Section 17

Compact Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a cover of a subset of a topological space? What is an open cover?
- What is a subcover of a cover?
- What is a compact subset of a topological space?
- What is one application of compactness?
- How do we characterize the compact subsets of $\mathbb{R}^n$? What theorem provides this characterization?

Introduction

Closed and bounded intervals have important properties in calculus. Recall, for example, that every real-valued function that is continuous on a closed interval $[a, b]$ attains a maximum and minimum value on that interval. The question we want to address in this section is if there is a corresponding characterization for subsets of topological spaces that ensure that continuous real-valued functions with domains in topological spaces attain maximum and minimum values. The property that we will develop is called compactness.

The word "compact" might bring to mind a notion of smallness, but we need to be careful with the term. We might think that the interval $(0, 0.5)$ is small, but $(0, 0.5)$ is homeomorphic to $\mathbb{R}$, which is not small. Similarly, we might think that the interval $[-10000, 10000]$ is large, but this interval is homeomorphic to the "small" interval $[-0.00001, 0.000001]$. As a result, the concept of compactness does not correspond to size, but rather structure, in a way. We will expand on this idea in this section.

Since a topology defines open sets, topological properties are often defined in terms of open sets. Let us consider an example to see if we can tease out some of the details we will need to get a useful notion of compactness. Consider the open interval $(0,1)$ in $\mathbb{R}$. Suppose we write $(0,1)$ as a union of open balls. For example, let $O_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ for $n \in \mathbb{Z}^+$ and $n \geq 3$. Notice that $(0,1) \subseteq \bigcup_{n\geq 3} O_n$. Any collection of open sets whose union contains $(0,1)$ is called an *open cover* of $(0,1)$. Working with a larger number of sets is generally more complicated than working with a smaller number, so it is reasonable to ask if we can reduce the number of sets in our open cover of $(0,1)$ and still cover $(0,1)$. In particular, working with a finite collection of sets is preferable to working with an infinite number of sets (we can exhaustively check all of the possibilities in a finite setting if necessary). Notice that $O_n \subset O_{n+1}$ for each n, so we can eliminate many of the sets in this cover. However, if we eliminate enough sets so that we are left with only finitely many, then there will be a maximum value of n so that O_n remains in our collection. But then $\frac{1}{2n}$ will not be in the union of our remaining collection of sets. As a result, we cannot find a finite collection of the O_n whose union contains $(0,1)$. Note that there may be some collections of open sets that cover of $(0,1)$ for which there is a finite subcollection of sets that also cover $(0,1)$. For example, if we let $U_n = \left(n - \frac{3}{4}, n + \frac{3}{4}\right)$, then $(0,1) \subseteq \bigcup_{n\in\mathbb{Z}} U_n$, and $(0,1) \subseteq U_0 \cup U_1$. The main point is that there is at least one collection of open sets that covers $(0,1)$ for which there is no finite subcollection of sets that covers $(0,1)$.

Let's apply the same idea now to the set $[0,1]$. Suppose we extend our open cover $\{O_n\}$ to be an open cover of the closed interval $[0,1]$ by including two additional open balls in $\mathbb{R}$: $O_0 = B(0, 0.5)$ and $O_1 = B(1, 0.5)$. Now the sets O_0, O_1, and O_4 form a finite collection of sets that covers $[0,1]$. So even though the interval $[0,1]$ is "larger" than $(0,1)$ in the sense that $(0,1) \subset [0,1]$ we can represent $[0,1]$ in a more efficient (that is finite) way in terms of open sets than we can the interval $(0,1)$. This is the basic idea behind compactness.

Definition 17.1. A subset A of a topological space X is **compact** if for every set I and *every* family of open sets $\{O_\alpha\}$ with $\alpha \in I$ such that $A \subseteq \bigcup_{\alpha\in I} O_\alpha$, there exists a finite subfamily $\{O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}\}$ such that $A \subseteq \bigcup_{i=1}^{n} O_{\alpha_i}$.

If (X,τ) is a topological space and X is a compact subset of X, then we say that X is a *compact topological space*. There is some terminology associated with Definition 17.1.

Definition 17.2. A **cover** of a subset A of a topological space X is a collection $\{S_\alpha\}$ of subsets of X for α in some indexing set I so that $A \subseteq \bigcup_{\alpha\in I} S_\alpha$. In addition, if each set S_α is an open set, then the collection $\{S_\alpha\}$ is an **open cover** for A.

Definition 17.3. A **subcover** of a cover $\{S_\alpha\}_{\alpha\in I}$ of a subset A of a topological space X is a collection $\{S_\beta\}$ for $\beta \in J$, where J is a subset of I such that $A \subseteq \bigcup_{\beta\in J} S_\beta$. In addition, if J is a finite set, the subcover $\{S_\beta\}_{\beta\in J}$ is a **finite subcover** of $\{S_\alpha\}_{\alpha\in I}$.

So the sets O_0, O_1, and O_4 in our previous example form a finite subcover of the open cover $\{O_n\}_{n\geq 3}$.

Using the terminology we have now established, we can restate the definition of compactness in the following way: a subset A of a topological space X is compact if every open cover of A has a finite subcover of A.

Preview Activity 17.1. Determine if the subset A of the topological space X is compact. Either prove A is compact by starting with an arbitrary infinite cover and demonstrating that there is a finite subcover, or find a specific infinite cover and prove that there is no finite subcover.

(1) $A = \{-2, 3, e, \pi, 456875\}$ in $X = \mathbb{R}$ with the Euclidean topology. Generalize this example.

(2) $A = (0, 1]$ in $X = \mathbb{R}$ with the Euclidean topology.

(3) $A = \left\{\frac{1}{n} \mid n \in \mathbb{Z}^+\right\}$ in $X = \mathbb{R}$ with the Euclidean topology.

(4) $A = \mathbb{Z}^+$ in $X = \mathbb{R}$ with the Euclidean topology.

(5) $A = \mathbb{Z}^+$ in $X = \mathbb{R}$ with the finite complement topology.

(6) $A = \mathbb{R}$ in $X = \mathbb{R}$ with the Euclidean topology.

There are two perspectives by which we can look at compactness. If (X, τ_X) is a topological space and A is a subset of X, then Definition 17.1 tells us what it means for A to be compact as a subset of X. From this perspective, we use open sets in X to make open covers of A. We can also consider A as a subspace of X using the subspace topology τ_A. From this perspective we can examine the compactness of A using relatively open sets for open covers. Exercise (14) tells us that these two perspectives are equivalent, so we will use whatever perspective is appropriate for a given situation.

Compactness and Continuity

In our preview activity we learned about compactness – the analog of closed intervals from $\mathbb{R}$ in topological spaces. Recall that a subset A of a topological space X is compact if every open cover of A has a finite sub-cover. As we will see, the definition of compactness is exactly what we need to ensure results of the type that continuous real-valued functions with domains in topological spaces attain maximum and minimum values on compact sets.

We might expect that compact sets have certain properties, but we must be careful which ones we assume.

Activity 17.1. Let $X = \{a, b, c, d\}$ and give X the topology $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$.

(a) Explain why every finite subset of a topological space must be compact.

(b) Find, if possible, a subset of X that is compact and open. If no such subset exists, explain why.

(c) If A is a compact subset of X, must A be open? Explain.

(d) Find, if possible, a subset of X that is compact and closed. If no such subset exists, explain why.

(e) If A is a compact subset of X, must A be closed? Explain.

The message of Activity 17.1 is that compactness by itself is not related to closed or open sets. We will see later, though, that in some reasonable circumstances, compact sets and closed sets are related. For the moment, we take a short detour and ask if compactness is a topological property.

Activity 17.2. Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : X \to Y$ be continuous. Assume that A is a compact subset of X. In this activity we want to determine if $f(A)$ must be a compact subset of Y.

(a) What do we need to show to prove that $f(A)$ is a compact subset of Y? Where do we start?

(b) If we have an open cover of $f(A)$ in Y, how can we find an open cover $\{U_\alpha\}$ for A? Be sure to verify that what you claim is actually an open cover of A.

(c) What do we know about any open cover of A?

(d) Complete the proof of the following theorem.

Theorem 17.4. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : X \to Y$ be continuous. If A is a compact subset of X, then $f(A)$ is a compact subset of Y.*

A consequence of Activity 17.2 is that compactness is a topological property.

Corollary 17.5. *Let (X, τ_X) and (Y, τ_Y) be homeomorphic topological spaces. Then a subset A of X is compact if and only if $f(A)$ is compact in Y.*

Compact Subsets of $\mathbb{R}^n$

The metric space $(\mathbb{R}^n, d_E)$ is not compact since the open cover $\{B(0, n)\}_{n \in \mathbb{Z}^+}$ has no finite subcover. Since we have already shown that $(\mathbb{R}, d_E)$ is homeomorphic to the topological subspaces (a, b), $(-\infty, b)$, and (a, ∞) for any $a, b \in \mathbb{R}$, we conclude that no open intervals are compact. Similarly, no half-closed intervals are compact. In fact, we will demonstrate in this section that the compact subsets of $(\mathbb{R}^n, d_E)$ are exactly the subsets that are closed and bounded. The first step is contained in the next activity.

Activity 17.3. We have seen that compact sets can be either open or closed. However, in certain situations compact sets must be closed. We investigate that idea in this activity. Let A be a compact subset of a Hausdorff topological space X. We will examine why A must be a closed set.

(a) To prove that A is a closed set, we consider the set $X \setminus A$. What property of $X \setminus A$ will ensure that A is closed? How do we prove that $X \setminus A$ has this property?

(b) Let $x \in X \setminus A$. Assume that A is a nonempty set (why can we make this assumption)? For each $a \in A$, why must there exist disjoint open sets O_{xa} and O_a with $x \in O_{xa}$ and $a \in O_a$?

(c) Why must there exist a positive integer n and elements a_1, a_2, …, a_n in A such that the sets O_{a_1}, O_{a_2}, …, O_{a_n} form an open cover of A?

(d) Now find an open subset of $X \setminus A$ that has x as an element. What does this tell us about A?

The result of Activity 17.3 is summarized in Theorem 17.6.

Theorem 17.6. *If A is a compact subset of a Hausdorff topological space, then A is closed.*

Theorem 17.6 tells us something about compact subsets of $(\mathbb{R}^n, d_E)$. Since every metric space is Hausdorff, we can conclude the following corollary.

Corollary 17.7. *If A is a compact subset of $(\mathbb{R}^n, d_E)$, then A is closed.*

To classify the compact subsets of $(\mathbb{R}^n, d_E)$ as closed and bounded, we need to discuss what it means for a set in $\mathbb{R}^n$ to be bounded. The basic idea is straightforward – a subset of $\mathbb{R}^n$ is bounded if it doesn't go off to infinity in any direction. In other words, a subset A of $\mathbb{R}^n$ is bounded if we can construct a box in $\mathbb{R}^n$ that is large enough to contain it. Thus, the following definition.

Definition 17.8. A subset A of $\mathbb{R}^n$ is **bounded** if there exists $M > 0$ such that $A \subseteq Q^n_M$, where

$$Q^n_M = \{(x_1, x_2, \ldots, x_n) \mid -M \leq x_i \leq M \text{ for every } 1 \leq i \leq n\}.$$

The set Q^n_M in Definition 17.8 is called the *standard n-dimensional cube of size M*. A standard 3-dimensional cube of size M is shown in Figure 17.1.

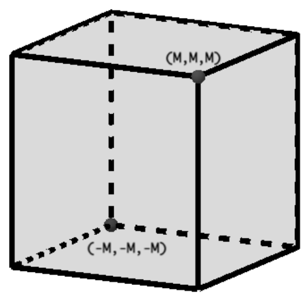

Figure 17.1: A standard 3-cube Q^3_M.

An important fact about standard n-cubes is that they are compact subsets of $\mathbb{R}^n$. Compactness is a complicated property – it is difficult to prove a result that is true about every open cover. As a result, the proof of Theorem 17.9 is quite technical, but it is a critical step to classifying the compact subsets of $\mathbb{R}^n$.

Theorem 17.9. *Let $n \in \mathbb{Z}^+$. The standard n-dimensional cube of size M is a compact subset of $\mathbb{R}^n$ for any $M > 0$.*

Proof. We proceed by contradiction and assume that there is an $n \in \mathbb{Z}^+$ and a positive real number M such that Q^n_M is not compact. So there exists an open cover $\{O_\alpha\}$ with α in some indexing set I of Q^n_M that has no finite sub-cover. Let $Q_0 = Q^n_M$ so that Q_0 is an n-cube with side length $2M$. Partition Q_0 into 2^n uniform sub-cubes of side length $M = \frac{2M}{2}$ (a picture for $n = 2$ is shown at left in Figure 17.2). Let Q'_0 be one of these sub-cubes. The collection $\{O_\alpha \cap Q'_0\}_{\alpha \in I}$ is an open cover of Q'_0 in the subspace topology. If each of these open covers has a finite sub-cover, then we can take the union of all of the O_αs over all of the finite sub-covers to obtain a finite sub-cover of $\{O_\alpha\}_{\alpha \in I}$ for Q_0. Since our cover $\{O_\alpha\}_{\alpha \in I}$ for Q_0 has no finite sub-cover, we conclude that there is one sub-cube, Q_1, for which the open cover $\{O_\alpha \cap Q_1\}_{\alpha \in I}$ has no finite sub-cover. Now we repeat the process and partition Q_1 into 2^n uniform sub-cubes of side length $\frac{M}{2} = \frac{2M}{2^2}$. The same argument we just made tells us that there is a sub-cube Q_2 of Q_1 for which the open cover $\{O_\alpha \cap Q_2\}_{\alpha \in I}$

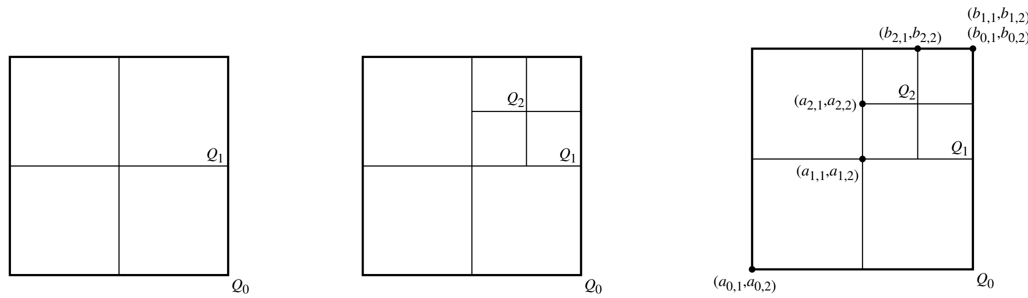

Figure 17.2: Left : Q_1. Middle: Q_2. Right: Labeling the corners.

has no finite sub-cover (an illustration for the $n = 2$ case is shown at middle in Figure 17.2). We proceed inductively to obtain an infinite nested sequence

$$Q_0 \supset Q_1 \supset Q_2 \supset Q_3 \supset \cdots \supset Q_k \supset \cdots$$

of cubes such that for each $k \in \mathbb{Z}$, the lengths of the sides of cube Q_k are $\frac{M}{2^{k-1}} = \frac{2M}{2^k}$ and the open cover $\{O_\alpha \cap Q_k\}_{\alpha\in I}$ of Q_k has no finite sub-cover. Now we show that $\bigcap_{k=1}^\infty Q_k \neq \emptyset$.

For $i \in \mathbb{Z}^+$, let $Q_i = [a_{i,1}, b_{i,1}] \times [a_{i,2}, b_{i,2}] \times \cdots [a_{i,n}, b_{i,n}]$. That is, think of the point $(a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ as a lower corner of the cube and the point $(b_{i,1}, b_{i,2}, \ldots, b_{i,n})$ as an upper corner of the n-cube Q_i (a labeling for $n = 2$ and i from 1 to 3 is shown at right in Figure 17.2). Let $q = (\sup\{a_{i,1}\}, \sup\{a_{i,2}\}, \ldots, \sup\{a_{i,n}\})$. We will show that $q \in \bigcap_{k=1}^\infty Q_k$. Fix $r \in \mathbb{Z}^+$. We need to demonstrate that

$$q \in Q_r = \{(x_1, x_2, \ldots, x_n) \mid a_{r,s} \leq x_s \leq b_{r,s} \text{ for each } 1 \leq s \leq n\}.$$

For each s between 1 and n we have

$$a_{r,s} \leq \sup\{a_{i,s}\} \tag{17.1}$$

because $\sup\{a_{i,s}\}$ is an upper bound for all of the $a_{i,s}$. The fact that our cubes are nested means that

$$\begin{aligned} a_{1,s} &\leq a_{2,s} \leq \cdots, \\ b_{1,s} &\geq b_{2,s} \geq \cdots, \\ a_{i,s} &\leq b_{i,s} \end{aligned} \tag{17.2}$$

for every i and s. Since $\sup\{a_{i,s}\}$ is the least upper bound of all of the $a_{i,s}$, property (17.2) shows that $\sup\{a_{i,s}\} \leq b_{i,s}$ for every i. Thus, $\sup\{a_{i,s}\} \leq b_{r,s}$ and so $a_{r,s} \leq \sup\{a_{i,s}\} \leq b_{r,s}$. This shows that $q \in Q_k$ for every k. Consequently, $q \in \bigcap_{k=1}^\infty Q_k$ and $\bigcap_{k=1}^\infty Q_k$ is not empty. (The fact that the side lengths of our cubes are converging to 0 implies that $\bigcap_{k=1}^\infty Q_k = \{q\}$, but we only need to know that $\bigcap_{k=1}^\infty Q_k$ is not empty for our proof.)

Since $\{O_\alpha\}_{\alpha\in I}$ is a cover for Q_0, there must exist an $\alpha_q \in I$ such that $q \in O_{\alpha_q}$. The set O_{α_q} is open, so there exists $\epsilon_q > 0$ such that $B(q, \epsilon_q) \subseteq O_{\alpha_q}$. The maximum distance between points in Q_k is the distance between the corner points $(a_{k,1}, a_{k,2}, \ldots, a_{k,n})$ and $(b_{k,1}, b_{k,2}, \ldots, b_{k,n})$, where each length $b_{k,s} - a_{k,s}$ is $\frac{M}{2^{k-1}}$. The distance formula tells us that this maximum distance between points in Q_k is

$$D_k = \sqrt{\sum_{s=1}^{n} \left(\frac{M}{2^{k-1}}\right)^2} = \sqrt{n\left(\frac{M}{2^{k-1}}\right)^2} = \frac{M}{2^{k-1}}\sqrt{n}.$$

Now choose $K \in \mathbb{Z}^+$ such that $D_K < \epsilon_q$. Then if $x \in Q_K$ we have $d_E(q,x) < D_K$ and $x \in B(q,\epsilon_q)$. So $Q_K \subseteq B(q,\epsilon_q)$. But $B(q,\epsilon_q) \subseteq O_{\alpha_q}$. So the collection $\{O_{\alpha_q} \cap Q_K\}$ is a sub-cover of $\{O_\alpha \cap Q_K\}_{\alpha\in I}$ for Q_K. But this contradicts the fact this open cover has no finite sub-cover. The assumption that led us to this contradiction was that Q_0 was not compact, so we conclude that the standard n-dimensional cube of size M is a compact subset of $\mathbb{R}^n$ for any $M > 0$. ■

One consequence of Theorem 17.9 is that any closed interval $[a,b]$ in $\mathbb{R}$ is a compact set. But we can say even more – that the compact subsets of $\mathbb{R}^n$ are the closed and bounded subsets. This will require one more intermediate result about closed subsets of compact topological spaces.

Activity 17.4. Let X be a compact topological space and C a closed subset of X. In this activity we will prove that C is compact.

(a) What does it take to prove that C is compact?

(b) Use an open cover for C and the fact that C is closed to make an open cover for X.

(c) Use the fact that X is compact to complete the proof of the following theorem.

Theorem 17.10. *Let X be a compact topological space. Then any closed subset of X is compact.*

Now we can prove a major result, that the compact subsets of $(\mathbb{R}^n, d_E)$ are the closed and bounded subsets. This result is important enough that it is given a name.

Theorem 17.11 (The Heine-Borel Theorem)**.** *A subset A of $(\mathbb{R}^n, d_E)$ is compact if and only if A is closed and bounded.*

Proof. Let A be a subset of $(\mathbb{R}^n, d_E)$. Assume that A is closed and bounded. Since A is bounded, there is a positive number M such that $A \subseteq Q^n_M$. Theorem 17.9 shows that Q^n_M is compact, and then Theorem 17.10 shows that A is compact.

For the converse, assume that A is a compact subset of $\mathbb{R}^n$. We must show that A is closed and bounded. Now $(\mathbb{R}^n, d_E)$ is a metric space, and so Hausdorff. Theorem 17.6 then shows that A is closed. To conclude our proof, we need to demonstrate that A is bounded. For each $k > 0$, let

$$O_k = \{(x_1, x_2, \ldots, x_n) \mid -k < x_i < k \text{ for every } 1 \leq i \leq n\}.$$

That is, O_k is the open k-cube in $\mathbb{R}^n$. Next let

$$U_k = O_k \cap A$$

for each k. Since $\bigcup_{k>0} O_k = \mathbb{R}^n$, it follows that $\{U_k\}_{k>0}$ is an open cover of A. The fact that A is compact means that there is a finite collection U_{k_1}, U_{k_2}, ..., U_{k_m} of sets in $\{U_k\}_{k>0}$ that cover A. Let $K = \max\{k_i \mid 1 \leq i \leq m\}$. Then $U_{k_i} \subseteq U_K$ for each i, and so $A \subseteq U_K \subset Q^m_K$. Thus, A is bounded. This completes the proof that if A is compact in $\mathbb{R}^n$, then A is closed and bounded. ■

You might wonder whether the Heine-Borel Theorem is true in any metric space.

Activity 17.5. A subset A of a metric space (X, d) is bounded if there exists a real number M such that $d(a_1, a_2) \leq M$ for all $a_1, a_2 \in A$. (This is equivalent to our definition of a bounded subset of $\mathbb{R}^n$ given earlier, but works in any metric space.) Explain why $\mathbb{Z}$ as a subset of $(\mathbb{R}, d)$, where d is the discrete metric, is closed and bounded but not compact.

An Application of Compactness

As mentioned at the beginning of this section, compactness is the quality we need to ensure that continuous functions from topological spaces to $\mathbb{R}$ attain their maximum and minimum values.

Activity 17.6. In this activity we prove the following theorem.

Theorem 17.12. *A continuous function from a compact topological space to the real numbers assumes a maximum and minimum value.*

(a) Let X be a compact topological space and $f : X \to \mathbb{R}$ a continuous function. What does the continuity of f tell us about $f(X)$ in $\mathbb{R}$?

(b) Why can we conclude that the set $f(X)$ has a least upper bound M? Why must M be an element of $f(X)$?

(c) Complete the proof of Theorem 17.12.

Summary

Important ideas that we discussed in this section include the following.

- A cover of a subset A of a topological space X is any collection of subsets of X whose union contains A. An open cover is a cover consisting of open sets.
- A subcover of a cover of a set A is a subset of the cover such that the union of the sets in the subcover also contains A.
- A subset A of a topological space is compact if every open cover of A has a finite subcover.
- A continuous function from a compact topological space to the real numbers must attain a maximum and minimum value.
- The Heine-Borel Theorem states that the compact subsets of $\mathbb{R}^n$ are exactly the subsets that are closed and bounded.

Exercises

(1) (a) Determine the compact subsets of a topological space X with the indiscrete topology.

(b) Determine the compact subsets of a topological space X with the indiscrete topology.

(2) Recall from Definition 12.13 on page 126 that if τ_1 and τ_2 are two topologies on a set X such that $\tau_1 \subseteq \tau_2$, then τ_1 is said to be a *coarser* (or *weaker*) topology than τ_2, or τ_2 is a *finer* (or *stronger*) topology than τ_1. In this exercise we explore the question of whether compactness is a property that is passed from weaker to stronger topologies or from stronger to weaker.

Let τ_1 and τ_2 be two topologies on a set X. If $\tau_1 \subseteq \tau_2$, what does compactness under τ_1 or τ_2 imply, if anything, about compactness under the other topology? Justify your answers.

(3) Let $\mathbb{E}$ be the set of even integers, and let $\tau = \{\mathbb{Z}\} \cup \{O \subseteq \mathbb{E}\}$. That is, τ is the collection of all subsets of $\mathbb{E}$ along with $\mathbb{Z}$.

(a) Prove that τ is a topology on $\mathbb{Z}$.

(b) Find all compact subsets of $(\mathbb{Z}, \tau)$. Verify your answer.

(c) Prove or disprove: If A and B are compact subsets of a topological space X, then $A \cap B$ is also a compact subset of X.

(4) Let (X, τ) be a topological space

(a) Prove that the union of any finite number of compact subsets of X is a compact subset of X.

(b) In Exercise (3) we should have seen that the intersection of compact sets is not necessarily compact. If X is Hausdorff, prove that the intersection of any finite number of compact subsets of X is a compact subset of X.

(5) Consider $\mathbb{Z}$ with the digital line topology (see Exercise (11) on page 127). Determine the compact subsets of $\mathbb{Z}$.

(6) For each $n \in \mathbb{Z}^+$, let $(-n, n)$ be the set of integers in the interval $(-n, n)$ (see Exercise (4) on 126.)

(a) Show that $\mathcal{B} = \{(-n, n)\}_{n \in \mathbb{Z}^+}$ is a basis for a topology τ on $\mathbb{Z}$

(b) Is the subset $(-2, 2)$ compact in this topology?

(c) Determine all of the compact subsets of $\mathbb{Z}$.

(7) Let X and Y be topological spaces, and let $f : X \to Y$ be a function.

(a) Suppose that f is a continuous function, and that X is compact and Y is Hausdorff. Prove that if C is a closed subset of X, then $f(C)$ is a closed subset of Y. (Thus, f is a *closed* function.) (Hint: Use Activity 17.3, Activity 17.4, and Theorem 17.6.)

(b) Suppose that f is a continuous bijection. Prove that if X is compact and Y is Hausdorff, then f is a homeomorphism.

(c) Give an example where f is a continuous bijection and X is compact, but f is not a homeomorphism.

(d) Give an example where f is a continuous bijection and Y is Hausdorff, but f is not a homeomorphism.

(8) The Either-Or topology on the interval $X = [-1, 1]$ has as its open sets all subsets of X that contain $(-1, 1)$ and any subset of X that doesn't contain 0.

(a) Describe the non-trivial closed subsets of X.

(b) Is X a Hausdorff topological space? Explain.

(c) Is X compact? Prove your answer.

(d) Are there any subsets of $\mathbb{Z}$ that are not compact? Justify your answer.

(9) Let $K = \left\{ \frac{1}{k} \mid k \text{ is a positive integer} \right\}$. Let $\mathcal{B}$ be the collection of all open intervals of the form (a, b) and all sets of the form $(a, b) \setminus K$, where $a < b$ are real numbers as in Example 13.13 on page 137. Let τ_K be the topology generated by $\mathcal{B}$.

(a) Show that $(\mathbb{R}, \tau_K)$ is not compact. (Hint: How is the K-topology related to the Euclidean topology?)

(b) Show that any subset of $\mathbb{R}$ that contains K is not a compact subset of $(\mathbb{R}, \tau_K)$. In particular, even though $[0, 1]$ is a closed and bounded subset of $\mathbb{R}$ in $(\mathbb{R}, \tau_K)$, we note that $[0, 1]$ is not compact. (Hint: Consider the sets $O_k = \left(\frac{1}{k}, 2\right) \cup (-1, 1) \setminus K$ for $k \in \mathbb{Z}^+$.)

(10) Let X be a topological space.

(a) Prove that if X is Hausdorff and C is a compact subset of X, then for each $x \in X \setminus C$ there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$.

(b) Prove that if X is a compact Hausdorff space, then X is normal.

(11) Let X be a nonempty set and let p be a fixed element in X. Let τ_p be the particular point topology and $\tau_{\overline{p}}$ the excluded point topology on X. That is

- τ_p is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that contain p.
- $\tau_{\overline{p}}$ is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that do not contain p.

That the particular point and excluded point topologies are topologies is the subject of Exercises (9) and (10) on page 127.

Determine, with proof, the compact subsets of X when

(a) X has the particular point topology τ_p

(b) X has the excluded point topology $\tau_{\overline{p}}$.

(12) In this exercise we encounter a non-Hausdorff topological space in which single points sets are closed, and in which compact subsets need not be closed. Consider the set $\mathbb{Z}$ with the finite complement topology τ_{FC}.

(a) Show that every single point set is closed.

(b) Explain why $(\mathbb{Z}, \tau_{FC})$ is not a Hausdorff space.

(c) Let U be an open set in $(\mathbb{Z}, \tau_{FC})$ that contains 0 and V an open set in $(\mathbb{Z}, \tau_{FC})$ that contains 1. Explain why it cannot be the case that U and V are disjoint – that is, $U \cap V$ must be non-empty.

(d) Show that the subset $\mathbb{E}$ of even integers is a compact subset of $(\mathbb{Z}, \tau_{FC})$ that is not closed. Verify your result.

(13) Let (X, τ) be a Hausdorff topological space.

(a) Let $x \in X$ and let A be a compact subset of X. Prove that there exist disjoint open subsets U and V of X such that $x \in U$ and $A \subseteq V$.

(b) Let A and B be disjoint compact subsets of X. Prove that there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(14) Let (X, τ_X) be a topological space and let A be a subset of X. Let τ_A be the subspace topology on A. Prove that A is a compact subset of X if and only if (A, τ_A) is a compact topological space.

(15) Let X be a topological space. A family $\{F_\alpha\}_{\alpha \in I}$ of subsets of X is said to have the *finite intersection property* if for each finite subset J of I, $\bigcap_{\alpha \in J} F_\alpha \neq \emptyset$. Prove that X is compact if and only if for each family $\{F_\alpha\}_{\alpha \in I}$ of closed subsets of X that has the finite intersection property, we have $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$.

(16) Even though $\mathbb{R}$ is not a compact space, if $x \in \mathbb{R}$, then $x \in [x-1, x+1]$ and so every point in $\mathbb{R}$ is contained in a compact subset of $\mathbb{R}$. So if we view $\mathbb{R}$ from the perspective of a point in $\mathbb{R}$, the space $\mathbb{R}$ looks compact around that point. This is the idea of local compactness. Locally compact spaces are important in the general topology of function spaces.

Definition 17.13. A topological space X is **locally compact** if for each $x \in X$ there is an open set O such that $p \in O$ and $\overline{O}$ is compact.

(a) Explain why $\mathbb{R}^n$ is locally compact for each $n \in \mathbb{Z}^+$.

(b) Show that any compact space is locally compact.

(c) Consider the Sorgenfrey line from Exercise 5 on page 140. Recall that the Sorgenfrey line is the space $\mathbb{R}$ with a basis $\mathcal{B} = \{[a, b) \mid a < b \text{ in } \mathbb{R}\}$ for its topology. Show that the Sorgenfrey line is Hausdorff but not locally compact.

(17) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) If X and Y are compact topological spaces and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

(b) If X is a compact topological space, then any closed subspace of X is compact.

(c) If X is a Hausdorff space, Y is a compact space, and $f : X \to Y$ is a continuous and bijective function, then f is a homeomorphism.

(d) If X is a compact space, Y is a Hausdorff space, and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

(e) Let C be a closed subset of a metric space (X, d) with the metric topology. Then C is compact.

(f) If A is a compact subset of a topological space X, then A is a closed subset of X.

(g) Let (X, τ) be a topological space with τ the discrete topology. Then X is compact if and only if X is finite.

An Application of Compactness: Fractals

Introduced by Felix Hausdorff in the early 20th century as a way to measure the distance between sets, the Hausdorff metric (also called the Pompeiu-Hausdorff metric) has since been widely studied and has many applications. For example, the United States military has used the Hausdorff distance in target recognition procedures. In addition, the Hausdorff metric has been used in image matching and visual recognition by robots, medicine, image analysis, and astronomy.

The basic idea in these applications is that we need a way to compare two shapes. For example, if a manufacturer needs to mill a specific product based on a template, there is usually some tolerance that is allowed. So the manufacturer needs a way to compare the milled parts to the template to determine if the tolerance has been met or exceeded.

The Hausdorff metric is also familiar to fractal aficionados for describing the convergence of sequences of compact sets to their attractors in iterated function systems. The variety of applications of this metric make it one that is worth studying.

To define the Hausdorff metric, we begin with the distance from a point x in a metric space X to a subset A of X as

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

Since images will be represented as compact sets, we restrict ourselves to compact subsets of a metric space. In this case the infimum becomes a minimum and we have

$$d(x, A) = \min\{d(x, a) \mid a \in A\}.$$

We now extend that idea to define the distance from one subset of X to another. Let A and B be nonempty compact subsets of X. To find the distance from the set A to the set B, it seems reasonable to consider how far each point in A is from the set B. Then the distance from A to B should measure how far we have to travel to get from *any* point in A to B.

Definition 17.14. Let (X, d) be a metric space and let A and B be nonempty compact subsets of X. Then **distance** $d(A, B)$ **from** A **to** B is

$$d(A, B) = \max_{a \in A} \left\{ \min_{b \in B} \{d(a, b)\} \right\}.$$

Note: since A and B are compact, $d(A, B)$ is guaranteed to exist.

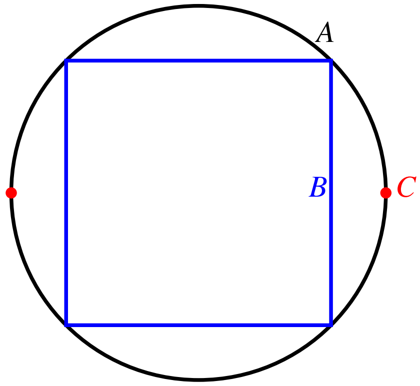

Figure 17.3: Sets A, B, and C.

Activity 17.7.

(a) A problem with d as in Definition 17.14 is that d is not symmetric. Find examples of compact subsets A and B of $\mathbb{R}^n$ with the Euclidean metric such that $d(A, B) \neq d(B, A)$.

(b) Even though the function d in Definition 17.14 is not a metric, we can define the Hausdorff distance in terms of d as follows.

Definition 17.15. Let (X, d) be a metric space and A and B nonempty compact subsets of X. Then **Hausdorff distance between** A **and** B is

$$h(A, B) = \max\{d(A, B), d(B, A)\}.$$

Let A be the circle in $\mathbb{R}^2$ centered at the origin with radius 1, let B be the inscribed square, and let $C = \{(1, 0), (-1, 0)\}$ as shown in Figure 17.3.

Determine $h(A, B)$, $h(A, C)$, and $h(B, C)$, and verify that $h(A, C) \leq h(A, B) + h(B, C)$.

(c) It may be surprising that h as in Definition 17.15 is actually a metric, but it is. The underlying space is the collection of nonempty compact subsets of X which we denote at $\mathcal{H}(X)$. Prove the following theorem.

Theorem 17.16. *Let X be a metric space. The Hausdorff distance function is a metric on $\mathcal{H}(X)$.*

(d) One fun application of the Hausdorff metric is in fractal geometry. You may be familiar with objects like the Sierpinski triangle or the Koch curve. These objects are limits of sequences of sets in $\mathcal{H}(\mathbb{R}^2)$. We illustrate with the Sierpinski triangle. Start with three points v_1, v_2, and v_3 that form the vertices of an equilateral triangle S_0. For i=1,2, or 3, let $v_i = \begin{bmatrix} a_i \\ b_i \end{bmatrix}$. For i=1,2, or 3, we define $\omega_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\omega_i\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a_i \\ b_i \end{bmatrix}.$$

Then ω_i, when applied to S_0, contracts S_0 by a factor of 2 and then translates the image of S_0 so that the i^{th} vertex of S_0 and the ith vertex of the image of S_0 coincide. Such a map is called a *contraction mapping* with *contraction factor* equal to $\frac{1}{2}$. Define $S_{1,i}$ to be $\omega_i(S_0)$. Then $S_{1,i}$ is the set of all points half way between any point in S_0 and v_i, or $S_{1,i}$ is a triangle half the size of the original translated to the i^{th} vertex of the original. Let $S_1 = \bigcup_{i=1}^{3} S_{1,i}$. Both S_0 and S_1 are shown in figure 17.4. We can continue this procedure replacing S_0 with S_1. In other words, for i = 1, 2, and 3, let $S_{2,i} = \omega_i(S_1)$. Then let $S_2 = \bigcup_{i=1}^{3} S_{2,i}$. A picture of S_2 is shown in figure 17.4. We can continue this procedure, each time replacing S_{j-1} with S_j. A picture of S_8 is shown in figure 17.4.

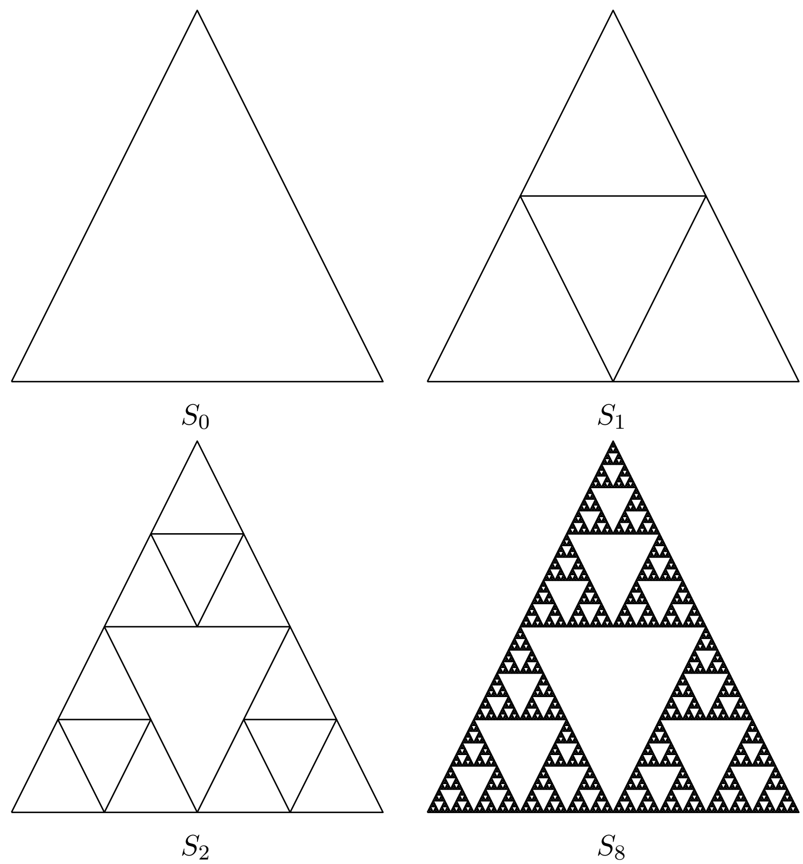

Figure 17.4: S_i for i equal to 0, 1, 2, and 8.

To continue this process, we need to take a limit. But the S_i are sets in $\mathcal{H}(\mathbb{R}^2)$, so the limit is taken with respect to the Hausdorff metric.

i. Assume that the length of a side of S_0 is 1. Determine $h(S_0, S_1)$. Then find $h(S_k, S_{k+1})$ for an arbitrary positive integer k.

ii. The Sierpinski triangle will exist if the sequence (S_n) converges to a set S (which would be the Sierpinski triangle). The question of convergence is not a trivial one.

A. Consider the sequence (a_n), where $a_n = \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{Z}^+$. Note that each a_n is a rational number. Explain why the terms in this sequence get arbitrarily close together, but the sequence does not converge in $\mathbb{Q}$. Explain why the sequence (a_n) converges in $\mathbb{R}$.

B. A sequence (x_n) in a metric space (X, d) is a *Cauchy sequence* if given $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$. Every convergent sequence is a Cauchy sequence. A metric space X is said to be *complete* if every Cauchy sequence in X converges to an element in X. For example, $(\mathbb{R}, d_E)$ is complete while $(\mathbb{Q}, d_E)$ is not. Although we will not prove it, the metric space $(\mathcal{H}(\mathbb{R}^2), h)$ is complete. Show that the sequence (S_n) is a Cauchy sequence in $\mathcal{H}(\mathbb{R}^2)$. The limit of this sequence is the famous Sierpinski triangle. The picture of S_8 in figure 17.4 is a close approximation of the Sierpinski triangle.