Section 14

Continuity and Homeomorphisms

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- How do we define a continuous function between topological spaces?
- What is the difference between metric equivalence and topological equivalence?
- What is a homeomorphism? What does it mean for two topological spaces to be homeomorphic?
- What is a topological invariant? Why are topological invariants useful?

Introduction

Recall that we could characterize a function f from a metric space (X, d_X) to a metric space (Y, d_Y) as continuous at $a \in X$ if $f^{-1}(N)$ is a neighborhood of a in X whenever N is a neighborhood of $f(a)$ in Y. We have defined neighborhoods in topological spaces, so we can use this characterization as our definition of a continuous function from one topological space to another.

Definition 14.1. A function f from a topological space (X, τ_X) to a topological space (Y, τ_Y) is **continuous at a point** $a \in X$ if $f^{-1}(N)$ is a neighborhood of a in X whenever N is a neighborhood of $f(a)$ in Y. The function f is **continuous** if f is continuous at each point in X.

We saw that in metric spaces, a useful characterization of continuity was in terms of open sets. It is not surprising that we have the same characterization in topological spaces. You may assume the result of Theorem 14.2 (the topological space version of Theorem 8.5 on 78 for metric spaces) for this activity.

Theorem 14.2. *Let f be a function from a topological space (X, τ_X) to a topological space (Y, τ_Y). Then f is continuous if and only if $f^{-1}(O)$ is an open set in X whenever O is an open set in Y.*

Preview Activity 14.1.

(1) Let
$$(X, \tau_X) = (\{1,2,3,4\}, \{\emptyset, \{1\}, \{2\}, \{1,2\}, X\})$$
and let
$$(Y, \tau_Y) = (\{2,4,6,8\}, \{\emptyset, \{4\}, \{6\}, \{4,6\}, Y\}).$$
Define $f : X \to Y$ by $f(x) = 2x$.

(a) Is f continuous at 4?

(b) Is f a continuous function?

(2) Let
$$(X, \tau_X) = (\{1,2,3,4\}, \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4\}, X\})$$
and let
$$(Y, \tau_Y) = (\{a,b,c\}, \{\emptyset, \{a\}, \{a,c\}, Y\}).$$
Define $f : X \to Y$ by $f(1) = a$, $f(2) = c$, $f(3) = f(4) = b$.

(a) Show that f is a continuous function.

(b) Even though f is continuous, it is possible that $f(O)$ may not be open for every open set in X. Find such an example for this function f.

Functions f that have the property that $f(O)$ is open whenever O is open in X are called *open* functions.

Definition 14.3. Let $f : X \to Y$ be a function from a topological space X to a topological space Y. Then f is an **open** function if $f(U)$ is open in Y whenever U is open in X.

There is a similar definition of a *closed* function.

(3) Let $X = \{1,2,3,4,5\}$ and $\tau = \{\emptyset, \{1\}, \{3,5\}, \{1,3,5\}, X\}$. Define $f : X \to X$ by $f(x) = |x - 3| + 1$. At which points is f continuous? Is f a continuous function?

(4) Let $f : (\mathbb{Z}, \tau_{FC}) \to (\mathbb{Z}, d_E)$ where $f(n) = n$ and τ_{FC} is the finite complement topology. Is f a continuous function? If f is not continuous, exhibit a specific point at which f fails to be continuous. Explain.

(5) Let $f : (\mathbb{Z}, d_E) \to (\mathbb{Z}, \tau_{FC})$ where $f(n) = n$ and τ_{FC} is the finite complement topology. Is f a continuous function? If f is not continuous, exhibit a specific point at which f fails to be continuous. Explain.

(6) It can sometimes be easier to show that a function f mapping a topological space (X, d_X) to a topological space (Y, d_Y) is continuous by working with a basis instead of all open sets. Let $\mathcal{B}$ be a basis for the topology on Y. Is it the case that if $f^{-1}(B)$ is open for every $B \in \mathcal{B}$, then f is continuous? Verify your result.

To complete the introduction to this section, we prove Theorem 14.2. We prove one direction now and leave the other for the next activity.

Let f be a function from a topological space (X, τ_X) to a topological space (Y, τ_Y). We first assume that f is continuous and show that $f^{-1}(O)$ is an open set in X whenever O is an open set in Y. Suppose that O is an open set in Y. To show that $f^{-1}O)$ is open in X, we will show that $f^{-1}(O)$ is a neighborhood of each of its points. Let $a \in f^{-1}(O)$. Then $f(a) \in O$. Since O is an open set, O is a neighborhood of $f(a)$. The fact that f is continuous means that $f^{-1}(O)$ is a neighborhood of a. So $f^{-1}(O)$ is a neighborhood of each of its points and $f^{-1}(O)$ is an open set.

Activity 14.1. Now we prove the remaining implication in Theorem 14.2. That is, let f be a function from a topological space (X, τ_X) to a topological space (Y, τ_Y), and assume that $f^{-1}(O)$ is open whenever O is open in Y. We will prove that f is a continuous function.

(a) Using the definition, what does it take to show that f is a continuous function?

(b) Let $a \in X$ and suppose that N is a neighborhood of $f(a)$ in Y. What can we conclude from N being a neighborhood?

(c) Use the assumption about f in this activity to explain why $f^{-1}(N)$ is a neighborhood of a in X.

(d) Explain how we have shown that f is a continuous function.

The following theorem is the topological analog of Theorem 10.5 on page 99. The proof is left for Exercise (4).

Theorem 14.4. *Let f be a function from a topological space (X, τ_X) to a topologicval space (Y, τ_Y). Then f is continuous if and only if $f^{-1}(C)$ is closed in X whenever C is a closed set in Y.*

Metric Equivalence

We have seen that we can make a set into a metric space with different metrics. For example, the spaces $(\mathbb{R}^2, d_E)$, $(\mathbb{R}^2, d_T)$, $(\mathbb{R}^2, d_M)$, and $(\mathbb{R}^2, d)$ are all metric spaces, where d_E is the Euclidean metric, d_T the taxicab metric, d_M the max metric, and d the discrete metric. But are these metric spaces really "different" metric spaces? What do we mean by "different"?

Activity 14.2. We might consider two metric spaces (X, d_X) and (Y, d_Y) to be equivalent if we can find a bijection between the two sets X and Y that preserves the metric properties. That is, find a bijective function $f : X \to Y$ such that $d_X(a, b) = d_Y(f(a), f(b))$ for all $a, b \in X$. In other words, f preserves distances.

(a) Let $X = ((0, 1), d_X)$ and $Y = ((0, 2), d_Y)$, with both d_X and d_Y the Euclidean metric. Is it possible to find a bijection $f : X \to Y$ that preserves the metric properties? Explain.

(b) Now let $X = ((0, 1), d_X)$ and $Y = ((0, 2), d_Y)$, where d_X is defined by $d_X(a, b) = 2|a-b|$ and $d_Y = d_E$. You may assume that d_X is a metric. Is it possible to find a bijection $f : X \to Y$ that preserves the metric properties? Explain.

If there is a bijection between metric spaces that preserves distances, we say that the metric spaces are *metrically equivalent*.

Definition 14.5. Two metric spaces (X, d_X) and (Y, d_Y) are **metrically equivalent** if there is a bijection $f : X \to Y$ such that

$$d_X(x, y) = d_Y(f(x), f(y))$$

for all $x, y \in X$.

Because f is a bijection, it will also be the case in Definition 14.5 that

$$d_Y(u, v) = d_X(f^{-1}(u), f^{-1}(v))$$

for all u and v in Y. The proof is left for Exercise (1).

Any function f that preserves distances (like the one in Definition 14.5) is called an *isometry*.

Definition 14.6. A function f from a metric space (X, d_X) to a metric space (Y, d_Y) is an **isometry** if f is a bijection and

$$d_Y(f(a), f(b)) = d_X(a, b) \tag{14.1}$$

for all $a, b \in X$.

Metric equivalence is a very strong type of equivalence – the existence of an isometry does not allow for much flexibility since distances must be preserved. From a topological perspective, we are only concerned about the open sets – there are no distances. The open unit ball in $(\mathbb{R}^2, d_E)$ and the open ball in $(\mathbb{R}^2, d_M)$ (where d_E is the Euclidean metric and d_M is the max metric) are not that different as we can see in Figure 14.1. If we don't worry about preserving distances, we can stretch the open ball $B_E = B((0, 0), 1)$ in $(\mathbb{R}^2, d_E)$ along the lines $y = x$ and $y = -x$ uniformly in a way to mold it onto the unit ball $B_M = B((0, 0), 1)$ in $(\mathbb{R}^2, d_M)$. The important thing is that this stretching will preserve the open sets. This is a much more forgiving type of equivalence and maintains the central idea of topology that we have discussed – what properties of a space are not altered by stretching and bending the space. This type of equivalence that allows us to manipulate a space without fundamentally changing the open sets is called *topological equivalence*.

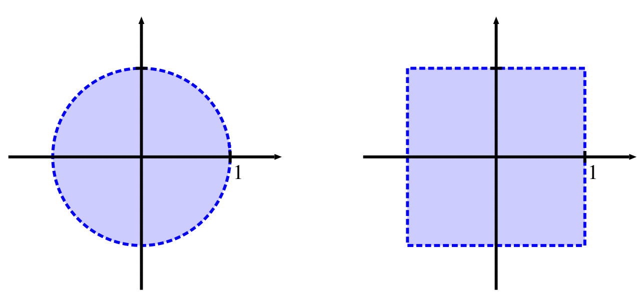

Figure 14.1: The open unit balls in $(\mathbb{R}^2, d_E)$ and $(\mathbb{R}^2, d_M)$.

Topological Equivalence

When we can deform one set into another without poking holes in the set, we consider the two sets to be equivalent from a topological perspective. Such a deformation f has to be a bijection to ensure that the two sets contain the same number of elements, continuous so that the inverse images of open sets are open, and f^{-1} must be continuous so images of open sets are open. Such a function provides a one-to-one correspondence between open sets in the two spaces. This leads to the next definition.

Definition 14.7. Two topological spaces (X, d_X) and (Y, d_Y) are **topologically equivalent** if there is a continuous bijection $f : X \to Y$ such that f^{-1} is also continuous.

Metric equivalence always implies topological equivalence (using the metric topologies), which is left for Exercise (3). So metric equivalence is a stronger condition than topological equivalence.

The function f (or f^{-1}) in Definition 14.7 is called a *homeomorphism*.

Definition 14.8. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is a **homeomorphism** if f is a continuous bijection such that f^{-1} is also continuous.

If there is a homeomorphism from (X, τ_X) to (Y, τ_Y) we say that the spaces (X, τ_X) to (Y, τ_Y) are *homeomorphic* topological spaces.

It can be difficult to show directly that two metric spaces are homeomorphic, but there are ways to make the process easier in metric spaces. If f is a homeomorphism from the metric space $(\mathbb{R}^2, d_E)$ to the metric space $(\mathbb{R}^2, d_M)$, the continuity of f ensures a smooth deformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. In terms of the metrics, this means that distances cannot get distorted too much – in fact, the amount distances are distorted should be bounded. In other words, we might expect that there is a constant K so that $d_E(x, y) \leq K d_M(f(x), f(y))$ for any $x, y \in \mathbb{R}^2$. The next theorem tells us that this is a sufficient condition for topological equivalence when we work in the same underlying space.

Theorem 14.9. *Let X be a set on which two metrics d and d' are defined. If there exist positive constants K and K' so that*

$$\begin{aligned} d'(x, y) &\leq K d(x, y) \\ d(x, y) &\leq K' d'(x, y) \end{aligned}$$

for all $x, y \in X$, then (X, d) is topologically equivalent to (X, d').

Proof. Let X be a set on which two metrics d and d' are defined. Suppose there exist positive constants K and K' so that

$$\begin{aligned} d'(x, y) &\leq K d(x, y) \\ d(x, y) &\leq K' d'(x, y) \end{aligned}$$

for all $x, y \in X$. Let $i_X : (X, d) \to (X, d')$ be the identity mapping. That is, $i_X(x) = x$ for all $x \in X$. We will prove that i_X is a homeomorphism. We know that i_X is a bijection, so we only need verify that i_X and i_X^{-1} are continuous. Let $\epsilon > 0$ be given, and let $a \in X$. Let $\delta = \frac{\epsilon}{K}$. Suppose $x \in X$ so that $d(x, a) < \delta$. Then

$$d'(i_X(x), i_X(a)) = d'(x, a) \leq Kd(x, a) < K\delta = K\left(\frac{\epsilon}{K}\right) = \epsilon.$$

Thus, i_X is continuous. The same argument shows that i_X^{-1} is also continuous. Therefore, i_X is a homeomorphism between (X, d) and (X, d'). ■

Activity 14.3.

(a) Are $(\mathbb{R}^2, d_T)$ and $(\mathbb{R}^2, d_M)$ topologically equivalent? Explain.

(b) Are $(\mathbb{R}^2, d_E)$ and $(\mathbb{R}^2, d_T)$ topologically equivalent? Explain.

(c) Do you expect that $(\mathbb{R}^2, d_E)$ and $(\mathbb{R}^2, d_M)$ are topologically equivalent. Explain without doing any calculations or comparisons.

Relations

We use the word "equivalent" deliberately when talking about metric or topological equivalence. Recall that equivalence is a word used with relations, and that a relation is a way to compare two elements from a set. We are familiar with many relations on sets, "$<$", "$=$", "$\geq$" on the integers, for example.

Definition 14.10. A *relation* on a set S is a subset R of $S \times S$.

For example, the subset $R = \{(a, a) \mid a \in \mathbb{Z}\}$ of $\mathbb{Z} \times \mathbb{Z}$ is the relation we call equals. If R is a relation on a set S, we usually suppress the set notation and write $a \sim b$ if $(a, b) \in R$ and say that a is related to b. In this case we often refer to $\sim$ as the relation instead of the set R. Sometimes we use familiar symbols for special relations. For example, we write $a = b$ if $(a, b) \in R = \{(a, a) \mid a \in \mathbb{Z}\}$.

When discussing relations, there are three specific properties that we consider.

- A relation $\sim$ on a set S is *reflexive* if $a \sim a$ for all $a \in S$.
- A relation $\sim$ on a set S is *symmetric* if whenever $a \sim b$ in S we also have $b \sim a$.
- A relation $\sim$ on a set S is *transitive* if whenever $a \sim b$ and $b \sim c$ in S we also have $a \sim c$.

When we use the word "equivalence", we are referring to an equivalence relation.

Definition 14.11. An **equivalence relation** is a relation on a set that is reflexive, symmetric, and transitive.

Activity 14.4.

(a) Explain why metric equivalence is an equivalence relation.

(b) Explain why topological equivalence is an equivalence relation.

Equivalence relations are important because an equivalence relation on a set S partitions the set into a disjoint union of equivalence classes. Since topological equivalence is an equivalence relation, we can treat the spaces that are topologically equivalent to each other as being essentially the same space from a topological perspective. Under the relation of homeomorphism we call the equivalence classes *homeomorphism* classes.

Topological Invariants

Homeomorphic topological spaces are essentially the same from a topological perspective, and they share many properties, but not all. The properties they share are called *topological invariants* or *topological properties*.

Definition 14.12. A property of a topological space X is a **topological property** (or **topological invariant**) if every topological space homeomorphic to X has the same property.

Activity 14.5. Which of the following are topological invariants? That is for topological spaces (X, τ_X) and (Y, τ_Y), if X and Y are homeomorphic space and X has the property, does it follow that Y must also have that property?

(a) X has the indiscrete topology

(b) X has the discrete topology

(c) X has the finite complement topology

(d) X contains the number 2

(e) X contains exactly 13 elements

Summary

Important ideas that we discussed in this section include the following.

- A function f from a topological space X to a topological space Y is continuous if $f^{-1}(O)$ is open in X whenever O is open in Y.
- Two metric spaces (X, d_X) and (Y, d_Y) are metrically equivalent if there is a bijection $f : X \to Y$ such that

$$d_X(x, y) = d_Y(f(x), f(y))$$
$$d_Y(u, v) = d_X(f^{-1}(u), f^{-1}(v))$$

  for all $x, y \in X$ and $u, v \in Y$. That is, X and Y are metrically equivalent if there is a isometry f from X to Y such that f^{-1} is also an isometry. Topological equivalence is a less stringent condition. Two topological spaces X and Y are topologically equivalent if there is a continuous function f from X to Y such that f^{-1} is also continuous. That is, X and Y are topologically equivalent if there is a homeomorphism between X to Y.

- A homeomorphism between topological spaces X and Y is a continuous function f from X to Y such that f^{-1} is also continuous. Two topological spaces X and Y are homeomorphic if there is a homeomorphism $f : X \to Y$.
- A topological invariant is any property that topological space X has that must also be a property of any topological space homeomorphic to X. We can sometimes use topological invariants to determine if two topological spaces are not homeomorphic.

Exercises

(1) Let (X, d_X) and (Y, d_Y) be metrically equivalent metric spaces, and let $f : X \to Y$ be a bijection such that

$$d_X(x, y) = d_Y(f(x), f(y))$$

for all $x, y \in X$. Prove that

$$d_Y(u, v) = d_X(f^{-1}(u), f^{-1}(v))$$

for all u and v in Y.

(2) Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : X \to Y$ be a homeomorphism. Let A be a subset of X.

(a) If x is a limit point of A, must $f(x)$ be a limit point of $f(A)$? Prove your answer.

(b) If x is an interior point of A, must $f(x)$ be an interior point of $f(A)$? Prove your answer.

(c) If x is a boundary point of A, must $f(x)$ be a boundary point of $f(A)$? Prove your answer.

(3) Let (X, d_X) and (Y, d_Y) be metrically equivalent metric spaces. Show that X and Y are topologically equivalent using the metric topologies.

(4) Prove theorem 14.4 that if (X, τ_X) and (Y, τ_Y) are topological spaces, and $f : X \to Y$ is a function, then f is continuous if and only if $f^{-1}(C)$ is a closed set in X whenever C is a closed set in Y.

(5) Let (X, τ_X), (Y, τ_Y), and (Z, τ_Z) be topological spaces.

(a) Let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Prove that $g \circ f : X \to Z$ is a continuous function. (Hint: Exercise (9) on page 26 could be helpful here.)

(b) Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a homeomorphism. Let h be a function from (Y, τ_Y) to (Z, τ_Z) and let k be a function from (Z, τ_Z) to (X, τ_X) .

i. Prove that h is continuous if and only if $h \circ f$ is continuous.

ii. Prove that k is continuous if and only if $f \circ k$ is continuous.

(6) Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{a, b, d\}, X\}$.

(a) Find a function $f : X \to X$ that is continuous at exactly one point, or show that no such function exists.

(b) Find a function $f : X \to X$ that is continuous at exactly two points, or show that no such function exists.

(c) Find a function $f : X \to X$ that is continuous at exactly three points, or show that no such function exists.

(7) Consider $\mathbb{R}$ and $\mathbb{R}^2$ equipped with the Euclidean topology. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let

$$\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{R}\}$$

be the graph of f. Note that Γ_f is a subspace of $\mathbb{R}^2$ and is a topological space using the subspace topology.

(a) Show that if f is a continuous function, then Γ_f is homeomorphic to $\mathbb{R}$.

(b) If we remove the condition that f is continuous, must it still be the case that Γ_f is homeomorphic to $\mathbb{R}$? Prove your conjecture.

(8) Let X be a nonempty set and let p be a fixed element in X. Let τ_p be the particular point topology and $\tau_{\overline{p}}$ the excluded point topology on X. That is

- τ_p is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that contain p.
- $\tau_{\overline{p}}$ is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that do not contain p.

That the particular point and excluded point topologies are topologies is the subject of Exercises (9) and (10) on page 127.

(a) Let p be a fixed point in $\mathbb{R}$. Is the identity function $i : \mathbb{R} \to \mathbb{R}$ defined by $i(x) = x$ for all $x \in \mathbb{R}$ a homeomorphism from $(\mathbb{R}, \tau_p)$ to $(\mathbb{R}, \tau_{\overline{p}})$? Prove your answer.

(b) Is $(\mathbb{R}, \tau_p)$ homeomorphic to $(\mathbb{R}, \tau_{\overline{p}})$ with the specific point $p = 0$? Prove your answer.

(9) A topological space X is *embedded* in a topological space Y if there is a homeomorphism from X to some subspace of Y. The homeomorphism is called an *embedding*.

(a) Show that if X is the open interval $(0, 1)$ with the Euclidean metric topology, then X can be embedded in the topological space $\mathbb{R}$ with the Euclidean metric topology.

(b) Show that there exist non-homeomorphic topological spaces A and B for which A can be embedded in B and B can be embedded in A.

(10) Let $X = \{a, b\}$ be a two element set.

(a) Find all of the distinct topologies on X. Be sure to explain how you know you have identified all of the topologies.

(b) Determine the distinct homeomorphism classes of topological spaces on two elements. Justify your response.

(11) Let $X = \{a, b, c\}$. There are 29 distinct topologies on X, shown below. Determine the number of distinct homeomorphism classes for these 29 topologies and identify the elements of each homeomorphism class. Justify your answers.

1. $\{\emptyset, X\}$
2. $\{\emptyset, \{a, b\}, X\}$
3. $\{\emptyset, \{a, c\}, X\}$
4. $\{\emptyset, \{b, c\}, X\}$
5. $\{\emptyset, \{a\}, X\}$
6. $\{\emptyset, \{b\}, X\}$
7. $\{\emptyset, \{c\}, X\}$
8. $\{\emptyset, \{a\}, \{a, b\}, X\}$
9. $\{\emptyset, \{a\}, \{a, c\}, X\}$
10. $\{\emptyset, \{a\}, \{b, c\}, X\}$
11. $\{\emptyset, \{b\}, \{a, b\}, X\}$
12. $\{\emptyset, \{b\}, \{a, c\}, X\}$
13. $\{\emptyset, \{b\}, \{b, c\}, X\}$
14. $\{\emptyset, \{c\}, \{a, b\}, X\}$
15. $\{\emptyset, \{c\}, \{a, c\}, X\}$
16. $\{\emptyset, \{c\}, \{b, c\}, X\}$
17. $\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$
18. $\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$
19. $\{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$
20. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$
21. $\{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$
22. $\{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$
23. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$
24. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$
25. $\{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, X\}$
26. $\{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$
27. $\{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}, X\}$
28. $\{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$
29. the discrete topology

(12) Show that property T_i is a topological property for each i. (See Section 13 for definitions of the separation axioms.)

(13) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) If $f : X \to Y$ is a continuous function between topological spaces X and Y, then for every open subset U of X, $f(U)$ is open in Y.

(b) If τ_{FC} is the finite complement topology, then $f(x) = x^2$ mapping $(\mathbb{R}, \tau_{FC})$ to $(\mathbb{R}, \tau_{FC})$ is continuous.

(c) If $f : X \to Y$ is a bijective function between topological spaces X and Y, and for every open subset U of X, $f(U)$ is open in Y, then f is a homeomorphism.

(d) If X and Y are topological space with the discrete topologies, and if $f : X \to Y$ is a bijection, then the spaces X and Y are homeomorphic.

(e) Let S be a set and let R_1 and R_2 be equivalence relations on S. Then $R = R_1 \cap R_2$ is also an equivalence relation on S.

(f) Let S be a set and let R_1 and R_2 be equivalence relations on S. Then $R = R_1 \cup R_2$ is also an equivalence relation on S.