Section 16

Quotient Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a quotient topology?
- What is a quotient space?
- What are two examples of familiar quotient spaces?

Introduction

We are familiar with the word *quotient* when working with rational numbers. That is, the fraction $\frac{1}{2}$ is the quotient of 1 by 2, and the set of rational numbers is the collection of all defined quotients of integers. The word quotient seems to have come from the latin word "quotiens", which can be translated as "how often" or "how many times". We can think of the fraction $\frac{1}{2}$ as dividing a unit (1) into two pieces. So we often apply the word quotient to any kind of construction that divides a set into pieces. Another familiar quotient construction is the set $\mathbb{Z}_n$, the set of quotients of integers after we divide by n. Another way to think of $\mathbb{Z}_n$ is as a quotient $\mathbb{Z}/n\mathbb{Z}$, where $n\mathbb{Z}$ is the set of multiples of n and two integers a and b are identified with each other (or are *equivalent*) if $b - a \in n\mathbb{Z}$. This defines the relation of congruence module n on $\mathbb{Z}$, and the elements of $\mathbb{Z}/n\mathbb{Z}$ are the equivalence classes for this relation. We make similar constructions in many branches of mathematics by defining an equivalence relation on a set, and we then divide the set into pieces (the equivalence classes) and call the set of equivalence classes a *quotient space*. We explore the concept of quotient spaces of topological spaces in this section.

As an example, take the interval $X = [0, 1]$ in $\mathbb{R}$ and bend it to be able to glue the endpoints together. The resulting object is a circle. By identifying the endpoints 0 and 1 of the interval, we are able to create a new topological space. We can view this gluing or identifying of points in the space X in a formal way that allows us to recognize the resulting space as a quotient space.

The Quotient Topology

Given a topological space X and a surjection p from X to a set Y, we can use the topology on X to define a topology on Y. This topology on Y identifies points in X through the function p. The resulting topology on Y is called a *quotient* topology. The quotient topology gives us a way of creating a topological space which models gluing and collapsing parts of a topological space.

Preview Activity 16.1. Let $X = \{1, 2, 3, 4, 5, 6\}$ and let $\tau = \{\emptyset, \{1, 2\}, \{4, 6\}, \{1, 2, 4, 6\}, X\}$. Let $Y = \{a, b, c, d\}$ and define $p : X \to Y$ by

$$p(1) = b, \ p(2) = a, \ p(3) = c, \ p(4) = d, \ p(5) = c, \ \text{ and } \ p(6) = a.$$

Our goal in this activity is to define a topology on Y that is related to the topology on X via p.

(1) We know the sets in X that are open. So let us consider the sets U in Y such that $p^{-1}(U)$ is open in X. Define σ to be this set. That is

$$\sigma = \{U \subseteq Y \mid p^{-1}(U) \in \tau\}.$$

Find all of the sets in σ.

(2) Show that σ is a topology on Y.

(3) Explain why $p : (X, \tau) \to (Y, \sigma)$ is continuous.

(4) Show that σ is the largest topology on Y for which p is continuous. That is, if σ' is a topology on Y with $\sigma \subsetneq \sigma'$, then $p : (X, \tau) \to (Y, \sigma')$ is not continuous.

Quotient Spaces

As we saw in our preview activity, if we have a surjection p from a topological space (X, τ) to a set Y, we were able to define a topology on Y by making the open sets the sets $U \subseteq Y$ such that $p^{-1}(U)$ is open in X. This is how we will create what is called the *quotient topology*. Before we can define the quotient topology, we need to know that this construction always makes a topology.

Activity 16.1. Let (X, τ_X) be a topological space, let Y be a set, and let $p : X \to Y$ be a surjection. Let

$$\tau_Y = \{U \subseteq Y \mid p^{-1}(U) \in \tau_X\}.$$

(a) Why are $\emptyset$ and Y in τ_Y?

(b) Let $\{U_\beta\}$ be a collection of sets in τ_Y for β in some indexing set J.

i. Show that $\bigcup_{\beta \in J} U_\beta$ is in τ_Y.

ii. If J is finite, show that $\bigcap_{\beta \in J} U_\beta$ is in τ_Y.

(c) What conclusion can we draw about τ_Y?

Activity 16.1 allows us to define the quotient topology.

Definition 16.1. Let (X, τ_X) be a topological space, let Y be a set, and let $p : X \to Y$ be a surjection.

(1) The **quotient topology** on Y is the set

$$\{U \subseteq Y \mid p^{-1}(U) \in \tau_X\}.$$

(2) Any function $p : X \to Y$ is a **quotient map** if p is surjective and for $U \subseteq Y$, U is open in Y if and only if $p^{-1}(U)$ is open in X.

(3) If $p : X \to Y$ is a quotient map, then the space Y is the **quotient space** of X determined by p.

Activity 16.2.

(a) Let $X = \mathbb{R}$ with standard topology, let $Y = \{-1, 0, 1\}$, and define $p : X \to Y$ by

$$p(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Find all of the open sets in the quotient topology.

(b) Let $X = \mathbb{R}$ with standard topology, let $Y = [0, 1)$, and define $p : X \to Y$ by

$$p(x) = x - \lfloor x \rfloor,$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x, (For example $\lfloor 1.2 \rfloor = 1$, and so $p(1.2) = 1.2 - 1 = 0.2$. The function defined by $\lfloor x \rfloor$ is also called the *floor* function. Be careful, note that $\lfloor -0.7 \rfloor = -1$.) Determine the sets in the quotient topology. (Hint: The graph of p on $[-2, 2]$ is shown in Figure 16.1.)

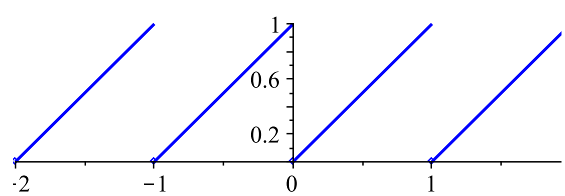

Figure 16.1: The graph of $p(x) = x - \lfloor x \rfloor$.

Another perspective of the quotient topology utilizes the fact that any equivalence relation $\sim$ on a set X partitions X into a union of disjoint equivalence classes $[x] = \{y \in X \mid y \sim x\}$. There is a natural surjection q from X to the space of equivalence classes given by $q(x) = [x]$. We investigate this perspective in the next activity.

Activity 16.3. Let $X = \{a, b, c, d, e, f\}$ and let $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$. Then (X, τ) is a topological space. Let $A = \{a, b, c\}$ and $B = \{d, e, f\}$. Define a relation $\sim$ on X such that $x \sim y$ if x and y are both in A or both in B. Assume that $\sim$ is an equivalence relation. The sets A and B are the equivalence classes for this relation. That is $A = [a] = [b] = [c]$ and $B = [d] = [e] = [f]$. Let $X^* = \{A, B\}$. Then we can define $p : X \to X^*$ by sending $x \in X$ to the set to which it belongs. That is, $p(x) = [x]$ for $x \in X$, or

$$p(a) = A, p(b) = A, p(c) = A, p(d) = B, p(e) = B, \text{ and } p(f) = B.$$

Determine the sets in the quotient topology on X^*.

The partition of X in Activity 16.3 into the disjoint union of sets A and B defines an equivalence relation on X where $x \sim y$ if x and y are both in the same set A or B. That is, $a \sim b \sim c$ and $d \sim e \sim f$. In this context, the sets A and B are equivalence classes – $A = [a]$ and $B = [d]$, where $[x]$ is the equivalence class of x. This leads to a general construction.

If (X, τ) is a topological space and $\sim$ is an equivalence relation on X, we can let $X/\sim$ be the set of distinct equivalence classes of X under $\sim$. Then $p : X \to X/\sim$ defined by $p(x) = [x]$ is a surjection and $X/\sim$ has the quotient topology. The space $X/\sim$ is called a *quotient space*. The space $X/\sim$ is also called an *identification* space because the equivalence relation identifies points in the set to be thought of as the same. This allows us to visualize quotient spaces as resulting from gluing or collapsing parts of the space X.

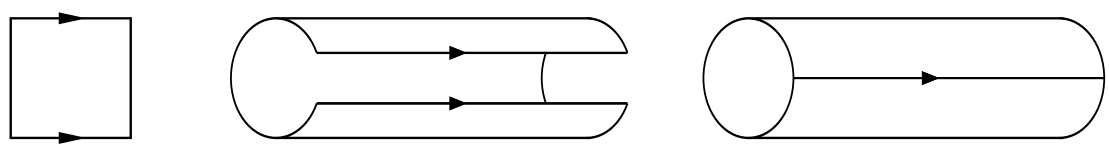

Figure 16.2: A tube as the identification space $X/\sim$.

Example 16.2. Let $I = [0, 1]$ and let $X = I \times I$ with standard topology. Define a relation $\sim$ on X by $(x, y) \sim (x, y)$ if $0 < y < 1$ and $0 \leq x \leq 1$, $(x, 0) \sim (x, 1)$ if $0 \leq x \leq 1$. It is straightforward to show that $\sim$ is an equivalence relation. Let us consider what the identification space $X/\sim$ looks like. The space $I \times I$ is the unit square as shown in Figure 16.2. All points in the interior of the square are identified only with themselves. However, the top side and bottom side are identified with each other in the same direction. Think of X as a piece of paper. We roll up the sides of the square to make the top and bottom sides coincide. The result is that $X/\sim$ is the cylinder as shown in Figure 16.2.

Activity 16.4. Quotient spaces can be difficult to describe. This activity presents a few more examples.

(a) Let $X = [0, 1]$ with standard topology and define an equivalence relation $\sim$ on X by $0 \sim 1$ and $x \sim x$ for all x not equal to 0 or 1. What does the quotient space $X/\sim$ look like? (Hint: Think about the relation $\sim$ as gluing the points 0 and 1 together.)

(b) Describe quotient spaces of $X = I \times I$ with standard topology given by the following equivalence relations $\sim$. Depictions of the identifications are shown in Figure 16.3. (Here I is the closed interval $[0, 1]$.)

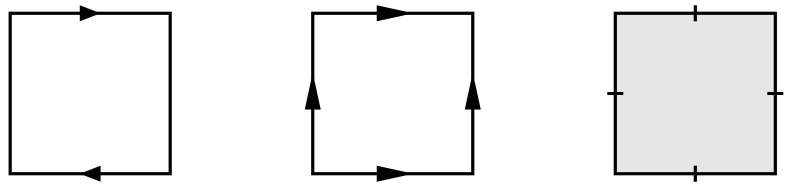

Figure 16.3: From left to right: the identifications for parts i., ii., and iii.

i. $(x, y) \sim (x, y)$ if $0 < y < 1$ and $0 \leq x \leq 1$ and $(x, 0) \sim (1-x, 0)$ when $0 \leq x \leq 1$. (This space is called a Möbius strip.)

ii. $(x, y) \sim (x, y)$ if $0 < x < 1$ and $0 < y < 1$, $(x, 0) \sim (x, 1)$ for $0 < x < 1$, $(0, y) \sim (1, y)$ for $0 < y < 1$, and $(0, 0) \sim (0, 1) \sim (1, 0) \sim (1, 1)$

iii. $(x, y) \sim (x, y)$ if $0 < x < 1$ and $0 < y < 1$ and $(x, y) \sim (u, v)$ if (x, y) and (u, v) are boundary points.

Many other interesting identification spaces can be made. For example, let $X = I \times I$ and define $\sim$ by $(x, y) \sim (x, y)$ if $0 < x < 1$ and $0 < y < 1$, $(0, y) \sim (1, y)$ for $0 < y < 1$, $(x, 0) \sim (1-x, 1)$ for $0 < x < 1$. This identification is illustrated in Figure 16.4. The resulting identification space $X/\sim$ is a Klein bottle. A nice illustration of this can be seen at https://plus.maths.org/content/introducing-klein-bottle.

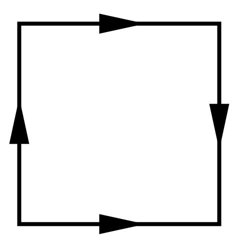

Figure 16.4: Identifications for the Klein Bottle.

Identifying Quotient Spaces

Suppose X is a topological space and Y is a set, and let $p : X \to Y$ be a surjection. We can define a relation $\sim_p$ on X by $x \sim_p y$ if and only if $p(x) = p(y)$. It is straightforward to show that $\sim_p$ is an equivalence relation, the details are left for Exercise (1). From this we can see that our two approaches to defining the quotient topology and quotient spaces are really the same.

Oftentimes we have a topological space X and a relation $\sim$ on X, and we would like to have an effective way to be able to identify the quotient space $X/\sim$ as homeomorphic to some familiar topological space Y. That is, we want to be able to show that there is a homeomorphism f from $X/\sim$ to Y.

Example 16.3. Consider the following situation. Let $X = \mathbb{R}$ with the standard topology and define the relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if $x - y \in \mathbb{Z}$. It is straightforward to show that $\sim$ is an equivalence relation. By this equivalence relation, we have $x - 1 \sim x$ for every real number x. This identifies $\mathbb{R}$ with the interval $[0, 1]$, where 0 and 1 are identified under the relation. So we might expect that $\mathbb{R}/\sim$ is homeomorphic to the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$ with the standard topology. Now the objective is to find a homeomorphism between S_1 and $\mathbb{R}/\sim$. Since every point on the unit circle has the form $(\cos(t), \sin(t))$ for some real number t, we might try defining $f : (\mathbb{R}/\sim) \to S^1$ by $f([t]) = (\cos(t), \sin(t))$. However, we have that $0 \sim 1$, which means that $[0] = [1]$, but $f([0]) \neq f([1])$ and so f is not well-defined. Another option might be $f([t]) = (\cos(2\pi t), \sin(2\pi t))$. In this case, if $x \sim y$, then $2\pi x$ and $2\pi y$ differ by a multiple of 2π and so $f([x]) = f([y])$. We could then show that f is a homeomorphism. We will continue this example shortly.

The following theorem encapsulates the above example.

Theorem 16.4. *Let X and Y be sets and let $\sim$ be an equivalence relation on X. Let f be a function from X to Y such that $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$ in X. Let $X/\sim$ be the set of equivalence classes of X under the relation $\sim$, and let $p : X \to (X/\sim)$ be the standard map defined by $p(x) = [x]$. The function $\overline{f}$ mapping $X/\sim$ to Y defined by $\overline{f}([x]) = f(x)$ for every $x \in X$ is the unique function that satisfies*

$$f = \overline{f} \circ p.$$

Activity 16.5. Theorem 16.4 is a statement about sets and functions, and there is no topology involved. We prove the theorem in this activity. Use the conditions stated in Theorem 16.4.

(a) Show that $\overline{f}$ is well-defined. That is, show that whenever $[x_1] = [x_2]$ in $X/\sim$, then $\overline{f}([x_1]) = \overline{f}([x_2])$.

(b) Prove that $f = \overline{f} \circ p$.

(c) Show that the uniqueness of $\overline{f}$ comes from the equation $f = \overline{f} \circ p$.

Now we present a final result that can be very helpful when working with quotient spaces.

Theorem 16.5. *Let X be a topological space and let $\sim$ be an equivalence relation on X. Consider the set $X/\sim$ to be a topological space with the quotient topology, and let $p : X \to (X/\sim)$ be the standard surjection defined by $p(x) = [x]$. Let Y be a topological space with $f : X \to Y$ a continuous function such that $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$ in X. Then $\overline{f} : (X/\sim) \to Y$ defined by $\overline{f}([x]) = f(x)$ is the unique continuous function satisfying $f = \overline{f} \circ p$.*

Proof. The existence of $\overline{f}$ as the unique function satisfying $f = \overline{f} \circ p$ was established in Theorem 16.4. All that remains is to show that $\overline{f}$ is continuous. Let O be an open set in Y. Since f is continuous, we know that $f^{-1}(O)$ is open in X. If $x_1 \in f^{-1}(O)$ and $x_1 \sim x_2$, then $x_2 \in f^{-1}(O)$ as well. Thus, we can write $f^{-1}(O)$ as

$$f^{-1}(O) = \bigcup_{x \in f^{-1}(O)} [x].$$

That is, $f^{-1}(O)$ is a union of equivalence classes. Now $\overline{f}([x]) = f(x)$, so if $x \in f^{-1}(O)$, then $[x] \in \overline{f}^{-1}(O)$. Thus,

$$f^{-1}(O) = \bigcup_{x \in f^{-1}(O)} [x] = \bigcup_{[x] \in \overline{f}^{-1}(O)} [x] = \overline{f}^{-1}(O).$$

We conclude that $\overline{f}^{-1}(O)$ is open in X and $\overline{f}$ is continuous. ∎

Now we will see how to use Theorem 16.5 to establish a homeomorphism from a quotient space of a given topological space to another topological space

Example 16.6. We return to the situation from Example 16.3 with $X = \mathbb{R}$ under the standard topology and equivalence relation $\sim$ defined by $x \sim y$ if $x - y \in \mathbb{Z}$. We will use Theorem 16.5 to show that $\mathbb{R}/\sim$ is homeomorphic to the circle $Y = S^1$.

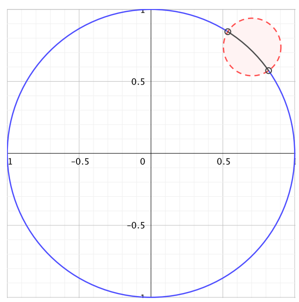

Figure 16.5: A basis element for S^1.

Step 1. Define a continuous surjection $f : X \to Y$ that respects the relation. That is, we need to ensure that $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$ in X. We saw earlier that the function f defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ respects the relation. Since every point on the unit circle is of the form $(\cos(\theta), \sin(\theta))$ for some real number θ, choosing $t = \frac{\theta}{2\pi}$ makes $f(t) = (\cos(\theta), \sin(\theta))$ and f is a surjection. Now we need to demonstrate that f is continuous. A collection of basic open sets in S^1 can be found by intersecting S^1 with open balls in $\mathbb{R}^2$ as illustrated in Figure 16.5. We can see that the basic open sets are arcs of the form $\widehat{ab}$ for a and b in S^1. Suppose $a = (\cos(2\pi A), \sin(2\pi A))$ and $b = (\cos(2\pi B), \sin(2\pi B))$ for angles A and B. Then $f^{-1}(\widehat{ab})$ is the union of intervals $(A + 2\pi k, B + 2\pi k)$ for $k \in \mathbb{Z}$. As a union of open intervals, we have that $f^{-1}(\widehat{ab})$ is open in X. We have now found a continuous surjection from X to Y that respects the relation.

Step 2. Find a continuous function from $X/\sim$ to Y. Theorem 16.5 tells us that the function $\overline{f} : (X/\sim) \to Y$ defined by $\overline{f}([t]) = f(t)$ is continuous. So $\overline{f}$ is our candidate to be a homeomorphism.

Step 3. Show that $\overline{f}$ is a bijection. Let $y \in Y$. The fact that f is a surjection means that there is a $t \in \mathbb{R}$ such that $f(t) = y$. It follows that $\overline{f}([t]) = f(t) = y$ and $\overline{f}$ is a surjection. To demonstrate that $\overline{f}$ is an injection, suppose $\overline{f}([s]) = \overline{f}([t])$ for some $s, t \in \mathbb{R}$. Then $(\cos(2\pi s), \sin(2\pi s)) = f(s) = f(t) = (\cos(2\pi t), \sin(2\pi t))$. It must be the case then that

$2\pi s$ and $2\pi t$ differ by a multiple of 2π. That is, $2\pi s - 2\pi t = 2\pi k$ for some integer k. From this we have $s - t = k \in \mathbb{Z}$, and so $s \sim t$. This makes $[s] = [t]$ and we conclude that $\overline{f}$ is an injection.

Step 4. Show that $\overline{f}$ is a homeomorphism. At this point we already know that $\overline{f}$ is a continuous bijection, so the only item that remains is to show that $\overline{f}(\overline{O})$ is open whenever $\overline{O}$ is open in $X/\sim$. Let $p : X \to (X/\sim)$ be the standard map. Let $\overline{O}$ be a nonempty open set in $X/\sim$. Then $O = p^{-1}(\overline{O})$ is open in X. Thus, O is a union of open intervals. Let (a, b) be an interval contained in O. From the definition of f we have that $f(a, b)$ is the open arc $\widehat{f(a)f(b)}$, which is open in Y. So $f(O)$ is a union of open arcs in Y, which makes $f(O)$ open in Y. Now $f(O) = (\overline{f} \circ p)(O) = \overline{f}(p(O)) = \overline{f}(\overline{O})$, and $\overline{f}(\overline{O})$ is open in Y. We conclude that $\overline{f}$ is a homeomorphism from $X/\sim$ to S^1, and so S^1 is a quotient space of $\mathbb{R}$.

Summary

Important ideas that we discussed in this section include the following.

- Let (X, τ_X) be a topological space, let Y be a set, and let $p : X \to Y$ be a surjection. The quotient topology on Y is the set

$$\{U \subseteq Y \mid p^{-1}(U) \in \tau_X\}.$$

- The function p is a quotient topology as in the previous bullet is called a quotient map and the space Y is a quotient space.
- A circle, a Möbius strip, a torus, and a sphere can all be realized as quotient spaces.

Exercises

(1) Let X be a topological space, Y a set, and $p : X \to Y$ a surjection. Define the relation $\sim_p$ on X by $x \sim_p y$ if and only if $p(x) = p(y)$. Prove that $\sim_p$ is an equivalence relation.

(2) Let X be the real numbers with the standard topology and let $p : X \to \{a, b, c\}$ be defined by

$$p(x) = \begin{cases} a & \text{if } x < 0 \\ b & \text{if } x = 0 \\ c & \text{if } x > 0. \end{cases}$$

What is the quotient topology?

(3) Define an equivalence relation $\sim$ on $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ whenever $x_2 - x_1 \in \mathbb{Z}$ and $y_2 - y_1 \in \mathbb{Z}$.

(a) Prove that $\sim$ is an equivalence relation on $\mathbb{R}^2$.

(b) The quotient space is a familiar space. Find that space and explain why it is the quotient space.

(4) Find an example of a continuous surjection that is not a quotient map.

(5) Let X be a topological space and let A be a subspace of X. Define a relation $\sim$ on X whose equivalence classes are A and $\{x\}$ if $x \notin A$. In this case the quotient space is denoted as X/A (think of this space as obtained by crushing A to a point and leaving everything else alone). Describe each of the following quotient spaces.

  (a) X is the closed interval $[0,1]$ in $\mathbb{R}$ and $A = \{0,1\}$

  (b) $X = \{(x,y) \mid x^2 + y^2 = 1\}$, $A = \{(-1,0),(1,0)\}$

  (c) If $X = \{(x,y) \mid x^2 + y^2 \leq 1\}$ and $A = \{(x,y) \mid x^2 + y^2 = 1\}$

(6) Let (X,τ) be the topological space where $X = \{1,2,3,4\}$ and

$$\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,3,4\}\}.$$

Let $Y = \{a,b,c\}$.

  (a) Let $p : X \to Y$ be defined by $p(1) = a$, $p(2) = a$, $p(3) = b$, and $p(4) = c$. Find the quotient topology τ_p on Y defined by the function p.

  (b) Let $q : X \to Y$ be defined by $q(1) = c$, $q(2) = c$, $q(3) = b$, and $q(4) = a$. Find the quotient topology τ_q on Y defined by the function q.

  (c) Are the spaces (Y,τ_p) and (Y,τ_q) homeomorphic? If yes, write down a specific homeomorphism and explain why your mapping is a homeomorphism. If not, explain why not.

(7) Let $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the unit disk in $\mathbb{R}^2$ with the standard topology, and let $S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$ be the boundary of D^2. Describe the quotient spaces $D^2/\sim$ for each equivalence relation (assume that points are similar to themselves). Let $x = (s_1,t_1)$ and $y = (s_2,t_2)$.

  (a) $x \sim y$ if $s_1 = s_2$ for x,y in D^2

  (b) $x \sim y$ if $s_1 = s_2$ for x,y in S^1

  (c) $x \sim y$ if x and y are diagonally opposite each other for x,y in D^2

(8) Let $X = I \times I$ where I is the interval $[0,1]$ with the standard metric topology, and define an equivalence relation on X by $(s_1,t_1) \sim (s_2,t_2)$ when $t_1 = t_2 > 0$ and $x \sim x$ for all other $x \in X$.

  (a) Describe the quotient space $X/\sim$, and describe the quotient topology.

  (b) Show that $X/\sim$ is not Hausdorff.

(9) Let X be a nonempty set and let p be a fixed element in X. Let τ_p be the particular point topology and $\tau_{\overline{p}}$ the excluded point topology on X. That is

- τ_p is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that contain p.
- $\tau_{\overline{p}}$ is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that do not contain p.

That the particular point and excluded point topologies are topologies is the subject of Exercises (9) and (10) on page 127.

Let $\sim$ be the equivalence relation on $\mathbb{Z}$ defined by $x \sim y$ if $x \equiv y \pmod 3$. Describe the quotient space $\mathbb{Z}/\sim$ and then determine, with justification, the quotient topology on $\mathbb{Z}/\sim$ when

(a) $\mathbb{Z}$ has the topology τ_p with $p = 1$

(b) $\mathbb{Z}$ has the topology $\tau_{\tau_{\overline{p}}}$ with $p = 1$.

(10) In the process of developing techniques of drawing in perspective, renaissance artists found it necessary to consider a point at infinity where all lines intersect. This creates a geometry that extends the concept of the real plane. This new geometry is the real projective plane $\mathbb{R}P^2$, which can be thought of as the quotient space of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ with the relation $\sim_P$ such that $(x_1, x_2, x_3) \sim_P (y_1, y_2, y_3)$ in $\mathbb{R}^3 \setminus \{(0,0,0)\}$ if and only if there is a nonzero real number k such that $y_1 = kx_1$, $y_2 = kx_2$, and $y_3 = kx_3$. In the projective plane, parallel lines intersect at a point at infinity, just as they seem to with our human vision.

(a) Show that $\sim_P$ is an equivalence relation.

(b) Give a geometric description of the elements in the quotient space $\mathbb{R}P^2$.

(c) There are other ways to visualize $\mathbb{R}P^2$. For example, explain why the real projective plane $\mathbb{R}P^2$ is homeomorphic to the quotient space $S^2/\sim$ of $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, where $\sim$ identifies antipodal points on S^2. No formal proofs are necessary, but a convincing explanation is in order.

(d) Since we identify antipodal points on S^2 in the space $S^2/\sim$, we can think of this space in another way. If P is a point on S^2 not on the equator, then its antipodal point is also not on the equator. So we can think of $S^2/\sim$ as the top hemisphere, along with the equator on which antipodal points are identified, as illustrated at left in Figure 16.6.

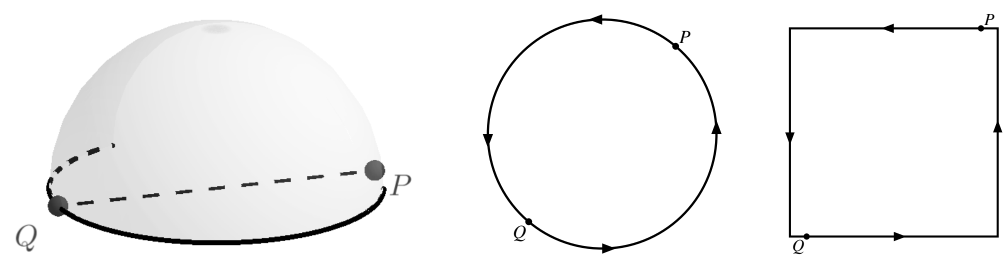

Figure 16.6: Three perspectives of $\mathbb{R}P^2$.

By projecting the points on the hemisphere down to the xy-plane, we can represent $S^2/\sim$ as a disk whose antipodal points are identified, as seen in the middle in Figure 16.6. Use this last perspective to explain why $\mathbb{R}P^2$ can be realized as a square where opposite sides are identified in opposite directions as shown at right in Figure 16.6.

(e) The projective plane $\mathbb{R}P^2$ is a complicated object – it cannot be embedded in $\mathbb{R}^3$ and so it is not something that can be easily visualized. The projective plane is a non-orientable surface and is also important in classifying surfaces – basically, every closed surface is made up of spheres, tori, and/or projective planes. In this part of the exercise we see how the projective plane itself is made by adjoining a Möbius strip to a disk (think of sewing the boundary of Möbius strip to the perimeter of a disk).

i. Start with the model of $\mathbb{R}P^2$ shown at left in Figure 16.7. Partition this object into three pieces as shown at right in Figure 16.7. Explain why the shaded region in the middle figure, separated out at right in Figure 16.7, is a Möbius strip.

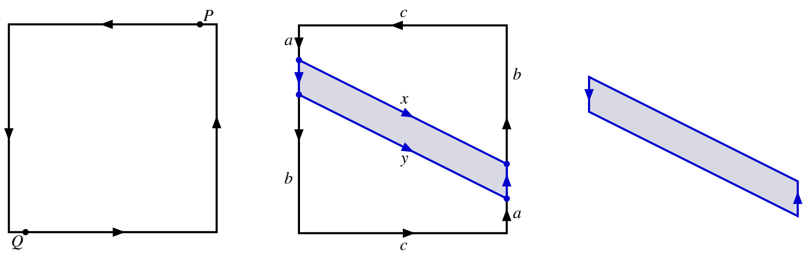

Figure 16.7: Splitting the real projective plane.

ii. The space S that remains after we remove the Möbius strip from $\mathbb{R}P^2$ is shown at left in Figure 16.8. The two spaces that follow are homeomorphic to S. Describe the homeomorphisms that produce the spaces from S. Then explain how $\mathbf{R}P^2$ is obtained by attaching a Möbius strip to a disk along its boundary.

(11) Let $X = \mathbb{R}^2$ with the standard topology, and let $Y = \{x \in \mathbb{R} \mid x \geq 0\}$ with the standard topology. Let $f : X \to Y$ be defined by $f((x, y)) = x^2 + y^2$.

(a) Show that f is a continuous surjective function.

(b) Prove that the quotient space X^* of X defined by f is homeomorphic to Y.

(12) Let (X, τ) be a topological space. A subspace A of X is a *retract* of X (or that X retracts onto A) if there is a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Such a map r is called a *retraction*. Intuitively, a subspace A of X is a retract of X if we can continually collapse (or retract) X onto A without moving any of the points in A. Certain types of retracts, namely deformation retracts, are important in algebraic topology.

(a) Show that every nonempty space can retract to a point.

(b) Show that $\{-1, 1\}$ is a retract of $\mathbb{R} \setminus \{0\}$.

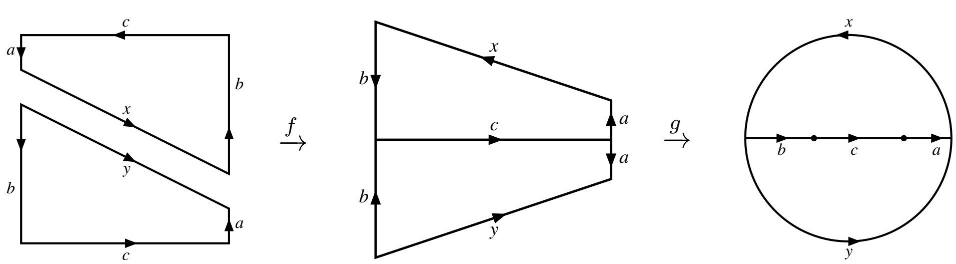

Figure 16.8: Recognizing the space S.

(c) Show that every retraction is a quotient map.

(d) Show that if X is Hausdorff and A is a retract of X, then A must be a closed subset of X.

(13) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) Let p be a surjection from a topological space X to a nonempty set Y. The quotient topology on Y is the largest topology on Y such that p is continuous.

(b) Let $X = [0, 1]$ with the Euclidean metric topology and define the relation $\sim$ on X as $x \sim y$ if x and y are either both rational or both irrational. Then the quotient space $X/\sim$ is a two point space with the indiscrete topology.

(c) Let p be a surjection from a topological space X to a nonempty set Y. The quotient topology on Y is the largest topology on Y such that p is continuous.

(d) Let $X = [0, 1]$ with the Euclidean metric topology and define the relation $\sim$ on X as $x \sim y$ if x and y are either both rational or both irrational. Then the quotient space $X/\sim$ is a two point space with the indiscrete topology.

(e) If X is a topological space, Y is a set, and $p : X \to Y$ is a surjection, then $p(U)$ is open in the quotient topology whenever U is open in X.

(f) If $\sim$ is an equivalence relation on a topological space X, then the quotient space $X/\sim$ is the set of all equivalence classes of X with the quotient topology.