Section 13

Closed Sets in Topological Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What does it mean for a set to be closed in a topological space?
- What important properties do closed sets have in relation to unions and intersections?
- What is a sequence in a topological space?
- What does it mean for a sequence to converge in a topological space?
- What is a limit of a sequence in a topological space?
- What is a limit point of a subset of a topological space? How are closed sets related to limit points?
- What is a boundary point of a subset of a topological space and what is the boundary of a subset of a topological space? How are closed sets related to boundary points?
- What does it mean for a space to be Hausdorff? What important properties do Hausdorff spaces have?
- What are the separation axioms T_1, T_2, T_3, and T_4. What is the underlying idea behind these properties?

Introduction

We defined a closed set in a metric space to be the complement of an open set. Since a topology is defined in terms of open sets, we can make the same definition of closed set in a topological space. With the definition of closed set in hand, we can then ask if it is possible to define limit points, boundary, and closure in topological spaces and determine if there are corresponding properties for these ideas in topological spaces.

Definition 13.1. A subset C of a topological space X is **closed** if its complement $X \setminus C$ is open.

Preview Activity 13.1.

(1) List all of the closed sets in the indicated topological space.

  (a) (X, τ) with $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.

  (b) (X, τ) with $X = \{a, b, c, d, e, f\}$ and $\tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}, X\}$.

  (c) (X, τ) with $X = \mathbb{R}$ and $\tau = \{\emptyset, \{0\}, \mathbb{R}\}$.

  (d) (X, τ) with $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. (What is the name of this topology?)

  (e) (X, τ) with $X = \mathbb{Z}^+$ and $\tau = \{\emptyset, X\}$ (this topology is called the *indiscrete* or *trivial* topology).

(2) Using the examples from part (1), find (if possible), a set that is

  (a) both closed and open (if possible, find one that is not the entire set or the empty set)

  (b) closed but not open

  (c) open but not closed

  (d) not open and not closed

(3) In $\mathbb{R}^n$ with the Euclidean metric, every single element set is closed. Does this property hold in the topological space (X, τ), where $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$? Explain.

Unions and Intersections of Closed Sets

Now we have defined open and closed sets in topological spaces. In our preview activity we saw that a set can be both open and closed. As we did in metric spaces, we will call any set that is both open and closed a *clopen* (for closed-open) set.

By definition, any union and any finite intersection of open sets in a topological space is open, so the fact that closed sets are complements of open sets implies the following theorem.

Theorem 13.2. *Let X be a topological space.*

(1) Any intersection of closed sets in X is a closed set in X.

(2) Any finite union of closed sets in X is a closed set in X.

Proof. Let X be a topological space. To prove part 1, assume that C_α is a collection of closed set in X for α in some indexing set I. Then

$$X \setminus \bigcap_{\alpha \in I} C_\alpha = \bigcup_{\alpha \in I} X \setminus C_\alpha.$$

The latter is an arbitrary union of open sets and so it an open set. By definition, then, $\bigcap_{\alpha \in I} C_\alpha$ is a closed set.

For part 2, assume that C_1, C_2, ..., C_n are closed sets in X for some $n \in \mathbb{Z}^+$. To show that $C = \bigcap_{k=1}^{n} C_k$ is a closed set, we will show that $X \setminus C$ is an open set. Now

$$X \setminus \bigcup_{\alpha \in I} C_\alpha = \bigcap_{\alpha \in I} X \setminus C_\alpha$$

is a finite intersection of open sets, and so is an open set. Therefore, $\bigcup_{\alpha \in I} C_\alpha$ is a closed set. ■

Theorem 13.2 tells us that any intersection of closed sets is closed, but only finite unions of closed sets are closed. How do we know that non-finite unions of closed sets aren't necessarily closed?

Activity 13.1. Let $\mathbb{Z}$ be a topological space with the finite complement topology τ_{FC}. That is, a non-empty set O is open in $\mathbb{Z}$ if $\mathbb{Z} \setminus O$ is finite.

(a) What must be true about the cardinality of the closed sets in $(\mathbb{Z}, \tau_{FC})$?

(b) Let $C_n = \{2, 3, \ldots, n\}$. Is the set $\bigcup_{n \geq 3} C_n$ a closed set in $(\mathbb{Z}, \tau_{FC})$? Explain.

Limit Points and Sequences in Topological Spaces

Recall that we defined a limit point of a set A in a metric space X to be a point $x \in X$ such that every neighborhood of x contains a point in A different from x. Since we have defined neighborhoods in topological spaces, we can make the same definition.

Definition 13.3. Let X be a topological space, and let A be a subset of X. A **limit point** of A is a point $x \in X$ such that every neighborhood of x contains a point in A different from x.

The set A' of limit points of A is called the *derived set* of A.

Activity 13.2. Find the limit point(s) of the following sets

(a) $\{c, d\}$ in (X, τ) with $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$

(b) $\{a, b\}$ in the set $X = \{a, b, c, d, e, f\}$ with topology

$$\tau = \{\emptyset, \{b\}, \{a, b, c\}, \{d, e, f\}, \{b, d, e, f\}, X\}.$$

(c) $\{a, b\} \subset X$ where $X = \{a, b, c\}$ in the discrete topology.

(d) $\{-1, 0, 1\} \subset \mathbb{Z}$ with τ the topology on $\mathbb{Z}$ with basis $\{B(n)\}$, where

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even.} \end{cases}$$

(This topology is called the *digital line topology* and has applications in digital processing. That the collection $\{B(n)\}$ is a basis for a topology on $\mathbb{Z}$ is the topic of Exercise (11) on page 127.)

In metric spaces, a set is closed if and only if it contains all of its limit points. So the corresponding result in topological spaces should be no surprise.

Theorem 13.4. *Let C be a subset of a topological space X, and let C' be the set of limit points of C. Then C is closed if and only if $C' \subseteq C$.*

Proof. Let X be a topological space, and let C be a subset of X. First we assume that C is closed and show that C contains all of its limit points. Let $x \in X$ be a limit point of C. We proceed by contradiction and assume that $x \notin C$. Then $x \in X \setminus C$, which is an open set. This means that there is a neighborhood (namely $X \setminus C$) of x that contains no points in C, which contradicts the fact that x is a limit point of C. We conclude that $x \in C$ and C contains all of its limit points.

For the converse, assume that C contains all of its limit points. To show that C is closed, we prove that $X \setminus C$ is open. We again proceed by contradiction and assume that $X \setminus C$ is not open. Then there exists $x \in X \setminus C$ such that no neighborhood of x is entirely contained in $X \setminus C$. This implies that every neighborhood of x contains a point in C and so x is a limit point of C. It follows that $x \in C$, contradicting the fact that $x \in X \setminus C$. We conclude that $X \setminus C$ is open and C is closed. ■

In metric spaces we saw that a limit point of a set is the limit of a sequence of points in the set. To explore this idea in topological spaces, we define sequences in the same way we did in metric spaces.

Definition 13.5. A **sequence** in a topological space X is a function $f : \mathbb{Z}^+$ to X.

We use the same notation and terminology related to sequences as we did in metric spaces: we will write (x_n) to represent a sequence f, where $x_n = f(n)$ for each $n \in \mathbb{Z}^+$. We can't define convergence in a topological space using a metric, but we can use open sets. Recall that a sequence (x_n) in a metric space (X, d) converges to a point x in the space if, given $\epsilon > 0$ there exists a positive integer N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In other words, every open ball centered at x contains all of the entries of the sequence past a certain point. We can replace open balls with open sets and make a similar definition of convergence in topological spaces.

Definition 13.6. A sequence (x_n) in a topological space X **converges** to the point $x \in X$ if, for each open set O that contains x there exists a positive integer N such that $x_n \in O$ for all $n \geq N$.

If a sequence (x_n) converges to a point x, we call x a *limit* of the sequence (x_n).

Activity 13.3. In metric spaces, limits of sequences are unique. We may wonder if the same result is true in topological spaces. Consider the topological space (X, τ), where $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. Find all limits of all constant sequences in X.

The result of Activity 13.3 is that sequences do not behave in topological spaces as we would expect them to. Consequently, sequences do not play the same important role in topological spaces as they do in metric spaces. However, the concept of limit point is important, as are the notions of boundary and closure in topological spaces.

Closure in Topological Spaces

Once we have a definition of limit point, we can define the closure of a set just as we did in metric spaces.

Definition 13.7. The **closure** of a subset A of a topological space X is the set

$$\overline{A} = A \cup A'.$$

In other words, the closure of a set is the collection of the elements of the set and the limit points of the set. The following theorem is the analog of the theorem in metric spaces about closures.

Theorem 13.8. *Let X be a topological space and A a subset of X. The closure of a A is a closed set. Moreover, the closure of A is the smallest closed subset of X that contains A.*

Proof. Let X be a topological space and A a subset of X. To prove that $\overline{A}$ is a closed set, we will prove that $\overline{A}$ contains its limit points. Let $x \in \overline{A}'$. To show that $x \in \overline{A}$, we proceed by contradiction and assume that $x \notin \overline{A}$. This implies that $x \notin A$ and $x \notin A'$. Since $x \notin A'$, there exists a neighborhood N of x that contains no points of A other than x. But $A \subseteq \overline{A}$ and $x \notin \overline{A}$, so it follows that $N \cap A = \emptyset$. This implies that there is an open set $O \subseteq N$ centered at x so that $O \cap A = \emptyset$. The fact that $x \in \overline{A}'$ means that $O \cap \overline{A}$ contains a point y in $\overline{A}$ different from x. Since $O \cap A = \emptyset$, we must have $y \in A'$. But the fact that O is a neighborhood of y means that O must contain a point of A different than y, which contradicts the fact that $O \cap A = \emptyset$. We conclude that $x \in \overline{A}$ and $\overline{A}' \subseteq \overline{A}$. This shows that $\overline{A}$ is a closed set.

The proof that $\overline{A}$ is the smallest closed subset of X that contains A is left for the next activity. ■

Activity 13.4. Let (X, d) be a topological space, and let A be a subset of X.

(a) What will we have to show to prove that $\overline{A}$ is the smallest closed subset of X that contains A?

(b) Suppose that C is a closed subset of X that contains A. To show that $\overline{A} \subseteq C$, why is it enough to demonstrate that $A' \subseteq C$?

(c) If $x \in A'$, what can we say about x?

(d) Complete the proof that $\overline{A} \subseteq C$.

One consequence of Theorem 13.8 is the following.

Corollary 13.9. *A subset C of a topological space X is closed if and only if $C = \overline{C}$.*

The Boundary of a Set

In addition to limit points, we also defined boundary points in metric spaces. Recall that a boundary point of a set A in a metric space X could be considered to be any point in $\overline{A} \cap \overline{X \setminus A}$. We make the same definition in a topological space.

Definition 13.10. Let (X, τ) be a topological space, and let A be a subset of X. A **boundary point** of A is a point $x \in X$ such that every neighborhood of x contains a point in A and a point in $X \setminus A$. The **boundary** of A is the set

$$\text{Bdry}(A) = \{x \in X \mid x \text{ is a boundary point of } A\}.$$

As with metric spaces, the boundary points of a set A are those points that are "between" A and its complement.

Activity 13.5. Find the boundaries of the following sets

(a) $\{c, d\}$ in (X, τ) with $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.

(b) $\{a, b\}$ in the set $X = \{a, b, c, d, e, f\}$ with topology

$$\tau = \{\emptyset, \{b\}, \{a, b, c\}, \{d, e, f\}, \{b, d, e, f\}, X\}.$$

(c) $\{a, b\} \subset X$ where $X = \{a, b, c\}$ in the discrete topology.

(d) $\mathbb{Z}$ in $\mathbb{R}$ with the finite complement topology τ_{FC}.

Just as with metric spaces, we can characterize the closed sets as the sets that contain their boundary.

Theorem 13.11. *A subset C of a topological space X is closed if and only if C contains its boundary.*

The proof of Theorem 13.11 is left to Exercise (10).

Separation Axioms

As we have seen, sequences in topological spaces do not generally behave as we would expect them to. As a result, we look for conditions on topological spaces under which sequences do exhibit some regular behavior. In our preview activity we saw that it is possible in a topological space that single point sets do not have to be closed. In Activity 13.3, we also saw that limits of sequences in topological spaces are not necessarily unique. This type of behavior limits the results that one can prove about such spaces. As a result, we define classes of topological spaces whose behaviors are closer to what our intuition suggests.

Activity 13.6.

(a) Consider a metric space (X, d), and let x and y be distinct points in X.

  i. Explain why x and y cannot both be limits of the same sequence if we can find disjoint open balls $B(x, r)$ centered at x and $B(y, s)$ centered at y such that $B_x \cap B_y = \emptyset$.

  ii. Now show that we can find disjoint open balls $B(x, r)$ centered at x and $B(y, s)$ centered at y such that $B(x, s) \cap B(y, r) = \emptyset$.

(b) Return to our example from Activity 13.3 with $X = \{a, b, c\}$ and topology

$$\tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}.$$

We saw that every point in X is a limit of the constant sequence (c). If $x \neq c$ in X, Explain why there are no disjoint open sets O_x containing x and O_c containing c.

It is the fact as described in Activity 13.6 that we can separate disjoint points by disjoint open sets that separates metric spaces from other spaces where limits are not unique. If we restrict ourselves to spaces where we can separate points like this, then we might expect to have unique limits. Such spaces are called *Hausdorff* spaces.

Definition 13.12. A topological space X is a **Hausdorff** space if for each pair x, y of distinct points in X, there exists open sets O_x of x and O_y of y such that $O_x \cap O_y = \emptyset$.

Activity 13.6 shows that every metric space is a Hausdorff space. Once we start imposing conditions on topological spaces, we restrict the number of spaces we consider.

Activity 13.7.

(a) Let X be any set and τ the discrete topology. Is (X, τ) Hausdorff? Justify your answer.

(b) Let (X, τ) be a Hausdorff topological space with $X = \{x, x_1, x_2, \ldots, x_n\}$ a finite set. Is $\{x\}$ an open set? Explain. What does this say about the topology τ? (Hint: Is x a limit point of $\{x_1, x_2, \ldots, x_n\}$?)

Example 13.13. There are examples of Hausdorff spaces that are not the standard metric spaces. For example, Let $K = \left\{\frac{1}{k} \mid k \text{ is a positive integer}\right\}$. We use K to make a topology on $\mathbb{R}$ with basis all open intervals of the form (a, b) and all sets of the form $(a, b) \setminus K$, where $a < b$ are real numbers. This topology adds the extra intervals of the form $(a, b) \setminus K$ to the standard open intervals to make a new topology. This topology is known as the K-topology on $\mathbb{R}$. Just as in $(\mathbb{R}, d_E)$, if x and y are distinct real numbers we can separate x and y with open intervals.

The reason we defined Hausdorff spaces is because they have familiar properties, as the next theorems illustrate.

Theorem 13.14. *Each single point subset of a Hausdorff topological space is closed.*

Proof. Let X be a Hausdorff topological space, and let $A = \{a\}$ for some $a \in X$. To show that A is closed, we prove that $X \setminus A$ is open. Let $x \in X \setminus A$. Then $x \neq a$. So there exist open sets O_x containing x and O_a containing a such that $O_x \cap O_a = \emptyset$. So $a \notin O_x$ and $O_x \subseteq X \setminus A$. Thus, every point of $X \setminus A$ is an interior point and $X \setminus A$ is an open set. This verifies that A is a closed set. ■

Theorem 13.15. *A sequence of points in a Hausdorff topological space can have at most one limit in the space.*

Proof. Let X be a Hausdorff topological space, and let (x_n) be a sequence in X. Suppose (x_n) converges to $a \in X$ and to $b \in X$. Suppose $a \neq b$. Then there exist open sets O_a of a and O_b of b such that $O_a \cap O_b = \emptyset$. But the fact that (x_n) converges to a implies that there is a positive integer

N such that $x_n \in O_a$ for all $n \geq N$. But then $x_n \notin O_b$ for any $n \geq N$. This contradicts the fact that (x_n) converges to b. We conclude that $a = b$ and that the sequence (x_n) can have at most one limit in X. ■

Hausdorff spaces are important because we can separate distinct points with disjoint open sets. It is also of interest to consider what other types of objects we can separate with disjoint open sets. For example, the indiscrete topology is quite bad in the sense that its open sets can't distinguish between distinct points. That is, if x and y are distinct points in a space with the indiscrete topology, then every open set that contains x also contains y. By contrast, in a Hausdorff space we can separate distinct points with disjoint open sets. This is an example of what is called a "separation" property. Other types of separation properties describe different types of topological spaces. These separation properties determine what kind of objects we can separate with disjoint open sets – e.g., points, points and closed sets, closed sets and closed sets. The following are the most widely used separation properties. These properties rule out kinds of unwelcome properties that a topological space might have. For example, recall that limits of sequences are unique in Hausdorff spaces. (We traditionally call these separation properties "axioms" because we generally assume that our topological spaces have these properties. However, these are not axioms in the usual sense of the word, but rather properties.)

Definition 13.16. Let X be a topological space.

(1) The space X is a $\boldsymbol{T_1}$**-space** or **Frechet space** if for every $x \neq y$ in X, there exist an open set U containing y such that $x \notin U$.

(2) The space X is a $\boldsymbol{T_2}$**-space** or a **Hausdorff space** if for every $x \neq y$ in X, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

(3) The space X is **regular** if for each closed set C of X and each point $x \in X \setminus C$, there exists disjoint open sets U and V in X such that $C \subseteq U$ and $x \in V$. The space X is a $\boldsymbol{T_3}$**-space** or a **regular Hausdorff space** if X is a regular T_1 space.

(4) The space X is a **normal** space if for each pair C and D of disjoint closed subsets of X there exist disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$. The space X is a $\boldsymbol{T_4}$**-space** or a **normal Hausdorff space** if X is a normal T_1 space.

Exercise (16) shows that every metric space is both regular and normal. The use of the variable T comes from the German "Trennungsaxiome" for separation axioms. Note again that these are not technically axioms, but rather properties. An interesting question is why we insist that T_3 and T_4-spaces also be T_1. We want these axioms to provide more separation as the index increases. Consider a space X with the indiscrete topology. In this space, nothing is separated. However, this space is vacuously regular and normal. To avoid this seeming incongruity, we insist on working only with T_1 spaces. Note that a space with the indiscrete topology is not T_1.

It is the case that every T_4-space is T_3, every T_3-space is T_2, and every T_2-space is T_1. Verification of these statements are left to Exercise (18). These properties are also all different. That is, there are T_1-spaces that are not T_2 and T_2-spaces that are not T_3. These problems are given in Exercise (19). The fact that there are T_3-spaces that are not T_4 is a bit difficult. An example is the *Niemytzki plane*. The Niemytzki plane is the upper half plane $X = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Let L be the boundary of X. That is, $L = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$. A basis for the topology on X consists of the standard open disks centered at points with $y > 0$ along with the open disks in $X \setminus L$ that are tangent to L together with their points of tangency. We won't verify that the Niemytzki plane is T_3 but not T_4. The interested reader can find an accessible proof in the article "Another Proof that the Niemytzki Plane is not Normal" by David H. Vetterlein in the *Pi Mu Epsilon Journal*, Vol. 10, No. 2 (SPRING 1995), pp. 119-121.

Summary

Important ideas that we discussed in this section include the following.

- A subset C of a topological space X is closed if $X \setminus C$ is open.
- Any intersection of closed sets is closed, while unions of finitely many closed sets are closed.
- A sequence in a topological space X is a function $f : \mathbb{Z}^+$ to X.
- A sequence (x_n) in a topological space X converges to a point x in X if for each open set O containing x, there exists a positive integer N such that $x_n \in O$ for all $n \geq N$.
- If a sequence (x_n) in a topological space X converges to a point x, then x is a limit of the sequence (x_n).
- A limit point of a subset A of a topological space X is a point $x \in X$ such that every neighborhood of x contains a point in A different from x. A subset C of a topological space X is closed if and only if C contains all of its limit points.
- A boundary point of a subset A of a topological space X is a point $x \in X$ such that every neighborhood of x contains a point in A and a point in $X \setminus A$. The boundary of A is the set

  $$\text{Bdry}(A) = \{x \in X \mid x \text{ is a boundary point of } A\}.$$

  A subset C of X is closed if and only if C contains its boundary.
- A topological space X is Hausdorff if we can separate distinct points with open sets in the space. That is, if for each pair x, y of distinct points in X, there exists open sets O_x of x and O_y of y such that $O_x \cap O_y = \emptyset$. Hausdorff spaces are important because sequences have unique limits in Hausdorff spaces and single point sets are closed.
- Separation axioms tell us what kinds of objects can be separated by open sets.
  - In a T_1-space, we can separate two distinct points with one open set. That is, given distinct points x and y in a T_1 topological space X, there is an open set U that separates y from x in the sense that $y \in U$ but $x \notin U$.
  - In a T_2-space X we can separate points more distinctly. That is, if x and y are different points in X, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.
  - In a T_3-space X we can separate a point from a closed set that does not contain that point. That is, if C is a closed subset of X and x is a point not in C, there exists disjoint open sets U and V in X such that $C \subseteq U$ and $x \in V$.
  - In a T_4-space X we can separate disjoint closed sets. That is, if C and D are disjoint closed subsets of X, there exist disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$.

Exercises

(1) Determine exactly which finite topological spaces are Hausdorff. Prove your result.

(2) Let (X, τ) be a topological space and let A be a subset of X. Prove that $\overline{A} = A \cup \text{Bdry}(A)$.

(3) Let A a subset of a topological space. Prove that $\text{Bdry}(A) = \emptyset$ if and only if A is open and closed.

(4) Let X be a nonempty set with at least two elements and let p be a fixed element in X. Let τ_p be the particular point topology and $\tau_{\overline{p}}$ the excluded point topology on X. That is

- τ_p is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that contain p.
- $\tau_{\overline{p}}$ is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that do not contain p.

That the particular point and excluded point topologies are topologies is the subject of Exercises (9) and (10) on page 127.

Let $A = (0, 1]$ be a subset of $\mathbb{R}$. Find, with proof, $\overline{A}$, $\text{Int}(A)$, and $\text{Bdry}(A)$ when

(a) $\mathbb{R}$ has the topology τ_p with $p = 0$

(b) $\mathbb{R}$ has the topology $\tau_{\overline{p}}$ with $p = 0$.

(5) Let $\mathcal{B} = \{[a, b) \mid a < b \text{ in } \mathbb{R}\}$.

(a) Show that $\mathcal{B}$ is a basis for a topology $\tau_{\ell\ell}$ on $\mathbb{R}$. This topology is called the *lower limit* topology on $\mathbb{R}$. The line $\mathbb{R}$ with the topology $\tau_{\ell\ell}$ is sometimes called the *Sorgenfrey line* (after the mathematician Robert Sorgenfrey).

(b) Show that every open interval (a, b) is also an open set in the lower limit topology.

(c) If τ_1 and τ_2 are topologies on a set X such that $\tau_1 \subseteq \tau_2$, then τ_1 is said to be a *coarser* topology that τ_2, or τ_2 is a *finer* topology that τ_1. Part (b) shows that the lower limit topology may be a finer topology than the Euclidean metric topology. Determine if this is true, that the lower limit topology is actually a finer topology than the Euclidean metric topology on $\mathbb{R}$. Justify your answer.

(d) Let $a < b$ be in $\mathbb{R}$. Is the set $[a, b)$ clopen in $(\mathbb{R}, \tau_{\ell\ell})$? Prove your answer.

(6) A subset A of a topological space X is said to be *dense* in X if $\overline{A} = X$.

(a) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$ using the Euclidean metric topology.

(b) Is $\mathbb{Z}$ dense in $\mathbb{R}$ using the Euclidean metric topology? Prove your answer.

(c) Let A be a subset of a topological space A. Prove that A is dense in X if and only if $A \cap O \neq \emptyset$ for every open set O.

(7) Let X be a topological space and let A be a subset of X.

(a) Show that the sets $\text{Int}(A)$, $\text{Bdry}(A)$, and $\text{Int}(A^c)$ are mutually disjoint (that is, the intersection of any two of these sets is empty).

(b) Prove that $X = \text{Int}(A) \cup \text{Bdry}(A) \cup \text{Int}(A^c)$.

(8) Prove that a subspace of a Hausdorff space is a Hausdorff space.

(9) Let X be a nonempty set with at least two elements and let p be a fixed element in X. Let τ_p be the particular point topology and $\tau_{\overline{p}}$ the excluded point topology on X. That is

- τ_p is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that contain p.
- $\tau_{\overline{p}}$ is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that do not contain p.

That the particular point and excluded point topologies are topologies is the subject of Exercises (9) and (10) on page 127.

Determine, with proof, if X is a Hausdorff space when

(a) X has the topology τ_p

(b) X has the topology $\tau_{\overline{p}}$.

(10) Prove that a subset C of a topological space X is closed if and only if C contains its boundary.

(11) Recall that a point a in a subset A of a metric space X is an isolated point of A if there is a neighborhood N of a in X such that $N \cap A = \{a\}$. We can make the same definition in any topological space.

Definition 13.17. A point a in a subset A of a topological space X is an isolated point of A if there is a neighborhood N of a such that $N \cap A = \{a\}$.

(a) If A is a subset of a topological space X, prove that a point $a \in A$ is an isolated point of A if and only if $\{a\}$ is an open set in A.

(b) We proved that in a metric space every boundary point of a set A is either a limit point or an isolated point of A. (See Exercise 12 on page 106.) Is the same statement true in a topological space? Prove your answer.

(12) For each integer a, let $a\mathbb{Z} = \{ka \mid k \in \mathbb{Z}\}$. That is, $a\mathbb{Z}$ is the set of all integer multiples of a. That $\{a\mathbb{Z} \mid a \in \mathbb{Z}\}$ is a basis for a topology τ on $\mathbb{Z}$ is the topic of Exercise (2) on page 125. In this exercise work in the topological space $(\mathbb{Z}, \tau)$

(a) Let $A = \mathbb{E}$, the set of even integers.

i. Find, with justification, $\text{Int}(A)$.

ii. Find, with justification, $\overline{A}$.

(b) Let $B = \mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 1\}$. That is, $\mathbb{N}$ is the set of natural numbers.

i. Find, with justification, $\text{Int}(B)$.

ii. Find, with justification, $\overline{B}$.

(13) Consider the Double Origin topology defined as follows. Let $X = \mathbb{R}^2 \cup \{0^*\}$, where 0^* is considered as a point that is not in $\mathbb{R}^2$ (0^* is our double origin). As a basis for the open sets, we use the standard open balls for every point except 0 and 0^*. For the point 0, we define open sets to be

$$N(0,r) = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < \frac{1}{r^2}, y > 0\right\} \cup \{0\}$$

and for 0^* we define open sets to be

$$N(0^*,r) = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < \frac{1}{r^2}, y < 0\right\} \cup \{0^*\}.$$

So $N(0,r)$ is the top half of a disk of radius $\frac{1}{r}$ centered at the origin, excluding the y-axis but including the origin, and $N(0^*,r)$ is the bottom half of a disk of radius $\frac{1}{r}$ centered at the origin, excluding the y-axis and including the point 0^*.

(a) Show that the collection of sets described as a basis for the Double Origin topology is actually a basis for a topology.

(b) Is X with the Double Origin topology Hausdorff? Prove your answer.

(14) (a) Show that finite sets are closed in $\mathbb{R}^n$ with the Zariski topology.

(b) Show that $\mathbb{R}^n$ with the Zariski topology is not Hausdorff. (Exercise 12 on page 128 shows that a basis for the Zariski topology on $\mathbb{R}^n$ is the collection of sets of the form $\mathbb{R}^n \setminus Z(f)$, where $Z(f)$ is the set of zeros of the polynomial f in n variables.)

(15) Consider the digital line topology τ_{dl} on $\mathbb{Z}$ with basis $\{B(n)\}$, where

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is an odd integer,} \\ \{n-1, n, n+1\} & \text{if } n \text{ is an even integer.} \end{cases}$$

(a) Let $A = \{-1, 0, 1\}$ of $(\mathbb{Z}, \tau_{dl})$.

i. Find the limit points and boundary points of A. Prove your conjectures. Is every limit point of A a boundary point of A? Is every boundary point of A a limit point of A?

ii. Find $\overline{A}$ and write $X \setminus \overline{A}$ as a union of open sets.

(b) Now consider the subset $B = \{0\}$ of $(\mathbb{Z}, \tau_{dl})$.

i. Find the limit points and boundary points of B. Prove your conjectures. Is every limit point of B a boundary point of B? Is every boundary point of B a limit point of B?

ii. Find $\overline{B}$ and write $X \setminus \overline{B}$ as a union of open sets.

(16) Let (X, d) be a metric space. Recall from Exercise 8 on page 105 in Section 10, that if C is a closed subset of a metric space X and x is an element of $X \setminus C$, then $d(x, C) = 0$ if and only if $x \in C$. Use this idea to do the following.

(a) Prove that every metric space is regular.

(b) Prove that every metric space is a normal space.

(17) Let $K = \left\{\frac{1}{k} \mid k \text{ is a positive integer}\right\}$. Let $\mathcal{B}$ be the collection of all open intervals of the form (a, b) and all sets of the form $(a, b) \setminus K$, where $a < b$ are real numbers as in Example 13.13 on page 137. That $\mathcal{B}$ generates a topology τ_K on $\mathbb{R}$ follows from the fact that τ_K is finer than the Euclidean topology.

(a) Show that $(\mathbb{R}, \tau_K)$ is a Hausdorff space.

(b) Exercise 16 shows that every metric space is regular. In this part of the exercise, show that (R, τ_K) is not a regular space. (Hint: Consider 0 and K.) We can conclude that $(\mathbb{R}, \tau_K)$ is not metrizable.

(18) (a) Prove that a topological space X is T_1 if and only if each singleton set is closed.

(b) Show that every T_2-space is T_1, that every T_3-space is T_2, and that every T_4-space is T_3.

(19) In this exercise we illustrate spaces that are T_1 but not T_2 and T_2 but not T_3.

(a) Show that $\mathbb{R}$ with the finite complement topology is T_1 but not T_2.

(b) Define the space $\mathbb{R}_K$ as in Example 13.13 to be the set of reals with topology τ with a basis that consists of the standard open intervals in $\mathbb{R}$ along with all sets of the form $(a, b) \setminus K$, where (a, b) is any open interval and $K = \left\{\frac{1}{k} \mid k \in \mathbb{Z}^+\right\}$. Show that $\mathbb{R}_K$ is T_2 but not T_3.

(20) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) Every limit point of a subset A of a topological space X is also a boundary point of A.

(b) Every boundary point of a subset A of a topological space X is also a limit point of A.

(c) If X is a topological space and $A \subseteq X$ such that $\text{Int}(A) = \overline{A}$, then A is both open and closed.

(d) If X is a topological space and A and B are subsets of X with $\overline{A} = \overline{B}$ and $\text{Int}(A) = \text{Int}(B)$, then $A = B$.

(e) If A and B are subsets of a topological space X, then $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

(f) If A and B are subsets of a topological space X, then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.