

# Chapter 1

## Introduction to Writing Proofs in Mathematics

### 1.1 Statements and Conditional Statements

Much of our work in mathematics deals with statements. In mathematics, a **statement** is a declarative sentence that is either true or false but not both. A statement is sometimes called a **proposition**. The key is that there must be no ambiguity. To be a statement, a sentence must be true or false, and it cannot be both. So a sentence such as “The sky is beautiful” is not a statement since whether the sentence is true or not is a matter of opinion. A question such as “Is it raining?” is not a statement because it is a question and is not declaring or asserting that something is true.

Some sentences that are mathematical in nature often are not statements because we may not know precisely what a variable represents. For example, the equation  $2x + 5 = 10$  is not a statement since we do not know what  $x$  represents. If we substitute a specific value for  $x$  (such as  $x = 3$ ), then the resulting equation,  $2 \cdot 3 + 5 = 10$  is a statement (which is a false statement). Following are some more examples:

- There exists a real number  $x$  such that  $2x + 5 = 10$ .  
This is a statement because either such a real number exists or such a real number does not exist. In this case, this is a true statement since such a real number does exist, namely  $x = 2.5$ .

- For each real number  $x$ ,  $2x + 5 = 2\left(x + \frac{5}{2}\right)$ .

This is a statement since either the sentence  $2x + 5 = 2\left(x + \frac{5}{2}\right)$  is true when any real number is substituted for  $x$  (in which case, the statement is true) or there is at least one real number that can be substituted for  $x$  and produce a false statement (in which case, the statement is false). In this case, the given statement is true.

- Solve the equation  $x^2 - 7x + 10 = 0$ .  
This is not a statement since it is a directive. It does not assert that something is true.
- $(a + b)^2 = a^2 + b^2$  is not a statement since it is not known what  $a$  and  $b$  represent. However, the sentence, “There exist real numbers  $a$  and  $b$  such that  $(a + b)^2 = a^2 + b^2$ ” is a statement. In fact, this is a true statement since there are such integers. For example, if  $a = 1$  and  $b = 0$ , then  $(a + b)^2 = a^2 + b^2$ .
- Compare the statement in the previous item to the statement, “For all real numbers  $a$  and  $b$ ,  $(a + b)^2 = a^2 + b^2$ .” This is a false statement since there are values for  $a$  and  $b$  for which  $(a + b)^2 \neq a^2 + b^2$ . For example, if  $a = 2$  and  $b = 3$ , then  $(a + b)^2 = 5^2 = 25$  and  $a^2 + b^2 = 2^2 + 3^2 = 13$ .

### Progress Check 1.1 (Statements)

Which of the following sentences are statements? Do not worry about determining whether a statement is true or false; just determine whether each sentence is a statement or not.

- $3 + 4 = 8$ .
- $2 \cdot 7 + 8 = 22$ .
- $(x - 1) = \sqrt{x + 11}$ .
- $2x + 5y = 7$ .
- There are integers  $x$  and  $y$  such that  $2x + 5y = 7$ .
- There are integers  $x$  and  $y$  such that  $23x + 37y = 52$ .
- Given a line  $L$  and a point  $P$  not on that line and in the same plane, there is a unique line in that plane through  $P$  that does not intersect  $L$ .
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ .
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$  for all real numbers  $a$  and  $b$ .



10. The derivative of the sine function is the cosine function.
  11. Does the equation  $3x^2 - 5x - 7 = 0$  have two real number solutions?
  12. If  $ABC$  is a right triangle with right angle at vertex  $B$ , and if  $D$  is the midpoint of the hypotenuse, then the line segment connecting vertex  $B$  to  $D$  is half the length of the hypotenuse.
  13. There do not exist three integers  $x$ ,  $y$ , and  $z$  such that  $x^3 + y^3 = z^3$ .
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### How Do We Decide If a Statement Is True or False?

In mathematics, we often establish that a statement is true by writing a mathematical proof. To establish that a statement is false, we often find a so-called counterexample. (These ideas will be explored later in this chapter.) So mathematicians must be able to discover and construct proofs. In addition, once the discovery has been made, the mathematician must be able to communicate this discovery to others who speak the language of mathematics. We will be dealing with these ideas throughout the text.

For now, we want to focus on what happens before we start a proof. One thing that mathematicians often do is to make a conjecture beforehand as to whether the statement is true or false. This is often done through exploration. The role of exploration in mathematics is often difficult because the goal is not to find a specific answer but simply to investigate. Following are some techniques of exploration that might be helpful.

### Techniques of Exploration

- **Guesswork and conjectures.** Formulate and write down questions and conjectures. When we make a guess in mathematics, we usually call it a conjecture.
- **Examples. Constructing appropriate examples is extremely important.** Exploration often requires looking at lots of examples. In this way, we can gather information that provides evidence that a statement is true, or we might find an example that shows the statement is false. This type of example is called a **counterexample**.

For example, if someone makes the conjecture that  $\sin(2x) = 2 \sin(x)$ , for all real numbers  $x$ , we can test this conjecture by substituting specific values



for  $x$ . One way to do this is to choose values of  $x$  for which  $\sin(x)$  is known. Using  $x = \frac{\pi}{4}$ , we see that

$$\begin{aligned}\sin\left(2\left(\frac{\pi}{4}\right)\right) &= \sin\left(\frac{\pi}{2}\right) = 1, \text{ and} \\ 2\sin\left(\frac{\pi}{4}\right) &= 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}.\end{aligned}$$

Since  $1 \neq \sqrt{2}$ , these calculations show that this conjecture is false. However, if we do not find a counterexample for a conjecture, we usually cannot claim the conjecture is true. The best we can say is that our examples indicate the conjecture is true. As an example, consider the conjecture that

If  $x$  and  $y$  are odd integers, then  $x + y$  is an even integer.

We can do lots of calculations, such as  $3 + 7 = 10$  and  $5 + 11 = 16$ , and find that every time we add two odd integers, the sum is an even integer. However, it is not possible to test every pair of odd integers, and so we can only say that the conjecture appears to be true. (We will prove that this statement is true in the next section.)

- **Use of prior knowledge.** This also is very important. We cannot start from square one every time we explore a statement. We must make use of our acquired mathematical knowledge. For the conjecture that  $\sin(2x) = 2\sin(x)$ , for all real numbers  $x$ , we might recall that there are trigonometric identities called “double angle identities.” We may even remember the correct identity for  $\sin(2x)$ , but if we do not, we can always look it up. We should recall (or find) that

$$\text{for all real numbers } x, \sin(2x) = 2\sin(x)\cos(x).$$

We could use this identity to argue that the conjecture “for all real numbers  $x$ ,  $\sin(2x) = 2\sin(x)$ ” is false, but if we do, it is still a good idea to give a specific counterexample as we did before.

- **Cooperation and brainstorming.** Working together is often more fruitful than working alone. When we work with someone else, we can compare notes and articulate our ideas. Thinking out loud is often a useful brainstorming method that helps generate new ideas.



**Progress Check 1.2 (Explorations)**

Use the techniques of exploration to investigate each of the following statements. Can you make a conjecture as to whether the statement is true or false? Can you determine whether it is true or false?

1.  $(a + b)^2 = a^2 + b^2$ , for all real numbers  $a$  and  $b$ .
2. There are integers  $x$  and  $y$  such that  $2x + 5y = 41$ .
3. If  $x$  is an even integer, then  $x^2$  is an even integer.
4. If  $x$  and  $y$  are odd integers, then  $x \cdot y$  is an odd integer.

**Conditional Statements**

One of the most frequently used types of statements in mathematics is the so-called conditional statement. Given statements  $P$  and  $Q$ , a statement of the form “If  $P$  then  $Q$ ” is called a **conditional statement**. It seems reasonable that the truth value (true or false) of the conditional statement “If  $P$  then  $Q$ ” depends on the truth values of  $P$  and  $Q$ . The statement “If  $P$  then  $Q$ ” means that  $Q$  must be true whenever  $P$  is true. The statement  $P$  is called the **hypothesis** of the conditional statement, and the statement  $Q$  is called the **conclusion** of the conditional statement. Since conditional statements are probably the most important type of statement in mathematics, we give a more formal definition.

**Definition.** A **conditional statement** is a statement that can be written in the form “If  $P$  then  $Q$ ,” where  $P$  and  $Q$  are sentences. For this conditional statement,  $P$  is called the **hypothesis** and  $Q$  is called the **conclusion**.

Intuitively, “If  $P$  then  $Q$ ” means that  $Q$  must be true whenever  $P$  is true. Because conditional statements are used so often, a symbolic shorthand notation is used to represent the conditional statement “If  $P$  then  $Q$ .” We will use the notation  $P \rightarrow Q$  to represent “If  $P$  then  $Q$ .” When  $P$  and  $Q$  are statements, it seems reasonable that the truth value (true or false) of the conditional statement  $P \rightarrow Q$  depends on the truth values of  $P$  and  $Q$ . There are four cases to consider:

- $P$  is true and  $Q$  is true.
- $P$  is true and  $Q$  is false.
- $P$  is false and  $Q$  is true.
- $P$  is false and  $Q$  is false.



The conditional statement  $P \rightarrow Q$  means that  $Q$  is true whenever  $P$  is true. It says nothing about the truth value of  $Q$  when  $P$  is false. Using this as a guide, we define the conditional statement  $P \rightarrow Q$  to be false only when  $P$  is true and  $Q$  is false, that is, only when the hypothesis is true and the conclusion is false. In all other cases,  $P \rightarrow Q$  is true. This is summarized in Table 1.1, which is called a **truth table** for the conditional statement  $P \rightarrow Q$ . (In Table 1.1, T stands for “true” and F stands for “false.”)

$P$	$Q$	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.1: Truth Table for  $P \rightarrow Q$

The important thing to remember is that the conditional statement  $P \rightarrow Q$  has its own truth value. It is either true or false (and not both). Its truth value depends on the truth values for  $P$  and  $Q$ , but some find it a bit puzzling that the conditional statement is considered to be true when the hypothesis  $P$  is false. We will provide a justification for this through the use of an example.

**Example 1.3** Suppose that I say

“If it is not raining, then Daisy is riding her bike.”

We can represent this conditional statement as  $P \rightarrow Q$  where  $P$  is the statement, “It is not raining” and  $Q$  is the statement, “Daisy is riding her bike.”

Although it is not a perfect analogy, think of the statement  $P \rightarrow Q$  as being *false* to mean that I lied and think of the statement  $P \rightarrow Q$  as being *true* to mean that I did not lie. We will now check the truth value of  $P \rightarrow Q$  based on the truth values of  $P$  and  $Q$ .

1. Suppose that both  $P$  and  $Q$  are true. That is, it is not raining and Daisy is riding her bike. In this case, it seems reasonable to say that I told the truth and that  $P \rightarrow Q$  is true.
2. Suppose that  $P$  is true and  $Q$  is false or that it is not raining and Daisy is not riding her bike. It would appear that by making the statement, “If it is not



raining, then Daisy is riding her bike,” that I have not told the truth. So in this case, the statement  $P \rightarrow Q$  is false.

3. Now suppose that  $P$  is false and  $Q$  is true or that it is raining and Daisy is riding her bike. Did I make a false statement by stating that if it is not raining, then Daisy is riding her bike? The key is that I did not make any statement about what would happen if it was raining, and so I did not tell a lie. So we consider the conditional statement, “If it is not raining, then Daisy is riding her bike,” to be true in the case where it is raining and Daisy is riding her bike.
4. Finally, suppose that both  $P$  and  $Q$  are false. That is, it is raining and Daisy is not riding her bike. As in the previous situation, since my statement was  $P \rightarrow Q$ , I made no claim about what would happen if it was raining, and so I did not tell a lie. So the statement  $P \rightarrow Q$  cannot be false in this case and so we consider it to be true.

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#### Progress Check 1.4 (Explorations with Conditional Statements)

1. Consider the following sentence:

If  $x$  is a positive real number, then  $x^2 + 8x$  is a positive real number.

Although the hypothesis and conclusion of this conditional sentence are not statements, the conditional sentence itself can be considered to be a statement as long as we know what possible numbers may be used for the variable  $x$ . From the context of this sentence, it seems that we can substitute any positive real number for  $x$ . We can also substitute 0 for  $x$  or a negative real number for  $x$  provided that we are willing to work with a false hypothesis in the conditional statement. (In Chapter 2, we will learn how to be more careful and precise with these types of conditional statements.)

- (a) Notice that if  $x = -3$ , then  $x^2 + 8x = -15$ , which is negative. Does this mean that the given conditional statement is false?
- (b) Notice that if  $x = 4$ , then  $x^2 + 8x = 48$ , which is positive. Does this mean that the given conditional statement is true?
- (c) Do you think this conditional statement is true or false? Record the results for at least five different examples where the hypothesis of this conditional statement is true.



2. “If  $n$  is a positive integer, then  $(n^2 - n + 41)$  is a prime number.” (Remember that a prime number is a positive integer greater than 1 whose only positive factors are 1 and itself.)

To explore whether or not this statement is true, try using (and recording your results) for  $n = 1$ ,  $n = 2$ ,  $n = 3$ ,  $n = 4$ ,  $n = 5$ , and  $n = 10$ . Then record the results for at least four other values of  $n$ . Does this conditional statement appear to be true?

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### Further Remarks about Conditional Statements

1. The conventions for the truth value of conditional statements may seem a bit strange, especially the fact that the conditional statement is true when the hypothesis of the conditional statement is false. The following example is meant to show that this makes sense.

Suppose that Ed has exactly \$52 in his wallet. The following four statements will use the four possible truth combinations for the hypothesis and conclusion of a conditional statement.

- If Ed has exactly \$52 in his wallet, then he has \$20 in his wallet. This is a true statement. Notice that both the hypothesis and the conclusion are true.
- If Ed has exactly \$52 in his wallet, then he has \$100 in his wallet. This statement is false. Notice that the hypothesis is true and the conclusion is false.
- If Ed has \$100 in his wallet, then he has at least \$50 in his wallet. This statement is true regardless of how much money he has in his wallet. In this case, the hypothesis is false and the conclusion is true.
- If Ed has \$100 in his wallet, then he has at least \$80 in his wallet. This statement is true regardless of how much money he has in his wallet. In this case, the hypothesis is false and the conclusion is false.

This is admittedly a contrived example but it does illustrate that the conventions for the truth value of a conditional statement make sense. The message is that in order to be complete in mathematics, we need to have conventions about when a conditional statement is true and when it is false.

2. The fact that there is only one case when a conditional statement is false often provides a method to show that a given conditional statement is false. In





Progress Check 1.4, you were asked if you thought the following conditional statement was true or false.

If  $n$  is a positive integer, then  $(n^2 - n + 41)$  is a prime number.

Perhaps for all of the values you tried for  $n$ ,  $(n^2 - n + 41)$  turned out to be a prime number. However, if we try  $n = 41$ , we get

$$n^2 - n + 41 = 41^2 - 41 + 41$$

$$n^2 - n + 41 = 41^2.$$

So in the case where  $n = 41$ , the hypothesis is true (41 is a positive integer) and the conclusion is false ( $41^2$  is not prime). Therefore, 41 is a counterexample for this conjecture and the conditional statement

“If  $n$  is a positive integer, then  $(n^2 - n + 41)$  is a prime number”

is false. There are other counterexamples (such as  $n = 42$ ,  $n = 45$ , and  $n = 50$ ), but only one counterexample is needed to prove that the statement is false.

3. Although one example can be used to prove that a conditional statement is false, in most cases, we cannot use examples to prove that a conditional statement is true. For example, in Progress Check 1.4, we substituted values for  $x$  for the conditional statement “If  $x$  is a positive real number, then  $x^2 + 8x$  is a positive real number.” For every positive real number used for  $x$ , we saw that  $x^2 + 8x$  was positive. However, this does not prove the conditional statement to be true because it is impossible to substitute every positive real number for  $x$ . So, although we may believe this statement is true, to be able to conclude it is true, we need to write a mathematical proof. Methods of proof will be discussed in Section 1.2 and Chapter 3.

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### Progress Check 1.5 (Working with a Conditional Statement)

Sometimes, we must be aware of conventions that are being used. In most calculus texts, the convention is that any function has a domain and a range that are subsets of the real numbers. In addition, when we say something like “the function  $f$  is differentiable at  $a$ ”, it is understood that  $a$  is a real number. With these conventions, the following statement is a true statement, which is proven in many calculus texts.

If the function  $f$  is differentiable at  $a$ , then the function  $f$  is continuous at  $a$ .



Using only this true statement, is it possible to make a conclusion about the function in each of the following cases?

1. It is known that the function  $f$ , where  $f(x) = \sin x$ , is differentiable at 0.
  2. It is known that the function  $f$ , where  $f(x) = \sqrt[3]{x}$ , is not differentiable at 0.
  3. It is known that the function  $f$ , where  $f(x) = |x|$ , is continuous at 0.
  4. It is known that the function  $f$ , where  $f(x) = \frac{|x|}{x}$  is not continuous at 0.
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### Closure Properties of Number Systems

The primary number system used in algebra and calculus is the **real number system**. We usually use the symbol  $\mathbb{R}$  to stand for the set of all real numbers. The real numbers consist of the rational numbers and the irrational numbers. The **rational numbers** are those real numbers that can be written as a quotient of two integers (with a nonzero denominator), and the **irrational numbers** are those real numbers that cannot be written as a quotient of two integers. That is, a rational number can be written in the form of a fraction, and an irrational number cannot be written in the form of a fraction. Some common irrational numbers are  $\sqrt{2}$ ,  $\pi$ , and  $e$ . We usually use the symbol  $\mathbb{Q}$  to represent the set of all rational numbers. (The letter  $\mathbb{Q}$  is used because rational numbers are quotients of integers.) There is no standard symbol for the set of all irrational numbers.

Perhaps the most basic number system used in mathematics is the set of **natural numbers**. The natural numbers consist of the positive whole numbers such as 1, 2, 3, 107, and 203. We will use the symbol  $\mathbb{N}$  to stand for the set of natural numbers. Another basic number system that we will be working with is the set of **integers**. The integers consist of zero, the natural numbers, and the negatives of the natural numbers. If  $n$  is an integer, we can write  $n = \frac{n}{1}$ . So each integer is a rational number and hence also a real number.

We will use the letter  $\mathbb{Z}$  to stand for the set of integers. (The letter  $\mathbb{Z}$  is from the German word, *Zahlen*, for numbers.) Three of the basic properties of the integers are that the set  $\mathbb{Z}$  is **closed under addition**, the set  $\mathbb{Z}$  is **closed under multiplication**, and the set of integers is **closed under subtraction**. This means that

- If  $x$  and  $y$  are integers, then  $x + y$  is an integer;



- If  $x$  and  $y$  are integers, then  $x \cdot y$  is an integer; and
- If  $x$  and  $y$  are integers, then  $x - y$  is an integer.

Notice that these so-called closure properties are defined in terms of conditional statements. This means that if we can find one instance where the hypothesis is true and the conclusion is false, then the conditional statement is false.

**Example 1.6 (Closure)**

1. In order for the set of natural numbers to be closed under subtraction, the following conditional statement would have to be true: If  $x$  and  $y$  are natural numbers, then  $x - y$  is a natural number. However, since 5 and 8 are natural numbers,  $5 - 8 = -3$ , which is not a natural number, this conditional statement is false. Therefore, the set of natural numbers is not closed under subtraction.
2. We can use the rules for multiplying fractions and the closure rules for the integers to show that the rational numbers are closed under multiplication. If  $\frac{a}{b}$  and  $\frac{c}{d}$  are rational numbers (so  $a$ ,  $b$ ,  $c$ , and  $d$  are integers and  $b$  and  $d$  are not zero), then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Since the integers are closed under multiplication, we know that  $ac$  and  $bd$  are integers and since  $b \neq 0$  and  $d \neq 0$ ,  $bd \neq 0$ . Hence,  $\frac{ac}{bd}$  is a rational number and this shows that the rational numbers are closed under multiplication.

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**Progress Check 1.7 (Closure Properties)**

Answer each of the following questions.

1. Is the set of rational numbers closed under addition? Explain.
  2. Is the set of integers closed under division? Explain.
  3. Is the set of rational numbers closed under subtraction? Explain.
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**Exercises for Section 1.1**

- \* 1. Which of the following sentences are statements?
- (a)  $3^2 + 4^2 = 5^2$ .
  - (b)  $a^2 + b^2 = c^2$ .
  - (c) There exist integers  $a$ ,  $b$ , and  $c$  such that  $a^2 = b^2 + c^2$ .
  - (d) If  $x^2 = 4$ , then  $x = 2$ .
  - (e) For each real number  $x$ , if  $x^2 = 4$ , then  $x = 2$ .
  - (f) For each real number  $t$ ,  $\sin^2 t + \cos^2 t = 1$ .
  - (g)  $\sin x < \sin\left(\frac{\pi}{4}\right)$ .
  - (h) If  $n$  is a prime number, then  $n^2$  has three positive factors.
  - (i)  $1 + \tan^2 \theta = \sec^2 \theta$ .
  - (j) Every rectangle is a parallelogram.
  - (k) Every even natural number greater than or equal to 4 is the sum of two prime numbers.
- \* 2. Identify the hypothesis and the conclusion for each of the following conditional statements.
- (a) If  $n$  is a prime number, then  $n^2$  has three positive factors.
  - (b) If  $a$  is an irrational number and  $b$  is an irrational number, then  $a \cdot b$  is an irrational number.
  - (c) If  $p$  is a prime number, then  $p = 2$  or  $p$  is an odd number.
  - (d) If  $p$  is a prime number and  $p \neq 2$ , then  $p$  is an odd number.
  - (e) If  $p \neq 2$  and  $p$  is an even number, then  $p$  is not prime.
- \* 3. Determine whether each of the following conditional statements is true or false.
- (a) If  $10 < 7$ , then  $3 = 4$ .
  - (b) If  $7 < 10$ , then  $3 = 4$ .
  - (c) If  $10 < 7$ , then  $3 + 5 = 8$ .
  - (d) If  $7 < 10$ , then  $3 + 5 = 8$ .
- \* 4. Determine the conditions under which each of the following conditional sentences will be a true statement.

- (a) If  $a + 2 = 5$ , then  $8 < 5$ .      (b) If  $5 < 8$ , then  $a + 2 = 5$ .

5. Let  $P$  be the statement “Student X passed every assignment in Calculus I,” and let  $Q$  be the statement “Student X received a grade of C or better in Calculus I.”

- (a) What does it mean for  $P$  to be true? What does it mean for  $Q$  to be true?
- (b) Suppose that Student X passed every assignment in Calculus I and received a grade of B–, and that the instructor made the statement  $P \rightarrow Q$ . Would you say that the instructor lied or told the truth?
- (c) Suppose that Student X passed every assignment in Calculus I and received a grade of C–, and that the instructor made the statement  $P \rightarrow Q$ . Would you say that the instructor lied or told the truth?
- (d) Now suppose that Student X did not pass two assignments in Calculus I and received a grade of D, and that the instructor made the statement  $P \rightarrow Q$ . Would you say that the instructor lied or told the truth?
- (e) How are Parts (5b), (5c), and (5d) related to the truth table for  $P \rightarrow Q$ ?

6. Following is a statement of a theorem which can be proven using calculus or precalculus mathematics. For this theorem,  $a$ ,  $b$ , and  $c$  are real numbers.

**Theorem.** If  $f$  is a quadratic function of the form  
 $f(x) = ax^2 + bx + c$  and  $a < 0$ , then the function  $f$  has a  
 maximum value when  $x = \frac{-b}{2a}$ .

Using **only** this theorem, what can be concluded about the functions given by the following formulas?

- \* (a)  $g(x) = -8x^2 + 5x - 2$       (d)  $j(x) = -\frac{71}{99}x^2 + 210$
- \* (b)  $h(x) = -\frac{1}{3}x^2 + 3x$       (e)  $f(x) = -4x^2 - 3x + 7$
- \* (c)  $k(x) = 8x^2 - 5x - 7$       (f)  $F(x) = -x^4 + x^3 + 9$

7. Following is a statement of a theorem which can be proven using the quadratic formula. For this theorem,  $a$ ,  $b$ , and  $c$  are real numbers.

**Theorem** If  $f$  is a quadratic function of the form  
 $f(x) = ax^2 + bx + c$  and  $ac < 0$ , then the function  $f$  has two  
 $x$ -intercepts.

Using **only** this theorem, what can be concluded about the functions given by the following formulas?



$$\begin{array}{ll} \text{(a)} \quad g(x) = -8x^2 + 5x - 2 & \text{(d)} \quad j(x) = -\frac{71}{99}x^2 + 210 \\ \text{(b)} \quad h(x) = -\frac{1}{3}x^2 + 3x & \text{(e)} \quad f(x) = -4x^2 - 3x + 7 \\ \text{(c)} \quad k(x) = 8x^2 - 5x - 7 & \text{(f)} \quad F(x) = -x^4 + x^3 + 9 \end{array}$$

8. Following is a statement of a theorem about certain cubic equations. For this theorem,  $b$  represents a real number.

**Theorem A.** If  $f$  is a cubic function of the form  $f(x) = x^3 - x + b$  and  $b > 1$ , then the function  $f$  has exactly one  $x$ -intercept.

Following is another theorem about  $x$ -intercepts of functions:

**Theorem B.** If  $f$  and  $g$  are functions with  $g(x) = k \cdot f(x)$ , where  $k$  is a nonzero real number, then  $f$  and  $g$  have exactly the same  $x$ -intercepts.

Using only these two theorems and some simple algebraic manipulations, what can be concluded about the functions given by the following formulas?

$$\begin{array}{ll} \text{(a)} \quad f(x) = x^3 - x + 7 & \text{(d)} \quad k(x) = 2x^3 + 2x + 3 \\ \text{(b)} \quad g(x) = x^3 + x + 7 & \text{(e)} \quad r(x) = x^4 - x + 11 \\ \text{(c)} \quad h(x) = -x^3 + x - 5 & \text{(f)} \quad F(x) = 2x^3 - 2x + 7 \end{array}$$

- \* 9. (a) Is the set of natural numbers closed under division?  
 (b) Is the set of rational numbers closed under division?  
 (c) Is the set of nonzero rational numbers closed under division?  
 (d) Is the set of positive rational numbers closed under division?  
 (e) Is the set of positive real numbers closed under subtraction?  
 (f) Is the set of negative rational numbers closed under division?  
 (g) Is the set of negative integers closed under addition?

## Explorations and Activities

10. **Exploring Propositions.** In Progress Check 1.2, we used exploration to show that certain statements were false and to make conjectures that certain statements were true. We can also use exploration to formulate a conjecture that we believe to be true. For example, if we calculate successive powers of 2 ( $2^1, 2^2, 2^3, 2^4, 2^5, \dots$ ) and examine the units digits of these numbers, we could make the following conjectures (among others):



- If  $n$  is a natural number, then the units digit of  $2^n$  must be 2, 4, 6, or 8.
  - The units digits of the successive powers of 2 repeat according to the pattern “2, 4, 8, 6.”
- (a) Is it possible to formulate a conjecture about the units digits of successive powers of 4 ( $4^1, 4^2, 4^3, 4^4, 4^5, \dots$ )? If so, formulate at least one conjecture.
- (b) Is it possible to formulate a conjecture about the units digit of numbers of the form  $7^n - 2^n$ , where  $n$  is a natural number? If so, formulate a conjecture in the form of a conditional statement in the form “If  $n$  is a natural number, then . . . .”
- (c) Let  $f(x) = e^{2x}$ . Determine the first eight derivatives of this function. What do you observe? Formulate a conjecture that appears to be true. The conjecture should be written as a conditional statement in the form, “If  $n$  is a natural number, then . . . .”
- 

## 1.2 Constructing Direct Proofs

### Preview Activity 1 (Definition of Even and Odd Integers)

Definitions play a very important role in mathematics. A direct proof of a proposition in mathematics is often a demonstration that the proposition follows logically from certain definitions and previously proven propositions. A **definition** is an agreement that a particular word or phrase will stand for some object, property, or other concept that we expect to refer to often. In many elementary proofs, the answer to the question, “How do we prove a certain proposition?”, is often answered by means of a definition. For example, in Progress Check 1.2 on page 5, all of the examples should have indicated that the following conditional statement is true:

If  $x$  and  $y$  are odd integers, then  $x \cdot y$  is an odd integer.

In order to construct a mathematical proof of this conditional statement, we need a precise definition of what it means to say that an integer is an even integer and what it means to say that an integer is an odd integer.

**Definition.** An integer  $a$  is an **even integer** provided that there exists an integer  $n$  such that  $a = 2n$ . An integer  $a$  is an **odd integer** provided there exists an integer  $n$  such that  $a = 2n + 1$ .



Using this definition, we can conclude that the integer 16 is an even integer since  $16 = 2 \cdot 8$  and 8 is an integer. By answering the following questions, you should obtain a better understanding of these definitions. These questions are not here just to have questions in the textbook. Constructing and answering such questions is a way in which many mathematicians will try to gain a better understanding of a definition.

1. Use the definitions given above to
  - (a) Explain why 28,  $-42$ , 24, and 0 are even integers.
  - (b) Explain why 51,  $-11$ , 1, and  $-1$  are odd integers.

It is important to realize that mathematical definitions are not made randomly. In most cases, they are motivated by a mathematical concept that occurs frequently.

2. Are the definitions of even integers and odd integers consistent with your previous ideas about even and odd integers?
- 

### Preview Activity 2 (Thinking about a Proof)

Consider the following proposition:

**Proposition.** If  $x$  and  $y$  are odd integers, then  $x \cdot y$  is an odd integer.

Think about how you might go about proving this proposition. A **direct proof** of a conditional statement is a demonstration that the conclusion of the conditional statement follows logically from the hypothesis of the conditional statement. Definitions and previously proven propositions are used to justify each step in the proof. To help get started in proving this proposition, answer the following questions:

1. The proposition is a conditional statement. What is the hypothesis of this conditional statement? What is the conclusion of this conditional statement?
2. If  $x = 2$  and  $y = 3$ , then  $x \cdot y = 6$ , and 6 is an even integer. Does this example prove that the proposition is false? Explain.
3. If  $x = 5$  and  $y = 3$ , then  $x \cdot y = 15$ . Does this example prove that the proposition is true? Explain.

In order to prove this proposition, we need to prove that whenever both  $x$  and  $y$  are odd integers,  $x \cdot y$  is an odd integer. Since we cannot explore all possible pairs of integer values for  $x$  and  $y$ , we will use the definition of an odd integer to help us construct a proof.





4. To start a proof of this proposition, we will assume that the hypothesis of the conditional statement is true. So in this case, we assume that both  $x$  and  $y$  are odd integers. We can then use the definition of an odd integer to conclude that there exists an integer  $m$  such that  $x = 2m + 1$ . Now use the definition of an odd integer to make a conclusion about the integer  $y$ .

**Note:** The definition of an odd integer says that a certain other integer exists. This definition may be applied to both  $x$  and  $y$ . However, do not use the same letter in both cases. To do so would imply that  $x = y$  and we have not made that assumption. To be more specific, if  $x = 2m + 1$  and  $y = 2m + 1$ , then  $x = y$ .

5. We need to prove that if the hypothesis is true, then the conclusion is true. So, in this case, we need to prove that  $x \cdot y$  is an odd integer. At this point, we usually ask ourselves a so-called **backward question**. In this case, we ask, “Under what conditions can we conclude that  $x \cdot y$  is an odd integer?” Use the definition of an odd integer to answer this question.

---

## Properties of Number Systems

At the end of Section 1.1, we introduced notations for the standard number systems we use in mathematics. We also discussed some closure properties of the standard number systems. For this text, it is assumed that the reader is familiar with these closure properties and the basic rules of algebra that apply to all real numbers. That is, it is assumed the reader is familiar with the properties of the real numbers shown in Table 1.2.

## Constructing a Proof of a Conditional Statement

In order to prove that a conditional statement  $P \rightarrow Q$  is true, we only need to prove that  $Q$  is true whenever  $P$  is true. This is because the conditional statement is true whenever the hypothesis is false. So in a direct proof of  $P \rightarrow Q$ , we assume that  $P$  is true, and using this assumption, we proceed through a logical sequence of steps to arrive at the conclusion that  $Q$  is true.

Unfortunately, it is often not easy to discover how to start this logical sequence of steps or how to get to the conclusion that  $Q$  is true. We will describe a method of exploration that often can help in discovering the steps of a proof. This method will involve working forward from the hypothesis,  $P$ , and backward from the conclusion,  $Q$ . We will use a device called the “**know-show table**” to help organize



For all real numbers  $x$ ,  $y$ , and  $z$

Identity Properties	$x + 0 = x$ and $x \cdot 1 = x$
Inverse Properties	$x + (-x) = 0$ and if $x \neq 0$ , then $x \cdot \frac{1}{x} = 1$ .
Commutative Properties	$x + y = y + x$ and $xy = yx$
Associative Properties	$(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$
Distributive Properties	$x(y + z) = xy + xz$ and $(y + z)x = yx + zx$

Table 1.2: Properties of the Real Numbers

our thoughts and the steps of the proof. This will be illustrated with the proposition from Preview Activity 2.

**Proposition.** *If  $x$  and  $y$  are odd integers, then  $x \cdot y$  is an odd integer.*

The first step is to identify the hypothesis,  $P$ , and the conclusion,  $Q$ , of the conditional statement. In this case, we have the following:

$P$ :  $x$  and  $y$  are odd integers.       $Q$ :  $x \cdot y$  is an odd integer.

We now treat  $P$  as what we know (we have assumed it to be true) and treat  $Q$  as what we want to show (that is, the goal). So we organize this by using  $P$  as the first step in the know portion of the table and  $Q$  as the last step in the show portion of the table. We will put the know portion of the table at the top and the show portion of the table at the bottom.

Step	Know	Reason
$P$	$x$ and $y$ are odd integers.	Hypothesis
$P1$		
$\vdots$	$\vdots$	$\vdots$
$Q1$		
$Q$	$x \cdot y$ is an odd integer.	?
Step	Show	Reason

We have not yet filled in the reason for the last step because we do not yet know how we will reach the goal. The idea now is to ask ourselves questions about what



we know and what we are trying to prove. We usually start with the conclusion that we are trying to prove by asking a so-called **backward question**. The basic form of the question is, “Under what conditions can we conclude that  $Q$  is true?” How we ask the question is crucial since we must be able to answer it. We should first try to ask and answer the question in an abstract manner and then apply it to the particular form of statement  $Q$ .

In this case, we are trying to prove that some integer is an odd integer. So our backward question could be, “How do we prove that an integer is odd?” At this time, the only way we have of answering this question is to use the definition of an odd integer. So our answer could be, “We need to prove that there exists an integer  $q$  such that the integer equals  $2q + 1$ .” We apply this answer to statement  $Q$  and insert it as the next to last line in the know-show table.

Step	Know	Reason
$P$	$x$ and $y$ are odd integers.	Hypothesis
$P1$		
$\vdots$	$\vdots$	$\vdots$
$Q1$	There exists an integer $q$ such that $xy = 2q + 1$ .	
$Q$	$x \cdot y$ is an odd integer.	Definition of an odd integer
Step	Show	Reason

We now focus our effort on proving statement  $Q1$  since we know that if we can prove  $Q1$ , then we can conclude that  $Q$  is true. We ask a backward question about  $Q1$  such as, “How can we prove that there exists an integer  $q$  such that  $x \cdot y = 2q + 1$ ?” We may not have a ready answer for this question, and so we look at the know portion of the table and try to connect the know portion to the show portion. To do this, we work forward from step  $P$ , and this involves asking a **forward question**. The basic form of this type of question is, “What can we conclude from the fact that  $P$  is true?” In this case, we can use the definition of an odd integer to conclude that there exist integers  $m$  and  $n$  such that  $x = 2m + 1$  and  $y = 2n + 1$ . We will call this Step  $P1$  in the know-show table. It is important to notice that we were careful not to use the letter  $q$  to denote these integers. If we had used  $q$  again, we would be claiming that the same integer that gives  $x \cdot y = 2q + 1$  also gives  $x = 2q + 1$ . This is why we used  $m$  and  $n$  for the integers  $x$  and  $y$  since there is no guarantee that  $x$  equals  $y$ . The basic rule of thumb is to use a different symbol for each new object we introduce in a proof. So at this point, we have:

- Step  $P1$ . We know that there exist integers  $m$  and  $n$  such that  $x = 2m + 1$  and  $y = 2n + 1$ .



- Step  $Q1$ . We need to prove that there exists an integer  $q$  such that  $x \cdot y = 2q + 1$ .

We must always be looking for a way to link the “know part” to the “show part”. There are conclusions we can make from  $P1$ , but as we proceed, we must always keep in mind the form of statement in  $Q1$ . The next forward question is, “What can we conclude about  $x \cdot y$  from what we know?” One way to answer this is to use our prior knowledge of algebra. That is, we can first use substitution to write  $x \cdot y = (2m + 1)(2n + 1)$ . Although this equation does not prove that  $x \cdot y$  is odd, we can use algebra to try to rewrite the right side of this equation  $(2m + 1)(2n + 1)$  in the form of an odd integer so that we can arrive at step  $Q1$ . We first expand the right side of the equation to obtain

$$\begin{aligned} x \cdot y &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \end{aligned}$$

Now compare the right side of the last equation to the right side of the equation in step  $Q1$ . Sometimes the difficult part at this point is the realization that  $q$  stands for some integer and that we only have to show that  $x \cdot y$  equals two times some integer plus one. Can we now make that conclusion? The answer is yes because we can factor a 2 from the first three terms on the right side of the equation and obtain

$$\begin{aligned} x \cdot y &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

We can now complete the table showing the outline of the proof as follows:

Step	Know	Reason
$P$	$x$ and $y$ are odd integers.	Hypothesis
$P1$	There exist integers $m$ and $n$ such that $x = 2m + 1$ and $y = 2n + 1$ .	Definition of an odd integer.
$P2$	$xy = (2m + 1)(2n + 1)$	Substitution
$P3$	$xy = 4mn + 2m + 2n + 1$	Algebra
$P4$	$xy = 2(2mn + m + n) + 1$	Algebra
$P5$	$(2mn + m + n)$ is an integer.	Closure properties of the integers
$Q1$	There exists an integer $q$ such that $xy = 2q + 1$ .	Use $q = (2mn + m + n)$
$Q$	$x \cdot y$ is an odd integer.	Definition of an odd integer



It is very important to realize that we have only constructed an outline of a proof. Mathematical proofs are not written in table form. They are written in narrative form using complete sentences and correct paragraph structure, and they follow certain conventions used in writing mathematics. In addition, most proofs are written only from the forward perspective. That is, although the use of the backward process was essential in discovering the proof, when we write the proof in narrative form, we use the forward process described in the preceding table. A completed proof follows.

---

**Theorem 1.8.** *If  $x$  and  $y$  are odd integers, then  $x \cdot y$  is an odd integer.*

**Proof.** We assume that  $x$  and  $y$  are odd integers and will prove that  $x \cdot y$  is an odd integer. Since  $x$  and  $y$  are odd, there exist integers  $m$  and  $n$  such that

$$x = 2m + 1 \text{ and } y = 2n + 1.$$

Using algebra, we obtain

$$\begin{aligned}x \cdot y &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1.\end{aligned}$$

Since  $m$  and  $n$  are integers and the integers are closed under addition and multiplication, we conclude that  $(2mn + m + n)$  is an integer. This means that  $x \cdot y$  has been written in the form  $(2q + 1)$  for some integer  $q$ , and hence,  $x \cdot y$  is an odd integer. Consequently, it has been proven that if  $x$  and  $y$  are odd integers, then  $x \cdot y$  is an odd integer. ■

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## Writing Guidelines for Mathematics Proofs

At the risk of oversimplification, doing mathematics can be considered to have two distinct stages. The first stage is to convince yourself that you have solved the problem or proved a conjecture. This stage is a creative one and is quite often how mathematics is actually done. The second equally important stage is to convince other people that you have solved the problem or proved the conjecture. This second stage often has little in common with the first stage in the sense that it does not really communicate the process by which you solved the problem or proved the conjecture. However, it is an important part of the process of communicating mathematical results to a wider audience.



A **mathematical proof** is a convincing argument (within the accepted standards of the mathematical community) that a certain mathematical statement is necessarily true. A proof generally uses deductive reasoning and logic but also contains some amount of ordinary language (such as English). A mathematical proof that you write should convince an appropriate audience that the result you are proving is in fact true. So we do not consider a proof complete until there is a well-written proof. So it is important to introduce some writing guidelines. The preceding proof was written according to the following basic guidelines for writing proofs. More writing guidelines will be given in Chapter 3.

- 1. Begin with a carefully worded statement of the theorem or result to be proven.** This should be a simple declarative statement of the theorem or result. Do not simply rewrite the problem as stated in the textbook or given on a handout. Problems often begin with phrases such as “Show that” or “Prove that.” This should be reworded as a simple declarative statement of the theorem. Then skip a line and write “Proof” in italics or boldface font (when using a word processor). Begin the proof on the same line. Make sure that all paragraphs can be easily identified. Skipping a line between paragraphs or indenting each paragraph can accomplish this.

As an example, an exercise in a text might read, “Prove that if  $x$  is an odd integer, then  $x^2$  is an odd integer.” This could be started as follows:

**Theorem.** If  $x$  is an odd integer, then  $x^2$  is an odd integer.

*Proof:* We assume that  $x$  is an odd integer . . .

- 2. Begin the proof with a statement of your assumptions.** Follow the statement of your assumptions with a statement of what you will prove.

**Theorem.** If  $x$  is an odd integer, then  $x^2$  is an odd integer.

*Proof.* We assume that  $x$  is an odd integer and will prove that  $x^2$  is an odd integer.

- 3. Use the pronoun “we.”** If a pronoun is used in a proof, the usual convention is to use “we” instead of “I.” The idea is to stress that you and the reader are doing the mathematics together. It will help encourage the reader to continue working through the mathematics. Notice that we started the proof of Theorem 1.8 with “We assume that . . .”
- 4. Use italics for variables when using a word processor.** When using a word processor to write mathematics, the word processor needs to be capable of producing the appropriate mathematical symbols and equations. The



mathematics that is written with a word processor should look like typeset mathematics. This means that italics is used for variables, boldface font is used for vectors, and regular font is used for mathematical terms such as the names of the trigonometric and logarithmic functions.

For example, we do not write  $\sin(x)$  or *sin(x)*. The proper way to typeset this is  $\sin(x)$ .

- 5. Display important equations and mathematical expressions.** Equations and manipulations are often an integral part of mathematical exposition. Do not write equations, algebraic manipulations, or formulas in one column with reasons given in another column. Important equations and manipulations should be displayed. This means that they should be centered with blank lines before and after the equation or manipulations, and if the left side of the equations does not change, it should not be repeated. For example,

Using algebra, we obtain

$$\begin{aligned}x \cdot y &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1.\end{aligned}$$

Since  $m$  and  $n$  are integers, we conclude that . . . .

- 6. Tell the reader when the proof has been completed.** Perhaps the best way to do this is to simply write, “This completes the proof.” Although it may seem repetitive, a good alternative is to finish a proof with a sentence that states precisely what has been proven. In any case, it is usually good practice to use some “end of proof symbol” such as ■.

---

### Progress Check 1.9 (Proving Propositions)

Construct a know-show table for each of the following propositions and then write a formal proof for one of the propositions.

1. If  $x$  is an even integer and  $y$  is an even integer, then  $x + y$  is an even integer.
2. If  $x$  is an even integer and  $y$  is an odd integer, then  $x + y$  is an odd integer.
3. If  $x$  is an odd integer and  $y$  is an odd integer, then  $x + y$  is an even integer.

### Some Comments about Constructing Direct Proofs

1. When we constructed the know-show table prior to writing a proof for Theorem 1.8, we had only one answer for the backward question and one answer for the forward question. Often, there can be more than one answer for these questions. For example, consider the following statement:

If  $x$  is an odd integer, then  $x^2$  is an odd integer.

The backward question for this could be, “How do I prove that an integer is an odd integer?” One way to answer this is to use the definition of an odd integer, but another way is to use the result of Theorem 1.8. That is, we can prove an integer is odd by proving that it is a product of two odd integers.

The difficulty then is deciding which answer to use. Sometimes we can tell by carefully watching the interplay between the forward process and the backward process. Other times, we may have to work with more than one possible answer.

2. Sometimes we can use previously proven results to answer a forward question or a backward question. This was the case in the example given in Comment (1), where Theorem 1.8 was used to answer a backward question.
3. Although we start with two separate processes (forward and backward), the key to constructing a proof is to find a way to link these two processes. This can be difficult. One way to proceed is to use the know portion of the table to motivate answers to backward questions and to use the show portion of the table to motivate answers to forward questions.
4. Answering a backward question can sometimes be tricky. If the goal is the statement  $Q$ , we must construct the know-show table so that if we know that  $Q1$  is true, then we can conclude that  $Q$  is true. It is sometimes easy to answer this in a way that if it is known that  $Q$  is true, then we can conclude that  $Q1$  is true. For example, suppose the goal is to prove

$$y^2 = 4,$$

where  $y$  is a real number. A backward question could be, “How do we prove the square of a real number equals four?” One possible answer is to prove that the real number equals 2. Another way is to prove that the real number equals  $-2$ . This is an appropriate backward question, and these are appropriate answers.





However, if the goal is to prove

$$y = 2,$$

where  $y$  is a real number, we could ask, “How do we prove a real number equals 2?” It is not appropriate to answer this question with “prove that the square of the real number equals 4.” This is because if  $y^2 = 4$ , then it is not necessarily true that  $y = 2$ .

5. Finally, it is very important to realize that not every proof can be constructed by the use of a simple know-show table. Proofs will get more complicated than the ones that are in this section. The main point of this section is not the know-show table itself, but the way of thinking about a proof that is indicated by a know-show table. In most proofs, it is very important to specify carefully what it is that is being assumed and what it is that we are trying to prove. The process of asking the “backward questions” and the “forward questions” is the important part of the know-show table. It is very important to get into the “habit of mind” of working backward from what it is we are trying to prove and working forward from what it is we are assuming. Instead of immediately trying to write a complete proof, we need to stop, think, and ask questions such as

- Just exactly what is it that I am trying to prove?
- How can I prove this?
- What methods do I have that may allow me to prove this?
- What are the assumptions?
- How can I use these assumptions to prove the result?

**Progress Check 1.10 (Exploring a Proposition)**

Construct a table of values for  $(3m^2 + 4m + 6)$  using at least six different integers for  $m$ . Make one-half of the values for  $m$  even integers and the other half odd integers. Is the following proposition true or false?

If  $m$  is an odd integer, then  $(3m^2 + 4m + 6)$  is an odd integer.

Justify your conclusion. This means that if the proposition is true, then you should write a proof of the proposition. If the proposition is false, you need to provide an example of an odd integer for which  $(3m^2 + 4m + 6)$  is an even integer.

**Progress Check 1.11 (Constructing and Writing a Proof)**

The **Pythagorean Theorem** for right triangles states that if  $a$  and  $b$  are the lengths of the legs of a right triangle and  $c$  is the length of the hypotenuse, then  $a^2 + b^2 = c^2$ . For example, if  $a = 5$  and  $b = 12$  are the lengths of the two sides of a right triangle and if  $c$  is the length of the hypotenuse, then the  $c^2 = 5^2 + 12^2$  and so  $c^2 = 169$ . Since  $c$  is a length and must be positive, we conclude that  $c = 13$ .

Construct and provide a well-written proof for the following proposition.

**Proposition.** If  $m$  is a real number and  $m$ ,  $m + 1$ , and  $m + 2$  are the lengths of the three sides of a right triangle, then  $m = 3$ .

Although this proposition uses different mathematical concepts than the one used in this section, the process of constructing a proof for this proposition is the same forward-backward method that was used to construct a proof for Theorem 1.8. However, the backward question, “How do we prove that  $m = 3$ ?” is simple but may be difficult to answer. The basic idea is to develop an equation from the forward process and show that  $m = 3$  is a solution of that equation.

---

**Exercises for Section 1.2**

1. Construct a know-show table for each of the following statements and then write a formal proof for one of the statements.
  - \* (a) If  $m$  is an even integer, then  $m + 1$  is an odd integer.
  - (b) If  $m$  is an odd integer, then  $m + 1$  is an even integer.
2. Construct a know-show table for each of the following statements and then write a formal proof for one of the statements.
  - (a) If  $x$  is an even integer and  $y$  is an even integer, then  $x + y$  is an even integer.
  - (b) If  $x$  is an even integer and  $y$  is an odd integer, then  $x + y$  is an odd integer.
  - \* (c) If  $x$  is an odd integer and  $y$  is an odd integer, then  $x + y$  is an even integer.
3. Construct a know-show table for each of the following statements and then write a formal proof for one of the statements.



- \* (a) If  $m$  is an even integer and  $n$  is an integer, then  $m \cdot n$  is an even integer.
  - \* (b) If  $n$  is an even integer, then  $n^2$  is an even integer.
  - (c) If  $n$  is an odd integer, then  $n^2$  is an odd integer.
4. Construct a know-show table and write a complete proof for each of the following statements:
- \* (a) If  $m$  is an even integer, then  $5m + 7$  is an odd integer.
  - (b) If  $m$  is an odd integer, then  $5m + 7$  is an even integer.
  - (c) If  $m$  and  $n$  are odd integers, then  $mn + 7$  is an even integer.
5. Construct a know-show table and write a complete proof for each of the following statements:
- (a) If  $m$  is an even integer, then  $3m^2 + 2m + 3$  is an odd integer.
  - \* (b) If  $m$  is an odd integer, then  $3m^2 + 7m + 12$  is an even integer.
6. In this section, it was noted that there is often more than one way to answer a backward question. For example, if the backward question is, “How can we prove that two real numbers are equal?”, one possible answer is to prove that their difference equals 0. Another possible answer is to prove that the first is less than or equal to the second and that the second is less than or equal to the first.
- \* (a) Give at least one more answer to the backward question, “How can we prove that two real numbers are equal?”
  - (b) List as many answers as you can for the backward question, “How can we prove that a real number is equal to zero?”
  - (c) List as many answers as you can for the backward question, “How can we prove that two lines are parallel?”
  - \* (d) List as many answers as you can for the backward question, “How can we prove that a triangle is isosceles?”
7. Are the following statements true or false? Justify your conclusions.
- (a) If  $a$ ,  $b$  and  $c$  are integers, then  $ab + ac$  is an even integer.
  - (b) If  $b$  and  $c$  are odd integers and  $a$  is an integer, then  $ab + ac$  is an even integer.

8. Is the following statement true or false? Justify your conclusion.

If  $a$  and  $b$  are nonnegative real numbers and  $a + b = 0$ , then  $a = 0$ .

Either give a counterexample to show that it is false or outline a proof by completing a know-show table.

9. An integer  $a$  is said to be a **type 0 integer** if there exists an integer  $n$  such that  $a = 3n$ . An integer  $a$  is said to be a **type 1 integer** if there exists an integer  $n$  such that  $a = 3n + 1$ . An integer  $a$  is said to be a **type 2 integer** if there exists an integer  $m$  such that  $a = 3m + 2$ .

- \* (a) Give examples of at least four different integers that are type 1 integers.
- (b) Give examples of at least four different integers that are type 2 integers.
- \* (c) By multiplying pairs of integers from the list in Exercise (9a), does it appear that the following statement is true or false?

If  $a$  and  $b$  are both type 1 integers, then  $a \cdot b$  is a type 1 integer.

10. Use the definitions in Exercise (9) to help write a proof for each of the following statements:

- \* (a) If  $a$  and  $b$  are both type 1 integers, then  $a + b$  is a type 2 integer.
- (b) If  $a$  and  $b$  are both type 2 integers, then  $a + b$  is a type 1 integer.
- (c) If  $a$  is a type 1 integer and  $b$  is a type 2 integer, then  $a \cdot b$  is a type 2 integer.
- (d) If  $a$  and  $b$  are both type 2 integers, then  $a \cdot b$  is type 1 integer.

11. Let  $a$ ,  $b$ , and  $c$  be real numbers with  $a \neq 0$ . The solutions of the **quadratic equation**  $ax^2 + bx + c = 0$  are given by the **quadratic formula**, which states that the solutions are  $x_1$  and  $x_2$ , where

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

- (a) Prove that the sum of the two solutions of the quadratic equation  $ax^2 + bx + c = 0$  is equal to  $-\frac{b}{a}$ .
- (b) Prove that the product of the two solutions of the quadratic equation  $ax^2 + bx + c = 0$  is equal to  $\frac{c}{a}$ .



12. (a) See Exercise (11) for the quadratic formula, which gives the solutions to a quadratic equation. Let  $a$ ,  $b$ , and  $c$  be real numbers with  $a \neq 0$ . The discriminant of the quadratic equation  $ax^2 + bx + c = 0$  is defined to be  $b^2 - 4ac$ . Explain how to use this discriminant to determine if the quadratic equation has two real number solutions, one real number solution, or no real number solutions.
- (b) Prove that if  $a$ ,  $b$ , and  $c$  are real numbers with  $a > 0$  and  $c < 0$ , then one solutions of the quadratic equation  $ax^2 + bx + c = 0$  is a positive real number.
- (c) Prove that if  $a$ ,  $b$ , and  $c$  are real numbers with  $a \neq 0$ ,  $b > 0$ , and  $b < 2\sqrt{ac}$ , then the quadratic equation  $ax^2 + bx + c = 0$  has no real number solutions.

### Explorations and Activities

13. **Pythagorean Triples.** Three natural numbers  $a$ ,  $b$ , and  $c$  with  $a < b < c$  are said to form a **Pythagorean triple** provided that  $a^2 + b^2 = c^2$ . For example, 3, 4, and 5 form a Pythagorean triple since  $3^2 + 4^2 = 5^2$ . The study of Pythagorean triples began with the development of the **Pythagorean Theorem** for right triangles, which states that if  $a$  and  $b$  are the lengths of the legs of a right triangle and  $c$  is the length of the hypotenuse, then  $a^2 + b^2 = c^2$ . For example, if the lengths of the legs of a right triangle are 4 and 7 units, then  $c^2 = 4^2 + 7^2 = 63$ , and the length of the hypotenuse must be  $\sqrt{63}$  units (since the length must be a positive real number). Notice that 4, 7, and  $\sqrt{63}$  are not a Pythagorean triple since  $\sqrt{63}$  is not a natural number.
- (a) Verify that each of the following triples of natural numbers form a Pythagorean triple.
- 3, 4, and 5
  - 8, 15, and 17
  - 12, 35, and 37
  - 6, 8, and 10
  - 10, 24, and 26
  - 14, 48, and 50
- (b) Does there exist a Pythagorean triple of the form  $m$ ,  $m + 7$ , and  $m + 8$ , where  $m$  is a natural number? If the answer is yes, determine all such Pythagorean triples. If the answer is no, prove that no such Pythagorean triple exists.
- (c) Does there exist a Pythagorean triple of the form  $m$ ,  $m + 11$ , and  $m + 12$ , where  $m$  is a natural number? If the answer is yes, determine all such

Pythagorean triples. If the answer is no, prove that no such Pythagorean triple exists.

**14. More Work with Pythagorean Triples.** In Exercise (13), we verified that each of the following triples of natural numbers are Pythagorean triples:

- 3, 4, and 5
- 8, 15, and 17
- 12, 35, and 37
- 6, 8, and 10
- 10, 24, and 26
- 14, 48, and 50

- (a) Focus on the least even natural number in each of these Pythagorean triples. Let  $n$  be this even number and find  $m$  so that  $n = 2m$ . Now try to write formulas for the other two numbers in the Pythagorean triple in terms of  $m$ . For example, for 3, 4, and 5,  $n = 4$  and  $m = 2$ , and for 8, 15, and 17,  $n = 8$  and  $m = 4$ . Once you think you have formulas, test your results with  $m = 10$ . That is, check to see that you have a Pythagorean triple whose smallest even number is 20.
- (b) Write a proposition and then write a proof of the proposition. The proposition should be in the form: If  $m$  is a natural number and  $m \geq 2$ , then .....
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## 1.3 Chapter 1 Summary

### Important Definitions

- Statement, page 1
- Conditional statement, pages ??, 5
- Even integer, page 15
- Odd integer, page 15
- Pythagorean triple, page 29

### Important Number Systems and Their Properties

- The natural numbers,  $\mathbb{N}$ ; the integers,  $\mathbb{Z}$ ; the rational numbers,  $\mathbb{Q}$ ; and the real numbers,  $\mathbb{R}$ . See page 10.
- Closure Properties of the Number Systems

Number System	Closed Under
Natural Numbers, $\mathbb{N}$	addition and multiplication
Integers, $\mathbb{Z}$	addition, subtraction, and multiplication
Rational Numbers, $\mathbb{Q}$	addition, subtraction, multiplication, and division by nonzero rational numbers
Real Numbers, $\mathbb{R}$	addition, subtraction, multiplication, and division by nonzero real numbers

- Inverse, commutative, associative, and distributive properties of the real numbers. See page 18.

### Important Theorems and Results

- **Exercise (1), Section 1.2**  
*If  $m$  is an even integer, then  $m + 1$  is an odd integer.*  
*If  $m$  is an odd integer, then  $m + 1$  is an even integer.*
- **Exercise (2), Section 1.2**  
*If  $x$  is an even integer and  $y$  is an even integer, then  $x + y$  is an even integer.*  
*If  $x$  is an even integer and  $y$  is an odd integer, then  $x + y$  is an odd integer.*  
*If  $x$  is an odd integer and  $y$  is an odd integer, then  $x + y$  is an even integer.*



- **Exercise (3), Section 1.2.** *If  $x$  is an even integer and  $y$  is an integer, then  $x \cdot y$  is an even integer.*
- **Theorem 1.8.** *If  $x$  is an odd integer and  $y$  is an odd integer, then  $x \cdot y$  is an odd integer.*
- **The Pythagorean Theorem,** page 26. *If  $a$  and  $b$  are the lengths of the legs of a right triangle and  $c$  is the length of the hypotenuse, then  $a^2 + b^2 = c^2$ .*