Section 15

Subspaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a subspace of a topological space?
- How do we define the subspace topology?
- What are relatively open and closed sets?
- To what kind of spaces is $\mathbb{R}$ with the standard topology homeomorphic?

Introduction

We have seen that a subset A of a metric space (X, d_X) is a subspace of X using the restriction of the metric d_X to A. We do not have a metric in general topological spaces, so that approach can't be duplicated. But, we proved that the open sets in a subspace A of a metric space (X, d_X) are exactly the intersections of open sets in X with A. That idea can be transferred to topological spaces.

To make a subspace A of a topological space (X, τ) into a topological space, we need to define a topology on A.

Preview Activity 15.1. Let (X, τ) be a topological space and A a nonempty subset of X. It is reasonable to use the open sets in X to define open sets in A. More specifically, we might consider a subset O_A of A to be open in A if O_A is the intersection of A with some open set in X, as illustrated in Figure 15.1. With this in mind we define τ_A as

$$\tau_A = \{O \cap A \mid O \in \tau\}.$$

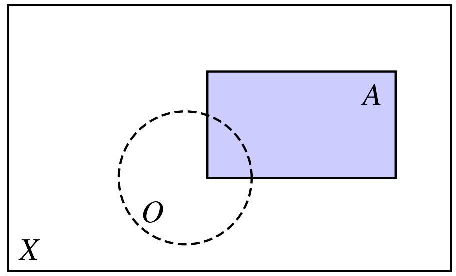

Figure 15.1: A potentially open subset in a subspace.

(1) Show that τ_A is a topology on A.

The result of item (1) is that any subset of a topological space (X, τ) is also a topological space with topology τ_A.

Definition 15.1. Let (X, τ) be a topological space. A **subspace** of (X, τ) is a nonempty subset A of X together with the topology

$$\tau_A = \{O \cap A \mid O \in \tau\}.$$

(2) For each of the following, X is a topological space and τ is a topology on X.

(a) Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Consider the subset $A = \{b, c\}$ and list the open sets in the subspace topology τ_A. Now consider $Z = \{a, b\}$. What is the name of the subspace topology τ_Z on this subset of X?

(b) Consider $X = \mathbb{R}$ with τ the indiscrete topology. What are the open sets in the subspace topology on $[1, 2]$? Now generalize to any nonempty set in the indiscrete topology.

(c) Let $X = \{a, b, c, d, e, f, g, h, i\}$ with τ the discrete topology. What are the open sets in the subspace topology on $\{a, b, d\}$. Now generalize to any nonempty set in the discrete topology.

(d) Let $X = \{a, b, c, d, e, f\}$ with $\tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}, X\}$. What are the open sets in the subspace $A = \{a, b, e\}$? Is every open set in A an open set in X? Explain.

(e) Let $X = \mathbb{Z}$ with $\tau = \tau_{FC}$ the finite complement topology. What are the open sets in the subspace topology on $A = \{0, 19, 37, 5284\}$? Can you generalize this to the subspace topology on any finite subset of $\mathbb{Z}$?

(f) Let $X = \mathbb{Z}$ with $\tau = \tau_{FC}$ the finite complement topology. What are the open sets in the subspace topology on the even integers? Can you generalize this to the subspace topology on any infinite subset of $\mathbb{Z}$?

The Subspace Topology

In our preview activity, we saw that the intersection of the open sets in a topological space X with any nonempty subset A of X forms a topology for A. We then have A as a subspace of X.

The topology τ_A in Definition 15.1 is called the *subspace topology*, the *induced topology*, or the *relative topology*. In our preview activity we saw that sets that are open in a subspace A of a topological space X need not be open in X. So we call the sets in τ_A *relatively open*.

Once we have defined relatively open sets, we can then consider how to define relatively closed sets.

Activity 15.1. Let (X, τ) be a topological space, and let A be a subset of X.

(a) Recall that a subset of a topological space is closed if its complement is open. Given that (A, τ_A) is a topological space, how is a closed set in A defined? Such a set will be called *relatively closed*.

(b) Recall that a subset U of A is relatively open if and only if $U = A \cap O$ for some open subset of X. With this in mind, how might we expect a relatively closed set in A to be related to a closed set in X? State and prove a theorem for this result.

Bases for Subspaces

Recall that a basis $\mathcal{B}$ for a topological space is a collection of sets that generate all of the open sets through unions. If we have a basis $\mathcal{B}$ for a topological space (X, τ), and if A is a subspace of X, we might ask if we can find a basis $\mathcal{B}_A$ from $\mathcal{B}$ in a natural way.

Activity 15.2. Let (X, τ) be a topological space with basis $\mathcal{B}$, and let A be a subspace of X.

(a) There is a natural candidate to consider as a basis $\mathcal{B}_A$ for A. How do you think we should define the elements in $\mathcal{B}_A$?

(b) Recall that a set $\mathcal{B}$ is a basis for a topological space X if

(1) For each $x \in X$, there is a set in $\mathcal{B}$ that contains x.

(2) If $x \in X$ is an element of $B_1 \cap B_2$ for some $B_1, B_2 \in B$, then there is a set $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Show that your set from (a) is a basis for the induced topology on A.

Open Intervals and $\mathbb{R}$

If we think of a homeomorphism as allowing us to stretch or bend a space, it is reasonable to think that we could stretch an open interval of the form (a, b) infinitely in both directions without altering the nature of the open sets. That is, we should expect that $\mathbb{R}$ with the standard topology is homeomorphic to (a, b) with the subspace topology.

Activity 15.3. Let a and b be real numbers with $a < b$. To show that $(\mathbb{R}, d_E)$ is homeomorphic to (a, b), we need a continuous bijection from $\mathbb{R}$ to (a, b) whose inverse is also continuous.

(a) First we demonstrate that $(0, 1)$ and $\mathbb{R}$ are homeomorphic using the Euclidean metric topology. Let $f : (0, 1) \to \mathbb{R}$ be defined by

$$f(x) = \tan\left(\pi\left(x - \frac{1}{2}\right)\right).$$

i. Explain why f maps $(0, 1)$ to $\mathbb{R}$.

ii. Explain why f is an injection.

iii. Explain why f is a surjection.

iv. Explain why f and f^{-1} are continuous. (Hint: Use a result from calculus.)

(b) The result of (a) is that $\mathbb{R}$ and $(0, 1)$ are homeomorphic spaces. To complete the argument that $\mathbb{R}$ is homeomorphic to (a, b), define a function $g : (0, 1) \to (a, b)$ and explain why your g is a homeomorphism.

It is left to Exercise (4) to show that $\mathbb{R}$ is also homeomorphic to any interval of the form (a, ∞) or $(-\infty, b)$. Later we will determine if $\mathbb{R}$ is homeomorphic to intervals of the form $[a, b)$, $(a, b]$, $[a, \infty)$ or $(-\infty, b]$.

Summary

Important ideas that we discussed in this section include the following.

- A subspace of a topological space is any nonempty subset of the topological space endowed with the subspace topology.

- An open subset in the subspace topology for a subset A of a topological space X is any set of the form $O \cap A$, where O is an open set in X.

- The relatively open sets are the open sets in a subspace topology. The relatively closed sets are complements of the relatively open sets in a subspace topology. That is, a relatively closed set in the subspace A of a topological space X are the sets of the form $A \cap C$, where C is a closed set in X.

- The topological space $\mathbb{R}$ with the standard topology is homeomorphic to any open interval as well as open intervals of the form (a, ∞) or $(-\infty, b)$ for any real numbers a and b.

Exercises

(1) Let X and Y be topological spaces and $f : X \to Y$ be a continuous function. If A is a subspace of X, prove that $f|_A : A \to Y$ is also continuous.

(2) Let X be a topological space, let A be a subspace of X, and let B be a subspace of A. Show that the subspace topology that B inherits from A is the same as the subspace topology that B inherits from X.

(3) Let A be a subspace of a topological space X and let B be a subset of A.

   (a) Prove that a point x in A is a limit point of B in the subspace topology for A if and only if x is a limit point of B in the topology on X.

   (b) Prove that the closure of B in the subspace topology for A is equal to $\overline{B} \cap A$, where $\overline{B}$ is the closure of B in X.

(4) Show that $\mathbb{R}$ is homeomorphic, with the standard topology, to any interval of the form (a, ∞) or $(-\infty, b)$.

(5) Let X be a topological space.

   (a) Let O be an open subset of X. Prove that a subset A of O is open in O if and only if A is open in X.

   (b) Let C be a closed subset of X. Prove that a subset B of C is closed in C if and only if B is closed in X.

(6) A property of a topological space is said to be hereditary if that property is inherited by every subspace. We state this more formally in the following definition.

   Definition 15.2. A property P of a topological space X is **hereditary** if every subspace of X also has property P.

   Show that properties T_1, T_2, and T_3 are hereditary. (The separation axioms T_i are found on page 131.) The fact that T_4 is not hereditary is somewhat difficult. One example is the Tychonoff plank (which is normal) with the Deleted Tychonoff plank (which is not normal) as subspace. An interested reader can consult *Counterexamples in Topology (2nd ed.)*, Lynn Arthur Steen and J. Arthur Seebach, Jr., Dover Publications, 1978.

(7) Suppose that $f : X \to Y$ is a homeomorphism from a topological space X to a topological space Y. Let $a \in X$. Must the subspace $X' = X \setminus \{a\}$ of X be homeomorphic to the subspace $Y' = Y \setminus \{f(a)\}$ of Y? Prove your conjecture.

(8) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

   (a) If X has the discrete topology, then every subspace of X has the discrete topology.

   (b) If X is a topological space that does not have the discrete topology, then no subspace of X has the discrete topology.

   (c) If $f : X \to Y$ is a continuous function between topological spaces X and Y, and X is Hausdorff, then the subspace $f(X)$ of Y is Hausdorff.

   (d) If A is a subspace of a topological space X and B is a subset of X, then the closure of $B \cap A$ in the subspace topology for A equals $\overline{B} \cap A$, where $\overline{B}$ is the closure of B in X.

(e) If A is a subspace of a topological space X and C is a subset of X, then the interior of $C \cap A$ in the subspace topology for A equals Int$(C) \cap A$, where Int(C) is the interior of C in X.