.2.

Note that because of property (1) of Definition 12.4, the union of the sets in the basis must contain X. In other words, the sets in a basis cover the space. The second property ensures that if a point is in the intersection of two basic open sets, then there is a smaller basic open set that contains x.

Definition 12.5. Let $\mathcal{B}$ be a basis for a topology on a set X. The **topology τ generated by $\mathcal{B}$** contains the empty set and all arbitrary unions of basis elements.

When the topology for a space X is clear from the context, we also call a basis for the topology a basis for X.

Activity 12.4.

(a) Let $X = \{a, b, c, d, e, f\}$ and $\tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}, X\}$.

i. Is the set

$$\mathcal{B} = \{\{a\}, \{a, c, d\}\}$$

a basis for τ? If not, add the smallest number of sets that you can to $\mathcal{B}$ to make a basis for this topology.

ii. Is the set

$$\mathcal{B} = \{\{a\}, \{c, d\}, \{b, c, d, e, f\}\}$$

a basis for τ? If not, add the smallest number of sets that you can to $\mathcal{B}$ to make a basis for this topology.

(b) Let $X = \{a, b, c\}$ and let X have the discrete topology (the topology consisting of all subsets of X). Is $\mathcal{B} = \{\{a\}, \{c\}, \{a, b\}, \{b, c\}\}$ a basis for τ in the the discrete topology? If not, add the smallest number of sets that you can to $\mathcal{B}$ to make a basis for this topology.

(c) Find a basis for the discrete topology on any set X.

Metric Spaces and Topological Spaces

Every metric space is a topological space, where the topology is the collection of open sets defined by the metric. This topology is called the *metric topology*. A natural question to ask is whether every topological space is a metric space. That is, given a topological space, can we define a metric on the space so that the open sets are exactly the sets in the topology? For example, any space with the discrete topology is a metric space with the discrete metric.

Activity 12.5. Let $X = \{a, b, c, e\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Explain why there cannot be a metric $d : X \times X \to \mathbb{R}$ so that the open sets in the metric topology are the sets in τ. (Hint: Assume that such a metric exists and consider the open balls centered at c.)

We conclude that every metric space is a topological space, but not every topological space is a metric space. The topological spaces that can be realized as metric spaces are called *metrizable*.

Neighborhoods in Topological Spaces

Recall that we defined a neighborhood of a point a in a metric space to be a subset of the space that contains an open ball centered at X. Every open ball is an open set, so we can extend the idea of neighborhood to topological spaces.

Definition 12.6. Let (X, τ) be a topological space, and let $a \in X$. A subset N of X is a **neighborhood** of a if N contains an open set that contains a.

Let's look at some examples.

Activity 12.6. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.

(a) Find all of the neighborhoods of the point a.

(b) Find all of the neighborhoods of the point c.

In metric spaces, an open set was a neighborhood of each of its points. This is also true in topological spaces.

Theorem 12.7. *Let (X, τ) be a topological space. A subset O of X is open if and only if O is a neighborhood of each of its points.*

Proof. Let (X, τ) be a topological space, and let O be a subset of X. First we demonstrate that if O is open, then O is a neighborhood of each of its points. Assume that O is an open set, and let $a \in O$. Then O contains the open set O that contains a, so O is a neighborhood of a.

The reverse containment is the subject of the next activity. ∎

Activity 12.7. Let (X, τ) be a topological space. Let O be a subset of X. Assume O is a neighborhood of each of its points.

(a) What do we need to do to show that O is an open set?

(b) Let $a \in O$. Why must there exist an open set O_a such that $a \in O_a \subseteq O$?

(c) Complete the proof that O is an open set.

The Interior of a Set in a Topological Space

We have seen that topologies define the open sets in a topological space. As in metric spaces, open sets can be characterized in terms of their interior points. We defined interior points in metric spaces in terms of neighborhoods – the same holds true in topological spaces.

Definition 12.8. Let A be a subset of a topological space X. A point $a \in A$ is an **interior point** of A if A is a neighborhood of a.

Remember that a set is a neighborhood of a point if the set contains an open set that contains the point. By definition, every open set is a neighborhood of each of its points, so every point of an open set O is an interior point of O. Conversely, if every point of a set O is an interior point, then O is a neighborhood of each of its points and is open. This argument is summarized in the next theorem.

Theorem 12.9. *Let X be a topological space. A subset O of X is open if and only if every point of O is an interior point of O.*

The collection of interior points in a set form a subset of that set, called the *interior* of the set.

Definition 12.10. The **interior** of a subset A of a topological space X is the set

$$\text{Int}(A) = \{a \in A \mid a \text{ is an interior point of } A\}.$$

Activity 12.8.

(a) Consider $(\mathbb{R}, \tau)$, where τ is the standard topology (by standard in this situation, we mean the metric topology determined by the Euclidean metric). Let $A = (-\infty, 0) \cup (1, 2] \cup \{3\}$ in $\mathbb{R}$. What is Int(A)? What is the largest open subset of $\mathbb{R}$ contained in A?

(b) Consider $(\mathbb{R}, \tau)$, where τ is the discrete topology (the one where all subsets are open). Let $A = (-\infty, 0) \cup (1, 2] \cup \{3\}$ in $\mathbb{R}$. What is Int(A)? What is the largest open subset of $\mathbb{R}$ contained in A?

(c) Consider $(\mathbb{R}, \tau)$, where τ is the finite complement topology (the one where the open sets are the empty set along with all subsets O of $\mathbb{R}$ such that $\mathbb{R} \setminus O$ is finite). Let $A = (-\infty, 0) \cup (1, 2] \cup \{3\}$ in $\mathbb{R}$. What is Int(A)? What is the largest open subset of $\mathbb{R}$ contained in A?

(d) Let $X = \{a, b, c, d\}$ and let

$$\tau = \{\emptyset, \{a\}, \{a, b\}, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, X\}.$$

Assume that τ is a topology on X. Let $A = \{b, c, d\}$. What is Int(A)? What is the largest open subset of X contained in A?

One might expect that the interior of a set is an open set, as it was in metric spaces. This is true, but we can say even more. In Activity 12.8 we saw that in our examples that Int(A) was the largest open subset of X contained in A. That this is always true is the subject of the next theorem.

Theorem 12.11. *Let (X, d) be a topological space, and let A be a subset of X. The interior of A is the largest open subset of X contained in A.*

Proof. Let X be a topological space, and let A be a subset of X. We need to prove that Int(A) is an open set in X, and that Int(A) is the largest open subset of X contained in A. First we demonstrate that Int(A) is an open set. Let $a \in$ Int(A). Then a is an interior point of A, so A is a neighborhood of a. This implies that there exists an open set O containing a so that $O \subseteq A$. But O is a neighborhood of each of its points, so every point in O is an interior point of A. It follows that $O \subseteq$ Int(A). Thus, Int(A) is a neighborhood of each of its points and, consequently, Int(A) is an open set.

The proof that Int(A) is the largest open subset of X contained in A is left for the next activity. ■

Activity 12.9. Let (X, d) be a topological space, and let A be a subset of X.

(a) What will we have to show to prove that Int(A) is the largest open subset of X contained in A?

(b) Suppose that O is an open subset of X that is contained in A, and let $x \in O$. What does the fact that O is open tell us? Then complete the proof that $O \subseteq$ Int(A).

One consequence of Theorem 12.11 is the following.

Corollary 12.12. *A subset O of a topological space X is open if and only if $O =$ Int(O).*

Summary

Important ideas that we discussed in this section include the following.

- A topology on a set X is a collection of open subsets of X. More specifically, a set τ of subsets of a set X is a topology on X if

(1) X and $\emptyset$ belong to τ,

(2) any union of sets in τ is a set in τ, and

(3) any finite intersection of sets in τ is a set in τ.

A topological space is a set along with a topology on the set.

- Any arbitrary union of open sets is open and any finite intersection of open sets is open in a topological space.
- It can be difficult to completely describe the open sets in a topology, and it can be difficult to work with arbitrary open sets. If a collection of simpler sets generate a topology, that collection of simpler sets is a basis for the topology. More formally set $\mathcal{B}$ is a basis for a topology on a set X if
  (1) For each $x \in X$, there is a set in $\mathcal{B}$ that contains x.
  (2) If $x \in X$ is an element of $B_1 \cap B_2$ for some $B_1, B_2 \in B$, then there is a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.
- A subset A of a topological space X is a neighborhood of a point $a \in A$ if there is an open set O contained in A such that $a \in O$.
- A point x in a subset A of a topological space X is an interior point of A if A is a neighborhood of x. The interior of set A is the collection of all interior points of A.
- A subset A of a topological space X is open if and only if A is equal to its interior.

Exercises

(1) You may wonder why we can't define a basis for a topology on a set X to be any collection of subsets whose union is X. Consider the example of $X = \{a, b, c\}$ and $S = \{\{a\}, \{c\}, \{a, b\}, \{b, c\}\}$.

(a) Determine the collection of all of the unions of elements of S.

(b) Explain why the collection of unions of the elements of S, along with the empty set, is not a topology on X. What property of a basis is not satisfied?

(2) For each integer a, let $a\mathbb{Z} = \{ka \mid k \in \mathbb{Z}\}$. That is, $a\mathbb{Z}$ is the set of all integer multiples of a.

(a) Show that $\{a\mathbb{Z} \mid a \in \mathbb{Z}\}$ is a basis for a topology τ on $\mathbb{Z}$. (Hint: What set is $m\mathbb{Z} \cap n\mathbb{Z}$?)

(b) Is the set of positive integers an open set in the topological space $(\mathbb{Z}, \tau)$? Explain.

(c) Is the set of odd integers open in the topological space $(\mathbb{Z}, \tau)$? Explain.

(d) Is the set $\{0\} \cup \{x \in Z \mid |x| \geq 5\}$ open in the topological space $(\mathbb{Z}, \tau)$? Explain.

(3) This exercise is a generalization of Exercise (2). Let a and b be integers with $a \neq 0$. Let $A_{a,b} = a\mathbb{Z} + b = \{ak + b \mid k \in \mathbb{Z}\}$.

(a) Show that $\{A_{a,b} \mid a,b \in \mathbb{Z}, a \neq 0\}$ is a basis for a topology τ on $\mathbb{Z}$. (Hint: If $B_1 = A_{a_1,b_1}$ and $B_2 = A_{a_2,b_2}$, and if $x \in B_1 \cap B_2$, what can we say about $A_{a_1a_2,x}$?)

(b) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = n + (-1)^n$.

i. Prove that f is a bijection.

ii. If O is an open set in $\mathbb{Z}$, is $f(O)$ an open set?

iii. If U is an open set in $\mathbb{Z}$, is $f^{-1}(U)$ an open set? (Hint: What is f^{-1}?)

(4) Let $\mathcal{B} = \{(-x,x) \mid x \in \mathbb{R}^+\}$.

(a) Show that $\mathcal{B}$ is a basis for a topology τ on $\mathbb{R}$.

(b) Every basis set is open in $(\mathbb{R}, \tau_E)$. So we can ask it the topology τ is different than the Euclidean topology generated by all open intervals in $\mathbb{R}$. Show that there are intervals of the form (a,b) that are open in $(\mathbb{R}, d_E)$ that are not open sets in τ.

(5) Let $X = \{a,b,c\}$, and let $\tau_1 = \{\emptyset, \{a\}, \{a,b,c\}\}$ and $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$. Both τ_1 and τ_2 are topologies in X, but every element in τ_1 is also an element in τ_2. Then this happens we say that τ_1 is a weaker topology than τ_2. Exercise (4 provides an example. More formally,

Definition 12.13. Let τ_1 and τ_2 be two topologies on a set X. If $\tau_1 \subseteq \tau_2$, then τ_1 is a **coarser** (or **weaker**) topology than τ_2. We also say that τ_2 is a **finer** (or **stronger**) topology than τ_1.

(a) What is the weakest topology on any set?

(b) What is the strongest topology on any set?

(c) If a topology on X contains all single point sets, then every subset is open and our our topology is the discrete topology. Also, if a topology on X contains all two-point sets, then if x, y, and z are in X it follows that $\{x,y\} \cap \{x,z\} = \{x\}$ is in the topology and we again have the.discrete topology. Consider the topology

$$\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\}.$$

The only sets not in σ are $\{c\}$ and $\{b,c\}$, but adding either set to σ will produce the discrete topology. So σ is a strongest topology possible other than the discrete topology.

(d) Let $X = \{a,b,c\}$. Are there any topologies σ on X such that σ is not the discrete topology but there are no stronger topologies on X other than the discrete topology? Explain.

(e) Let $X = \{a,b,c\}$. Are there any topologies γ on X such that γ is not the indiscrete topology but there are no weaker topologies on X other than the indiscrete topology? Explain.

(f) In general, there may be many different bases for a given topology, and two different bases can have the same cardinality. This is not the case for finite topological space. Let X be a finite set and let τ be a topology on X. In this exercise we will show that there is a minimal basis for the topology τ. That is, there is a basis $\mathcal{B}_{\text{min}}$ of τ such that if $\mathcal{B}$ is any other basis for τ, then $\mathcal{B}_{\text{min}} \subseteq \mathcal{B}$.

i. If $x \in X$, let U_x be the intersection of all open sets that contain x. Explain why U_x is an open set.

ii. Let $\mathcal{B}_{\text{min}} = \{U_x \mid x \in X\}$. Show that $\mathcal{B}_{\text{min}}$ is a basis for τ.

iii. Show that if $\mathcal{B}$ is a basis for τ, then $\mathcal{B}_{\text{min}} \subseteq \mathcal{B}$.

iv. Let $X = \{a, b, c, d\}$ and let

$$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}.$$

You may assume that τ is a topology on X. Find the unique minimal basis for τ.

(g) Below is a list of 9 distinct topologies on $X = \{a, b, c\}$. Each topology lies in one or more sequences of topologies ordered by coarseness. For each topology τ, list the longest sequence(s) of topologies that start $\{\emptyset, X\} \subset \tau$, ordered by coarseness.

1. $\{\emptyset, X\}$
2. $\{\emptyset, \{a\}, X\}$
3. $\{\emptyset, \{a, b\}, X\}$
4. $\{\emptyset, \{a\}, \{a, b\}, X\}$
5. $\{\emptyset, \{a\}, \{b, c\}, X\}$
6. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$
7. $\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$
8. $\{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$
9. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$
10. the discrete topology

(6) Find all of the topologies on X if

(a) X is a single point set

(b) X is a two point set

(c) X is a three point set (Hint: there are 29 distinct topologies).

(7) For each $n \in \mathbb{Z}^+$, let $O_n = \{n, n+1, n+2, \ldots\}$. Let $\tau = \{\emptyset, O_1, O_2, O_3, \ldots\}$. Show that $(\mathbb{Z}^+, \tau)$ is a topological space.

(8) Let A, B be two subsets in a topological space X. What can you say about the relationships between $\text{Int}(A \cap B), \text{Int}(A \cup B)$ and $\text{Int}(A) \cap \text{Int}(B), \text{Int}(A) \cup \text{Int}(B)$, respectively? Verify your results.

(9) Let X be a nonempty set and let p be an element in X. Let τ_p be the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that contain p. Show that τ_p is a topology on X. (This topology is called the *particular point topology*).

(10) Let X be a nonempty set and let p be an element in X. Let $\tau_{\overline{p}}$ be the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that do not contain p. Show that $\tau_{\overline{p}}$ is a topology on X. (This topology is called the *excluded point topology*.)

(11) One application of topology is in digital image displays, such as a computer screen. A digital image display is a rectangular array of pixels and can be modeled using a digital plane. In this exercise we consider a simplification of the digital plane – the digital line – which we consider as an infinite length one-dimensional collection of pixels. For each $n \in \mathbb{Z}$ we define

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even.} \end{cases}$$

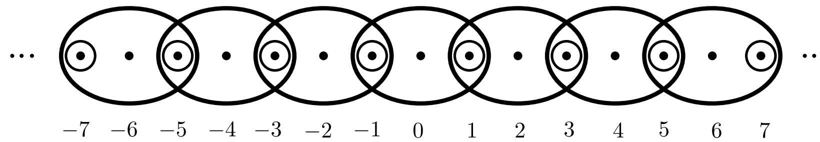

Figure 12.1: The digital line topology.

The sets $B(n)$ are illustrated in Figure 12.1.

In this exercise we explore the collection $\mathcal{B} = \{B(n)\}$.

(a) Show that the collection $\mathcal{B} = \{B(n)\}$ is a basis for a topology on $\mathbb{Z}$. (The resulting topology is called the *digital line topology* τ_{dl}.[2] (The digital line models a one-dimensional array of pixels, where the even integers are the pixels and the odd integers are boundaries between the pixels. Information about the *digital plane* can be found in Section 20.)

(b) Determine which of the following sets are open in the digital line topology:

i. $\{0\}$
ii. $\{1\}$
iii. $\{0, 2\}$
iv. $\{1, 2, 3, 4, 5\}$
v. $\mathbb{Z}^+$
vi. The set of odd integers.

(12) Let n be a positive integer and let $\mathcal{P}_n$ be the collection of all polynomials in n real variables x_1, x_2, ..., x_n. As a specific example, the polynomial

$$f(x_1, x_2, x_3) = 2x_1x_3 + 5x_1x_2^2x_3^4 - x_2 + 10x_1^5x_3$$

is in $\mathcal{P}_3$. If $f(x_1, x_2, \ldots, x_n)$ is in $\mathcal{P}_n$, let $Z(f)$ be the set of zeros of the polynomial f. That is,

$$Z(f) = \{(x_1, x_2, \ldots, x_n) \mid f(x_1, x_2, \ldots, x_n) = 0\}.$$

Note that $Z(f)$ is a subset of $\mathbb{R}^n$. For example, if $n = 2$ and $f(x_1, x_2) = x_1^2 - x_2$ then $Z(f)$ is the set of ordered pairs in $\mathbb{R}^2$ satisfying $x_1^2 - x_2 = 0$, or $x_2 = x_1^2$. This is the graph of the parabola $y = x^2$ in the plane.

(a) Describe $Z(f)$ in $\mathbb{R}^2$ if $f(x_1, x_2) = x_1^2 - 1$.

(b) If E is a set of polynomials in $\mathcal{P}_n$, we let $Z(E) = \bigcap_{f \in E} Z(f)$ be the set of common zeros of all of the polynomials in E. Describe $Z(E)$ if $E = \{x_1 + x_2 + x_3, x_1 - x_2 - x_3, 3x_1 + x_2 + x_3\}$ in $\mathbb{R}^3$.

(c) Let $\mathcal{B}$ be the set of complements of the sets $Z(f)$ for $f \in \mathcal{P}_n$. Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^n$. The resulting topology is called the *Zariski* topology.

[2] This digital line topology has applications in digital processing – see *Introduction to Topology: Pure and Applied* by Colin Adams and Robert Franzosa , Pearson Education, Inc., 2008, Sections 1.4 and 11.3. The set $\mathbb{Z}$ with the digital line topology is called the *digital line*.

(d) Is the set $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_2 = 0\}$ an open set in $\mathbb{R}^2$ with the Zariski topology? Explain.

(e) Explain why the Zariski topology when $n = 1$ is just the cofinite topology on $\mathbb{R}$. That is, show that every set that is open in the cofinite topology is open in the Zariski topology and that every set that is open in the Zaariski topology is open in the cofinite topology.

(13) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) The set $\{\emptyset, \{a, b\}, \{a, b, d, f\}, \{d, f\}, X\}$ is a topology on the set $X = \{a, b, c, d, e, f\}$.

(b) The set $\mathbb{Z}$ is an open subset of $\mathbb{R}$ using the finite complement topology τ_{FC} on $\mathbb{R}$.

(c) The set $\mathcal{B} = \{\{b\}, \{c\}, \{a, b\}, \{b, c, d\}\}$ is a basis for the topology τ on the set $X = \{a, b, c, d\}$, where

$$\tau = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.$$

(d) Let X be a nonempty set. If τ is the discrete topology, then the topological set (X, τ) is metrizable.

(e) The point b is an interior point of the subset $A = \{a, b, d\}$ in the topological space (X, τ), where $X = \{a, b, c, d\}$ and

$$\tau = \{\emptyset, \{a\}, \{a, b\}, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, \{a, c\}, \{a, d\}, \{a, b, c, \}, \{a, b, d\}, X\}.$$

(f) If τ_1 and τ_2 are topologies on a space X, then $\tau_1 \cup \tau_2$ is also a topology on X.

(g) If τ_1 and τ_2 are topologies on a space X, then $\tau_1 \cap \tau_2$ is also a topology on X.