Appendix C

Answers and Hints for Selected Exercises

Section 1.1

1. Sentences (a), (c), (e), (f), (j) and (k) are statements. Sentence (h) is a statement if we are assuming that n is a prime number means that n is a natural number.

2.

	Hypothesis	Conclusion
(a)	n is a prime number.	n^2 has three positive divisors.
(b)	a is an irrational number and b is an irrational number.	$a \cdot b$ is an irrational number.
(c)	p is a prime number.	$p = 2$ or p is an odd number.
(d)	p is a prime number and $p \neq 2$.	p is an odd number.
(e)	$p \neq 2$ and p is an even number	p is not prime.

3. Statements (a), (c), and (d) are true.

4. **(a)** True when $a \neq 3$. **(b)** True when $a = 3$.

6. **(a)** This function has a maximum value when $x = \dfrac{5}{16}$.

   (b) The function h has a maximum value when $x = \dfrac{9}{2}$.

   (c) No conclusion can be made about this function from this theorem.

9. **(a)** The set of natural numbers is not closed under division.

(b) The set of rational numbers is not closed under division since division by zero is not defined.

(c) The set of nonzero rational numbers is closed under division.

(d) The set of positive rational numbers is closed under division.

(e) The set of positive real numbers is not closed under subtraction.

(f) The set of negative rational numbers is not closed under division.

(g) The set of negative integers is closed under addition.

Section 1.2

1. **(a)**

Step	**Know**	**Reason**
P	m is an even integer.	Hypothesis
$P1$	There exists an integer k such that $m = 2k$.	Definition of an even integer
$P2$	$m + 1 = 2k + 1$	Algebra
$Q1$	There exists an integer q such that $m + 1 = 2q + 1$.	Substitution of $k = q$
Q	$m + 1$ is an odd integer.	Definition of an odd integer

2. **(c)** We assume that x and y are odd integers and will prove that $x + y$ is an even integer. Since x and y are odd, there exist integers m and n such that $x = 2m + 1$ and $y = 2n + 1$. Then

$$\begin{aligned} x + y &= (2m + 1) + (2n + 1) \\ &= 2m + 2n + 2 \\ &= 2\left(m + n + 1\right). \end{aligned}$$

Since the integers are closed under addition, $(m + n + 1)$ is an integer, and hence the last equation shows that $x + y$ is even. Therefore, we have proven that if x and y are odd integers, then $x + y$ is an even integer.

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3. **(a)**

Step	**Know**	**Reason**
P	m is an even integer and n is an integer.	Hypothesis
$P1$	There exists an integer k $m = 2k$.	Definition of an even integer.
$P2$	$m \cdot n = (2k)\, n$	Substitution
$P3$	$m \cdot n = 2\,(kn)$	Algebra
$P4$	(kn) is an integer	Closure properties of the integers
$Q1$	There exists an integer q such that $m \cdot n = 2q$	$q = kn$.
Q	$m \cdot n$ is an even integer.	Definition of an even integer.

(b) Use Part (a) to prove this.

4. **(a)** We assume that m is an even integer and will prove that $5m + 7$ is an odd integer. Since m is an even integer, there exists an integer k such that $m = 2k$. Using substitution and algebra, we see that

$$\begin{aligned} 5m + 7 &= 5(2k) + 7 \\ &= 10k + 6 + 1 \\ &= 2(5k + 3) + 1 \end{aligned}$$

By the closure properties of the integers, we conclude that $5k + 3$ is an integer, and so the last equation proves that $5m + 7$ is an odd integer.

Another proof. By Part (a) of Exercise 2, $5m$ is an even integer. Hence, by Part (b) of Exercise 2, $5m + 7$ is an even integer.

5. **(b)** We assume that m is an odd integer and will prove that $3m^2 + 7m + 12$ is an even integer. Since m is odd, there exists an integer k such that $m = 2k + 1$. Hence,

$$\begin{aligned} 3m^2 + 7m + 12 &= 3\,(2k + 1)^2 + 7\,(2k + 1) + 12 \\ &= 12k^2 + 26k + 22 \\ &= 2\left(6k^2 + 13k + 11\right) \end{aligned}$$

By the closure properties of the integers, $\left(6k^2 + 13k + 11\right)$ is an integer. Hence, this proves that if m is odd, then $3m^2 + 7m + 12$ is an even integer.

6. **(a)** Prove that they are not zero and their quotient is equal to 1.

(d) Prove that two of the sides have the same length. Prove that the triangle has two congruent angles. Prove that an altitude of the triangle is a perpendicular bisector of a side of the triangle.

9. **(a)** Some examples of type 1 integers are $-5, -2, 1, 4, 7, 10$.

(c) All examples should indicate the proposition is true.

10. **(a)** Let a and b be integers and assume that a and b are both type 1 integers. Then, there exist integers m and n such that $a = 3m + 1$ and $b = 3n + 1$. Now show that

$$a + b = 3(m + n) + 2.$$

The closure properties of the integers imply that $m + n$ is an integer. Therefore, the last equation tells us that $a + b$ is a type 2 integer. Hence, we have proved that if a and b are both type 1 integers, then $a + b$ is a type 2 integer.

Section 2.1

1. The statement was true. When the hypothesis is false, the conditional statement is true.

2. **(a)** P is false. **(b)** $P \wedge Q$ is false. **(c)** $P \vee Q$ is false.

4. **(c)** Statement P is true but since we do not know if R is true or false, we cannot tell if $P \wedge R$ is true or false.

5. Statements (a) and (d) have the same truth table. Statements (b) and (c) have the same truth table.

P	Q	$P \to Q$	$Q \to P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

7.

P	Q	R	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	F	F
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

The two statements have the same truth table.

9. **(c)** The integer x is even only if x^2 is even.

(d) For the integer x to be even, it is necessary that x^2 be even.

11. **(a)** $\neg Q \vee (P \to Q)$ is a tautology.

(b) $Q \wedge (P \wedge \neg Q)$ is a contradiction.

(c) $(Q \wedge P) \wedge (P \to \neg Q)$ is a contradiction.

(d) $\neg Q \to (P \wedge \neg P)$ is neither a tautology nor a contradiction.

Section 2.2

1. **(a)** Converse: If $a^2 = 25$, then $a = 5$. Contrapositive: If $a^2 \neq 25$, then $a \neq 5$.

(b) Converse: If Laura is playing golf, then it is not raining. Contrapositive: If Laura is not playing golf, then it is raining.

(c) Converse: If $a^4 \neq b^4$, then $a \neq b$. Contrapositive: If $a^4 = b^4$, then $a = b$.

(d) Converse: If $3a$ is an odd integer, then a is an odd integer. Contrapositive: If $3a$ is an even integer, then a is an even integer.

2. **(a)** Disjunction: $a \neq 5$ or $a^2 = 25$. Negation: $a = 5$ and $a^2 \neq 25$.

(b) Disjunction: It is raining or Laura is playing golf.
Negation: It is not raining and Laura is not playing golf.

(c) Disjunction: $a = b$ or $a^4 \neq b^4$. Negation: $a \neq b$ and $a^4 = b^4$.

(d) Disjunction: a is an even integer or $3a$ is an odd integer.
Negation: a is an odd integer and $3a$ is an even integer.

3. **(a)** We will not win the first game or we will not win the second game.

(b) They will not lose the first game and they will not lose the second game.

(c) You mow the lawn and I will not pay you \$20.

(d) We do not win the first game and we will play a second game.

(e) I will not wash the car and I will not mow the lawn.

7. **(a)** In this case, it may be better to work with the right side first.

$$\begin{aligned}(P \to R) \vee (Q \to R) &\equiv (\neg P \vee R) \vee (\neg Q \vee R)\\ &\equiv (\neg P \vee \neg Q) \vee (R \vee R)\\ &\equiv (\neg P \vee \neg Q) \vee R\\ &\equiv \neg (P \wedge Q) \vee R\\ &\equiv (P \wedge Q) \to R.\end{aligned}$$

(b) In this case, we start with the left side.

$$\begin{aligned}[P \to (Q \wedge R)] &\equiv \neg P \vee (Q \wedge R)\\ &\equiv (\neg P \vee Q) \wedge (\neg P \vee R)\\ &\equiv (P \to Q) \wedge (P \to R)\end{aligned}$$

10. Statements (c) and (d) are logically equivalent to the given conditional statement. Statement (f) is the negation of the given conditional statement.

11. **(d)** This is the contrapositive of the given statement and hence, it is logically equivalent to the given statement.

Section 2.3

1. **(a)** The set of all real number solutions of the equation $2x^2 + 3x - 2 = 0$, which is $\left\{\frac{1}{2}, -2\right\}$.

(b) The set of all integer solutions of the equation $2x^2 + 3x - 2 = 0$, which is $\{-2\}$.

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(c) The set of all integers whose square is less than 25, which is $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

(d) The set of all natural numbers whose square is less than 25, which is $\{1, 2, 3, 4\}$

(e) The set of all rational numbers that are 2 units from 2.5 on the number line, which is $\{-0.5, 4.5\}$.

(f) The set of all integers that are less than or equal to 2.5 units from 2 on the number line, which is $\{0, 1, 2, 3, 4\}$.

2. Possible answers for **(b)**:

$$A = \left\{n^2 \mid n \in \mathbb{N}\right\} \qquad B = \{-\pi^n \mid n \text{ is a nonnegative integer}\}$$
$$C = \{6n + 3 \mid n \text{ is a nonnegative integer}\} = \{6n - 3 \mid n \in \mathbb{N}\}$$
$$D = \{4n \mid n \in \mathbb{Z} \text{ and } 0 \le n \le 25\}$$

3. The sets in (b) and (c) are equal to the given set.

4. **(a)** $\{-3\}$ **(b)** $\{-8, 8\}$

5. **(a)** $\{x \in \mathbb{Z} \mid x \ge 5\}$ **(c)** $\{x \in \mathbb{Q} \mid x > 0\}$ **(e)** $\left\{x \in \mathbb{R} \mid x^2 > 10\right\}$

Section 2.4

1. **(a)** There exists a rational number x such that $x^2 - 3x - 7 = 0$. This statement is false since the solutions of the equation are $x = \dfrac{3 \pm \sqrt{37}}{2}$, which are irrational numbers.

(b) There exists a real number x such that $x^2 + 1 = 0$. This statement is false since the only solutions of the equation are i and $-i$, which are not real numbers.

(c) There exists a natural number m such that $m^2 < 1$. This statement is false because if m is a natural number, then $m \ge 1$ and hence, $m^2 \ge 1$.

2. **(a)** $m = 1$ is a counterexample. The negation is: There exists a natural number m such that m^2 is not even or there exists a natural number m such that m^2 is odd.

(b) $x = 0$ is a counterexample. The negation is: There exists a real number x such that $x^2 \le 0$.

(f) $x = \dfrac{\pi}{2}$ is a counterexample. The negation is: There exists a real number x such that $\tan^2 x + 1 \neq \sec^2 x$.

3. **(a)** There exists a rational number x such that $x > \sqrt{2}$.
The negation is $(\forall x \in \mathbb{Q})\left(x \leq \sqrt{2}\right)$, which is, For each rational number x, $x \leq \sqrt{2}$.

(c) For each integer x, x is even or x is odd.
The negation is $(\exists x \in \mathbb{Z})$ (x is odd and x is even), which is, There exists an integer x such that x is odd and x is even.

(e) For each integer x, if x^2 is odd, then x is odd.
The negation is $(\exists x \in \mathbb{Z})$ $\left(x^2 \text{ is odd and } x \text{ is even}\right)$, which is, There exists an integer x such that x^2 is odd and x is even.

(h) There exists a real number x such that $\cos(2x) = 2(\cos x)$.
The negation is $(\forall x \in \mathbb{R})\,(\cos(2x) \neq 2(\cos x))$, which is, For each real number x, $\cos(2x) \neq 2(\cos x)$.

4. **(a)** There exist integers m and n such that $m > n$.

(e) There exists an integer n such that for each integer m, $m^2 > n$.

5. **(a)** $(\forall m)\,(\forall n)\,(m \leq n)$.
For all integers m and n, $m \leq n$.

(e) $(\forall n)\,(\exists m)\left(m^2 \leq n\right)$.
For each integer n, there exists an integer m such that $m^2 \leq n$.

6. **(a)** It is not a statement since x is an unquantified variable.

(b) It is a true statement.

(c) It is a false statement.

(d) It is a true statement.

(e) $\{-20, -10, -5, -4, -2, -1, 1, 2, 4, 5, 10, 20\}$

10. **(a)** A function f with domain $\mathbb{R}$ is strictly increasing provided that $(\forall x, y \in \mathbb{R})\,[(x < y) \to (f(x) < f(y))]$.

Section 3.1

1. **(a)** Since $a \mid b$ and $a \mid c$, there exist integers m and n such that $b = am$ and $c = an$. Hence,

$$\begin{aligned} b - c &= am - an \\ &= a(m - n) \end{aligned}$$

Since $m - n$ is an integer (by the closure properties of the integers), the last equation implies that a divides $b - c$.

(b) What do you need to do in order to prove that n^3 is odd? Notice that if n is an odd integer, then there exists an integer k such that $n = 2k + 1$. Remember that to prove that n^3 is an odd integer, you need to prove that there exists an integer q such that $n^3 = 2q + 1$.

This can also be approached as follows: If n is odd, then by Theorem 1.8, n^2 is odd. Now use the fact that $n^3 = n \cdot n^2$.

(c) If 4 divides $(a - 1)$, then there exists an integer k such that $a - 1 = 4k$ and so $a = 4k + 1$. Use algebra to rewrite $\left(a^2 - 1\right) = (4k + 1)^2 - 1$.

2. **(a)** The natural number $n = 9$ is a counterexample since n is odd, $n > 3$, $n^2 - 1 = 80$ and 3 does not divide 80.

(d) The integer $a = 3$ is a counterexample since $a^2 - 1 = 8$ and $a - 1 = 2$. Since 4 divides 8 and 4 does not divide 2, this is an example where the hypothesis of the conditional statement is true and the conclusion is false.

3. **(b)** This statement is false. One counterexample is $a = 3$ and $b = 2$ since this is an example where the hypothesis is true and the conclusion is false.

(d) This statement is false. One counterexample is $n = 5$. Since $n^2 - 4 = 21$ and $n - 2 = 3$, this is an example where the hypothesis of the conditional statement is true and the conclusion is false.

(e) Make sure you first try some examples. How do you prove that an integer is an odd integer?

(f) The following algebra may be useful.

$$4\,(2m + 1)^2 + 7\,(2m + 1) + 6 = 16m^2 + 30m + 17.$$

(g) This statement is false. One counterexample is $a = 7$, $b = 1$, and $d = 2$. Why is this a counterexample?

4. **(a)** If $xy = 1$, then x and y are both divisors of 1, and the only divisors of 1 are -1 and 1.

(b) Part (a) is useful in proving this.

5. **Another hint**: $(4n + 3) - 2(2n + 1) = 1$.

8. Assuming a and b are both congruent to 2 modulo 3, there exist integers m and n such that $a = 3m + 2$ and $b = 3n + 2$.

For part (a), show that

$$a + b - 1 = 3(m + n + 1).$$

We can then conclude that 3 divides $(a+b)-1$ and this proves that $a + b \equiv 1 \pmod{3}$.

For part (b), show that

$$a \cdot b - 1 = 3(3mn + 2m + 2n + 1).$$

We can then conclude that 3 divides $a \cdot b - 1$ and this proves that $a \cdot b \equiv 1 \pmod{3}$.

11. **(a)** Let $n \in \mathbb{N}$. If a is an integer, then $a - a = 0$ and n divides 0. Therefore, $a \equiv a \pmod{n}$.

(b) Let $n \in \mathbb{N}$, let $a, b \in \mathbb{Z}$ and assume that $a \equiv b \pmod{n}$. We will prove that $b \equiv a \pmod{n}$. Since $a \equiv b \pmod{n}$, n divides $(a - b)$ and so there exists an integer k such that $a - b = nk$. From this, we can show that $b - a = n(-k)$ and so n divides $(b - a)$. Hence, if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$.

12. The assumptions mean that $n \mid (a - b)$ and that $n \mid (c - d)$. Use these divisibility relations to obtain an expression that is equal to a and to obtain an expression that is equal to c. Then use algebra to rewrite the resulting expressions for $a + c$ and $a \cdot c$.

Section 3.2

1. **(a)** Let n be an even integer. Since n is even, there exists an integer k such that $n = 2k$. Now use this to prove that n^3 must be even.

(b) Prove the contrapositive.

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(c) Explain why Parts (a) and (b) prove this.

(d) Explain why Parts (a) and (b) prove this.

2. (a) The contrapositive is, For all integers a and b, if $ab \equiv 0 \pmod 6$, then $a \equiv 0 \pmod 6$ or $b \equiv 0 \pmod 6$.

3. (a) The contrapositive is: For all positive real numbers a and b, if $a = b$, then $\sqrt{ab} = \dfrac{a+b}{2}$.

(b) The statement is true. If $a = b$, then $\dfrac{a+b}{2} = \dfrac{2a}{2} = a$, and $\sqrt{ab} = \sqrt{a^2} = a$. This proves the contrapositive.

4. (a) True. If $a \equiv 2 \pmod 5$, then there exists an integer k such that $a-2 = 5k$. Then,

$$a^2 - 4 = (2+5k)^2 - 4 = 20k + 25k^2.$$

This means that $a^2 - 4 = 5\left(4k + 5k^2\right)$, and hence, $a^2 \equiv 4 \pmod 5$.

(b) False. A counterexample is $a = 3$ since $3^2 \equiv 4 \pmod 5$ and $3 \not\equiv 2 \pmod 5$.

(c) False. Part (b) shows this is false.

6. (a) For each integer a, if $a \equiv 3 \pmod 7$, then $(a^2 + 5a) \equiv 3 \pmod 7$, and for each integer a, if $(a^2+5a) \equiv 3 \pmod 7$, then $a \equiv 3 \pmod 7$.

(b) For each integer a, if $a \equiv 3 \pmod 7$, then $(a^2 + 5a) \equiv 3 \pmod 7$ is true. To prove this, if $a \equiv 3 \pmod 7$, then there exists an integer k such that $a = 3 + 7k$. We can then prove that

$$\left(a^2 + 5a\right) - 3 = 21 + 77k + 49k^2 = 7(3 + 11k + 7k^2).$$

This shows that $(a^2 + 5a) \equiv 3 \pmod 7$.

For each integer a, if $(a^2 + 5a) \equiv 3 \pmod 7$, then $a \equiv 3 \pmod 7$ is false. A counterexample is $a = 6$. When $a = 6$, $a^2 + 5a = 66$ and $66 \equiv 3 \pmod 7$ and $6 \not\equiv 3 \pmod 7$.

(c) Since one of the two conditional statements in Part (b) is false, the given proposition is false.

8. Prove both of the conditional statements: (1) If the area of the right triangle is $c^2/4$, then the right triangle is an isosceles triangle. (2) If the right triangle is an isosceles triange, then the area of the right triangle is $c^2/4$.

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9. The statement is true. It is easier to prove the contrapositive, which is:

For each positive real number x, if $\sqrt{x}$ is rational, then x is rational.

Let x be a positive real number. If there exist positive integers m and n such that $\sqrt{x} = \dfrac{m}{n}$, then $x = \dfrac{m^2}{n^2}$.

10. Remember that there are two conditional statements associated with this biconditional statement. Be willing to consider the contrapositive of one of these conditional statements.

15. Define an appropriate function and use the Intermediate Value Theorem.

17. **(b)** Since 4 divides a, there exist an integer n such that $a = 4n$. Using this, we see that $b^3 = 16n^2$. This means that b^3 is even and hence by Exercise (1), b is even. So there exists an integer m such that $b = 2m$. Use this to prove that m^3 must be even and hence by Exercise (1), m is even.

18. It may be necessary to factor a sum of cubes. Recall that

$$u^3 + v^3 = (u + v)(u^2 - uv + v^2).$$

Section 3.3

1. **(a)** $P \vee C$

2. **(a)** This statement is true. Use a proof by contradiction. So assume that there exist integers a and b such that a is even, b is odd, and 4 divides $a^2 + b^2$. So there exist integers m and n such that

$$a = 2m \qquad \text{and} \qquad a^2 + b^2 = 4n.$$

Substitute $a = 2m$ into the second equation and use algebra to rewrite in the form $b^2 = 4(n - m^2)$. This means that b^2 is even and hence, that b is even. This is a contradiction to the assumption that b is odd.

(b) This statement is true. Use a proof by contradiction. So assume that there exist integers a and b such that a is even, b is odd, and 6 divides $a^2 + b^2$. So there exist integers m and n such that

$$a = 2m \qquad \text{and} \qquad a^2 + b^2 = 6n.$$

Substitute $a = 2m$ into the second equation and use algebra to rewrite in the form $b^2 = 2(3n - 2m^2)$. This means that b^2 is even and hence, that b is even. This is a contradiction.

(d) This statement is true. Use a direct proof. Let a and b be integers and assume they are odd. So there exist integers m and n such that

$$a = 2m + 1 \qquad \text{and} \qquad b = 2n + 1.$$

We then see that

$$\begin{aligned} a^2 + 3b^2 &= 4m^2 + 4m + 1 + 12n^2 + 12n + 3 \\ &= 4\left(m^2 + m + 3n^2 + 3n + 1\right). \end{aligned}$$

This shows that 4 divides $a^2 + 3b^2$.

3. **(a)** We would assume that there exists a positive real number r such that $r^2 = 18$ and r is a rational number.

(b) Do not attempt to mimic the proof that the square root of 2 is irrational (Theorem 3.20). You should still use the definition of a rational number but then use the fact that $\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$. So, if we assume that $r = \sqrt{18} = 3\sqrt{2}$ is rational, then $\dfrac{r}{3} = \dfrac{\sqrt{18}}{3}$ is rational since the rational numbers are closed under division. Hence, $\sqrt{2}$ is rational and this is a contradiction to Theorem 3.20.

5. **(a)** Use a proof by contradiction. So, we assume that there exist real numbers x and y such that x is rational, y is irrational, and $x + y$ is rational. Since the rational numbers are closed under addition, this implies that $(x + y) - x$ is a rational number. Since $(x + y) - x = y$, we conclude that y is a rational number and this contradicts the assumption that y is irrational.

(b) Use a proof by contradiction. So, we assume that there exist nonzero real numbers x and y such that x is rational, y is irrational, and xy is rational. Since the rational numbers are closed under division by nonzero rational numbers, this implies that $\dfrac{xy}{x}$ is a rational number. Since $\dfrac{xy}{x} = y$, we conclude that y is a rational number and this contradicts the assumption that y is irrational.

6. **(a)** This statement is false. A counterexample is $x = \sqrt{2}$.

(**b**) This statement is true since the contrapositive is true. The contrapositive is:

For any real number x, if $\sqrt{x}$ is rational, then x is rational.

If there exist integers a and b with $b \neq 0$ such that $\sqrt{x} = \dfrac{a}{b}$, then $x^2 = \dfrac{a^2}{b^2}$ and hence, x^2 is rational.

11. Recall that $\log_2(32)$ is the real number a such that $2^a = 32$. That is, $a = \log_2(32)$ means that $2^a = 32$. If we assume that a is rational, then there exist integers m and n, with $n \neq 0$, such that $a = \dfrac{m}{n}$.

12. **Hint**: The only factors of 7 are $-1, 1, -7$, and 7.

13. (**a**) What happens if you expand $[\sin(\theta) + \cos(\theta)]^2$? Don't forget your trigonometric identities.

14. **Hint**: Three consecutive natural numbers can be represented by $n, n+1$, and $n+2$, where $n \in \mathbb{N}$, or three consecutive natural numbers can be represented by $m-1$, m, and $m+1$, where $m \in \mathbb{N}$.

Section 3.4

1. Use the fact that $n^2 + n = n(n+1)$.

2. Do not use the quadratic formula. Try a proof by contradiction. If there exists a solution of the equation $x^2 + x - u = 0$ that is an integer, then we can conclude that there exists an integer n such that $n^2 + n - u = 0$. Then,

$$u = n\,(n+1)\,.$$

From Exercise (1), we know that $n(n+1)$ is even and hence, u is even. This contradicts the assumption that u is odd.

3. If n is an odd integer, then there exists an integer m such that $n = 2m + 1$. Use two cases: (1) m is even; (2) m is odd. If m is even, then there exists an integer k such that $m = 2k$ and this means that $n = 2(2k)+1$ or $n = 4k+1$. If m is odd, then there exists an integer k such that $m = 2k + 1$. Then $n = 2(2k+1) + 1$ or $n = 4k + 3$.

4. If $a \in \mathbb{Z}$ and $a^2 = a$, then $a(a-1) = 0$. Since the product is equal to zero, at least one of the factors must be zero. In the first case, $a = 0$. In the second case, $a - 1 = 0$ or $a = 1$.

5. **(c)** For all integers a, b, and d with $d \neq 0$, if d divides the product ab, then d divides a or d divides b.

6. **(a)** The statement, for all integers m and n, if 4 divides $\left(m^2 + n^2 - 1\right)$, then m and n are consecutive integers, is false. A counterexample is $m = 2$ and $n = 5$.

The statement, for all integers m and n, if m and n are consecutive integers, then 4 divides $\left(m^2 + n^2 - 1\right)$, is true. To prove this, let $n = m + 1$. Then

$$m^2 + n^2 - 1 = 2m^2 + 2m = 2m(m + 1).$$

We have proven the $m(m + 1)$ is even. (See Exercise (1).) So this can be used to prove that 4 divides $\left(m^2 + n^2 - 1\right)$.

8. Try a proof by contradiction with two cases: a is even or a is odd.

10. **(a)** One way is to use three cases: (i) $x > 0$; (ii) $x = 0$; and $x < 0$. For the first case, $-x < 0$ and $|-x| = -(-x) = x = |x|$.

11. **(a)** For each real number x, $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$.

Section 3.5

2. **(a)** The first case is when $n \equiv 0 \pmod 3$. We can then use Theorem 3.28 to conclude that $n^3 \equiv 0^3 \pmod 3$ or that $n^3 \equiv 0 \pmod 3$. So in this case, $n^3 \equiv n \pmod 3$.

For the second case, $n \equiv 1 \pmod 3$. We can then use Theorem 3.28 to conclude that $n^3 \equiv 1^3 \pmod 3$ or that $n^3 \equiv 1 \pmod 3$. So in this case, $n^3 \equiv n \pmod 3$.

The last case is when $n \equiv 2 \pmod 3$. We then get $n^3 \equiv 2^3 \pmod 3$ or $n^3 \equiv 8 \pmod 3$. Since $8 \equiv 2 \pmod 3$, we can use the transitive property to conclude that $n^3 \equiv 2 \pmod 3$, and so $n^3 \equiv n \pmod 3$.

Since we have proved it in all three cases, we conclude that for each integer n, $n^3 \equiv n \pmod 3$.

(b) Since $n^3 \equiv n \pmod 3$, we use the definition of congruence to conclude that 3 divides $\left(n^3 - n\right)$.

3. Let $n \in \mathbb{N}$. For $a, b \in \mathbb{Z}$, you need to prove that if $a \equiv b \pmod n$, then $b \equiv a \pmod n$. So let $a, b \in \mathbb{Z}$ and assume that $a \equiv b \pmod n$. So

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$n \mid (a-b)$ and there exists an integer k such that $a - b = nk$. Then, $b - a = n(-k)$ and $b \equiv a \pmod{n}$.

4. (**a**) The contrapositive is: For each integer a, if 3 does not divide a, then 3 divides a^2.

(**b**) To prove the contrapositive, let $a \in \mathbb{Z}$ and assume that 3 does not divide a. So using the Division Algorithm, we can consider two cases: (1) There exists a unique integer q such that $a = 3q + 1$. (2) There exists a unique integer q such that $a = 3q + 2$.

For the first case, show that $a^2 = 3\left(3q^2 + 2q\right) + 1$. For the second case, show that $a^2 = 3\left(3q^2 + 4q + 1\right) + 1$. Since the Division Algorithm states that the remainder is unique, this shows that in both cases, the remainder is 1 and so 3 does not divide a^2.

5. (**a**) $a \equiv 0 \pmod{n}$ if and only if $n \mid (a - 0)$.

(**b**) Let $a \in \mathbb{Z}$. Corollary 3.32 tell us that if $a \not\equiv 0 \pmod{3}$, then $a \equiv 1 \pmod{3}$ or $a \equiv 2 \pmod{3}$.

(**c**) Part (b) tells us we can use a proof by cases using the following two cases: (1) $a \equiv 1 \pmod{3}$; (2) $a \equiv 2 \pmod{3}$.

So, if $a \equiv 1 \pmod{3}$, then by Theorem 3.28, $a \cdot a \equiv 1 \cdot 1 \pmod{3}$, and hence, $a^2 \equiv 1 \pmod{3}$.

If $a \equiv 2 \pmod{3}$, then by Theorem 3.28, $a \cdot a \equiv 2 \cdot 2 \pmod{3}$, and hence, $a^2 \equiv 4 \pmod{3}$. Since $4 \equiv 1 \pmod{3}$, this implies that $a^2 \equiv 1 \pmod{3}$.

6. The contrapositive is: Let a and b be integers. If $a \not\equiv 0 \pmod{3}$ and $b \not\equiv 0 \pmod{3}$, then $ab \not\equiv 0 \pmod{3}$.

Using Exercise 5(b), we can use the following four cases:
(1) $a \equiv 1 \pmod{3}$ and $b \equiv 1 \pmod{3}$;
(2) $a \equiv 1 \pmod{3}$ and $b \equiv 2 \pmod{3}$;
(3) $a \equiv 2 \pmod{3}$ and $b \equiv 1 \pmod{3}$;
(4) $a \equiv 2 \pmod{3}$ and $b \equiv 2 \pmod{3}$.
In all four cases, we use Theorem 3.28 to conclude that $ab \not\equiv 0 \pmod{3}$. For example, for the third case, we see that $ab \equiv 2 \cdot 1 \pmod{3}$. That is, $ab \equiv 2 \pmod{3}$.

7. (**a**) This follows from Exercise (5) and the fact that $3 \mid k$ if and only if $k \equiv 0 \pmod{3}$.

(**b**) This follows directly from Part (a) using $a = b$.

8. **(a)** Use a proof similar to the proof of Theorem 3.20. The result of Exercise (7) may be helpful.

9. The result in Part (c) of Exercise (5) may be helpful in a proof by contradiction.

10. **(b)** Factor $n^3 - n$.

(c) Consider using cases based on congruence modulo 6.

Section 4.1

1. The sets in Parts (a) and (b) are inductive.

2. A finite nonempty set is not inductive (why?) but the empty set is inductive (why?).

3. **(a)** For each $n \in \mathbb{N}$, let $P(n)$ be, $2+5+8+\cdots+(3n-1) = \dfrac{n(3n+1)}{2}$. When we use $n = 1$, the summation on the left side of the equation is 2, and the right side is $\dfrac{1(3 \cdot 1 + 1)}{2} = 2$. Therefore, $P(1)$ is true. For the inductive step, let $k \in \mathbb{N}$ and assume that $P(k)$ is true. Then,

$$2 + 5 + 8 + \cdots + (3k - 1) = \frac{k\,(3k+1)}{2}.$$

We now add $3\,(k+1) - 1$ to both sides of this equation. This gives

$$\begin{aligned} 2 + 5 + 8 + \cdots + (3k - 1) & \\ +3\,(k+1) - 1 &= \frac{k\,(3k+1)}{2} + (3\,(k+1) - 1) \\ &= \frac{k\,(3k+1)}{2} + (3k+2) \end{aligned}$$

If we now combine the terms on the right side of the equation into a

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single fraction, we obtain

$$\begin{aligned} 2+5+\cdots+(3k-1)+3(k+1)-1 &= \frac{k(3k+1)+6k+4}{2} \\ &= \frac{3k^2+7k+4}{2} \\ &= \frac{(k+1)(3k+4)}{2} \\ &= \frac{(k+1)(3(k+1)+1)}{2} \end{aligned}$$

This proves that if $P(k)$ is true, then $P(k+1)$ is true.

6. The conjecture is that for each $n \in \mathbb{N}$, $1+3+5+\cdots+(2n-1) = n^2$. The key to the inductive step is that

$$\begin{aligned} 1+3+5+\cdots+(2k-1) & \\ +(2(k+1)-1) &= [1+3+5+\cdots+(2k-1)] + (2(k+1)-1) \\ &= k^2 + (2k+1) \\ &= (k+1)^2 \end{aligned}$$

7. **(e)** For each natural number n, $4^n \equiv 1 \pmod 3$.

(f) For each natural number n, let $P(n)$ be, "$4^n \equiv 1 \pmod 3$." Since $4^1 \equiv 1 \pmod 3$, we see that $P(1)$ is true. Now let $k \in \mathbb{N}$ and assume that $P(k)$ is true. That is, assume that

$$4^k \equiv 1 \pmod 3.$$

Multiplying both sides of this congruence by 4 gives

$$4^{k+1} \equiv 4 \pmod 3.$$

However, $4 \equiv 1 \pmod 3$ and so by using the transitive property of congruence, we see that $4^{k+1} \equiv 1 \pmod 3$. This proves that if $P(k)$ is true, then $P(k+1)$ is true.

8. **(a)** The key to the inductive step is that if $4^k = 1 + 3m$, then $4^k \cdot 4 = 4(1+3m)$, which implies that

$$4^{k+1} - 1 = 3(1+4m).$$

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13. Let k be a natural number. If $a^k \equiv b^k \pmod{n}$, then since we are also assuming that $a \equiv b \pmod{n}$, we can use Part (2) of Theorem 3.28 to conclude that $a \cdot a^k \equiv b \cdot b^k \pmod{n}$.

14. Three consecutive natural numbers may be represent by n, $n+1$, and $n+2$, where n is a natural number. For the inductive step, think before you try to do a lot of algebra. You should be able to complete a proof of the inductive step by expanding the cube of only one expression.

Section 4.2

1. **(a)** Let $P(n)$ be, "$3^n > 1 + 2^n$." $P(2)$ is true since $3^2 = 9$, $1 + 2^2 = 5$, and $9 > 5$.

For the inductive step, we assume that $P(k)$ is true and so

$$3^k > 1 + 2^k. \tag{1}$$

To prove that $P(k+1)$ is true, we must prove that $3^{k+1} > 1 + 2^{k+1}$. Multiplying both sides inequality (1) by 3 gives

$$3^{k+1} > 3 + 3 \cdot 2^k.$$

Now, since $3 > 1$ and $3 \cdot 2^k > 2^{k+1}$, we see that $3 + 3 \cdot 2^k > 1 + 2^{k+1}$ and hence, $3^{k+1} > 1 + 2^{k+1}$. Thus, if $P(k)$ is true, then $P(k+1)$ is true. This proves the inductive step.

2. If $n \geq 5$, then $n^2 < 2^n$. To prove this, we let $P(n)$ be $n^2 < 2^n$. For the basis step, when $n = 5$, $n^2 = 25$, $2^n = 32$, and $25 < 32$. For the inductive step, we assume that $k \geq 5$ and that $P(k)$ is true or that $k^2 < 2^k$. With these assumptions, we need to prove that $P(k+1)$ is true or that $(k+1)^2 < 2^{k+1}$. We first note that

$$(k+1)^2 = k^2 + 2k + 1 < 2^k + 2k + 1. \tag{1}$$

Since $k \geq 5$, we see that $5k < k^2$ and so $2k + 3k < k^2$. However, $3k > 1$ and so $2k + 1 < 2k + 3k < k^2$. Combinining this with inequality (1), we obtain $(k+1)^2 < 2^k + k^2$. Using the assumption that $P(k)$ is true $\left(k^2 < 2^k\right)$, we obtain

$$(k+1)^2 < 2^k + 2^k = 2 \cdot 2^k$$
$$(k+1)^2 < 2^{k+1}$$

This proves that if $P(k)$ is true, then $P(k+1)$ is true.

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5. Let $P(n)$ be the predicate, "$8^n \mid (4n)!$." Verify that $P(0)$, $P(1)$, $P(2)$, and $P(3)$ are true. For the inductive step, the following fact about factorials may be useful:

$$\begin{aligned}[4(k+1)]! &= (4k+4)! \\ &= (4k+4)(4k+3)(4k+2)(4k+1)(4k)!.\end{aligned}$$

8. Let $P(n)$ be, "The natural number n can be written as a sum of natural numbers, each of which is a 2 or a 3." Verify that $P(4)$, $P(5)$, $P(6)$, and $P(7)$ are true.

To use the Second Principle of Mathematical Induction, assume that $k \in \mathbb{N}$, $k \geq 5$ and that $P(4), P(5), \ldots, P(k)$ are true. Then notice that

$$k + 1 = (k - 1) + 2.$$

Since $k - 1 \geq 4$, we have assumed that $P(k-1)$ is true. This means that $(k-1)$ can be written as a sum of natural numbers, each of which is a 2 or a 3. Since $k + 1 = (k-1) + 2$, we can conclude that $(k+1)$ can be written as a sum of natural numbers, each of which is a 2 or a 3. This completes the proof of the inductive step.

12. Let $P(n)$ be, "Any set with n elements has $\dfrac{n(n-1)}{2}$ 2-element subsets." $P(1)$ is true since any set with only one element has no 2-element subsets.

Let $k \in \mathbb{N}$ and assume that $P(k)$ is true. This means that any set with k elements has $\dfrac{k(k-1)}{2}$ 2-element subsets. Let A be a set with $k+1$ elements, and let $x \in A$. Now use the inductive hypothesis on the set $A - \{x\}$, and determine how the 2-element subsets of A are related to the set $A - \{x\}$.

16. **(a)** Use Theorem 4.9.

(b) Assume $k \neq q$ and consider two cases: (i) $k < q$; (ii) $k > q$.

Section 4.3

1. Let $P(n)$ be $a_n = n!$. Since $a_0 = 1$ and $0! = 1$, we see that $P(0)$ is true. For the inductive step, we assume that $k \in \mathbb{N} \cup \{0\}$ and that $P(k)$ is true or

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that $a_k = k!$.

$$\begin{aligned} a_{k+1} &= (k+1)\,a_k \\ &= (k+1)\,k! \\ &= (k+1)!. \end{aligned}$$

This proves the inductive step that if $P(k)$ is true, then $P(k+1)$ is true.

2. **(a)** Let $P(n)$ be, "f_{4n} is a multiple of 3." Since $f_4 = 3$, $P(1)$ is true. If $P(k)$ is true, then there exists an integer m such that $f_{4k} = 3m$. We now need to prove that $P(k+1)$ is true or that $f_{4(k+1)}$ is a multiple of 3. We use the following:

$$\begin{aligned} f_{4(k+1)} &= f_{4k+4} \\ &= f_{4k+3} + f_{4k+2} \\ &= (f_{4k+2} + f_{4k+1}) + (f_{4k+1} + f_{4k}) \\ &= f_{4k+2} + 2f_{4k+1} + f_{4k} \\ &= (f_{4k+1} + f_{4k}) + 2f_{4k+1} + f_{4k} \\ &= 3f_{4k+1} + 2f_{4k} \end{aligned}$$

We now use the assumption that $f_{4k} = 3m$ and the last equation to obtain $f_{4(k+1)} = 3f_{4k+1} + 2 \cdot 3m$ and hence, $f_{4(k+1)} = 3\,(f_{4k+1} + 2m)$. Therefore, $f_{4(k+1)}$ is a multiple of 3 and this completes the proof of the inductive step.

(c) Let $P(n)$ be, "$f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - 1$." Since $f_1 = f_3 - 1$, $P(2)$ is true. For $k \geq 2$, if $P(k)$ is true, then $f_1 + f_2 + \cdots + f_{k-1} = f_{k+1} - 1$. Then

$$\begin{aligned} (f_1 + f_2 + \cdots + f_{k-1}) + f_k &= (f_{k+1} - 1) + f_k \\ &= (f_{k+1} + f_k) - 1 \\ &= f_{k+2} - 1. \end{aligned}$$

This proves that if $P\,(k)$ is true, then $P\,(k+1)$ is true.

(f) Let $P(n)$ be, "$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$." For the basis step, we notice that $f_1^2 = 1$ and $f_1 \cdot f_2 = 1$ and hence, $P(1)$ is true. For the inductive step, we need to prove that if $P(k)$ is true, then $P(k+1)$ is true. That is, we need to prove that if $f_1^2 + f_2^2 + \cdots + f_k^2 = f_k f_{k+1}$,

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then $f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 = f_{k+1} f_{k+2}$. To do this, we can use

$$\begin{aligned}\left(f_1^2 + f_2^2 + \cdots + f_k^2\right) + f_{k+1}^2 &= f_k f_{k+1} + f_{k+1}^2 \\ f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 &= f_{k+1}\left(f_k + f_{k+1}\right) \\ &= f_{k+1} f_{k+2}.\end{aligned}$$

6. For the inductive step, if $a_k = a \cdot r^{k-1}$, then

$$\begin{aligned}a_{k+1} &= r \cdot a_k \\ &= r\left(a \cdot r^{k-1}\right) \\ &= a \cdot r^k.\end{aligned}$$

8. For the inductive step, use the assumption that $S_k = a\left(\dfrac{1-r^k}{1-r}\right)$ and the recursive definiton to write $S_{k+1} = a + r \cdot S_k$.

9. **(a)** $a_2 = 7, a_3 = 12, a_4 = 17, a_5 = 22, a_6 = 27$.

(b) One possibility is: For each $n \in \mathbb{N}$, $a_n = 2 + 5(n-1)$.

12. **(a)** $a_2 = \sqrt{6}$, $a_3 = \sqrt{\sqrt{6}+5} \approx 2.729$, $a_4 \approx 2.780$, $a_5 \approx 2.789$, $a_6 \approx 2.791$

(b) Let $P(n)$ be, "$a_n < 3$." Since $a_1 = 1$, $P(1)$ is true. For $k \in \mathbb{N}$, if $P(k)$ is true, then $a_k < 3$. Now

$$a_{k+1} = \sqrt{5 + a_k}.$$

Since $a_k < 3$, this implies that $a_{k+1} < \sqrt{8}$ and hence, $a_{k+1} < 3$. This proves that if $P(k)$ is true, then $P(k+1)$ is true.

13. **(a)** $a_3 = 7, a_4 = 15, a_5 = 31, a_6 = 63$

(b) Think in terms of powers of 2.

14. **(a)** $a_3 = \dfrac{3}{2}, a_4 = \dfrac{7}{4}, a_5 = \dfrac{37}{24}, a_6 = \dfrac{451}{336}$

16. **(b)**

$a_2 = 5$	$a_5 = 719$	$a_8 = 362879$
$a_3 = 23$	$a_6 = 5039$	$a_9 = 3628799$
$a_4 = 119$	$a_7 = 40319$	$a_{10} = 39916799$

18. **(a)** Let $P(n)$ be, "$L_n = 2f_{n+1} - f_n$." First, verify that $P(1)$ and $P(2)$ are true. Now let k be a natural number with $k \geq 2$ and assume that $P(1), P(2), \ldots, P(k)$ are all true. Since $P(k)$ and $P(k-1)$ are both assumed to be true, we can use them to help prove that $P(k+1)$ must then be true as follows:

$$\begin{aligned} L_{k+1} &= L_k + L_{k-1} \\ &= (2f_{k+1} - f_k) + (2f_k - f_{k-1}) \\ &= 2\,(f_{k+1} + f_k) - (f_k + f_{k-1}) \\ &= 2f_{k+2} - f_{k+1}. \end{aligned}$$

Section 5.1

1. **(a)** $A = B$ **(b)** $A \subseteq B$ **(c)** $C \neq D$ **(d)** $C \subseteq D$ **(e)** $A \not\subseteq D$

2. In both cases, the two sets have preceisely the same elements.

3.

A	$\subset, \subseteq, \neq$	B	$\emptyset$	$\subset, \subseteq, \neq$	A
5	$\in$	C	$\{5\}$	$\subset, \subseteq, \neq$	C
A	$\subset, \subseteq, \neq$	C	$\{1,2\}$	$\subset, \subseteq, \neq$	B
$\{1,2\}$	$\not\subseteq, \neq$	A	$\{3,2,1\}$	$\subset, \subseteq, \neq$	D
4	$\notin$	B	D	$\not\subseteq, \neq$	$\emptyset$
$\text{card}\,(A)$	$=$	$\text{card}\,(D)$	$\text{card}\,(A)$	$\neq$	$\text{card}\,(B)$
A	$\in$	$\mathcal{P}\,(A)$	A	$\in$	$\mathcal{P}\,(B)$

4. $\mathbb{N} \subset \mathbb{Z}$ $\mathbb{N} \subset \mathbb{Q}$ $\mathbb{N} \subset \mathbb{R}$
$\mathbb{Z} \subset \mathbb{Q}$ $\mathbb{Z} \subset \mathbb{R}$ $\mathbb{Q} \subset \mathbb{R}$

5. **(a)** The set $\{a, b\}$ is a not a subset of $\{a, c, d, e\}$ since $b \in \{a, b\}$ and $b \notin \{a, c, d, e\}$.

(b) $\{-2, 0, 2\} = \left\{x \in \mathbb{Z} \mid x \text{ is even and } x^2 < 5\right\}$ since both sets have precisely the same elements.

(c) $\emptyset \subseteq \{1\}$ since the following statement is true: For every $x \in U$, if $x \in \emptyset$, then $x \in \{1\}$.

(d) The statement is false. The set $\{a\}$ is an element of $\mathcal{P}(A)$.

6. **(a)** $x \notin A \cap B$ if and only if $x \notin A$ or $x \notin B$.

7. **(a)** $A \cap B = \{5, 7\}$

(b) $A \cup B = \{1, 3, 4, 5, 6, 7, 9\}$

(c) $(A \cup B)^c = \{2, 8, 10\}$

(d) $A^c \cap B^c = \{2, 8, 10\}$

(e) $(A \cup B) \cap C = \{3, 6, 9\}$

(f) $A \cap C = \{3, 6\}$

(g) $B \cap C = \{9\}$

(h) $(A \cap C) \cup (B \cap C) = \{3, 6, 9\}$

(i) $B \cap D = \emptyset$

(j) $(B \cap D)^c = U$

(k) $A - D = \{3, 5, 7\}$

(l) $B - D = \{1, 5, 7, 9\}$

(m) $(A - D) \cup (B - D) = \{1, 3, 5, 7, 9\}$

(n) $(A \cup B) - D = \{1, 3, 5, 7, 9\}$

9. **(b)** There exists an $x \in U$ such that $x \in (P - Q)$ and $x \notin (R \cap S)$. This can be written as, There exists an $x \in U$ such that $x \in P$, $x \notin Q$, and $x \notin R$ or $x \notin S$.

10. **(a)** The given statement is a conditional statement. We can rewrite the subset relations in terms of conditional sentences: $A \subseteq B$ means, "For all $x \in U$, if $x \in A$, then $x \in B$," and $B^c \subseteq A^c$ means, "For all $x \in U$, if $x \in B^c$, then $x \in A^c$."

Section 5.2

1. **(a)** The set A is a subset of B. To prove this, we let $x \in A$. Then $-2 < x < 2$. Since $x < 2$, we conclude that $x \in B$ and hence, we have proved that A is a subset of B.

(b) The set B is not a subset of A. There are many examples of a real number that is in B but not in A. For example, -3 is in A, but -3 is not in B.

3. **(a)** $A = \{\ldots, -9, -1, 7, 15, 23, \ldots\}$ and $B = \{\ldots, -9, -5, -1, 3, 7, 11, 15, \ldots\}$.

(b) To prove that $A \subseteq B$, let $x \in A$. Then, $x \equiv 7 \pmod 8$ and so, $8 \mid (x - 7)$. This means that there exists an integer m such that $x - 7 = 8m$. By adding 4 to both sides of this equation, we see that $x - 3 =$

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$8m+4$, or $x-3 = 4(2m+1)$. From this, we conclude that $4 \mid (x-3)$ and that $x \equiv 3 \pmod 4$. Hence, $x \in B$.

(c) $B \not\subseteq A$. For example, $3 \in B$ and $3 \notin A$.

5. **(a)** We will prove that $A = B$. Notice that if $x \in A$, then there exists an integer m such that $x - 2 = 3m$. We can use this equation to see that $2x - 4 = 6m$ and so 6 divides $(2x-4)$. Therefore, $x \in B$ and hence, $A \subseteq B$.

Conversely, if $y \in B$, then there exists an integer m such that $2y - 4 = 6m$. Hence, $y - 2 = 3m$, which implies that $y \equiv 2 \pmod 3$ and $y \in A$. Therefore, $B \subseteq A$.

(c) $A \cap B = \emptyset$. To prove this, we will use a proof by contradiction and assume that $A \cap B \neq \emptyset$. So there exists an x in $A \cap B$. We can then conclude that there exist integers m and n such that $x - 1 = 5m$ and $x - y = 10n$. So $x = 5m + 1$ and $x = 10n + 7$. We then see that

$$\begin{aligned} 5m + 1 &= 10n + 7 \\ 5(m - 2n) &= 6 \end{aligned}$$

The last equation implies that 5 divides 6, and this is a contradiction.

7. **(a)** Let $x \in A \cap B$. Then, $x \in A$ and $x \in B$. This proves that if $x \in A \cap B$, then $x \in A$, and hence, $A \cap B \subseteq A$.

(b) Let $x \in A$. Then, the statement "$x \in A$ or $x \in B$" is true. Hence, $A \subseteq A \cup B$.

(e) By Theorem 5.1, $\emptyset \subseteq A \cap \emptyset$. By Part (a), $A \cap \emptyset \subseteq \emptyset$. Therefore, $A \cap \emptyset = \emptyset$.

10. Start with, "Let $x \in A$." Then use the assumption that $A \cap B^c = \emptyset$ to prove that x must be in B.

12. **(a)** Let $x \in A \cap C$. Then $x \in A$ and $x \in C$. Since we are assuming that $A \subseteq B$, we see that $x \in B$ and $x \in C$. This proves that $A \cap C \subseteq B \cap C$.

15. **(a)** This is Proposition 5.14. (See Exercise 10 .)

(b) To prove "If $A \subseteq B$, then $A \cup B = B$," first note that if $x \in B$, then $x \in A \cup B$ and, hence, $B \subseteq A \cup B$. Now let $x \in A \cup B$ and note that since $A \subseteq B$, if $x \in A$, then $x \in B$. Use this to argue that under the assumption that $A \subseteq B$, $A \cup B \subseteq B$.

To prove "If $A \cup B = B$, then $A \subseteq B$," start with, Let $x \in A$ and use this assumption to prove that x must be an element of B.

Section 5.3

1. **(a)** Let $x \in (A^c)^c$. Then $x \notin A^c$, which means $x \in A$. Hence, $(A^c)^c \subseteq A$. Now let $y \in A$. Then, $y \notin A^c$ and hence, $y \in (A^c)^c$. Therefore, $A \subseteq (A^c)^c$.

(c) Let $x \in U$. Then $x \notin \emptyset$ and so $x \in \emptyset^c$. Therefore, $U \subseteq \emptyset^c$. Also, since every set we deal with is a subset of the universal set, $\emptyset^c \subseteq U$.

2. To prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, we let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. So we will use two cases: (1) $x \in B$; (2) $x \in C$.

In Case (1), $x \in A \cap B$ and, hence, $x \in (A \cap B) \cup (A \cap C)$. In Case (2), $x \in A \cap C$ and, hence, $x \in (A \cap B) \cup (A \cap C)$. This proves that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

To prove that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, let $y \in (A \cap B) \cup (A \cap C)$. Then, $y \in A \cap B$ or $y \in A \cap C$. If $y \in A \cap B$, then $y \in A$ and $y \in B$. Therefore, $y \in A$ and $y \in B \cup C$. So, we may conclude that $y \in A \cap (B \cup C)$. In a similar manner, we can prove that if $y \in A \cap C$, then $y \in A \cap (B \cup C)$. This proves that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, and hence that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. **(a)** $A - (B \cup C) = (A - B) \cap (A - C)$.

(c) Using the algebra of sets, we obtain

$$\begin{aligned}(A - B) \cap (A - C) &= (A \cap B^c) \cap (A \cap C^c)\\ &= (A \cap A) \cap (B^c \cap C^c)\\ &= A \cap (B \cup C)^c\\ &= A - (B \cup C).\end{aligned}$$

6. **(a)** Using the algebra of sets, we see that

$$\begin{aligned}(A - C) \cap (B - C) &= (A \cap C^c) \cap (B \cap C^c)\\ &= (A \cap B) \cap (C^c \cap C^c)\\ &= (A \cap B) \cap C^c\\ &= (A \cap B) - C.\end{aligned}$$

9. (a) Use a proof by contradiction. Assume the sets are not disjoint and let $x \in A \cap (B - A)$. Then $x \in A$ and $x \in B - A$, which implies that $x \notin A$.

Section 5.4

1. (a) $A \times B = \{(1, a), (1, b), (1, c), (1, d), (2, a), (2, b), (2, c), (2, d)\}$.
   (b) $B \times A = \{(a, 1), (b, 1), (c, 1), (d, 1), (a, 2), (b, 2), (c, 2), (d, 2)\}$.
   (c) $A \times C = \{(1, 1), (1, a), (1, b), (2, 1), (2, a), (2, b)\}$.
   (d) $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.
   (e) $A \times (B \cap C) = \{(1, a), (1, b), (2, a), (2, b)\}$.
   (f) $(A \times B) \cap (A \times C) = \{(1, a), (1, b), (2, a), (2, b)\}$.
   (g) $A \times \emptyset = \emptyset$.
   (h) $B \times \{2\} = \{(a, 2), (b, 2), (c, 2), (d, 2)\}$.

3. Let $u \in A \times (B \cap C)$. Then, there exists $x \in A$ and there exists $y \in B \cap C$ such that $u = (x, y)$. Since $y \in B \cap C$, we know that $y \in B$ and $y \in C$. So, we have:

$$u = (x, y) \in A \times B \text{ and } u = (x, y) \in A \times C.$$

Hence, $u \in (A \times B) \cap (A \times C)$ and this proves that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

Now let $v \in (A \times B) \cap (A \times C)$. Then, $v \in A \times B$ and $v \in A \times C$. So, there exists an s in A and a t in B such that $v = (s, t)$. But, since v is also in $A \times C$, we see that t must also be in C. Thus, $t \in B \cap C$ and so, $v \in A \times (B \cap C)$. This proves that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

4. Let $u \in (A \cup B) \times C$. Then, there exists $x \in A \cup B$ and there exists $y \in C$ such that $u = (x, y)$. Since $x \in A \cup B$, we know that $x \in A$ or $x \in B$. In the case where $x \in A$, we see that $u \in A \times C$, and in the case where $x \in B$, we see that $u \in B \times C$. Hence, $u \in (A \times C) \cup (B \times C)$. This proves that $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$.

We still need to prove that $(A \times C) \cup (B \times C) \subseteq A \times (B \cup C)$.

Section 5.5

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1. **(a)** $\{3, 4\}$ **(d)** $\{3, 4, 5, 6, 7, 8, 9, 10\}$

2. **(a)** $\{5, 6, 7, \ldots\}$ **(d)** $\{1, 2, 3, 4\}$

(c) $\emptyset$ **(f)** $\emptyset$

3. **(a)** $\{x \in \mathbb{R} \mid -100 \le x \le 100\}$ **(b)** $\{x \in \mathbb{R} \mid -1 \le x \le 1\}$

4. **(a)** We let $\beta \in \Lambda$ and let $x \in A_\beta$. Then $x \in A_\alpha$, for at least one $\alpha \in \Lambda$ and, hence, $x \in \bigcup_{\alpha \in \Lambda} A_\alpha$. This proves that $A_\beta \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha$.

5. **(a)** We first let $x \in B \cap \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right)$. Then $x \in B$ and $x \in \bigcup_{\alpha \in \Lambda} A_\alpha$. This means that there exists an $\alpha \in \Lambda$ such that $x \in A_\alpha$. Hence, $x \in B \cap A_\alpha$, which implies that $x \in \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha)$. This proves that $B \cap \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right) \subseteq \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha)$.

We now let $y \in \bigcup_{\alpha \in \Lambda} (B \cap A_\alpha)$. So there exists an $\alpha \in \Lambda$ such that $y \in B \cap A_\alpha$. Then $y \in B$ and $y \in A_\alpha$, which implies that $y \in B$ and $y \in \bigcup_{\alpha \in \Lambda} A_\alpha$. Therefore, $y \in B \cap \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right)$, and this proves that $\bigcup_{\alpha \in \Lambda} (B \cap A_\alpha) \subseteq B \cap \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right)$.

8. **(a)** Let $x \in B$. For each $\alpha \in \Lambda$, $B \subseteq A_\alpha$ and, hence, $x \in A_\alpha$. This means that for each $\alpha \in \Lambda$, $x \in A_\alpha$ and, hence, $x \in \bigcap_{\alpha \in \Lambda} A_\alpha$. Therefore, $B \subseteq \bigcap_{\alpha \in \Lambda} A_\alpha$.

12. **(a)** We first rewrite the set difference and then use a distributive law.

$$\begin{aligned}
\left(\bigcup_{\alpha \in \Lambda} A_\alpha \right) - B &= \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right) \cap B^c \\
&= \bigcup_{\alpha \in \Lambda} \left(A_\alpha \cap B^c \right) \\
&= \bigcup_{\alpha \in \Lambda} \left(A_\alpha - B \right)
\end{aligned}$$

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Section 6.1

1. **(a)** $f(-3) = 15$, $f(-1) = 3$, $f(1) = -1$, $f(3) = 3$.

(b) The set of preimages of 0 is $\{0, 2\}$. The set of preimages of 4 is $\left\{\dfrac{2-\sqrt{20}}{2}, \dfrac{2+\sqrt{20}}{2}\right\}$. (Use the quadratic formula.)

(d) $\text{range}(f) = \{y \in \mathbb{R} \mid y \geq -1\}$

3. **(a)** $f(-7) = 10$, $f(-3) = 6$, $f(3) = 0$, $f(7) = -4$.

(b) The set of preimages of 5 is $\{-2\}$. There set of preimages of 4 is $\{-1\}$.

(c) $\text{range}(f) = \mathbb{Z}$. Notice that for all $y \in \mathbb{Z}$ (codomain), $f(3-y) = y$ and $(3-y) \in \mathbb{Z}$ (domain).

4. **(b)** The set of preimages of 5 is $\{2\}$. There set of preimages of 4 is $\emptyset$.

(c) The range of the function f is the set of all odd integers.

(d) The graph of the function f consists of an infinite set of discrete points.

5. **(b)** $\text{dom}(F) = \left\{x \in \mathbb{R} \mid x > \dfrac{1}{2}\right\}$, $\text{range}(F) = \mathbb{R}$

(d) $\text{dom}(g) = \{x \in \mathbb{R} \mid x \neq 2 \text{ and } x \neq -2\}$,
$\text{range}(g) = \{y \in \mathbb{R} \mid y > 0\} \cup \{y \in \mathbb{R} \mid y \leq -1\}$

6. **(a)** $d(1) = 1$, $d(2) = 2$, $d(3) = 2$, $d(4) = 3$, $d(5) = 2$, $d(6) = 4$, $d(7) = 2$, $d(8) = 4$, $d(9) = 3$, $d(10) = 4$, $d(11) = 2$, $d(12) = 6$.

(b) There is no natural number n other than 1 such that $d(n) = 1$ since every natural number greater than one has at least two divisors. The set of preimages of 1 is $\{1\}$.

(c) The only natural numbers n such that $d(n) = 2$ are the prime numbers. The set of preimages of the natural number 2 is the set of prime numbers.

(d) The statement is false. A counterexample is $m = 2$ and $n = 3$ since $d(2) = 2$ and $d(3) = 2$.

(e) $d\left(2^0\right) = 1$, $d\left(2^1\right) = 2$, $d\left(2^2\right) = 3$, $d\left(2^3\right) = 4$, $d\left(2^4\right) = 5$,
$d\left(2^5\right) = 6$, and $d\left(2^6\right) = 7$.

(f) For each nonnegative integer n, the divisors of 2^n are $2^0, 2^1, \ldots, 2^{n-1}$, and 2^n. This is a list of $n+1$ natural numbers and so $d(2^n) = n + 1$.

(g) The statement is true. To prove this, let n be a natural number. Then $2^{n-1} \in \mathbb{N}$ and $d\left(2^{n-1}\right) = (n-1) + 1 = n$.

7. **(a)** The domain of S is $\mathbb{N}$. The power set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, can be the codomain. The rule for determining outputs is that for each $n \in \mathbb{N}$, $S(n)$ is the set of all distinct natural number factors of n.

(b) For example, $S(8) = \{1, 2, 4, 8\}$, $S(15) = \{1, 3, 5, 15\}$.

(c) For example, $S(2) = \{1, 2\}$, $S(3) = \{1, 3\}$, $S(31) = \{1, 31\}$.

Section 6.2

1. **(a)** $f(0) = 4,\ f(1) = 0,\ f(2) = 3,\ f(3) = 3,\ f(4) = 0$

(b) $g(0) = 4, g(1) = 0, g(2) = 3, g(3) = 3, g(4) = 0$

(c) The two functions are equal.

3. **(a)** $f(2) = 9,\ f(-2) = 9,\ f(3) = 14,\ f(\sqrt{2} = 7$

(b) $g(0) = 5, g(2) = 9, g(-2) = 9, g(3) = 14, g(\sqrt{2}) = 7$

(c) The function f is not equal to the function g since they do not have the same domain.

(d) The function h is equal to the function f since if $x \neq 0$, then $\dfrac{x^2 + 5x}{x} = x^2 + 5$.

4. **(a)** $\langle a_n \rangle$, where $a_n = \dfrac{1}{n^2}$ for each $n \in \mathbb{N}$. The domain is $\mathbb{N}$, and $\mathbb{Q}$ can be the codomain.

(d) $\langle a_n \rangle$, where $a_n = \cos(n\pi)$ for each $n \in \mathbb{N} \cup 0$. The domain is $\mathbb{N} \cup 0$, and $\{-1, 1\}$ can be the codomain. This sequence is equal to the sequence in Part (c).

5. **(a)** $p_1(1, x) = 1,\ p_1(1, y) = 1,\ p_1(1, z) = 1,\ p_1(2, x) = 2,\ p_1(2, y) = 2,\ p_1(2, z) = 2$

(c) $\text{range}(p_1) = A$, $\text{range}(p_2) = B$

6. Start of the inductive step: Let $P(n)$ be "A convex polygon with n sides has $\dfrac{n(n-3)}{2}$ diagonals." Let $k \in D$ and assume that $P(k)$ is true, that is, a convex polygon with k sides has $\dfrac{k(k-3)}{2}$ diagonals. Now let Q be convex

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polygon with $(k+1)$ sides. Let v be one of the $(k+1)$ vertices of Q and let u and w be the two vertices adjacent to v. By drawing the line segment from u to w and omitting the vertex v, we form a convex polygon with k sides. Now complete the inductive step.

7. **(a)** $f(-3,4) = 9$, $f(-2,-7) = -23$

(b) $\text{range}(f) = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid m = 4 - 3n\}$

8. **(a)** $g(3,5) = (6,-2)$, $g(-1,4) = (-2,-5)$.

(c) The set of preimages of $(8,-3)$ is $\{(4,7)\}$.

9. **(a)** $\det\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} = -17$, $\det\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} = 7$, and $\det\begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix} = 10$

Section 6.3

2. **(a)** Notice that $f(0) = 4$, $f(1) = 0$, $f(2) = 3$, $f(3) = 3$, and $f(4) = 0$. So the function f is not an injection and is not a surjection.

(c) Notice that $F(0) = 4$, $F(1) = 0$, $F(2) = 2$, $F(3) = 1$, and $F(4) = 3$. So the function F is an injection and is a surjection.

3. **(a)** The function f is an injection. To prove this, let $x_1, x_2 \in \mathbb{Z}$ and assume that $f(x_1) = f(x_2)$. Then,

$$\begin{aligned} 3x_1 + 1 &= 3x_2 + 1 \\ 3x_1 &= 3x_2 \\ x_1 &= x_2. \end{aligned}$$

Hence, f is an injection. Now, for each $x \in \mathbb{Z}$, $3x + 1 \equiv 1 \pmod 3$, and hence $f(x) \equiv 1 \pmod 3$. This means that there is no integer x such that $f(x) = 0$. Therefore, f is not a surjection.

(b) The proof that F is an injection is similar to the proof in Part (a) that f is an injection. To prove that F is a surjection, let $y \in \mathbb{Q}$. Then, $\dfrac{y-1}{3} \in \mathbb{Q}$ and $F\left(\dfrac{y-1}{3}\right) = y$ and hence, F is a surjection.

(h) Since $h(1) = h(4)$, the function h is not an injection. Using calculus, we can see that the function h has a maximum when $x = 2$ and a minimum when $x = -2$, and so for each $x \in \mathbb{R}$, $h(-2) \leq h(x) \leq h(2)$ or

$$-\frac{1}{2} \leq h(x) \leq \frac{1}{2}.$$

This can be used to prove that h is not a surjection.

We can also prove that there is no $x \in \mathbb{R}$ such that $h(x) = 1$ using a proof by contradiction. If such an x were to exist, then $\dfrac{2x}{x^2+4} = 1$ or $2x = x^2 + 4$. Hence, $x^2 - 2x + 4 = 0$. We can then use the quadratic formula to prove that x is not a real number. Hence, there is no real number x such that $h(x) = 1$ and so h is not a surjection.

4. **(a)** Let $F\colon \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = 5x + 3$ for all $x \in \mathbb{R}$. Let $x_1, x_2 \in \mathbb{R}$ and assume that $F(x_1) = F(x_2)$. Then $5x_1 + 3 = 5x_2 + 3$. Show that this implies that $x_1 = x_2$ and, hence, F is an injection.

Now let $y \in \mathbb{R}$. Then $\dfrac{y-3}{5} \in \mathbb{R}$. Prove that $F\left(\dfrac{y-3}{5}\right) = y$. Thus, F is a surjection and hence F is a bijection.

(b) The proof that G is an injection is similar to the proof in Part (a) that F is an injection. Notice that for each $x \in \mathbb{Z}$, $G(x) \equiv 3 \pmod 5$. Now explain why G is not a surjection.

7. The birthday function is not an injection since there are two different people with the same birthday. The birthday function is a surjection since for each day of the year, there is a person that was born on that day.

9. **(a)** The function f is an injection and a surjection. To prove that f is an injection, we assume that $(a, b) \in \mathbb{R} \times \mathbb{R}$, $(c, d) \in \mathbb{R} \times \mathbb{R}$, and that $f(a, b) = f(c, d)$. This means that

$$(2a, a + b) = (2c, c + d).$$

Since this equation is an equality of ordered pairs, we see that

$$\begin{aligned} 2a &= 2c, \text{ and} \\ a + b &= c + d. \end{aligned}$$

The first equation implies that $a = c$. Substituting this into the second equation shows that $b = d$. Hence,

$$(a, b) = (c, d),$$

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and we have shown that if $f(a, b) = f(c, d)$, then $(a, b) = (c, d)$. Therefore, f is an injection.

Now, to determine if f is a surjection, we let $(r, s) \in \mathbb{R} \times \mathbb{R}$. To find an ordered pair $(a, b) \in \mathbb{R} \times \mathbb{R}$ such that $f(a, b) = (r, s)$, we need

$$(2a, a + b) = (r, s).$$

That is, we need

$$\begin{aligned} 2a &= r, \text{ and} \\ a + b &= s. \end{aligned}$$

Solving this system for a and b yields

$$a = \frac{r}{2} \text{ and } b = \frac{2s - r}{2}.$$

Since $r, s \in \mathbb{R}$, we can conclude that $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and hence that $(a, b) \in \mathbb{R} \times \mathbb{R}$. So,

$$\begin{aligned} f(a, b) &= f\left(\frac{r}{2}, \frac{2s - r}{2}\right) \\ &= \left(2\left(\frac{r}{2}\right), \frac{r}{2} + \frac{2s - r}{2}\right) \\ &= (r, s). \end{aligned}$$

This proves that for all $(r, s) \in \mathbb{R} \times \mathbb{R}$, there exists $(a, b) \in \mathbb{R} \times \mathbb{R}$ such that $f(a, b) = (r, s)$. Hence, the function f is a surjection. Since f is both an injection and a surjection, it is a bijection.

(b) The proof that the function g is an injection is similar to the proof that f is an injection in Part (a). Now use the fact that the first coordinate of $g(x, y)$ is an even integer to explain why the function g is not a surjection.

Section 6.4

2. $(g \circ h) : \mathbb{R} \to \mathbb{R}$ by $(g \circ h)(x) = g(h(x)) = g\left(x^3\right) = 3x^3 + 2$.

$(h \circ g) : \mathbb{R} \to \mathbb{R}$ by $(h \circ g)(x) = h(g(x)) = h(3x + 2) = (3x + 2)^3$.

This shows that $h \circ g \neq g \circ h$ or that composition of functions is not commutative.

3. **(a)** $F(x) = (g \circ f)(x)$, where $f(x) = e^x$ and $g(x) = \cos x$.

(b) $G(x) = (g \circ f)(x)$ where $f(x) = \cos x$ and $g(x) = e^x$.

(c) $H(x) = (g \circ f)(x)$, $f(x) = \sin x$, $g(x) = \frac{1}{x}$.

(d) $K(x) = (g \circ f)(x)$, $f(x) = e^{-x^2}$, $g(x) = \cos x$.

4. **(a)** For each $x \in A$, $(f \circ I_A)(x) = f(I_A(x)) = f(x)$. Therefore, $f \circ I_A = f$.

5. **(a)** $[(h \circ g) \circ f](x) = \sqrt[3]{\sin(x^2)}$; $[h \circ (g \circ f)](x) = \sqrt[3]{\sin(x^2)}$. This proves that $(h \circ g) \circ f = h \circ (g \circ f)$ for these particular functions.

6. Start of a proof: Let A, B, and C be nonempty sets and let $f\colon A \to B$ and $g\colon B \to C$. Assume that f and g are both injections. Let $x, y \in A$ and assume that $(g \circ f)(x) = (g \circ f)(y)$.

7. **(a)** $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x$, $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2$. Then $g \circ f : \mathbb{R} \to \mathbb{R}$ by $(g \circ f)(x) = x^2$. The function f is a surjection, but $g \circ f$ is not a surjection.

(b) $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$, $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2$. Then $g \circ f : \mathbb{R} \to \mathbb{R}$ by $(g \circ f)(x) = x^2$. The function f is an injection, but $g \circ f$ is not an injection.

(f) By Part (1) of Theorem 6.21, this is not possible since if $g \circ f$ is an injection, then f is an injection.

Section 6.5

2. **(b)** $f^{-1} = \{(c, a), (b, b), (d, c), (a, d)\}$

(d) For each $x \in S$, $\left(f^{-1} \circ f\right)(x) = x = (f \circ f^{-1})(x)$. This illustrates Corollary 6.28.

3. **(a)** This is a use of Corollary 6.28 since the cube root function and the cubing function are inverse functions of each other and consequently, the composition of the cubing function with the cube root function is the identity function.

(b) This is a use of Corollary 6.28 since the natural logarithm function and the exponential function with base e are inverse functions of each other and consequently, the composition of the natural logarithm function with the exponential function with base e is the identity function.

(c) They are similar because they both use the concept of an inverse function to "undo" one side of the equation.

4. Using the notation from Corollary 6.28, if $y = f(x)$ and $x = f^{-1}(y)$, then

$$\begin{aligned}\left(f \circ f^{-1}\right)(y) &= f(f^{-1}(y)) \\ &= f(x) \\ &= y\end{aligned}$$

6. **(a)** Let $x, y \in A$ and assume that $f(x) = f(y)$. Apply g to both sides of this equation to prove that $(g \circ f)(x) = (g \circ f)(y)$. Since $g \circ f = I_A$, this implies that $x = y$ and hence that f is an injection.

   (b) Start by assuming that $f \circ g = I_B$, and then let $y \in B$. You need to prove there exists an $x \in A$ such that $f(x) = y$.

7. **(a)** $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = e^{-x^2}$. Since this function is not an injection, the inverse of f is not a function.

   (b) $g : \mathbb{R}^* \to (0, 1]$ is defined by $g(x) = e^{-x^2}$. In this case, g is a bijection and hence, the inverse of g is a function.

   To see that g is an injection, assume that $x, y \in \mathbb{R}^*$ and that $e^{-x^2} = e^{-y^2}$. Then, $x^2 = y^2$ and since $x, y \geq 0$, we see that $x = y$. To see that g is a surjection, let $y \in (0, 1]$. Then, $\ln y < 0$ and $-\ln y > 0$, and $g\left(\sqrt{-\ln y}\right) = y$.

Section 6.6

1. **(a)** There exists an $x \in A \cap B$ such that $f(x) = y$.

   (d) There exists an $a \in A$ such that $f(a) = y$ or there exists a $b \in B$ such that $f(b) = y$.

   (f) $f(x) \in C \cup D$

   (h) $f(x) \in C$ or $f(x) \in D$

2. **(b)** $f^{-1}(f(A)) = [2, 5]$.

   (e) $f(A \cap B) = [-5, -3]$

   (d) $f(f^{-1}(C)) = [-2, 3]$

   (f) $f(A) \cap f(B) = [-5, -3]$

3. **(a)** $g(A \times A) = \{6, 12, 18, 24, 36, 54, 72, 108, 216\}$

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(b) $g^{-1}(C) = \{(1,1), (2,1), (1,2)\}$

4. **(a)** $\text{range}(F) = F(S) = \{1, 4, 9, 16\}$

5. To prove $f(A \cup B) \subseteq f(A) \cup f(B)$, let $y \in f(A \cup B)$. Then there exists an $x \in A \cup B$ such that $f(x) = y$. Since $x \in A \cup B$, $x \in A$ or $x \in B$. We first note that if $x \in A$, then $y = f(x)$ is in $f(A)$. In addition, if $x \in B$, then $y = f(x)$ is in $f(B)$. In both cases, $y = f(x) \in f(A) \cup f(B)$ and hence, $f(A \cup B) \subseteq f(A) \cup f(B)$.

Now let $y \in f(A) \cup f(B)$. If $y \in f(A)$, then there exists an $x \in A$ such that $y = f(x)$. Since $A \subseteq A \cup B$, this implies that $y = f(x) \in f(A \cup B)$. In a similar manner, we can prove that if $y \in f(B)$, then $y \in f(A \cup B)$. Therefore, $f(A) \cup f(B) \subseteq f(A \cup B)$.

6. To prove that $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$, let $x \in f^{-1}(C \cap D)$. Then $f(x) \in C \cap D$. How do we prove that $x \in f^{-1}(C) \cap f^{-1}(D)$?

9. Statement (a) is true and Statement (b) is false.

Section 7.1

1. **(a)** The set $A \times B$ contains nine ordered pairs. The set $A \times B$ is a relation from A to B since $A \times B$ is a subset of $A \times B$.

(b) The set R is a relation from A to B since $R \subseteq A \times B$.

(c) $\text{dom}(R) = A$, $\text{range}(R) = \{p, q\}$

2. **(a)** The statement is false since $(c, c) \notin R$, which can be written as $c \not R\, d$.

(b) The statement is true since whenever $(x, y) \in R$, (y, x) is also in R. That is, whenever $x \mathrel{R} y$, $y \mathrel{R} x$.

(c) The statement is false since $(a, c) \in R$, $(c, b) \in R$, but $(a, b) \notin R$. That is, $a \mathrel{R} c$, $c \mathrel{R} b$, but $a \not R\, b$.

(d) The statement is false since $(a, a) \in R$ and $(a, c) \in R$.

3. **(a)** The domain of D consists of the female citizens of the United States whose mother is a female citizen of the United States.

(b) The range of D consists of those female citizens of the United States who have a daughter that is a female citizen of the United States.

4. **(a)** $(S, T) \in R$ means that $S \subseteq T$.

(b) The domain of the subset relation is $\mathcal{P}(U)$.

(c) The range of the subset relation is $\mathcal{P}(U)$.

(d) The relation R is not a function from $\mathcal{P}(U)$ to $\mathcal{P}(U)$ since any proper subset of U is a subset of more than one subset of U.

6. **(a)** $\{x \in \mathbb{R} \mid (x, 6) \in S\} = \{-8, 8\}$.
$\{x \in \mathbb{R} \mid (x, 9) \in S\} = \left\{-\sqrt{19}, \sqrt{19}\right\}$.

(b) The domain of the relation S is the closed interval $[-10, 10]$.
The range of the relation S is the closed interval $[-10, 10]$.

(c) The relation S is not a function from $\mathbb{R}$ to $\mathbb{R}$.

(d) The graph of the relation S is the circle of radius 10 whose center is at the origin.

9. **(a)** $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid |a - b| \leq 2\}$

(b) $\text{dom}(R) = \mathbb{Z}$ and $\text{range}(R) = \mathbb{Z}$

Section 7.2

1. The relation R is not reflexive on A and is not symmetric. However, it is transitive since the conditional statement "For all $x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$" is a true conditional statement since the hypothesis will always be false.

3. There are many possible equivalence relations on this set. Perhaps one of the easier ways to determine one is to first decide what elements will be equivalent. For example, suppose we say that we want 1 and 2 to be equivalent (and of course, all elements will be equivalent to themselves). So if we use the symbol $\sim$ for the equivalence relation, then we need $1 \sim 2$ and $2 \sim 1$. Using set notation, we can write this equivalence relation as

$$\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 2), (2, 1)\}.$$

4. The relation R is not reflexive on A. For example, $(4, 4) \notin R$. The relation R is symmetric. If $(a, b) \in R$, then $|a| + |b| = 4$. Therefore, $|b| + |a| = 4$, and hence, $(b, a) \in R$. The relation R is not transitive. For example, $(4, 0) \in R$, $(0, 4) \in R$, and $(4, 4) \notin R$. The relation R is not an equivalence relation.

6. **(a)** The relation $\sim$ is an equivalence relation.

For $a \in \mathbb{R}$, $a \sim a$ since $f(a) = f(a)$. So, $\sim$ is reflexive.

For $a, b \in \mathbb{R}$, if $a \sim b$, then $f(a) = f(b)$. So, $f(b) = f(a)$. Hence, $b \sim a$ and $\sim$ is symmetric.

For $a, b, c \in \mathbb{R}$, if $a \sim b$ and $b \sim c$, then $f(a) = f(b)$ and $f(b) = f(c)$. So, $f(a) = f(c)$. Hence, $a \sim c$ and $\sim$ is transitive.

(b) $C = \{-5, 5\}$

10. **(a)** The relation $\sim$ is an equivalence relation on $\mathbb{Z}$. It is reflexive since for each integer a, $a + a = 2a$ and hence, 2 divides $a + a$. Now let $a, b \in \mathbb{Z}$ and assume that 2 divides $a + b$. Since $a + b = b + a$, 2 divides $b + a$ and hence, $\sim$ is symmetric. Finally, let $a, b, c \in \mathbb{Z}$ and assume that $a \sim b$ and $b \sim c$. Since 2 divides $a + b$, a and b must both be odd or both be even. In the case that a and b are both odd, then $b \sim c$ implies that c must be odd. Hence, $a + c$ is even and $a \sim c$. A similar proof shows that if a and b are both even, then $a \sim c$. Therefore, $\sim$ is transitive.

15. **(c)** The set C is a circle of radius 5 with center at the origin.

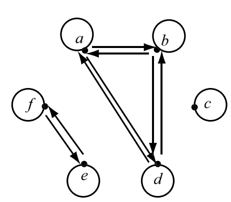
Section 7.3

1. Use the directed graph to examine all the cases necessary to prove that $\sim$ is reflexive, symmetric, and transitive. The distinct equivalence classes are: $[a] = [b] = \{a, b\}$; $[c] = \{c\}$; $[d] = [e] = \{d, e\}$

2. The equivalence class are

$$[a] = [b] = [d] = \{a, b, d\}, \quad [c] = \{c\}, \quad [e] = [f] = \{e, f\}.$$

Following is the directed graph for this equivalence relation.



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3. Let $x \in A$. Since x has the same number of digits as itself, the relation R is reflexive. Now let $x, y, z \in A$. If $x \mathrel{R} y$, then x and y have the same number of digits. Hence, y and x have the same number of digits and $y \mathrel{R} x$, and so R is symmetric.

If $x \mathrel{R} y$ and $y \mathrel{R} z$, then x and y have the same number of digits and y and z have the same number of digits. Hence, x and z have the same number of digits, and so $x \mathrel{R} z$. Therefore, R is transitive.

The equivalence classes are: $\{0, 1, 2, \ldots, 9\}$, $\{10, 11, 12, \ldots, 99\}$, $\{100, 101, 102, \ldots, 999\}$, $\{1000\}$.

4. The congruence classes for the relation of congruence modulo 5 on the set of integers are

$[0] = \{5n \mid n \in \mathbb{Z}\}$	$[3] = \{5n + 3 \mid n \in \mathbb{Z}\}$
$[1] = \{5n + 1 \mid n \in \mathbb{Z}\}$	
$[2] = \{5n + 2 \mid n \in \mathbb{Z}\}$	$[4] = \{5n + 4 \mid n \in \mathbb{Z}\}$.

5. **(a)** Let $a, b, c \in \mathbb{Z}_9$. Since $a^2 \equiv a^2 \pmod 9$, we see that $a \sim a$ and $\sim$ is reflexive. Let $a, b, c \in \mathbb{Z}_9$. If $a \sim b$, then $a^2 \equiv b^2 \pmod 9$ and hence, by the symmetric property of congruence, $b^2 \equiv a^2 \pmod 9$. This proves that $\sim$ is symmetric. Finally, if $a \sim b$ and $b \sim c$, then $a^2 \equiv b^2 \pmod 9$ and $b^2 \equiv c^2 \pmod 9$. By the transitive property of congruence, we conclude that $a^2 \equiv c^2 \pmod 9$ and hence, $a \sim c$. This proves that $\sim$ is transitive. The distinct equivalence classes are $\{0, 3, 6\}$, $\{1, 8\}$, $\{2, 7\}$, and $\{4, 5\}$.

6. **(a)** Let $x \in \left[\frac{5}{7}\right]$. Then $x - \frac{5}{7} \in \mathbb{Z}$, which means that there is an integer m such that $x - \frac{5}{7} = m$, or $x = \frac{5}{7} + m$. This proves that $x \in \left\{ m + \frac{5}{7} \,\middle|\, m \in \mathbb{Z} \right\}$ and, hence, that $\left[\frac{5}{7}\right] \subseteq \left\{ m + \frac{5}{7} \,\middle|\, m \in \mathbb{Z} \right\}$. We still need to prove that $\left\{ m + \frac{5}{7} \,\middle|\, m \in \mathbb{Z} \right\} \subseteq \left[\frac{5}{7}\right]$.

9. **(a)** To prove the relation is symmetric, note that if $(a, b) \approx (c, d)$, then $ad = bc$. This implies that $cb = da$ and, hence, $(c, d) \approx (a, b)$.

(c) $3a = 2b$

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Section 7.4

1. (a)

$\oplus$	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

$\odot$	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[0]	[2]
[3]	[0]	[3]	[2]	[1]

(b)

$\oplus$	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[0]
[2]	[2]	[3]	[4]	[5]	[6]	[0]	[1]
[3]	[3]	[4]	[5]	[6]	[0]	[1]	[2]
[4]	[4]	[5]	[6]	[0]	[1]	[2]	[3]
[5]	[5]	[6]	[0]	[1]	[2]	[3]	[4]
[6]	[6]	[0]	[1]	[2]	[3]	[4]	[5]

$\odot$	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[0]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[0]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[0]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[0]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[0]	[6]	[5]	[4]	[3]	[2]	[1]

2. **(a)** $[x] = [1]$ or $[x] = [3]$

   (e) $[x] = [2]$ or $[x] = [3]$

   (g) The equation has no solution.

3. **(a)** The statement is false. By using the multiplication table for $\mathbb{Z}_6$, we see that a counterexample is $[a] = [2]$.

   (b) The statement is true. By using the multiplication table for $\mathbb{Z}_5$, we see that:

$$[1] \odot [1] = [1]. \qquad [3] \odot [2] = [1].$$
$$[2] \odot [3] = [1]. \qquad [4] \odot [4] = [1].$$

5. **(a)** The proof consists of the following computations:

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$$[1]^2 = [1] \qquad [3]^2 = [9] = [4]$$
$$[2]^2 = [4] \qquad [4]^2 = [16] = [1].$$

17. **(a)** Prove the contrapositive by calculating $[a]^2 + [b]^2$ for all nonzero $[a]$ and $[b]$ in $\mathbb{Z}_3$.

Section 8.1

1. **(a)** The set of positive common divisors of 21 and 28 is $\{1, 7\}$. So $\gcd(21, 28) = 7$.

(b) The set of positive common divisors of -21 and 28 is $\{1, 7\}$. So $\gcd(-21, 28) = 7$.

(c) The set of positive common divisors of 58 and 63 is $\{1\}$. So $\gcd(58, 63) = 1$.

(d) The set of positive common divisors of 0 and 12 is $\{1, 2, 3, 4, 6, 12\}$. So $\gcd(0, 12) = 12$.

2. **(a)** **Hint**: Prove that $k \mid [(a + 1) - a]$.

4. **(a)** $|b|$ is the largest natural number that divides 0 and b.

(b) The integers b and $-b$ have the same divisors. Therefore, $\gcd(a, -b) = \gcd(a, b)$.

5. **(a)** $\gcd(36, 60) = 12$ $\qquad 12 = 36 \cdot 2 + 60 \cdot (-1)$
(b) $\gcd(901, 935) = 17$ $\qquad 17 = 901 \cdot 27 + 935 \cdot (-26)$
(e) $\gcd(901, -935) = 17$ $\qquad 17 = 901 \cdot 27 + (-935) \cdot (26)$

6. **(a)** One possibility is $u = -3$ and $v = 2$. In this case, $9u + 14v = 1$. We then multiply both sides of this equation by 10 to obtain

$$9 \cdot (-30) + 14 \cdot 20 = 10.$$

So we can use $x = -30$ and $y = 20$.

7. **(a)** $11 \cdot (-3) + 17 \cdot 2 = 1$ $\qquad$ **(b)** $\dfrac{m}{11} + \dfrac{n}{17} = \dfrac{17m + 11n}{187}$

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Section 8.2

1. For both parts, use the fact that the only natural number divisors of a prime number p are 1 and p.

2. Use cases: (1) p divides a; (2) p does not divide a. In the first case, the conclusion is automatically true. For the second case, use the fact that $\gcd(p, a) = 1$ and so we can use Theorem 8.12 to conclude that p divides b. Another option is to write the number 1 as a linear combination of a and p and then multiply both sides of the equation by b.

3. A hint for the inductive step: Write $p \mid (a_1 a_2 \cdots a_m) a_{m+1}$. Then look at two cases: (1) $p \mid a_{m+1}$; (2) p does not divide a_{m+1}.

4. **(a)** $\gcd(a, b) = 1$. Why?

(b) $\gcd(a, b) = 1$ or $\gcd(a, b) = 2$. Why?

7. **(a)** $\gcd(16, 28) = 4$. Also, $\dfrac{16}{4} = 4$, $\dfrac{28}{4} = 7$, and $\gcd(4, 7) = 1$.

(b) $\gcd(10, 45) = 5$. Also, $\dfrac{10}{5} = 2$, $\dfrac{45}{5} = 9$, and $\gcd(2, 9) = 1$.

9. Part (b) of Exercise (8) can be helpful.

11. The statement is true. Start of a proof: If $\gcd(a, b) = 1$ and $c \mid (a + b)$, then there exist integers x and y such that $ax + by = 1$ and there exists an integer m such that $a + b = cm$.

Section 8.3

3. **(a)** $x = -3 + 14k$, $y = 2 - 9k$

(b) $x = -1 + 11k$, $y = 1 - 9k$

(c) No solution

(d) $x = 2 + 3k$, $y = -2 - 4k$

4. There are several possible solutions to this problem, each of which can be generated from the solutions of the Diophantine equation $27x + 50y = 25$.

5. This problem can be solved by finding all solutions of a linear Diophantine equation $25x + 16y = 1461$, where both x and y are positive. The mininum number of people attending the banquet is 66.

6. **(a)** $y = 12 + 16k$, $x_3 = -1 - 3k$

(b) If $3y = 12x_1 + 9x_2$ and $3y + 16x_3 = 20$, we can substitute for $3y$ and obtain $12x_1 + 9x_2 + 16x_3 = 20$.

(c) Rewrite the equation $12x_1 + 9x_2 = 3y$ as $4x_1 + 3x_2 = y$. A general solution for this linear Diophantine equation is

$$\begin{aligned} x_1 &= y + 3n \\ x_2 &= -y - 4n. \end{aligned}$$

Section 9.1

2. One way to do this is to prove that the following function is a bijection: $f: A \times \{x\} \to A$ by $f(a, x) = a$, for all $(a, x) \in A \times \{x\}$.

3. One way to prove that $\mathbb{N} \approx E^+$ is to find a bijection from $\mathbb{N}$ to E^+. One possibility is $f : \mathbb{N} \to E^+$ by $f(n) = 2n$ for all $n \in \mathbb{N}$. (We must prove that this is a bijection.)

4. Notice that $A = (A - \{x\}) \cup \{x\}$. Use Theorem 9.6 to conclude that $A - \{x\}$ is finite. Then use Lemma 9.4.

5. **(a)** Since $A \cap B \subseteq A$, if A is finite, then Theorem 9.6 implies that $A \cap B$ is finite.

(b) The sets A and B are subsets of $A \cup B$. So if $A \cup B$ is finite, then A and B are finite.

7. **(a)** Remember that two ordered pairs are equal if and only if their corresponding coordinates are equal. So if $(a_1, c_1), (a_2, c_2) \in A \times C$ and $h(a_1, c_1) = h(a_2, c_2)$, then $(f(a_1), g(c_1)) = (f(a_2), g(c_2))$. We can then conclude that $f(a_1) = f(a_2)$ and $g(c_1) = g(c_2)$. Since f and g are both injections, this means that $a_1 = a_2$ and $c_1 = c_2$ and therefore, $(a_1, c_1) = (a_2, c_2)$. This proves that f is an injection.

Now let $(b, d) \in B \times D$. Since f and g are surjections, there exists $a \in A$ and $c \in C$ such that $f(a) = b$ and $g(c) = d$. Therefore, $h(a, c) = (b, d)$. This proves that f is a surjection.

8. **(a)** If we define the function f by $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = a$, and $f(5) = b$, then we can use $g(a) = 1$, $g(b) = 2$, and $g(3) = c$. The function g is an injection.

Section 9.2

1. All except Part (d) are true.

2. **(a)** Prove that the function $f : \mathbb{N} \to F^+$ defined by $f(n) = 5n$ for all $n \in \mathbb{N}$ is a bijection.

(e) One way is to define $f : \mathbb{N} \to \mathbb{N} - \{4, 5, 6\}$ by

$$f(n) = \begin{cases} n & \text{if } n = 1, n = 2, \text{or } n = 3 \\ f(n+3) & \text{if } n \geq 4. \end{cases}$$

and then prove that the function f is a bijection.

It is also possible to use Corollary 9.20 to conclude that $\mathbb{N} - \{4, 5, 6\}$ is countable, but it must also be proved that $\mathbb{N} - \{4, 5, 6\}$ cannot be finite. To do this, assume that $\mathbb{N} - \{4, 5, 6\}$ is finite and then prove that $\mathbb{N}$ is finite, which is a contradiction.

(f) Let $A = \{m \in \mathbb{Z} \mid m \equiv 2 \pmod 3\} = \{3k + 2 \mid k \in \mathbb{Z}\}$. Prove that the function $f : \mathbb{Z} \to A$ is a bijection, where $f(x) = 3x + 2$ for all $x \in \mathbb{Z}$. This proves that $\mathbb{Z} \approx A$ and hence, $\mathbb{N} \approx A$.

5. For each $n \in \mathbb{N}$, let $P(n)$ be "If $\text{card}(B) = n$, then $A \cup B$ is a countably infinite set."

Note that if $\text{card}(B) = k + 1$ and $x \in B$, then $\text{card}(B - \{x\}) = k$. Apply the inductive assumption to $B - \{x\}$.

6. Let $m, n \in \mathbb{N}$ and assume that $h(n) = h(m)$. Then since A and B are disjoint, either $h(n)$ and $h(m)$ are both in A or are both in B. If they are both in A, then both m and n are odd and

$$f\left(\frac{n+1}{2}\right) = h(n) = h(m) = f\left(\frac{m+1}{2}\right).$$

Since f is an injection, this implies that $\dfrac{n+1}{2} = \dfrac{m+1}{2}$ and hence that $n = m$. Similary, if both $h(n)$ and $h(m)$ are in B, then m and n are even and $g\left(\dfrac{n}{2}\right) = g\left(\dfrac{m}{2}\right)$, and since g is an injection, $\dfrac{n}{2} = \dfrac{m}{2}$ and $n = m$. Therefore, h is an injection.

Now let $y \in A \cup B$. There are only two cases to consider: $y \in A$ or $y \in B$. If $y \in A$, then since f is a surjection, there exists an $m \in \mathbb{N}$ such that $f(m) = y$. Let $n = 2m - 1$. Then n is an odd natural number, $m = \dfrac{n+1}{2}$, and

$$h(n) = f\left(\frac{n+1}{2}\right) = f(m) = y.$$

Now assume $y \in B$ and use the fact that g is a surjection to help prove that there exists a natural number n such that $h(n) = y$.

We can then conclude that h is a surjection.

7. By Theorem 9.14, the set $\mathbb{Q}^+$ of positive rational numbers is countably infinite. So by Theorem 9.15, $\mathbb{Q}^+ \cup \{0\}$ is countably infinite. Now prove that the set $\mathbb{Q}^-$ of all negative rational numbers is countably infinite and then use Theorem 9.17 to prove that $\mathbb{Q}$ is countably infinite.

8. Since $A - B \subseteq A$, the set $A - B$ is countable. Now assume $A - B$ is finite and show that this leads to a contradiction.

Section 9.3

1. **(a)** One such bijection is $f\colon (0, \infty) \to \mathbb{R}$ by $f(x) = \ln x$ for all $x \in (0, \infty)$

(b) One such bijection is $g\colon (0, \infty) \to (a, \infty)$ by $g(x) = x + a$ for all $x \in (0, \infty)$. The function g is a bijection and so $(0, \infty) \approx (a, \infty)$. Then use Part (a).

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2. Use a proof by contradiction. Let $\mathbb{H}$ be the set of irrational numbers and assume that $\mathbb{H}$ is countable. Then $\mathbb{R} = \mathbb{Q} \cup \mathbb{H}$ and $\mathbb{Q}$ and $\mathbb{H}$ are disjoint. Use Theorem 9.17, to obtain a contradiction.

3. By Corollary 9.20, every subset of a countable set is countable. So if B is countable, then A is countable.

4. By Cantor's Theorem (Theorem 9.27), $\mathbb{R}$ and $\mathcal{P}(\mathbb{R})$ do not have the same cardinality.

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