Section 19

Path Connected Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a path in a topological space?
- What is a path connected subset of a topological space?
- What is a path connected component of a topological space?
- What is a locally path connected space?
- What connections are there between connected spaces and path connected spaces?

Introduction

We defined connectedness in terms of separability by open sets. There are other ways to look at connectedness. For example, the subset $(0, 1)$ is connected in $\mathbb{R}$ because we can draw a line segment (which we will call a *path*) between any two points in $(0, 1)$ and remain in the set $(0, 1)$. So we might alternatively consider a topological space to be connected if there is always a path from one point in the spaced to another. Although this is a new notion of connectedness, we will see that path connectedness and connectedness are related.

Intuitively, a space is path connected if there is a path in the space between any two points in the space. To formalize this idea, we need to define what we mean by a path. Simply put, a path is a continuous curve between two points. We can therefore define a path as a continuous function.

Definition 19.1. Let X be a topological space. A **path** from point a to point b in X is a continuous function $p : [0, 1] \to X$ such that $p(0) = a$ and $p(1) = b$.

With the notion of path, we can now define path connectedness.

Definition 19.2. A subspace A of a topological space X is **path connected** if, given any $a, b \in A$ there is a path in A from a to b.

Preview Activity 19.1.

(1) Is $\mathbb{R}$ with the Euclidean metric topology path connected? Explain.

(2) Is $\mathbb{R}$ with the finite complement topology path connected? Explain.

(3) Let $A = \{b, c\}$ in (X, τ) with $X = \{a, b, c, d, e, f\}$ and

$$\tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}, X\}.$$

Is A connected? Is A path connected? Explain.

Path Connectedness

As with every new property we define, it is natural to ask if path connectedness is a topological property.

Activity 19.1. In this activity we prove Theorem 19.3.

Theorem 19.3. *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. If A is a path connected subspace of X, then $f(A)$ is a path connected subspace of Y.*

Assume that X and Y are topological spaces, $f : X \to Y$ is a continuous function, and $A \subseteq X$ is path connected. To prove that $f(A)$ is path connected, we choose two elements u and v in $f(A)$. It follows that there exist elements a and b in A such that $f(a) = u$ and $f(b) = v$.

(a) Explain why there is a continuous function $p : [0, 1] \to A$ such that $p(0) = a$ and $p(1) = b$.

(b) Determine how p and f can be used to define a path $q : [0, 1] \to f(A)$ from u to v. Be sure to explain why q is a path. Conclude that $f(A)$ is path connected.

A consequence of Theorem 19.3 is the following.

Corollary 19.4. *Path connectedness is a topological property.*

Path Connectedness as an Equivalence Relation

We saw that we could define an equivalence relation using connected subsets of a topological space, which partitions the space into a disjoint union of connected components. We might expect to be able to do something similar with path connectedness. The main difficulty will be transitivity. As illustrated in Figure 19.1, if we have a path p from a to b and a path q from b to c, it appears that we can just follow the path p from a to b, then path q from b to c to have a path from a to c. But there are two problems to consider: how do we define this path as a function from $[0, 1]$ into our space, and how do we know the resulting function is continuous. The next lemma will help.

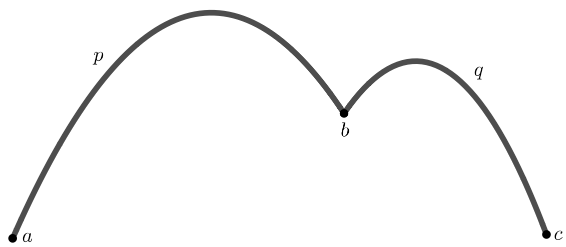

Figure 19.1: A path from a to c.

Lemma 19.5 (The Gluing Lemma)**.** *Let A and B be closed subsets of a space $X = A \cup B$, and let $f : A \to Y$ and $g : B \to Y$ be continuous functions into a space Y such that $f(x) = g(x)$ for all $x \in (A \cap B)$. Then the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

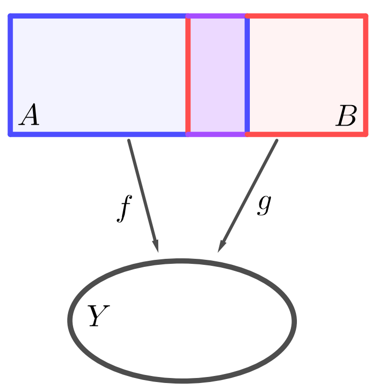

Figure 19.2: The Gluing Lemma.

Proof. Let A and B be closed subsets of a space $X = A \cup B$, and let $f : A \to Y$ and $g : B \to Y$ be continuous functions into a space Y such that $f(x) = g(x)$ for all $x \in (A \cap B)$ as illustrated in Figure 19.2. Define $h : X \to Y$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}$$

To show that h is continuous, let C be a closed subset of Y. Then

$$h^{-1}(C) = \{x \in X \mid h(x) \in C\} = \{x \in A \mid f(x) \in C\} \cup \{x \in B \mid g(x) \in C\} = f^{-1}(C) \cup g^{-1}(C).$$

Since f is continuous, $f^{-1}(C)$ is closed in the subspace topology on A and since g is continuous $g^{-1}(C)$ is closed in the subspace topology on B. So $f^{-1}(C) = A \cap D$ and $g^{-1}(C) = B \cap E$ for some closed sets D and E of X. The fact that A is closed in X implies that $A \cap D$ is closed in X. Similarly, the fact that B is closed in X implies that $B \cap E$ is closed in X. Thus,

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C) = (A \cap D) \cup (B \cap E)$$

is a finite union of closed sets in X and so is closed in X. Since $h^{-1}(C)$ is closed for every closed set in Y, it follows that h is continuous. ∎

We can use the Gluing Lemma to create a path from a to c given a path from a to b and a path from b to c.

Activity 19.2. Use the Gluing Lemma to explain why the path product given in the following definition is actually a path from $p(0)$ to $q(1)$.

Definition 19.6. Let p be a path from a to b and q a path from b to c in a space X. The **path product** $q * p$ is the path in X defined by

$$(q * p)(x) = \begin{cases} p(2x) & \text{for } 0 \leq x \leq \frac{1}{2} \\ q(2x-1) & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Now we can show that path connectedness defines an equivalence relation on a topological space.

Activity 19.3. Let (X, τ) be a topological space. Define a relation on X as follows:

$$a \sim b \text{ if there is a path in } X \text{ from } a \text{ to } b. \tag{19.1}$$

(a) Explain why $\sim$ is a reflexive relation.

(b) Explain why $\sim$ is a symmetric relation.

(c) Explain why $\sim$ is a transitive relation.

Since $\sim$ as defined in (19.1) is an equivalence relation, the relation partitions X into a union of disjoint equivalence classes. The equivalence class of an element $[a]$ is called a *path component* of X, and is the largest path connected subset of X that contains a.

Definition 19.7. The **path component** of an element a in a topological space (X, τ) is the largest path connected subset of X that contains a.

Path Connectedness and Connectedness

Path connectedness and connectedness are different concepts, but they are related. In this section we will show that any path connected space must also be connected. We will see later that the converse is not true except in finite topological spaces.

Theorem 19.8. *If a topological space X is path connected, then X is connected.*

Proof. Suppose that X is path connected. Let $a \in X$ and for any $x \in X$ let p_x be a path from a to x. Let $C_x = p_x([0, 1])$. Now C_x is the continuous image of the connected set $[0, 1]$ in $\mathbb{R}$, so C_x is connected. Also, $p_x(0) = a \in C_x$ and $p_x(1) = x \in C_x$. Thus, every set C_x contains a and so $\bigcap_{x\in X} C_x$ is not empty. Therefore,

$$C = \bigcup_{x \in X} C_x$$

is a connected subset of X. But every $x \in X$ is in a C_x, so $C = X$. We conclude that X is connected. ■

In the following sections we explore the reverse implication in Theorem 19.8 – that is, does connectedness imply path connectedness.

Path Connectedness and Connectedness in Finite Topological Spaces

In this section we will demonstrate that connectedness and path connectedness are equivalent concepts in finite topological spaces. In the following section, we prove that path connectedness and connectedness are not equivalent in infinite topological spaces. Throughout this section, we assume that X is a finite topological space. We begin with an example to motivate the main ideas.

Activity 19.4. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$. Assume that τ is a topology on X.

(a) Is X connected? Explain.

(b) For each $x \in X$, let U_x be the intersection of all open sets that contain x (we call U_x a *minimal neighborhood* of x).

Definition 19.9. For $x \in X$, the **minimal neighborhood** U_x of x is the intersection of all open sets that contain x.

Find U_x for each $x \in X$.

(c) We will see that the minimal neighborhoods of X are path connected. Here we will illustrate with U_d.

i. Let $p : [0, 1] \to X$ be defined by

$$p(t) = \begin{cases} b & \text{if } 0 \leq t < \frac{1}{2} \\ d & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Show that p is a path in U_d from b to d.

ii. Let $p : [0, 1] \to X$ be defined by

$$p(t) = \begin{cases} c & \text{if } 0 \leq t < \frac{1}{2} \\ d & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Show that p is a path in U_d from c to d.

iii. Explain why U_d is path connected.

The terminology in Definition 19.9 is apt. Since every neighborhood N of a point $x \in X$ must contain an open set O with $x \in O$, it follows that $U_x \subseteq O \subseteq N$. So every neighborhood of $x \in X$ has U_x as a subset. In addition, when X is finite, the set U_x is a finite intersection of open sets, so the sets U_x are open sets (this is not true in general in infinite topological spaces – you are asked to find an example in Exercise (1)). In Activity 19.4 we saw that U_x was path connected for a particular x in one example. The next activity shows that this result is true in general in finite topological spaces.

Activity 19.5. Let X be a finite topological space, and let $x \in X$. In this activity we demonstrate that U_x is path connected. Let $y \in U_x$ and define $p : [0, 1] \to X$ by

$$p(t) = \begin{cases} y & \text{if } 0 \leq t < \frac{1}{2} \\ x & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

To prove that p is continuous, let O be an open set in X. We either have $x \in O$ or $x \notin O$.

(a) Suppose $x \in O$. Why must y also be in O? What, then, is $p^{-1}(O)$?

(b) Now suppose $x \notin O$. There are two cases to consider.

i. What is $p^{-1}(O)$ if $y \in O$?

ii. What is $p^{-1}(O)$ if $y \notin O$?

(c) Explain why p is a path from y to x.

(d) Show that we can find a path between any two points in U_x. Conclude that U_x is path connected.

The sets U_x collectively form the space X, and each of the U_x is a path connected subspace. So every point in X is contained in some neighborhood with a path connected subset containing x. Spaces with this property are called *locally path connected*.

Definition 19.10. A topological space (X, τ) is **locally path connected at** x if every neighborhood of x contains a path connected open neighborhood with x as an element. The space (X, τ) is **locally path connected** if X is locally path connected at every point.

If X is a finite topological space, for any $x \in X$ the set U_x is the smallest open set containing x. This means that any neighborhood of N of x will contain U_x as a subset. Thus, a finite topological space is locally path connected (this is not true in general of infinite topological spaces, see Exercise (4) for example). One consequence of a locally path connected space is the following.

Lemma 19.11. *A space X is locally path connected if and only if for every open set O of X, each path component of O is open in X.*

Proof. Let X be a locally path connected topological space. We first show that for every open set O in X, every path component of O is open in X. Let O be an open set in X and let P be a path component of O. Let $p \in P$. Since X is locally path connected, the neighborhood O of x contains an open path connected neighborhood Q of p. The fact that $p \in Q$ and P is a path component of O implies that $Q \subseteq P$. Thus, P contains a neighborhood of p and P is open.

Now we show that if for every open set O in X the path components of O are open in X, then X is locally path connected. Let $x \in X$ and let N be a neighborhood of x. Then N contains an open set U with $x \in U$. Let P be the path component in U that contains x. Now P is path connected and, by hypothesis, P is open in X and so is an open path connected neighborhood of x. Thus, N contains a path connected neighborhood of x and X is locally path connected at every point.

■

Since X is open in X whenever X is a topological space, a natural corollary of Lemma 19.11 is the following.

Corollary 19.12. *Let X be a locally path connected topological space. Then every path component of X is open in X.*

Since there are only finitely many open sets in the finite space X, any arbitrary intersection of open sets in X just reduces to a finite intersection. So the intersection of any collection of open sets in X is again an open set in X. We will show that X is a union of path connected components, which will ultimately allow us to prove that if X is connected, then X is also path connected.

Activity 19.6. Let X be a locally path connected topological space. In this activity we will prove that the components and path components of X are the same.

(a) Let $x \in X$, and let C be the component of X containing x and P be the path component of X containing x. Show that $P \subseteq C$.

(b) To complete the proof that $P = C$, proceed by contradiction and assume that $C \neq P$. Let Q be the union of all path components of X that are different from P and that intersect C. Each such path component is connected, and is therefore a subset of C. So $C = P \cup Q$. Explain why P and Q form a separation of C. (Hint: How do we use the fact that X is locally path connected?)

We can now complete our main result of this section.

Theorem 19.13. *Let X be a finite topological space. Then X is connected if and only if X is path connected.*

Proof. Let X be a finite topological space. Theorem 19.8 demonstrates that if X is path connected, then X is connected. For the reverse implication, assume that X is path connected. Then X is composed of a single path component, $P = X$. Since the path components and components of X are the same, we conclude that $P = X$ is a component of X and that X is connected. ■

Path Connectedness and Connectedness in Infinite Topological Spaces

Given that connectedness and path connectedness are equivalent in finite topological spaces, a reasonable question now is whether the converse of Theorem 19.8 is true in arbitrary topological spaces. As we will see, the answer is no. To find a counterexample, we need to look in an infinite topological space. There are many examples, but a standard example to consider is the *topologist's sine curve*. This curve S is defined as the union of the sets

$$S_1 = \{(0, y) \mid -1 \le y \le 1\} \text{ and } S_2 = \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \,\middle|\, 0 < x \le 1 \right\}.$$

A picture of S is shown in Figure 19.3.

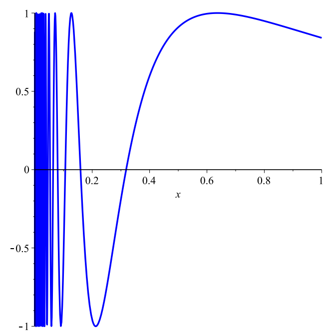

Figure 19.3: The topologist's sine curve.

To understand if S is connected, let us consider the relationship between S and S_2. Figure 19.3 seems to indicate that $S = \overline{S_2}$. To see if this is true, let $q = (0, y) \in S_1$, and let N be a neighborhood of q. Then there is an $\epsilon > 0$ such that $B = B(q, \epsilon) \subseteq N$. Choose $K \in \mathbb{Z}^+$ such that $\frac{1}{\arcsin(y)+2\pi K} < \epsilon$, and let $z = \frac{1}{\arcsin(y)+2\pi K}$. Then

$$\begin{aligned} d_E\left(q, \left(z, \sin\left(\frac{1}{z}\right)\right)\right) &= d_E\left((0, y), (z, \sin(\arcsin(y) + 2\pi K)\right) \\ &= d_E((0, y), (z, \sin(\arcsin(y)))) \\ &= d_E((0, y), (z, y)) \\ &= |z| \\ &< \epsilon, \end{aligned}$$

and so $(z, \arcsin(z)) \in B(q, \epsilon)$ and every neighborhood of q contains a point in S_2. Therefore, $S_1 \subseteq S_2' \subseteq \overline{S_2}$ and $\overline{S_2} = S$ in S. The fact that S is connected follows from Theorem 18.8.

Now that we know that S is connected, the following theorem demonstrates that S is a connected space that is not path connected.

Theorem 19.14. *The topologist's sine curve is connected but not path connected.*

Proof. We know that S is connected, so it remains to show that S is not path connected. The sets S_1 and S_2 are connected (as continuous images of the interval $[0, 1]$ and $(0, 1]$, respectively). We will prove that there is no path p in S from $p(0) = (0, 0)$ to $p(1) = b$ for any point $b \in S_2$ by contradiction. Assume the existence of such a path p. Let $U = p^{-1}(S_1)$ and $V = p^{-1}(S_2)$. Then

$$[0, 1] = p^{-1}(S) = p^{-1}(S_1 \cup S_2) = p^{-1}(S_1) \cup p^{-1}(S_2) = U \cup V. \tag{19.2}$$

Note that S_2 is an open subset of S, since $S_2 = \left(\bigcup_{z=(x,y)\in S_2} B\left(z, \frac{x}{2}\right)\right) \cap S$. So the continuity of p implies that V is an open subset of $[0,1]$. Also, the fact that $p(0) \in S_1$ means that $U \neq \emptyset$, and the fact that $p(1) \in S_2$ means that $V \neq \emptyset$. If we demonstrate that U is an open subset of $[0,1]$, then Equation (19.2) will imply that $[0,1]$ is not connected, a contradiction. So we proceed to prove that U is open in $[0,1]$.

Let $x \in U$, and so $p(x)$ in S_1. The set $O = B_S\left(p(x), \frac{1}{2}\right) \cap S$ is open in S. The continuity of p then tells us that $p^{-1}(O)$ is open in $[0,1]$. So there is a $\delta > 0$ such that the open ball $B = B_{[0,1]}(x, \delta)$ is a subset of $p^{-1}(O)$. We will prove that $p(B) \subseteq S_1$. This will imply that $B \subseteq U$ and so U is a neighborhood of each of its points, and U is therefore an open set.

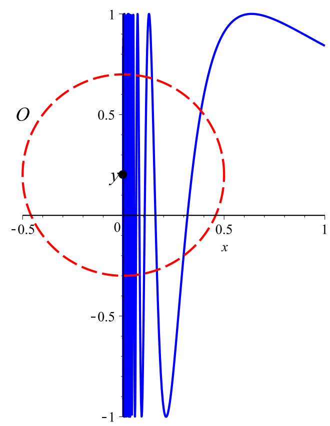

Figure 19.4: The set O.

Every element in B is mapped into O by the path p. The set O is complicated, consisting of infinitely many sub-curves of the curve S_2, along with points in S_1, as illustrated in Figure 19.4. To simplify our analysis, let us consider the projection onto the x-axis. The function $P_x : \mathbb{R}^2 \to \mathbb{R}$ defined by $P_x(x,y) = x$ is a continuous function. Let $I = P_x(p(B))$. Since $p(B) \subseteq O$, we know that $I \subseteq P_x(O)$. Let $Z = P_x(O)$. So $I \subseteq Z$. Since B is a connected set (B is an interval), we know that $p(B)$ is a connected set. The fact that P_x is continuous means that $I = P_x(p(B))$ is connected as well. Now I is a bounded subset of $\mathbb{R}$, so I must be a bounded interval. Recall that $x \in B$ and so $p(x) \in p(B)$. The fact that $p(x) \in S_1$ tells us that $0 = P_x(p(x)) \in P_x(p(B)) = I$. So $I \neq \emptyset$. There are two possibilities for I: either $I = \{0\}$, or I is an interval of positive length. We consider the cases.

Suppose $I = \{0\}$. Then the projection of $p(B)$ onto the x-axis is the single point 0 and $p(B) \subseteq S_1$ as desired. Suppose that I is an interval of the form $[0,d]$ or $[0,d)$ for some positive number d. The structure of O would indicate that there must be some gaps in the set Z, the projection of O onto the x-axis. This implies that I cannot be a connected interval. We proceed to show this. In other words, we will prove that $I \setminus Z \neq \emptyset$ (which is impossible since $I \subseteq Z$). Remember that $p(x) \in S_1$, so let $p(x) = (0,q)$. We consider what happens if $q < \frac{1}{2}$ and when $q \geq \frac{1}{2}$.

Suppose $q < \frac{1}{2}$. Then the ball $B_S\left(p(x), \frac{1}{2}\right)$ contains only points with y value less than 1. Let $N \in \mathbb{Z}^+$ so that $t = \frac{1}{\pi/2 + 2N\pi} < d$. Then $t \in I$. But $\sin\left(\frac{1}{t}\right) = \sin(\pi/2 + 2N\pi) = \sin(\pi/2) = 1$, and so $\left(t, \sin\left(\frac{1}{t}\right)\right)$ is not in O. Thus, $t \notin Z$. Thus we have found a point in $I \setminus Z$.

Finally, suppose $q \geq \frac{1}{2}$. Then the ball $B_S\left(p(x), \frac{1}{2}\right)$ contains only points with y value greater than -1. Let $N \in \mathbb{Z}^+$ so that $t = \frac{1}{3\pi/2+2N\pi} < d$. Then $t \in I$. But $\sin\left(\frac{1}{t}\right) = \sin(3\pi/2 + 2N\pi) = \sin(3\pi/2) = -1$, and so $t \notin Z$. Thus we have found a point in $I \setminus Z$.

We conclude that there can be no path in S from $(0,0)$ to any point in S_2, completing our proof that S is not path connected. (In fact, the argument given shows that there is no path in S from any point in S_1 to any point in S_2. ■

Summary

Important ideas that we discussed in this section include the following.

- A path in a topological space X is a continuous function p from the interval $[0,1]$ to X. If $p(0) = a$ and $p(1) = b$, then p is a path from a to b.
- A subspace A of a topological space X is path connected if, given any $a, b \in A$ there is a path in A from a to b.
- The path component of an element a in a topological space (X, τ) is the largest path connected subset of X that contains a.
- A topological space (X, τ) is locally path connected at x if every neighborhood of x contains a path connected subset with x as an element. The space (X, τ) is locally path connected if X is locally path connected at every point.
- Connectedness and path connectedness are equivalent in finite topological spaces, and path connectedness implies connectedness in general. However, there are topological spaces that are connected but not path connected. One example is the topologist's sine curve.

Exercises

(1) Find a topological space X and a point $x \in X$ such that the minimal neighborhood of x is not an open set.

(2) Let X be a topological space and for each $x \in X$ let $PC(x)$ denote the path component of x. Prove the following.

  (a) If A is a path connected subset of X, then $A \subseteq PC(x)$ for some $x \in X$.

  (b) The space X is path connected if and only if $X = PC(x)$ for some $x \in X$.

(3) In Activity 18.6 of Section 18 we showed that an arbitrary union of connected sets is connected provided the intersection of those sets is not empty. Is the same result true for path connected sets. That is, if X is a topological space and $\{A_\alpha\}$ for α in some indexing set I is a collection of path connected subsets of X and $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$, must it be the case that $A = \bigcup_{\alpha \in I} A_\alpha$ is path connected? Prove your answer.

(4) Let X be the subspace of $(\mathbb{R}^2, d_E)$ consisting of the line segments joining the point $(0,1)$ to every point in the set $\left\{\left(\frac{1}{n}, 0\right) \mid n \in \mathbb{Z}^+\right\}$ as illustrated in Figure 19.5. This space is called the *harmonic broom*.

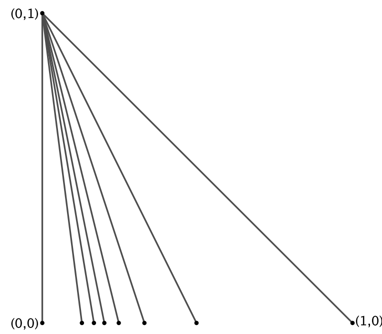

Figure 19.5: The harmonic broom.

(a) Show that the harmonic broom is connected.

(b) Show that the harmonic broom is path connected.

(c) Show that the harmonic broom is not locally connected.

(d) Show that the harmonic broom is not locally path connected. So path connectedness does not imply local path connectedness.

(5) In Exercise 4 we see an example of a space that is path connected but not locally path connected. Is it possible to find a space that is locally path connected but not path connected? Verify your answer.

(6) Let $K = \left\{\frac{1}{k} \mid k \text{ is a positive integer}\right\}$. Let $\mathcal{B}$ be the collection of all open intervals of the form (a, b) and all sets of the form $(a, b) \setminus K$, where $a < b$ are real numbers as in Example 13.13 on page 137. Let τ_K be the topology generated by $\mathcal{B}$. Show that $(\mathbb{R}, \tau_K)$ is not path connected. (Hint: Suppose there is a path between a and b where $a < 0$ and $b > 1$.)

(7) We know that a space can be connected but not path connected. We also know that local path connectedness does not imply connectedness. However, if we combine these conditions then a space must be path connected. That is, show that if a topological space X is connected and locally path connected, then X is path connected.

(8) Let X be a nonempty set and let p be a fixed element in X. Let τ_p be the particular point topology and $\tau_{\overline{p}}$ the excluded point topology on X. That is

- τ_p is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that contain p.
- $\tau_{\overline{p}}$ is the collection of subsets of X consisting of $\emptyset$, X, and all of the subsets of X that do not contain p.

That the particular point and excluded point topologies are topologies is the subject of Exercises (9) and (10) on page 127.

Determine, with proof, the path connected subsets of X when

(a) X has the particular point topology τ_p

(b) X has the excluded point topology $\tau_{\overline{p}}$.

(9) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) If X is a path connected topological space, then any subspace of X is path connected.

(b) If A and B are path connected subspaces of a topological space X, then $A \cap B$ is path connected.

(c) There is no path from a to b in (X, τ), where τ is the discrete topology.

(d) If X is a compact locally path connected topological space, then X has only finitely many path components.

(e) Every locally path connected space is locally connected.