Section 18

Connected Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a connected subset of a topological space?
- What is a separation of a subset of a topological space? Why are separations useful?
- What are the connected subsets of $\mathbb{R}$?
- What is a connected component of a topological space?
- What is an application of connectedness?
- What is a cut set of a topological space? Why are cut sets useful?

Introduction

The term "connected" should bring up images of something that is one piece, not separated. There is more than one way we can interpret the notion of connectedness in topological spaces. For example, we might consider a topological space to be connected if we can't separate it into disjoint pieces in any non-trivial way. As another possibility, we might consider a topological space to be connected if there is always a path from one point in the spaced to another, provided we define what "path' means. These are different notions of connectedness, and we focus on the first notion in this section.

Connectedness is an important property, and one that we encounter in the calculus. For example, we will see in this section that the Intermediate Value Theorem relies on connected subsets of $\mathbb{R}$. To define a connected set, we will need to have a way to understand when and how a set can be separated into different pieces. Since a topology is defined by open sets, when we want to separate objects we will do so with open sets. This is similar to the idea behind Hausdorff spaces, except that we now want to know if a set can be separated in some way rather than separating points.

As an example to motivate the definition, consider the sets $X = (0,1) \cup (1,2)$ and $Y = [1,2]$ in $\mathbb{R}$ with the Euclidean topology. Notice that we can write X as the union of two disjoint open sets $X_1 = (0,1)$ and $X_2 = (1,2)$. So we shouldn't think of X as being connected. However, if we attempt to write Y as a union of two subsets, say $Y_1 = [1,1.5)$ and $Y_2 = [1.5,2]$, it is impossible for both of these subsets to be open and disjoint. So Y is a set we should consider to be connected. This is the notion of connectedness that we wish to investigate.

Definition 18.1. A topological space (X,τ) is **connected** if X cannot be written as the union of two disjoint, nonempty, open subsets. A subset A of a topological space topological space (X,τ) is connected if A is connected in the subspace topology.

If a set X is not connected, we say that X is disconnected. If X is a disconnected topological space, then there exist disjoint nonempty open sets U and V such that $X = U \cup V$.

Preview Activity 18.1. Can the subset A of the topological space X be written as the union of two disjoint nonempty relatively open sets?

(1) The set $A = \{a,b\}$ in (X,τ) with $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$.

(2) The set $A = \{a,b,c\}$ in (X,τ) with $X = \{a,b,c,d,e,f\}$ and

$$\tau = \{\emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e,f\}, X\}.$$

(3) The set $A = X$ with $X = \{a,b,c,d\}$ and

$$\tau = \{\emptyset, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X\}.$$

(4) The set $A = \{d,f\}$ in $X = \{a,b,c,d,e,f\}$ with the discrete topology. Generalize your findings.

(5) The set $A = \{a,c,d\}$ in $X = \{a,b,c,d,e\}$ with the indiscrete topology. Generalize your findings.

(6) The set $A = \mathbb{Z}$ in $X = \mathbb{R}$ with the finite complement topology. Generalize your findings.

(7) The set $A = X$ in $X = \{x \in \mathbb{R} \mid 1 \leq x \leq 2 \text{ or } 3 < x < 4\}$ with the subspace metric topology from $(\mathbb{R}, d_E)$.

(8) The set $A = X$ in $X = \left\{(x,y) \in \mathbb{R}^2 \mid y = \frac{1}{x} \text{ or } y = 0\right\}$ with open sets

$$\tau = \{U \cap X \mid U \text{ is open in the Euclidean Topology on } \mathbb{R}^2\}.$$

Connected Sets

As we learned in our preview activity, connected sets are those sets that cannot be separated into a union of disjoint open sets. Another characterization of connectedness is established in the next activity.

Activity 18.1. Let (X, τ) be a topological space.

(a) Assume that X is a connected space, and let A be a subset of X that is both open and closed. What happens if we combine A and $X \setminus A$? What does the fact that X is connected tell us about A?

(b) Now assume that the only subsets of X that are both open and closed are $\emptyset$ and X. Must it follow that X is connected? Prove your assertion.

(c) Summarize the result of this activity into a theorem of the form "A topological space (X, τ) is connected if and only if ... ".

A standard example of a connected topological space is the metric space $(\mathbb{R}, d_E)$.

Theorem 18.2. *The metric space $(\mathbb{R}, d_E)$ is a connected topological space.*

Proof. We proceed by contradiction and assume that there are nonempty open sets U and V such that $\mathbb{R} = U \cup V$ and $U \cap V = \emptyset$. Let $a \in U$ and $b \in V$. Since $U \cap V = \emptyset$, we know that $a \neq b$. Without loss of generality we can assume $a < b$. Let $U' = U \cap [a, b]$ and let $V' = V \cap [a, b]$. The set V' is bounded below by a, so $x = \inf\{v \mid v \in V'\}$ exists. Since $\mathbb{R} = U \cup V$ it must be the case that $x \in U$ or $x \in V$.

Suppose $x \in U$. The fact that U is an open set implies that there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. But then $B(x, \epsilon) \cap V = \emptyset$ and so $d(x, v) \geq \epsilon$ for every $v \in V$. This means that $x + \epsilon < v$ for every $v \in V'$, contradicting the fact that x is the greatest lower bound. We conclude that $x \notin U$.

It follows that $x \in V$. Since $a \in U$, we know that $x \neq a$. The fact that V is an open set tells us that there exists $\delta > 0$ such that $B(x, \delta) \subseteq V$. We can choose δ to ensure that $\delta < x - a$. Since $x > a$, the interval $(x - \delta, x)$ is a subset of V', and so x is not a lower bound for V.

Each possibility leads to a contradiction, so we conclude that the sets U and V cannot exist. Therefore, $(\mathbb{R}, d_E)$ is a connected topological space. ■

As you might expect, connectedness is a topological property.

Activity 18.2. Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : X \to Y$ be a continuous function. Assume that X is a connected subset of X. Our goal is to prove that $f(X)$ is a connected subspace of Y.

Let $Z = f(X)$ and define $g : X \to Z$ by $g(x) = f(x)$. Then g is a continuous function that maps X onto Z. So we consider g instead of f.

(a) Assume to the contrary that Z is not connected. What do we then assume about Z?

(b) Suppose that U and V are disjoint nonempty open sets in Z such that $U \cup V = Z$. Let $R = g^{-1}(U)$ and $S = g^{-1}(V)$.

(i) Explain why R and S are open sets in X.

(ii) Show that $R \cup S = X$. (Hint: $X = g^{-1}(Z)$.)

(iii) Show R and S are nonempty sets. (Hint: Use the fact that g is a surjection.)

(iv) Now show that $R \cap S = \emptyset$. (Hint: $R \cap S = g^{-1}(U) \cap g^{-1}(V)$.)

(c) Explain how we have proved the following.

Theorem 18.3. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, and let $f : X \to Y$ be a continuous function. If X is connected, then $f(X)$ is connected.*

The fact that connectedness is preserved by continuous functions means that connectedness is a property that is shared by any homeomorphic topological spaces, as the next corollary indicates.

Corollary 18.4. *Let (X, τ_X) and (Y, τ_Y) be homeomorphic topological spaces. Then X is connected if and only if Y is connected.*

Proof. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$ be a homeomorphism. Assume that X is connected. Since f is continuous, Theorem 18.3 shows that $f(X) = Y$ is connected. The reverse implication follows from the fact that f^{-1} is a homeomorphism. ■

Recall that $(\mathbb{R}, d_E)$ is homeomorphic to the topological subspaces (a, b), $(-\infty, b)$, and (a, ∞) for any $a, b \in \mathbb{R}$. The fact that $(\mathbb{R}, d_E)$ is connected (Theorem 18.2) allows us to conclude that all open intervals are connected. It would seem natural that all closed (or half-closed) intervals should also be connected. We address this question next. Before we get to this result, we consider an alternate formulation of connected subsets.

Consider the set $A = (-1, 0) \cup (4, 5)$ in $\mathbb{R}$. Let $U = (-2, 3)$ and $V = (2, 6)$ in $\mathbb{R}$. Note that $U' = U \cap A = (-1, 0)$ and $V' = V \cap A = (4, 5)$, and so U and V are open sets in $\mathbb{R}$ that separate the set A into two disjoint pieces. We know that U' and V' are open in A and $A = U' \cup V'$ with $U' \cap V' = \emptyset$. So to show that a subset of a topological space X is not connected, this example suggests that it suffices to find nonempty open sets U and V in X with $U \cap V \cap A = \emptyset$ and $A \subseteq (U \cup V)$. Note that it is not necessary to have $U \cap V = \emptyset$. That this works in general is the result of the next theorem.

Theorem 18.5. *Let X be a topological space. A subset A of X is disconnected if and only if there exist open sets U and V in X with*

- $A \subseteq (U \cup V)$,
- $U \cap A \neq \emptyset$,
- $V \cap A \neq \emptyset$, *and*
- $U \cap V \cap A = \emptyset$.

Proof. Let X be a topological space, and let A be a subset of X. We first assume that A is disconnected and show that there are open sets U and V in X that satisfy the given conditions. Since A is disconnected, there are nonempty open sets U' and V' in A such that $U' \cup V' = A$ and $U' \cap V' = \emptyset$. Since U' and V' are open in A, there exist open sets U and V in X so that $U' = U \cap A$ and $V' = V \cap A$. Now

$$A = U' \cup V' = (U \cap A) \cup (V \cap A) = (U \cup V) \cap A,$$

and so $A \subseteq U \cup V$. By construction, $U \cap A = U'$ and $V \cap A = V'$ are not empty. Finally,

$$U \cap V \cap A = (U \cap A) \cap (V \cap A) = U' \cap V' = \emptyset.$$

So we have found sets U and V that satisfy the conditions of our theorem.

The proof of the reverse implication is left to the next activity.

■

Activity 18.3. Let X be a topological space, and let A be a subset of X. Assume that there exist open sets U and V in X with $A \subseteq U \cup V$, $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, and $U \cap V \cap A = \emptyset$. Prove that A is disconnected.

The conditions in Theorem 18.5 provide a convenient way to show that a set is disconnected, and so any pair of sets U and V that satisfy the conditions of Theorem 18.5 is given a special name.

Definition 18.6. Let X be a topological space, and let A be a subset of X. A **separation** of A is a pair of nonempty open subsets U and V of X such that

- $A \subseteq (U \cup V)$,
- $U \cap A \neq \emptyset$,
- $V \cap A \neq \emptyset$, and
- $U \cap V \cap A = \emptyset$.

If X is a disconnected topological space, then a separation of X is a pair U, V of disjoint nonempty open sets such that $U \cup V = X$.

Connected Subsets of $\mathbb{R}$

With Theorem 18.5 in hand, we are just about ready to show that any interval in $\mathbb{R}$ is connected. Let us return for a moment to our example of $A = (-1, 0) \cup (4, 5)$ in $\mathbb{R}$. It is not difficult to see that if U and V are a separation of A, then the subset $(-1, 0)$ must be entirely contained in either U or in V. The reason for this is that $(-1, 0)$ is a connected subset of A. This result is true in general.

Activity 18.4. Let X be a topological space, and let A be a subset of X. Assume that U and V form a separation of A. Let C be a connected subset of A. In this activity we want to prove that $C \subseteq U$ or $C \subseteq V$.

(a) Use the fact that U and V form a separation to A to wxplain why $C \subseteq U \cup V$ and $C \cap U \cap V = \emptyset$.

(b) Given that C is connected, what conclusion can we draw about the sets $U' = U \cap C$ and $V' = V \cap C$?

(c) Complete the proof of the following lemma.

Lemma 18.7. *Let X be a topological space, and let A be a subset of X. Assume that U and V form a separation of A. If C is a connected subset of A, then $C \subseteq U$ or $C \subseteq V$.*

Now we can prove that any interval in $\mathbb{R}$ is connected. Since $[a, b]$, $[a, b)$, and $(a, b]$ are all sets that lie between (a, b) and $\overline{(a, b)}$, we can address their connectedness all at once with the next result.

Theorem 18.8. *Let X be a topological space and C a connected subset of X. If A is a subset of X and $C \subseteq A \subseteq \overline{C}$, then A is connected in X.*

Proof. Let X be a topological space and C a connected subset of X. Let A be a subset of X such that $C \subseteq A \subseteq \overline{C}$. To show that A is connected, assume to the contrary that A is disconnected. Then there are nonempty open subsets U and V of X that form a separation of A. Lemma 18.7 shows that $C \subseteq U$ or $C \subseteq V$. Without loss of generality we assume that $C \subseteq U$. Since $U \cap V \cap A = \emptyset$, it follows that

$$C \cap V = (C \cap A) \cap V = C \cap (A \cap V) \subseteq U \cap A \cap V = \emptyset.$$

Since $A \cap V \neq \emptyset$, there is an element $x \in A \cap V$. Since $x \notin C$ and $x \in A \subseteq \overline{C}$, it must be the case that x is a limit point of C. Since V is an open neighborhood of x, it follows that $V \cap C \neq \emptyset$. This contradiction allows us to conclude that A is connected. ■

One consequence of Theorem 18.8 is that any interval of the form $[a, b)$, $(a, b]$, $[a, b]$, $(-\infty, b]$, or $[a, \infty)$ in $\mathbb{R}$ is connected. This prompts the question, are there any other subsets of $\mathbb{R}$ that are connected?

Activity 18.5. Let A be a subset of $\mathbb{R}$.

(a) Let $A = \{a\}$ be a single point subset of $\mathbb{R}$. Is A connected? Explain.

(b) Now suppose that A is a subset of $\mathbb{R}$ that contains two or more points. Assume that A is not an interval. Then there must exist points a and b in A and a point c in $\mathbb{R} \setminus A$ between a and b. Use this idea to find a separation of A. What can we conclude about A?

(c) Explicitly describe the connected subsets of $(\mathbb{R}, d_E)$.

Components

As Activity 18.5 demonstrates, spaces like $A = (1, 2) \cup (3, 4)$ are not connected. Even so, A is made of two connected subsets $(1, 2)$ and $(3, 4)$. These connected subsets are called *components*.

Definition 18.9. A subspace C of a topological space X is a **component** (or **connected component**) of X if C is connected and there is no larger connected subspace of X that contains C.

As an example, if $X = (1, 2) \cup [4, 10) \cup \{-1, 15\}$, then the components of X are $(1, 2)$, $[4, 10)$, $\{-1\}$ and $\{15\}$. As the next activity shows, we can always partition a topological space into a disjoint union of compenents.

Activity 18.6. Let (X, τ) be a nonempty topological space.

(a) Show that if $x \in X$, then $\{x\}$ is connected.

(b) Suppose that X is a topological space and $\{A_\alpha\}$ for α in some indexing set I is a collection of connected subsets of X. Let $A = \bigcup_{\alpha \in I} A_\alpha$. Suppose that $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$. Show that A is a connected subset of X. (Hint: Assume a separation and use Lemma 18.7.)

(c) Part (a) shows that every element in x belongs to some connected subset of X. So we can write X as a union of connected subsets. But there is probably overlap. To remove the overlap, we define the following relation $\sim$ on X:

For x and y in X, $x \sim y$ if x and y are contained in the same connected subset of X.

As with any relation, we ask if $\sim$ is an equivalence relation.

i. Is $\sim$ a reflexive relation? Why or why not?
ii. Is $\sim$ a symmetric relation? Why or why not?
iii. Is $\sim$ a transitive relation? Why or why not?

Activity 18.6 shows that the relation $\sim$ is an equivalence relation, and so partitions the underlying topological space X into disjoint sets. If $x \in X$, then the equivalence class of x is a connected subset of X. There can be no larger connected subset of X that contains x, since equivalence classes are disjoint or the same. So the equivalence classes are exactly the connected components of X. The components of a topological space X satisfy several properties.

- Each $a \in X$ is an element of exactly one connected component C_a of X.
- A component C_a contains all connected subsets of X that contain a. Thus, C_a is the union of all connected subsets of X that contain a.
- If a and b are in X, then either $C_a = C_b$ or $C_a \cap C_b = \emptyset$.
- Every connected subset of X is a subset of a connected component.
- Every connected component of X is a closed subset of X.
- The space X is connected if and only if X has exactly one connected component.

Cut Sets

It can be difficult to determine if two topological spaces are homeomorphic. We can sometimes use topological invariants to determine if spaces are not homeomorphic. For example, if X is connected and Y is not, then X and Y are not homeomorphic. But just because two spaces are connected, it does not automatically follow that the spaces are homeomorphic. For example consider the spaces $(0, 2)$ and $[0, 2)$. Both are connected subsets of $\mathbb{R}$. If we remove a point, say 1, from the set $(0, 2)$ the resulting space $(0, 1) \cup (1, 2)$ is no longer connected. The same result is true if we remove any point from $(0, 2)$. However, if we remove the point 0 from $[0, 2)$ the resulting space $(0, 2)$ is connected. So the spaces $(0, 2)$ and $[0, 2)$ are fundamentally different in this respect, and so are not homeomorphic. Any set that we can remove from a connected set to obtain a disconnected set is called a *cut set*.

Definition 18.10. A subset S of a connected topological space X is a **cut set** of X if the set $X \setminus S$ is disconnected. A point p in a connected topological space X is a **cut point** if $X \setminus \{p\}$ is disconnected.

Example 18.11. (1) The point 1 is a cut point of the space $(0, 2)$. In fact, every point in $(0, 2)$ is a cut point of $(0, 2)$.

(2) Let $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$ in $(\mathbb{R}^2, d_E)$. That is, D is the closed disk of radius 2 in the plane. The set D has no cut points. However, if $S = \{(x, y) \mid x^2 + y^2 = 1\}$, then $D \setminus S$ consists of two connected components: the open ball $B((0, 0), 1)$ and the annulus $\{(x, y) \mid 1 < x^2 + y^2 \leq 4\}$ as illustrated in Figure 18.1. So S is a cut set of D.

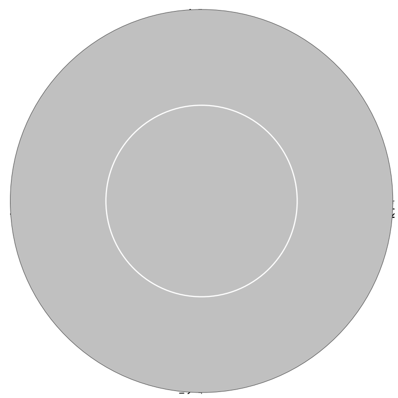

Figure 18.1: The disk D and cut set S.

Once we have a new property, we then ask if that property is a topological invariant.

Theorem 18.12. *Let X and Y be connected topological spaces and let $f : X \to Y$ be a homeomorphism. If $S \subset X$ is a cut set, then $f(S)$ is a cut set of Y.*

Proof. Let X and Y be topological spaces with $f : X \to Y$ a homeomorphism. Let S be a cut set of X. Let U and V form a separation of $X \setminus S$. We will demonstrate that $f(U)$ and $f(V)$ form a separation of $Y \setminus f(S)$, which will prove that $f(S)$ is a cut set of Y. Since f^{-1} is continuous, the sets $f(U)$ and $f(V)$ are open sets in Y. Next we prove that $(Y \setminus f(S)) \subseteq (f(U) \cup f(V))$. Let $y \in Y \setminus f(S)$. Since f is a surjection, there exists an $x \in X$ with $f(x) = y$. The fact that $y \notin f(S)$ means that $x \notin S$. So $x \in (X \setminus S) \subseteq (U \cup V)$. If $x \in U$, then $f(x) = y \in f(U)$ and if $x \in V$, then $x = f(y) \in f(V)$. So $(Y \setminus f(S)) \subseteq (f(U) \cup f(V))$.

Now we demonstrate that $f(U) \cap (Y \setminus f(S)) \neq \emptyset$ and $f(V) \cap (Y \setminus f(S)) \neq \emptyset$. Since U and V form a separation of $X \setminus S$, we know that $U \cap (X \setminus S) \neq \emptyset$ and $V \cap (X \setminus S) \neq \emptyset$. Let $x \in U \cap (X \setminus S)$. Then $x \in U$ and $x \notin S$. So $f(x) \in f(U)$ and the fact that f is an injection implies that $f(x) \notin f(S)$. Thus, $f(x) \in f(U) \cap (Y \setminus f(S))$. The same argument shows that $x \in V \cap (X \setminus S)$ implies that $f(x) \in f(V) \cap (Y \setminus f(S))$. So $f(U) \cap (Y \setminus f(S)) \neq \emptyset$ and $f(V) \cap (Y \setminus f(S)) \neq \emptyset$.

Finally, we show that $f(U) \cap f(V) \cap (Y \setminus f(S)) = \emptyset$. Suppose $y \in f(U) \cap f(V) \cap (Y \setminus f(S))$. Let $x \in X$ such that $f(x) = y$. Since f is an injection, we know that $f(x) \in f(U)$ means $x \in U$. so $x \in U \cap V$. The fact that $y \in Y \setminus f(S)$ means that $y \notin f(S)$. Thus, $x \notin S$. So $x \in X \setminus S$. We then have $x \in U \cap V \cap (X \setminus S) = \emptyset$. It follows that $f(U) \cap f(V) \cap (Y \setminus f(S)) = \emptyset$. Therefore, $f(U)$ and $f(V)$ form a separation of $Y \setminus f(S)$ and $f(S)$ is a cut set of Y.

■

Activity 18.7.

(a) Use the idea of cut sets/points to explain why the unit circle in $\mathbb{R}^2$ is not homeomorphic to the interval $[0, 1]$ in $\mathbb{R}$. Note: the unit circle is the set $\{(x, y) \mid x^2 + y^2 = 1\}$. Draw pictures to illustrate your explanation. (A formal proof is not necessary, but you need to provide a convincing justification.)

(b) Consider the following subsets of $\mathbb{R}^2$ in the subspace topology:

$$A = \{(x, y) \mid x^2 + y^2 = 1\} \quad \text{and} \quad B = \{(x, 0) \mid -1 \leq x \leq 1\}.$$

Is $A \cup B$ homeomorphic to A? (A formal proof is not necessary, but you need to provide a convincing justification.)

We have seen that topological equivalence is an equivalence relation, which partitions the collection of all topological spaces into disjoint homeomorphism classes. Topological invariants can then help us identify the classes to which different spaces belong. In general, though, it can be more difficult to prove that two spaces are homeomorphic than not homeomorphic.

Activity 18.8. Consider the spaces $S_1 = \mathbb{R}$, $S_2 = (0, 1)$ in $\mathbb{R}$, $S_3 = [-1, 1]$ in $\mathbb{R}$, the line segment S_4 in $\mathbb{R}^2$ between the points $(0, 0)$ and $(2, 2)$, the space S_5 determined by the letter X, and the space S_6 determined by the letter Y in $\mathbb{R}^2$. Identify the distinct homeomorphism classes determined by these six spaces. No formal proofs are necessary, but you need to give convincing arguments.

The Intermediate Value Theorem and a Fixed Point Theorem

In this section we present two important consequences of connectedness. The first consequence is the Intermediate Value Theorem. In calculus, the Intermediate Value Theorem tells us that if f is a continuous function on a closed interval $[a, b]$, then f assumes all values between $f(a)$ and $f(b)$. This result seems straightforward if one just draws a graph of a continuous function on a closed interval. But a picture is not a proof. We state and then prove a more general version of the Intermediate Value Theorem.

Theorem 18.13 (The Intermediate Value Theorem)**.** *Let X be a topological space and A a connected subset of X. If $f : A \to \mathbb{R}$ is a continuous function, then for any $a, b \in A$ and any $y \in \mathbb{R}$ between $f(a)$ and $f(b)$, there is a point $x \in A$ such that $f(x) = y$.*

Activity 18.9. In this activity we prove the Intermediate Value Theorem. Let X be a topological space and A a connected subset of X. Assume that $f : A \to \mathbb{R}$ is a continuous function, and let $a, b \in A$.

(a) Explain why we can assume that $a \neq b$.

(b) Explain what happens if $y = f(a)$ or $y = f(b)$.

(c) Now assume that $f(a) \neq f(b)$. Without loss of generality, assume that $f(a) < f(b)$. Why can we say that $f(A)$ is an interval?

(d) How does the fact that $f(A)$ is an interval complete the proof?

Our second consequence of connectedness involves a fixed point theorem. Fixed point theorems are important in mathematics. A fixed point of a function f is an input c so that $f(c) = c$. There are many fixed point theorems – one of the most well-known is the Brouwer Fixed Point Theorem that shows that every continuous function from a closed ball B in $\mathbb{R}^n$ to itself must have a fixed point. We can use the Intermediate Value Theorem to prove this result in $\mathbb{R}$.

Activity 18.10. In this activity we prove the following theorem.

Theorem 18.14. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \to [a, b]$ be a continuous function. Then there is a number $c \in [a, b]$ such that $f(c) = c$.*

So let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \to [a, b]$ be a continuous function.

(a) Explain why we can assume that $a < f(a)$ and $f(b) < b$.

(b) Define $g : [a, b] \to \mathbb{R}$ by $g(x) = x - f(x)$.

   i. Why is g a continuous function?

   ii. What can we say about $g(a)$ and $g(b)$? Use the Intermediate Value Theorem to complete the proof.

(c) Let $f(x) = \frac{1}{6}x^3 + \frac{1}{4}x$.

   i. Show that f maps the interval $[-1, 2]$ into the interval $[-1, 2]$. (Hint: Use Theorem 17.12 on page 180 that a continuous function from a compact topological space to $\mathbb{R}$ assumes a maximum and minimum value.)

   ii. Find all values of c in $[-1, 2]$ such that $f(c) = c$.

Summary

Important ideas that we discussed in this section include the following.

- A topological space (X, τ) is **connected** if X cannot be written as the union of two disjoint, nonempty, open subsets. A subset A of a topological space topological space (X, τ) is connected if A is connected in the subspace topology.

- A separation of a subset A of a topological space X is a pair of nonempty open subsets U and V of X such that
  - $A \subseteq (U \cup V)$,
  - $U \cap A \neq \emptyset$,
  - $V \cap A \neq \emptyset$, and

  - $U \cap V \cap A = \emptyset$.

  Showing that a set has a separation can be a convenient way to show that the set is disconnected.

- The connected subsets of $\mathbb{R}$ are the intervals and the single point sets.
- A subspace C of a topological space X is a connected component of X if C is connected and there is no larger connected subspace of X that contains C.
- One application of connectedness is the Intermediate Value Theorem that tells us that if A is a connected subset of a topological space X and if $f : A \to \mathbb{R}$ is a continuous function, then for any $a, b \in A$ and any $y \in \mathbb{R}$ between $f(a)$ and $f(b)$, there is a point $x \in A$ such that $f(x) = y$.
- A subset S of a connected topological space X is a cut set of X if the set $X \setminus S$ is disconnected, while a point p in X is a cut point if $X \setminus \{p\}$ is disconnected. The property of being a cut set or a cut point is a topological invariant, so we can sometimes use cut sets and cut points to show that topological spaces are not homeomorphic.

Exercises

(1) Recall from Definition 12.13 on page 126 that if τ_1 and τ_2 are two topologies on a set X such that $\tau_1 \subseteq \tau_2$, then τ_1 is said to be a *coarser* (or *weaker*) topology than τ_2, or τ_2 is a *finer* (or *stronger*) topology than τ_1. In this exercise we explore the question of whether compactness is a property that is passed from weaker to stronger topologies or from stronger to weaker.

Let τ_1 and τ_2 be two topologies on a set X. If $\tau_1 \subseteq \tau_2$, what does connectedness under τ_1 or τ_2 imply, if anything, about compactness under the other topology? Justify your answers.

(2) Let $f : [a, b] \to \mathbb{R}$ be a continuous function from a closed interval into the reals. Let $U = f(u)$ and $V = f(v)$ be such that $U \leq f(x) \leq V$ for all $x \in [a, b]$. Prove that there is a w between u and v such that $f(w)(b - a) = \int_a^b f(t)\,dt$.

(3) Let A be a connected subset of a topological space X. Prove or disprove:

  (a) $\text{Int}(A)$ is connected

  (b) $\overline{A}$ is connected

  (c) $\text{Bdry}(A)$ is connected

(4) Let $X = \mathbb{R}$ with the finite complement topology. We have shown that every subset of any topological space with the finite complement topology is compact. Now find all of the connected subsets of X. Prove your result.

(5) Give examples, with justification, of subsets A and B of a topological space to illustrate each of the following, or explain why no such example exists:

  (a) A and B are connected, but $A \cap B$ is disconnected

  (b) A and B are connected, but $A \setminus B$ is disconnected

(c) A and B are disconnected, but $A \cup B$ is connected

(d) A and B are connected and $A \cap B \neq \emptyset$, but $A \cup B$ is disconnected.

(e) A and B are connected and $\overline{A} \cap \overline{B} \neq \emptyset$, but $A \cup B$ is disconnected.

(6) Let $f : S^1 \to \mathbb{R}$ be a continuous function. Show that there is a point $x \in S^1$ with $f(x) = f(-x)$.

(7) Let $a, b \in \mathbb{R}$ with $a < b$. Explain why no two of the sets (a, b), $(a, b]$, and $[a, b]$ homeomorphic.

(8) Let $K = \left\{\frac{1}{k} \mid k \text{ is a positive integer}\right\}$. Let $\mathcal{B}$ be the collection of all open intervals of the form (a, b) and all sets of the form $(a, b) \setminus K$, where $a < b$ are real numbers as in Example 13.13 on page 137. Let τ_K be the topology generated by $\mathcal{B}$. Show that $(\mathbb{R}, \tau_K)$ is a connected space.

(9) Even though $X = (0, 1) \cup (1, 2)$ is not a connected space, if x is any element in X then we can surround x with a connected subset of X. This is the idea of local connectedness.

Definition 18.15. A topological space X is **locally connected at a point** $x \in X$ if every neighborhood U of x contains an open connected neighborhood of x. A topological space X is **locally connected** if X is locally connected at each point in X.

(a) Give an example of a locally connected space that is not connected.

(b) It would be reasonable to believe that a connected space is locally connected. However, that is not the case. Consider the space $X = A \cup B$ as a subspace of $\mathbb{R}^2$ with the standard Euclidean metric topology, where $A = \{(x, y) \mid x \text{ is irrational and } 0 \leq y \leq 1\}$ and $B = \{(x, y) \mid x \text{ is rational and } -1 \leq y \leq 0\}$.

i. Explain why X is connected.

ii. Show that X is not locally connected. (Hint: Let x be a point not on the x-axis and find an open ball around x that doesn't intersect the x-axis.)

(c) Prove that a topological space X is locally connected if and only if for every open set O in X, the connected components of O are open in X.

(10) Let A and B be nonempty subsets of a topological space X.

(a) Prove that $A \cup B$ is disconnected if $(\overline{A} \cap B) \cup (A \cap \overline{B}) = \emptyset$.

(b) Prove that X is connected if and only if for every pair of nonempty subsets A and B of X such that $X = A \cup B$ we have $(\overline{A} \cap B) \cup (A \cap \overline{B}) \neq \emptyset$.

(11) Give examples of the following.

(a) A topological space with exactly one cut point.

(b) A topological space with exactly two cut points.

(c) A topological space with infinitely many cut points.

(d) A topological space with no cut points.

(12) Let $a, b \in \mathbb{R}$ with $a < b$. Prove that a homeomorphism $f : [a, b] \to [a, b]$ carries end points into end points.

(13) Let X and Y be a topological spaces.

(a) Assume that X and Y homeomorphic spaces. Prove that there is a one-to-one correspondence between the connected components of X and the connected components of Y.

(b) Assume that X and Y homeomorphic spaces. Prove that there is a one-to-one correspondence between the set of cut points of X and the set of cut points of Y.

(c) Consider each letter in the statement as a topological space with the standard Euclidean metric topology.

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Group the letters in the statement into disjoint homeomorphism classes. Explain in detail the reasons for your groupings.

(14) Let (X, τ) be a topological space.

(a) Prove that X is disconnected if and only if X has a proper subset that is both open and closed.

(b) Prove that X is disconnected if and only if there is a continuous function from X onto a discrete two-point topological space.

(15) Let X be the set of real numbers.

(a) Consider X with the topology $\tau_1 = \{\emptyset, [0, 1], X\}$. Prove or disprove: X is connected.

(b) Consider X with the topology $\tau_2 = \{U \subseteq X \mid 0 \in U\} \cup \{\emptyset\}$.

i. Prove or disprove: X is connected.

ii. Prove or disprove: $X \setminus \{0\}$ is connected.

(16) Let X be a nonempty set and let p be a fixed element in X. Let τ_p be the particular point topology and $\tau_{\overline{p}}$ the excluded point topology on X. That is

- τ_p is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that contain p.
- $\tau_{\overline{p}}$ is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that do not contain p.

That the particular point and excluded point topologies are topologies is the subject of Exercises (9) and (10) on page 127.

Determine, with proof, the connected subsets of X when

(a) X has the particular point topology τ_p

(b) X has the excluded point topology $\tau_{\overline{p}}$.

(17) Let (X, τ) be a topological space and A a connected subset of X.

(a) Show that if X is Hausdorff, then A' is connected.

(b) Let $(X, \tau) = (\mathbb{R}, \tau_0)$, where τ_0 is the particular point topology on X. Explain why $A = \mathbb{Z}$ is a connected subset of X. Find $\mathbb{Z}'$ in $(\mathbb{R}, \tau_0)$. Is it true that in any topological space, if A is connected, then so is A'? Explain. (See Exercise 16.)

(18) Let X be a topological space. Prove each of the following.

(a) Each $a \in X$ is an element of exactly one connected component C_a of X.

(b) A component C_a contains all connected subsets of X that contain a. Thus, C_a is the union of all connected subsets of X that contain a.

(c) If a and b are in X, then either $C_a = C_b$ or $C_a \cap C_b = \emptyset$.

(d) Every connected subset of X is a subset of a connected component.

(e) Every connected component of X is a closed subset of X.

(f) The space X is connected if and only if X has exactly one connected component.

(19) Let X be a topological space with only a finite number of connected components. Show that each component of X is open.

(20) Let X and Y be connected spaces with $f : X \to Y$ a continuous function. Is it the case that if S is a cut set of X, then $f(S)$ is a cut set of Y? Prove your answer.

(21) Let $X = \{a, b, c, d\}$. There are 355 distinct topologies on X, but they fit into the 33 distinct homeomorphism classes listed below. The list is ordered by decreasing number of singleton sets in the topology, and, when that is fixed, by increasing number of two-point subsets and then by increasing number of three-point subsets. Under which topologies is X connected? Prove your answer.

1. the discrete topology
2. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\}$
3. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, X\}$
4. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$
5. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$
6. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$
7. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}, X\}$
8. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$
9. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$
10. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$
11. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, X\}$
12. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$
13. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, X\}$
14. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$
15. $\{\emptyset, \{a\}, X\}$

16. $\{\emptyset, \{a\}, \{a,b\}, X\}$
17. $\{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, X\}$
18. $\{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$
19. $\{\emptyset, \{a\}, \{a,b\}, \{a,c,d\}, X\}$
20. $\{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$
21. $\{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X\}$
22. $\{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, X\}$
23. $\{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}, X\}$
24. $\{\emptyset, \{a\}, \{c,d\}, \{a,b\}, \{a,c,d\}, X\}$
25. $\{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, X\}$
26. $\{\emptyset, \{a\}, \{a,b,c\}, X\}$
27. $\{\emptyset, \{a\}, \{b,c,d\}, X\}$
28. $\{\emptyset, \{a,b\}, X\}$
29. $\{\emptyset, \{a,b\}, \{c,d\}, X\}$
30. $\{\emptyset, \{a,b\}, \{a,b,c\}, X\}$
31. $\{\emptyset, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$
32. $\{\emptyset, \{a,b,c\}, X\}$
33. $\{\emptyset, X\}$

(22) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) If A is a connected subset of a topological space X with $|A| \geq 2$, then every point of A is a limit point of A.

(b) If A is a compact subspace of a Hausdorff space, then A is connected.

(c) If A is a connected subspace of a Hausdorff space, then A is compact.

(d) Every subset of a topological space with the discrete topology is disconnected.

(e) The set $\{a,b\}$ is a connected component of the topological space $X = \{a,b,c,d\}$ with topology

$$\tau = \{\emptyset, \{a\}, \{a,b\}, \{c\}, \{d\}, \{c,d\}, \{a,c,d\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, X\}.$$

(f) The sets $U = \{a,c,d\}$ and $V = \{a,b,c\}$ form a separation of the set $A = \{c,d\}$ in the topological space $X = \{a,b,c,d\}$ with topology

$$\tau = \{\emptyset, \{a\}, \{a,b\}, \{c\}, \{d\}, \{c,d\}, \{a,c,d\}, \{a,c\}, \{a,d\}, \{a,b,c,\}, \{a,b,d\}, X\}.$$

(g) The connected topological space $X = \{a,b,c,d\}$ with topology

$$\tau = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X\}$$

has no cut points.