# Part II

# Matrices

# Section 8

# Matrix Operations

**Focus Questions**

*By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.*

- Under what conditions can we add two matrices and how is the matrix sum defined?
- Under what conditions can we multiply a matrix by a scalar and how is a scalar multiple of a matrix defined?
- Under what conditions can we multiply two matrices and how is the matrix product defined?
- What properties do matrix addition, scalar multiplication of matrices and matrix multiplication satisfy? Are these properties similar to properties that are satisfied by vector operations?
- What are two properties that make matrix multiplication fundamentally different than our standard product of real numbers?
- What is the interpretation of matrix multiplication from the perspective of linear transformations?
- How is the transpose of a matrix defined?

## Application: Algorithms for Matrix Multiplication

Matrix multiplication is widely used in applications ranging from scientific computing and pattern recognition to counting paths in graphs. As a consequence, much work is being done in developing efficient algorithms for matrix multiplication.

We will see that a matrix product can be calculated through the row-column method. Recall that the product of two $2 \times 2$ matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ is given by

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix},$$

This product involves eight scalar multiplications and some scalar additions. As we will see, multiplication is more computationally expensive than addition, so we will focus on multiplication. In 1969, a German mathematician named Volker Strassen showed[1] that the product of two $2 \times 2$ matrices can be calculated using only seven multiplications. While this is not much of an improvement, the Strassen algorithm can be applied to larger matrices, using matrix partitions (which allow for parallel computation), and its publication led to additional research on faster algorithms for matrix multiplication. More details are provided later in this section.

## Introduction

A vector is a list of numbers in a specified order and a matrix is an ordered array of objects. In fact, a vector can be thought of as a matrix of size $n \times 1$. Vectors and matrices are so alike in this way that it would seem natural that we can define operations on matrices just as we did with vectors.

Recall that a matrix is made of rows and columns – the entries reading from left to right form the *rows* of the matrix and the entries reading from top to bottom form the *columns*. The number of rows and columns of a matrix is called the *size* of the matrix, so an $m \times n$ matrix has $m$ rows and $n$ columns. If we label the entry in the $i$th row and $j$th column of a matrix $A$ as $a_{ij}$, then we write $A = [a_{ij}]$.

We can generalize the operations of addition and scalar multiplication on vectors to matrices similarly. Given two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, we define the sum $A + B$ by

$$A + B = [a_{ij} + b_{ij}]$$

when the sizes of the matrices $A$ and $B$ match. In other words, for matrices of the same size the matrix addition is defined by adding corresponding entries in the matrices. For example,

$$\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 7 \end{bmatrix}.$$

We define the scalar multiple of a matrix $A = [a_{ij}]$ by scalar $c$ to be the matrix $cA$ defined by

$$cA = [ca_{ij}].$$

This means that we multiply each entry of the matrix $A$ by the scalar $c$. As an example,

$$3\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -6 & 9 \end{bmatrix}.$$

Even though we did not have a multiplication operation on vectors, we had a matrix-vector product, which is a special case of a matrix-matrix product since a vector is a matrix with one column. However, generalizing the matrix-vector product to a matrix-matrix product is not immediate

[1] Strassen, Volker, Gaussian Elimination is not Optimal, Number. Math. 13, p. 354-356, 1969

as it is not immediately clear what we can do with the other columns. We will consider this question in this section.

Note that all of the matrix operations can be performed on a calculator. After entering each matrix in the calculator, just use $+$, $-$ and $\times$ operations to find the result of the matrix operation. (Just for fun, try using $\div$ with matrices to see if it will work.)

**Preview Activity 8.1.**

(1) Pick three different varying sizes of pairs of $A, B$ matrices which can be added. For each pair:

(a) Find the matrices $A + B$ and $B + A$.

(b) How are the two matrices $A + B$ and $B + A$ related? What does this tell us about matrix addition?

(2) Let $A = \begin{bmatrix} 1 & 0 \\ -2 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & -5 \\ 1 & 6 \end{bmatrix}$. Determine the entries of the matrix $A + 2B - 7C$.

(3) Now we turn to multiplication of matrices. Our first goal is to find out what conditions we need on the sizes of matrices $A$ and $B$ if the matrix-matrix product $AB$ is defined and what the size of the resulting product matrix is. We know the condition and the size of the result in the special case of $B$ being a vector, i.e., a matrix with one column. So our conjectures for the general case should match what we know in the special case.

In each part of this problem, use any appropriate tool (e.g., your calculator, Maple, Mathematica, Wolfram|Alpha) to determine the matrix product $AB$, if it exists. If you obtain a product, write it down and explain how its size is related to the sizes of $A$ and $B$. If you receive an error, write down the error and guess why the error occurred and/or what it means.

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 5 & -2 \\ 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}$

(e) Make a guess for the condition on the sizes of two matrices $A, B$ for which the product $AB$ is defined. How is the size of the product matrix related to the sizes of $A$ and $B$?

(4) The final matrix products, when defined, in problem 3 might seem unrelated to the individual matrices at first. In this problem, we will uncover this relationship using our knowledge of the matrix-vector product.

Let $A = \begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$.

(a) Calculate $AB$ using any tool.

(b) Using the matrix-vector product, calculate $A\mathbf{x}$ where $\mathbf{x}$ is the first column (i.e., calculate $A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$), and then the second column of $B$ (i.e., calculate $A\begin{bmatrix} 2 \\ 3 \end{bmatrix}$), and then the third column of $B$ (i.e., calculate $A\begin{bmatrix} 1 \\ 2 \end{bmatrix}$). Do you notice these output vectors within $AB$?

(c) Describe as best you can a definition of $AB$ using the matrix-vector product.

## Properties of Matrix Addition and Multiplication by Scalars

Just as we were able to define an algebra of vectors with addition and multiplication by scalars, we can define an algebra of matrices. We will see that the properties of these operations on matrices are immediate generalizations of the properties of the operations on vectors. We will then see how the matrix product arises through the connection of matrices to linear transformations. Finally, we define the transpose of a matrix. The transpose of a matrix will be useful in applications such as graph theory and least-squares fitting of curves, as well as in advanced topics as inner product spaces and the dual space of a vector space.

We learned in Preview Activity 8.1 that we can add two matrices of the same size together by adding corresponding entries and we can multiply any matrix by a scalar by multiplying each entry of the matrix by that scalar. More generally, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices and $c$ is any scalar, then

$$A + B = [a_{ij} + b_{ij}] \quad \text{and} \quad cA = [ca_{ij}].$$

As we have done each time we have introduced a new operation, we ask what properties the operation has. For example, you determined in Preview Activity 8.1 that addition of matrices is a commutative operation. More specifically, for every two $m \times n$ matrices $A$ and $B$, $A + B = B + A$. We can use similar arguments to verify the following properties of matrix addition and multiplication by scalars. Notice that these properties are very similar to the properties of addition and scalar multiplication of vectors we discussed earlier. This should come as no surprise since the $n$-dimensional vectors are $n \times 1$ matrices. In a strange twist, we will see that matrices themselves can be considered as vectors when we discuss vector spaces in a later section.

**Theorem 8.1.** *Let $A$, $B$, and $C$ be $m \times n$ matrices and let $a$ and $b$ be scalars. Then*

*(1)* $A + B = B + A$ *(this property tells us that matrix addition is* commutative*)*

*(2)* $(A + B) + C = A + (B + C)$ *(this property tells us that matrix addition is* associative*)*

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*(3) The $m \times n$ matrix $0$ whose entries are all 0 has the property that $A + 0 = A$. The matrix $0$ is called the **zero matrix** (It is generally clear from the context what the size of the 0 matrix is.).*

*(4) The scalar multiple $(-1)A$ of the matrix $A$ has the property that $(-1)A + A = 0$. The matrix $(-1)A = -A$ is called the **additive inverse** of the matrix $A$.*

*(5)* $(a + b)A = aA + bA$ *(this property tells us that* scalar multiplication of matrices distributes over scalar addition*)*

*(6)* $a(A + B) = aA + aB$ *(this property tells us that* scalar multiplication of matrices distributes over matrix addition*)*

*(7)* $(ab)A = a(bA)$

*(8)* $1A = A$.

Later on, we will see that these properties define the set of all $m \times n$ matrices as a *vector space*. These properties just say that, regarding addition and multiplication by scalars, we can manipulate matrices just as we do real numbers. Note, however, we have not yet defined an operation of multiplication on matrices. That is the topic for the next section.

## A Matrix Product

As we saw in Preview Activity 8.1, a matrix-matrix product can be found in a way which makes use of and also generalizes the matrix-vector product.

**Definition 8.2.** The **matrix product** of a $k \times m$ matrix $A$ and an $m \times n$ matrix $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$ with columns $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_n$ is the $k \times n$ matrix

$$[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_n].$$

We now consider the motivation behind this definition by thinking about the matrix transformations corresponding to each of the matrices $A, B$ and $AB$. Recall that left multiplication by an $m \times n$ matrix $B$ defines a transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ by $T(\mathbf{x}) = B\mathbf{x}$. The domain of $T$ is $\mathbb{R}^n$ because the number of components of $\mathbf{x}$ have to match the number of entries in each of row of $B$ in order for the matrix-vector product $B\mathbf{x}$ to be defined. Similarly, a $k \times m$ matrix $A$ defines a transformation $A$ from $\mathbb{R}^m$ to $\mathbb{R}^k$. Since transformations are functions, we can compose them as long as the output vectors of the inside transformation lie in the domain of the outside transformation. Therefore if $T$ is the inside transformation and $S$ is the outside transformation, the composition $S \circ T$ is defined. So a natural question to ask is if we are given

- a transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ where $T(\mathbf{x}) = B\mathbf{x}$ for an $m \times n$ matrix $B$ and
- a transformation $S$ from $\mathbb{R}^m$ to $\mathbb{R}^k$ with $S(\mathbf{y}) = A\mathbf{y}$ for some $k \times m$ matrix $A$,

is there a matrix that represents the transformation $S \circ T$ defined by $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$? We investigate this question in the next activity in the special case of a $2 \times 3$ matrix $A$ and a $3 \times 2$ matrix $B$.

**Activity 8.1.** In this activity, we look for the meaning of the matrix product from a transformation perspective. Let $S$ and $T$ be matrix transformations defined by

$$S(\mathbf{y}) = A\mathbf{y} \quad \text{and} \quad T(\mathbf{x}) = B\mathbf{x},$$

where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 \\ 5 & -2 \\ 0 & 1 \end{bmatrix}.$$

(a) What are the domains and codomains of $S$ and $T$? Why is the composite transformation $S \circ T$ defined? What is the domain of $S \circ T$? What is the codomain of $S \circ T$? (Recall that $S \circ T$ is defined by $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$, i.e., we substitute the output $T(\mathbf{x})$ as the input into the transformation $S$.)

(b) Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Determine the components of $T(\mathbf{x})$.

(c) Find the components of $S \circ T(\mathbf{x}) = S(T(\mathbf{x}))$.

(d) Find a matrix $C$ so that $S(T(\mathbf{x})) = C\mathbf{x}$.

(e) Use the definition of composition of transformations and the definitions of the $S$ and $T$ transformations to explain why it is reasonable to define $AB$ to be the matrix $C$. Does the matrix $C$ agree with the

$$AB = \begin{bmatrix} 13 & -4 \\ 5 & -1 \end{bmatrix}$$

you found in Preview Activity 8.1 using technology?

We now consider this result in the general case of a $k \times m$ matrix $A$ and an $m \times n$ matrix $B$, where $A$ and $B$ define matrix transformations $S$ and $T$, respectively. In other words, $S$ and $T$ are matrix transformations defined by $S(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$. The domain of $S$ is $\mathbb{R}^m$ and the codomain is $\mathbb{R}^k$. The domain of $T$ is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. The composition $S \circ T$ is defined because the output vectors of $T$ are in $\mathbb{R}^m$ and they lie in the domain of $S$. The domain of $S \circ T$ is the same as the domain of $T$ since the input vectors first go through the $T$ transformation. The codomain of $S \circ T$ is the same as the codomain of $S$ since the final output vectors are produced by applying the $S$ transformation.

Let us see how we can obtain the matrix corresponding to the transformation $S \circ T$. Let $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$, where $\mathbf{b}_j$ is the $j$th column of $B$, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Recall that the matrix vector product $B\mathbf{x}$ is the linear combination of the columns of $B$ with the corresponding weights from $\mathbf{x}$. So

$$T(\mathbf{x}) = B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n.$$

Note that each of the $\mathbf{b}_j$ vectors are in $\mathbb{R}^m$ since $B$ is an $m \times n$ matrix. Therefore, each of these vectors can be multiplied by matrix $A$ and we can evaluate $S(B\mathbf{x})$. Therefore, $S \circ T$ is defined and

$$(S \circ T)(\mathbf{x}) = S(T(\mathbf{x})) = A(B\mathbf{x}) = A\left(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n\right). \tag{8.1}$$

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The properties of matrix-vector products show that

$$A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n) = x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_nA\mathbf{b}_n. \tag{8.2}$$

This expression is a linear combination of $A\mathbf{b}_i$'s with $x_i$'s being the weights. Therefore, if we let $C$ be the matrix with columns $A\mathbf{b}_1$, $A\mathbf{b}_2$, ..., $A\mathbf{b}_n$, that is

$$C = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_n],$$

then

$$x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_nA\mathbf{b}_n = C\mathbf{x} \tag{8.3}$$

by definition of the matrix-vector product. Combining equations (8.1), (8.2), and (8.3) shows that

$$(S \circ T)(\mathbf{x}) = C\mathbf{x}$$

where $C = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_n]$.

Also note that since $T(\mathbf{x}) = B\mathbf{x}$ and $S(\mathbf{y}) = A\mathbf{y}$, we find

$$(S \circ T)(\mathbf{x}) = S(T(\mathbf{x})) = S(B\mathbf{x}) = A(B(\mathbf{x})). \tag{8.4}$$

Since the matrix representing the transformation $S \circ T$ is the matrix

$$[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_n]$$

where $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_n$ are the columns of the matrix $B$, it is natural to define $AB$ to be this matrix in light of equation (8.4).

Matrix multiplication has some properties that are unfamiliar to us as the next activity illustrates.

**Activity 8.2.** Let $A = \begin{bmatrix} 3 & -1 \\ -2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(a) Find the indicated products (by hand or using a calculator).
$AB$ $BA$ $DC$ $AC$ $BC$ $AE$ $EB$

(b) Is matrix multiplication commutative? Explain.

(c) Is there an identity element for matrix multiplication? In other words, is there a matrix $I$ for which $AI = IA = A$ for any matrix $A$? Explain.

(d) If $a$ and $b$ are real numbers with $ab = 0$, then we know that either $a = 0$ or $b = 0$. Is this same property true with matrix multiplication? Explain.

(e) If $a$, $b$, and $c$ are real numbers with $c \neq 0$ and $ac = bc$, we know that $a = b$. Is this same property true with matrix multiplication? Explain.

As we saw in Activity 8.2, there are matrices $A, B$ for which $AB \neq BA$. On the other hand, there are matrices for which $AB = BA$. For example, this equality will always hold for a square matrix $A$ and if $B$ is the identity matrix of the same size. It also holds if $A = B$. If the equality

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$AB = BA$ holds, we say that matrices $A$ and $B$ *commute*. So the identity matrix commutes with all square matrices of the same size and every matrix $A$ commutes with $A^k$ for any power $k$.

There is an alternative method of calculating a matrix product that we will often use that we illustrate in the next activity. This alternate version depends on the product of a row matrix with a vector. Suppose $A = [a_1 \ a_2 \ \cdots \ a_n]$ is a $1 \times n$ matrix and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is an $n \times 1$ vector. Then the product $A\mathbf{x}$ is the $1 \times 1$ vector

$$[a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1x_1 + a_2x_2 + \cdots + a_nx_n].$$

In this situation, we usually identify the $1 \times 1$ matrix with its scalar entry and write

$$[a_1 \ a_2 \ \cdots \ a_n] \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \cdots + a_nx_n. \tag{8.5}$$

The product $\cdot$ in (8.5) is called the *scalar* or *dot* product of $[a_1 \ a_2 \ \cdots \ a_n]$ with $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

**Activity 8.3.** Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -4 \\ 2 & -5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 \\ 6 & 0 \\ 1 & 3 \end{bmatrix}$.

Let $\mathbf{a}_i$ be the $i$th row of $A$ and $\mathbf{b}_j$ the $j$th column of $B$. For example, $\mathbf{a}_1 = [1 \ -1 \ 2]$ and $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$.

Calculate the entries of the matrix $C$, where

$$C = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 \\ \mathbf{a}_3 \cdot \mathbf{b}_1 & \mathbf{a}_3 \cdot \mathbf{b}_2 \end{bmatrix},$$

where $\mathbf{a}_i \cdot \mathbf{b}_j$ refers to the scalar product of row $i$ of $A$ with column $j$ of $B$.[2] Compare your result with the result of $AB$ calculated via the product of $A$ with the columns of $B$.

[2]Recall from Exercise 5 of Section 5 that the scalar product $\mathbf{u} \cdot \mathbf{v}$ of a $1 \times n$ matrix $\mathbf{u} = [u_1 \ u_2 \ \ldots \ u_n]$ and an $n \times 1$ vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n$.

Activity 8.3 shows that these is an alternate way to calculate a matrix product. To see how this works in general, let $A = [a_{ij}]$ be a $k \times m$ matrix and $B = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_n]$ an $m \times n$ matrix. We know that

$$AB = [A\mathbf{b}_1\ A\mathbf{b}_2\ \cdots\ A\mathbf{b}_n].$$

Now let $\mathbf{r}_1$, $\mathbf{r}_2$, …, $\mathbf{r}_k$ be the rows of $A$ so that $A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_k \end{bmatrix}$. First we argue that if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$, then

$$A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_k \cdot \mathbf{x} \end{bmatrix}.$$

This is the *scalar product* (or *dot product*) definition of the matrix-vector product.

To show that this definition gives the same result as the linear combination definition of matrix-vector product, we first let $A = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_m]$, where $\mathbf{c}_1$, $\mathbf{c}_2$, …, $\mathbf{c}_m$ are the columns of $A$. By our linear combination definition of the matrix-vector product, we obtain

$$\begin{aligned} A\mathbf{x} &= x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_m\mathbf{c}_m \\ &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{bmatrix} + \cdots + x_m \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{km} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{km}x_m \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_k \cdot \mathbf{x} \end{bmatrix}. \end{aligned}$$

Therefore, the above work shows that both linear combination and scalar product definitions give the same matrix-vector product.

Applying this to the matrix product $AB$ defined in terms of the matrix-vector product, we see that

$$A\mathbf{b}_j = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{b}_j \\ \mathbf{r}_2 \cdot \mathbf{b}_j \\ \vdots \\ \mathbf{r}_k \cdot \mathbf{b}_j \end{bmatrix}.$$

So the $i, j$th entry of the matrix product $AB$ is found by taking the scalar product of the $i$th row of $A$ with the $j$th column of $B$. In other words,

$$(AB)_{ij} = \mathbf{r}_i \cdot \mathbf{b}_j$$

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where $\mathbf{r}_i$ is the $i$th row of $A$ and $\mathbf{b}_j$ is the $j$th column of $B$.

### Properties of Matrix Multiplication

Activity 8.2 shows that we must be very careful not to assume that matrix multiplication behaves like multiplication of real numbers. However, matrix multiplication does satisfy some familiar properties. For example, we now have an addition and multiplication of matrices under certain conditions, so we might ask if matrix multiplication distributes over matrix addition. To answer this question we take two *arbitrary* $k \times m$ matrices $A$ and $B$ and an *arbitrary* $m \times n$ matrix $C = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n]$. Then

$$\begin{aligned}(A+B)C &= [(A+B)\mathbf{c}_1 \ (A+B)\mathbf{c}_2 \ \cdots \ (A+B)\mathbf{c}_n] \\ &= [A\mathbf{c}_1 + B\mathbf{c}_1 \ A\mathbf{c}_2 + B\mathbf{c}_2 \ \cdots \ A\mathbf{c}_n + B\mathbf{c}_n] \\ &= [A\mathbf{c}_1 \ A\mathbf{c}_2 \ \cdots \ A\mathbf{c}_n] + [B\mathbf{c}_1 \ B\mathbf{c}_2 \ \cdots \ B\mathbf{c}_n] \\ &= AC + BC.\end{aligned}$$

Similar arguments can be used to show the following properties of matrix multiplication.

**Theorem 8.3.** *Let $A$, $B$, and $C$ be matrices of the appropriate sizes for all sums and products to be defined and let $a$ be a scalar. Then*

*(1)* $(AB)C = A(BC)$ *(this property tells us that matrix multiplication is* associative*)*

*(2)* $(A + B)C = AC + BC$ *(this property tells us that matrix multiplication on the right* distributes over matrix addition*)*

*(3)* $A(B+C) = AB+AC$ *(this property tells us that matrix multiplication on the left* distributes over matrix addition*)*

*(4) There is a square matrix $I_n$ with the property that $AI_n = A$ or $I_nA = A$ for whichever product is defined.*

*(5)* $a(AB) = (aA)B = A(aB)$

We verified the second part of this theorem and will assume that all of the properties of this theorem hold. The matrix $I_n$ introduced in Theorem 8.3 is called the *(multiplicative) identity matrix*. We usually omit the word multiplicative and refer to the $I_n$ simply as the identity matrix. This does not cause any confusion since we refer to the additive identity matrix as simply the zero matrix.

**Definition 8.4.** Let $n$ be a positive integer. The $n \times n$ **identity matrix** $I_n$ is the matrix $I_n = [a_{ij}]$, where $a_{ii} = 1$ for each $i$ and $a_{ij} = 0$ if $i \neq j$.

We also write the matrix $I_n$ as

$$I_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

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The matrix $I_n$ has the property that for any $n \times n$ matrix $A$,

$$AI_n = I_nA = A\,.$$

so $I_n$ is a multiplicative identity in the set of all $n \times n$ matrices. More generally, for an $m \times n$ matrix $A$,

$$AI_n = I_mA = A\,.$$

## The Transpose of a Matrix

One additional operation on matrices is the transpose. The transpose of a matrix occurs in many useful formulas in linear algebra and in applications of linear algebra.

**Definition 8.5.** The **transpose** of an $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix $A^{\mathsf{T}}$ whose $i, j$th entry is $a_{ji}$.

Written out, the transpose of the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ \vdots & & \ddots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn-1} & a_{mn} \end{bmatrix}$$

is the $n \times m$ matrix

$$A^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m-11} & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m-12} & a_{m2} \\ \vdots & & \ddots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{m-1n} & a_{mn} \end{bmatrix}.$$

In other words, the transpose of a matrix $A$ is the matrix $A^{\mathsf{T}}$ whose rows are the columns of $A$. Alternatively, the transpose of $A$ is the matrix $A^{\mathsf{T}}$ whose columns are the rows of $A$. We can also view the transpose of $A$ as the reflection of $A$ across its main diagonal, where the *diagonal* of a matrix $A = [a_{ij}]$ consists of the entries of the form $[a_{ii}]$.

**Activity 8.4.**

(a) Find the transpose of each of the indicated matrices.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 4 & -3 \\ 0 & -1 \end{bmatrix}$$

(b) Find the transpose of the new matrix for each part above. What can you conjecture based on your results?

(c) There are certain special types of matrices that are given names.

**Definition 8.6.** Let $A$ be a square matrix whose $ij$th entry is $a_{ij}$.

(1) The matrix $A$ is a **diagonal matrix** if $a_{ij} = 0$ whenever $i \neq j$.

(2) The matrix $A$ is a **symmetric** matrix if $A^{\mathsf{T}} = A$.

(3) The matrix $A$ is an **upper triangular** if $a_{ij} = 0$ whenever $i > j$.

(4) The matrix $A$ is a **lower triangular** if $a_{ij} = 0$ whenever $i < j$.

i. Find an example of a diagonal matrix $A$. What can you say about $A^{\mathsf{T}}$?

ii. Find an example of a non-diagonal symmetric matrix $B$. If $B^{\mathsf{T}} = B$, must $B$ be a square matrix?

iii. Find an example of an upper triangular matrix $C$. What kind of a matrix is $C^{\mathsf{T}}$?

We will see later that diagonal matrices are important in that their powers are easy to calculate. Symmetric matrices arise frequently in applications such as in graph theory as adjacency matrices and in quantum mechanics as observables, and have many useful properties including being diagonalizable and having real eigenvalues, as we will also see later.

## Properties of the Matrix Transpose

As with every other operation, we want to understand what properties the matrix transpose has. Properties of transposes are shown in the following theorem.

**Theorem 8.7.** *Let $A$ and $B$ be matrices of the appropriate sizes and let $a$ be a scalar. Then*

*(1)* $\left(A^{\mathsf{T}}\right)^{\mathsf{T}} = A$

*(2)* $(A+B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}}$

*(3)* $(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}$

*(4)* $(aA)^{\mathsf{T}} = aA^{\mathsf{T}}$

The one property that might seem strange is the third one. To understand this property, suppose $A$ is an $m \times n$ matrix and $B$ an $n \times k$ matrix so that the product $AB$ is defined. We will argue that $(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}$ by comparing the $i, j$th entry of each side.

- First notice that the $i, j$th entry of $(AB)^{\mathsf{T}}$ is the $j, i$th entry of $AB$. The $j, i$th entry of $AB$ is found by taking the scalar product of the $j$th row of $A$ with the $i$th column of $B$. Thus,

  the $i, j$th entry of $(AB)^{\mathsf{T}}$ is the scalar product of the $j$th row of $A$ with the $i$th column of $B$.

- The $i, j$th entry of $B^{\mathsf{T}}A^{\mathsf{T}}$ is the scalar product of the $i$th row of $B^{\mathsf{T}}$ with the $j$th column of $A^{\mathsf{T}}$. But the $i$th row of $B^{\mathsf{T}}$ is the $i$th column of $B$ and the $j$th column of $A^{\mathsf{T}}$ is the $j$th row of $A$. So

  the $i, j$th entry of $B^{\mathsf{T}}A^{\mathsf{T}}$ is the scalar product of the $j$th row of $A$ with the $i$th column of $B$.

Since the two matrices $(AB)^{\mathsf{T}}$ and $B^{\mathsf{T}}A^{\mathsf{T}}$ have the same size and same corresponding entries, they are the same matrix.

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# Examples

What follows are worked examples that use the concepts from this section.

**Example 8.8.** Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -4 & 5 \\ 7 & 6 & -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 4 & -3 \\ 5 & 1 & 9 \\ 1 & 1 & -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -1 & 6 \\ 3 & -2 & 5 \\ 1 & 0 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 10 & -4 \\ 5 & 2 \\ 8 & -1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 \\ 4 & -3 \\ 5 & -1 \end{bmatrix} \qquad \text{and} \qquad F = \begin{bmatrix} -2 & 1 & 5 \\ 6 & 3 & -8 \\ 1 & 0 & -1 \\ 7 & 0 & -5 \end{bmatrix}.$$

Determine the results of the following operations, if defined. If not defined, explain why.

(a) $AF$ (b) $A(BC)$ (c) $(BC)A$

(d) $(B+C)D$ (e) $D^{\mathsf{T}}E$ (f) $\left(A^{\mathsf{T}}+F\right)^{\mathsf{T}}$

**Example Solution.**

(a) Since $A$ is a $3 \times 4$ matrix and $F$ is a $4 \times 3$ matrix, the number of columns of $A$ equals the number of rows of $F$ and the matrix produce $AF$ is defined. Recall that if $F = [\mathbf{f}_1\ \mathbf{f}_2\ \mathbf{f}_3]$, where $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ are the columns of $F$, then $AF = [A\mathbf{f}_1\ A\mathbf{f}_2\ A\mathbf{f}_3]$. Recall also that $A\mathbf{f}_1$ is the linear combination of the columns of $A$ with weights from $\mathbf{f}_1$, so

$$\begin{aligned} A\mathbf{f}_1 &= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -4 & 5 \\ 7 & 6 & -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \\ 1 \\ 7 \end{bmatrix} \\ &= (-2)\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} + (6)\begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix} + (7)\begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -2+12+0+7 \\ -6+0-4+35 \\ -14+36-1+0 \end{bmatrix} \\ &= \begin{bmatrix} 17 \\ 25 \\ 21 \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
A\mathbf{f}_2 &= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -4 & 5 \\ 7 & 6 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} \\
&= (1) \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} + (3) \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} + (0) \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix} + (0) \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1+6+0+0 \\ 3+0+0+0 \\ 7+18+0+0 \end{bmatrix} \\
&= \begin{bmatrix} 7 \\ 3 \\ 25 \end{bmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
A\mathbf{f}_3 &= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -4 & 5 \\ 7 & 6 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \\ -1 \\ -5 \end{bmatrix} \\
&= (5) \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} - (8) \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} - (1) \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix} - (5) \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 5-16-0-5 \\ 15-0+4-25 \\ 35-48+1-0 \end{bmatrix} \\
&= \begin{bmatrix} -16 \\ -6 \\ -12 \end{bmatrix}.
\end{aligned}
$$

So $AF = \begin{bmatrix} 17 & 7 & -16 \\ 25 & 3 & -6 \\ 21 & 25 & -12 \end{bmatrix}$.

Alternatively, if $A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{bmatrix}$, then the matrix product $AF$ is the matrix whose $ij$ entry is $\mathbf{a}_i \cdot \mathbf{f}_j$. Using this method we have

$$
AF = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{f}_1 & \mathbf{a}_1 \cdot \mathbf{f}_2 & \mathbf{a}_1 \cdot \mathbf{f}_3 \\ \mathbf{a}_2 \cdot \mathbf{f}_1 & \mathbf{a}_2 \cdot \mathbf{f}_2 & \mathbf{a}_2 \cdot \mathbf{f}_3 \\ \mathbf{a}_3 \cdot \mathbf{f}_1 & \mathbf{a}_3 \cdot \mathbf{f}_2 & \mathbf{a}_3 \cdot \mathbf{f}_3 \end{bmatrix}.
$$

Now

$$\begin{aligned}
\mathbf{a}_1 \cdot \mathbf{f}_1 &= (1)(-2) + (2)(6) + (0)(1) + (1)(7) = 17 \\
\mathbf{a}_1 \cdot \mathbf{f}_2 &= (1)(1) + (2)(3) + (0)(0) + (1)(0) = 7 \\
\mathbf{a}_1 \cdot \mathbf{f}_3 &= (1)(5) + (2)(-8) + (0)(-1) + (1)(-5) = -16 \\
\mathbf{a}_2 \cdot \mathbf{f}_1 &= (3)(-2) + (0)(6) + (-4)(1) + (5)(7) = 25 \\
\mathbf{a}_2 \cdot \mathbf{f}_2 &= (3)(1) + (0)(3) + (-4)(0) + (5)(0) = 3 \\
\mathbf{a}_2 \cdot \mathbf{f}_3 &= (3)(5) + (0)(-8) + (-4)(-1) + (5)(-5) = -6 \\
\mathbf{a}_3 \cdot \mathbf{f}_1 &= (7)(-2) + (6)(6) + (-1)(1) + (0)(7) = 21 \\
\mathbf{a}_3 \cdot \mathbf{f}_2 &= (7)(1) + (6)(3) + (-1)(0) + (0)(0) = 25 \\
\mathbf{a}_3 \cdot \mathbf{f}_3 &= (7)(5) + (6)(-8) + (-1)(-1) + (0)(-5) = -12,
\end{aligned}$$

so $AF = \begin{bmatrix} 17 & 7 & -16 \\ 25 & 3 & -6 \\ 21 & 25 & -12 \end{bmatrix}$.

(b) Since $BC$ is a $3 \times 3$ matrix but $A$ is $3 \times 4$, the number of columns of $A$ is not equal to the number of rows of $BC$. We conclude that $A(BC)$ is not defined.

(c) Since $BC$ is a $3 \times 3$ matrix and $A$ is $3 \times 4$, the number of columns of $BC$ is equal to the number of rows of $A$. Thus, the quantity $(BC)A$ is defined. First we calculate $BC$ using the dot product of the rows of $B$ with the columns of $C$. Letting $B = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}$ and $C = [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3]$, where $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are the rows of $B$ and $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are the columns of $C$, we have

$$BC = \begin{bmatrix} \mathbf{b}_1 \cdot \mathbf{c}_1 & \mathbf{b}_1 \cdot \mathbf{c}_2 & \mathbf{b}_1 \cdot \mathbf{c}_3 \\ \mathbf{b}_2 \cdot \mathbf{c}_1 & \mathbf{b}_2 \cdot \mathbf{c}_2 & \mathbf{b}_2 \cdot \mathbf{c}_3 \\ \mathbf{b}_3 \cdot \mathbf{c}_1 & \mathbf{b}_3 \cdot \mathbf{c}_2 & \mathbf{b}_3 \cdot \mathbf{c}_3 \end{bmatrix}.$$

Now

$$\begin{aligned}
\mathbf{b}_1 \cdot \mathbf{c}_1 &= (-2)(0) + (4)(3) + (-3)(1) = 9 \\
\mathbf{b}_1 \cdot \mathbf{c}_1 &= (-2)(-1) + (4)(-2) + (-3)(0) = -6 \\
\mathbf{b}_1 \cdot \mathbf{c}_1 &= (-2)(6) + (4)(5) + (-3)(4) = -4 \\
\mathbf{b}_1 \cdot \mathbf{c}_1 &= (5)(0) + (1)(3) + (9)(1) = 12 \\
\mathbf{b}_1 \cdot \mathbf{c}_1 &= (5)(-1) + (1)(-2) + (9)(0) = -7 \\
\mathbf{b}_1 \cdot \mathbf{c}_1 &= (5)(6) + (1)(5) + (9)(4) = 71 \\
\mathbf{b}_1 \cdot \mathbf{c}_1 &= (1)(0) + (1)(3) + (-2)(1) = 1 \\
\mathbf{b}_1 \cdot \mathbf{c}_1 &= (1)(-1) + (1)(-2) + (-2)(0) = -3 \\
\mathbf{b}_1 \cdot \mathbf{c}_1 &= (1)(6) + (1)(5) + (-2)(4) = 3,
\end{aligned}$$

so $BC = \begin{bmatrix} 9 & -6 & -4 \\ 12 & -7 & 71 \\ 1 & -3 & 3 \end{bmatrix}$. If $BC = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$ and $A = [\mathbf{s}_1\ \mathbf{s}_2\ \mathbf{s}_3\ \mathbf{s}_4]$, where $\mathbf{r}_1$, $\mathbf{r}_2$, and

$\mathbf{r}_3$ are the rows of $BC$ and $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, and $\mathbf{s}_4$ are the columns of $A$, then

$$(BC)A = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{s}_1 & \mathbf{r}_1 \cdot \mathbf{s}_2 & \mathbf{r}_1 \cdot \mathbf{r}_3 & \mathbf{r}_1 \cdot \mathbf{s}_4 \\ \mathbf{r}_2 \cdot \mathbf{s}_1 & \mathbf{r}_2 \cdot \mathbf{s}_2 & \mathbf{r}_2 \cdot \mathbf{s}_3 & \mathbf{r}_2 \cdot \mathbf{s}_4 \\ \mathbf{r}_3 \cdot \mathbf{s}_1 & \mathbf{r}_3 \cdot \mathbf{s}_2 & \mathbf{r}_3 \cdot \mathbf{s}_3 & \mathbf{r}_3 \cdot \mathbf{s}_4 \end{bmatrix}.$$

Now

$$\begin{aligned}
\mathbf{r}_1 \cdot \mathbf{s}_1 &= (9)(1) + (-6)(3) + (-4)(7) = -37 \\
\mathbf{r}_1 \cdot \mathbf{s}_2 &= (9)(2) + (-6)(0) + (-4)(6) = -6 \\
\mathbf{r}_1 \cdot \mathbf{s}_3 &= (9)(0) + (-6)(-4) + (-4)(-1) = 28 \\
\mathbf{r}_1 \cdot \mathbf{s}_4 &= (9)(1) + (-6)(5) + (-4)(0) = -21 \\
\mathbf{r}_2 \cdot \mathbf{s}_1 &= (12)(1) + (-7)(3) + (71)(7) = 488 \\
\mathbf{r}_2 \cdot \mathbf{s}_2 &= (12)(2) + (-7)(0) + (71)(6) = 450 \\
\mathbf{r}_2 \cdot \mathbf{s}_3 &= (12)(0) + (-7)(-4) + (71)(-1) = -43 \\
\mathbf{r}_2 \cdot \mathbf{s}_4 &= (12)(1) + (-7)(5) + (71)(0) = -23 \\
\mathbf{r}_3 \cdot \mathbf{s}_1 &= (1)(1) + (-3)(3) + (3)(7) = 13 \\
\mathbf{r}_3 \cdot \mathbf{s}_2 &= (1)(2) + (-3)(0) + (3)(6) = 20 \\
\mathbf{r}_3 \cdot \mathbf{s}_3 &= (1)(0) + (-3)(-4) + (3)(-1) = 9 \\
\mathbf{r}_3 \cdot \mathbf{s}_4 &= (1)(1) + (-3)(5) + (3)(0) = -14,
\end{aligned}$$

so $(BC)A = \begin{bmatrix} -37 & -6 & 28 & -21 \\ 488 & 450 & -43 & -23 \\ 13 & 20 & 9 & -14 \end{bmatrix}$.

(d) Since $B$ and $C$ are both $3 \times 3$ matrices, their sum is defined and is a $3 \times 3$ matrix. Because $D$ is $3 \times 2$ matrix, the number of columns of $B + C$ is equal to the number of rows of $D$. Thus, the quantity $(B + C)D$ is defined and, using the row-column method of matrix multiplication as earlier,

$$\begin{aligned}
(B+C)D &= \left( \begin{bmatrix} -2 & 4 & -3 \\ 5 & 1 & 9 \\ 1 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 6 \\ 3 & -2 & 5 \\ 1 & 0 & 4 \end{bmatrix} \right) \begin{bmatrix} 10 & -4 \\ 5 & 2 \\ 8 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -2+0 & 4-1 & -3+6 \\ 5+3 & 1-2 & 9+5 \\ 1+1 & 1+0 & -2+4 \end{bmatrix} \begin{bmatrix} 10 & -4 \\ 5 & 2 \\ 8 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 3 & 3 \\ 8 & -1 & 14 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 10 & -4 \\ 5 & 2 \\ 8 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 19 & 11 \\ 187 & -48 \\ 41 & -8 \end{bmatrix}.
\end{aligned}$$

(e) Since $D^{\mathsf{T}}$ is a $2 \times 3$ matrix and $E$ is $3 \times 2$, the number of columns of $D^{\mathsf{T}}$ is equal to the

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number of rows of $E$. Thus, $D^{\mathsf{T}}E$ is defined and

$$D^{\mathsf{T}}E = \begin{bmatrix} 10 & -4 \\ 5 & 2 \\ 8 & -1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 1 & 0 \\ 4 & -3 \\ 5 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 10 & 5 & 8 \\ -4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & -3 \\ 5 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 70 & -23 \\ -1 & -5 \end{bmatrix}.$$

(f) The fact that $A$ is a $3 \times 4$ matrix means that $A^{\mathsf{T}}$ is a $4 \times 3$ matrix. Since $F$ is also a $4 \times 3$ matrix, the sum $A^{\mathsf{T}} + F$ is defined. The transpose of any matrix is also defined, so $\left(A^{\mathsf{T}} + F\right)^{\mathsf{T}}$ is defined and

$$\left(A^{\mathsf{T}} + F\right)^{\mathsf{T}} = \left( \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 0 & -4 & 5 \\ 7 & 6 & -1 & 0 \end{bmatrix}^{\mathsf{T}} + \begin{bmatrix} -2 & 1 & 5 \\ 6 & 3 & -8 \\ 1 & 0 & -1 \\ 7 & 0 & -5 \end{bmatrix} \right)^{\mathsf{T}}$$
$$= \left( \begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & 6 \\ 0 & -4 & -1 \\ 1 & 5 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 & 5 \\ 6 & 3 & -8 \\ 1 & 0 & -1 \\ 7 & 0 & -5 \end{bmatrix} \right)^{\mathsf{T}}$$
$$= \left( \begin{bmatrix} 1-2 & 3+1 & 7+5 \\ 2+6 & 0+3 & 6-8 \\ 0+1 & -4+0 & -1-1 \\ 1+7 & 5+0 & 0-5 \end{bmatrix} \right)^{\mathsf{T}}$$
$$= \left( \begin{bmatrix} -1 & 4 & 12 \\ 8 & 3 & -2 \\ 1 & -4 & -2 \\ 8 & 5 & -5 \end{bmatrix} \right)^{\mathsf{T}}$$
$$= \begin{bmatrix} -1 & 8 & 1 & 8 \\ 4 & 3 & -4 & 5 \\ 12 & -2 & -2 & -5 \end{bmatrix}.$$

**Example 8.9.** Let $A = \begin{bmatrix} 2 & -1 \\ 7 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ -3 & 5 \end{bmatrix}$.

(a) Determine the matrix sum $A + B$. Then use this sum to calculate $(A + B)^2$.

(b) Now calculate $(A + B)^2$ in a different way. Use the fact that matrix multiplication distributes over matrix addition to expand (like foiling) $(A + B)^2$ into a sum of matrix products. The calculate each summand and add to find $(A + B)^2$. You should obtain the same result as part (a). If not, what could be wrong?

**Example Solution.**

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(a) Adding corresponding terms shows that $A + B = \begin{bmatrix} 6 & 5 \\ 4 & 3 \end{bmatrix}$. Squaring this sum yields the result $(A+B)^2 = \begin{bmatrix} 56 & 45 \\ 36 & 29 \end{bmatrix}$.

(b) Expanding $(A+B)^2$ (remember that matrix multiplication is not commutative) gives us

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B) \\ &= A^2 + AB + BA + B^2 \\ &= \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 11 & 7 \\ 34 & 32 \end{bmatrix} + \begin{bmatrix} 50 & -16 \\ 29 & -7 \end{bmatrix} + \begin{bmatrix} -2 & 54 \\ -27 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 56 & 45 \\ 36 & 29 \end{bmatrix}\end{aligned}$$

just as in part (a). If instead you obtained the matrix $\begin{bmatrix} 17 & 68 \\ 41 & 68 \end{bmatrix}$ you likely made the mistake of equating $(A+B)^2$ with $A^2 + 2AB + B^2$. These two matrices are not equal in general, because we cannot say that $AB$ is equal to $BA$.

## Summary

In this section we defined a matrix sum, scalar multiples of matrices, the matrix product, and the transpose of a matrix.

- The sum of two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the $m \times n$ matrix $A + B$ whose $i, j$th entry is $a_{ij} + b_{ij}$.
- If $A = [a_{ij}]$ is an $m \times n$ matrix, the scalar multiple $kA$ of $A$ by the scalar $k$ is the $m \times n$ matrix whose $i, j$th entry is $ka_{ij}$.
- If $A$ is a $k \times m$ matrix and $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$ is an $m \times n$ matrix, then the matrix product $AB$ of the matrices $A$ and $B$ is the $k \times n$ matrix

  $$[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_n].$$

  The matrix product is defined in this way so that the matrix of a composite $S \circ T$ of linear transformations is the product of matrices of $S$ and $T$.
- An alternate way of calculating the product of an $k \times m$ matrix $A$ with rows $\mathbf{r}_1$, $\mathbf{r}_2$, ..., $\mathbf{r}_k$ and an $m \times n$ matrix $B$ with columns $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_n$ is that the product $AB$ is the $k \times n$ matrix whose $i, j$th entry is $\mathbf{r}_i \cdot \mathbf{b}_j$.
- Matrix multiplication does not behave as the standard multiplication on real numbers. For example, we can have a product of two non-zero matrices equal to the zero matrix and there is no cancellation law for matrix multiplication.
- The transpose of an $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix $A^{\mathsf{T}}$ whose $i, j$th entry is $a_{ji}$.

## Exercises

(1) Calculate $AB$ for each of the following matrix pairs by hand in two ways.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$

(2) For each of the following $A$ matrices, find all $2 \times 2$ matrices $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which commute with the given $A$. (Two matrices $A$ and $B$ commute with each other if $AB = BA$.)

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (c) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(3) Find all possible, if any, $X$ matrices satisfying each of the following matrix equations.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} X = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$

(4) For each of the following $A$ matrices, compute $A^2 = AA$, $A^3 = AAA$, $A^4$. Use your results to conjecture a formula for $A^m$. Interpret your answer geometrically using the transformation interpretation.

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(5) If $A\mathbf{v} = 2\mathbf{v}$ for unknown $A$ matrix and $\mathbf{v}$ vector, determine an expression for $A^2\mathbf{v}$, $A^3\mathbf{v}$, $\ldots$, $A^m\mathbf{v}$.

(6) If $A\mathbf{v} = 2\mathbf{v}$ and $A\mathbf{u} = 3\mathbf{u}$, find an expression for $A^m(a\mathbf{v} + b\mathbf{u})$ in terms of $\mathbf{v}$ and $\mathbf{u}$.

(7) A matrix $A$ is a **nilpotent** matrix if $A^m = 0$, i.e., $A^m$ is the zero matrix, for some positive integer $m$. Explain why the matrices

$$A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$$

are nilpotent matrices.

(8) Suppose $A$ is an $n \times n$ matrix for which $A^2 = 0$. Show that there is a matrix $B$ for which $(I_n + A)B = I_n$ where $I_n$ is the identity matrix of size $n$.

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(9) Let $A$, $B$, and $C$ be $m \times n$ matrices and let $a$ and $b$ be scalars. Verify Theorem 8.1. That is, show that

(a) $A + B = B + A$

(b) $(A + B) + C = A + (B + C)$

(c) The $m \times n$ matrix 0 whose entries are all 0 has the property that $A + 0 = A$.

(d) The scalar multiple $(-1)A$ of the matrix $A$ has the property that $(-1)A + A = 0$.

(e) $(a + b)A = aA + bA$

(f) $a(A + B) = aA + aB$

(g) $(ab)A = a(bA)$

(h) $1A = A$.

(10) Let $A$, $B$, and $C$ be matrices of the appropriate sizes for all sums and products to be defined and let $a$ be a scalar. Verify the remaining parts of Theorem 8.3. That is, show that

(a) $(AB)C = A(BC)$

(b) $A(B + C) = AB + AC$

(c) There is a square matrix $I_n$ with the property that $AI_n = A$ or $I_nA = A$ for whichever product is defined.

(d) $a(AB) = (aA)B = A(aB)$

(11) Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the appropriate sizes, and let $a$ be a scalar. Verify the remaining parts of Theorem 8.7. That is, show that

(a) $\left(A^{\mathsf{T}}\right)^{\mathsf{T}} = A$

(b) $(A + B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}}$

(c) $(aA)^{\mathsf{T}} = aA^{\mathsf{T}}$

(12) The *matrix exponential* is an important tool in solving differential equations. Recall from calculus that the Taylor series expansion for $e^x$ centered at $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

and that this Taylor series converges to $e^x$ for every real number $x$. We extend this idea to define the matrix exponential $e^A$ for any square matrix $A$ with real entries as

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!}A^n = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

We explore this idea with an example. Let $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

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(a) Calculate $B^2$, $B^3$, $B^4$. Explain why $B^n = \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix}$ for any positive integer $n$.

(b) Show that $I_2 + B + B^2 + B^3 + B^4$ is equal to

$$\begin{bmatrix} 1 + 2 + \frac{2^2}{2} + \frac{2^3}{3!} + \frac{2^4}{4!} & 0 \\ 0 & 1 + (-1) + \frac{(-1)^2}{2} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} \end{bmatrix}.$$

(c) Explain why $e^B = \begin{bmatrix} e^2 & 0 \\ 0 & e^{-1} \end{bmatrix}$.

(13) Show that if $A$ and $B$ are $2 \times 2$ rotation matrices, then $AB$ is also a $2 \times 2$ rotation matrix.

(14) Label each of the following statements as True or False. Provide justification for your response. Throughout, assume that matrices are of the appropriate sizes so that any matrix sums or products are defined.

(a) **True/False** For any three matrices $A, B, C$ with $A \neq 0$, $AB = AC$ implies $B = C$.

(b) **True/False** For any three matrices $A, B, C$ with $A \neq 0$, $AB = CA$ implies $B = C$.

(c) **True/False** If $A^2$ is the zero matrix, then $A$ itself is the zero matrix.

(d) **True/False** If $AB = BA$ for every $n \times n$ matrix $B$, then $A$ is the identity matrix $I_n$.

(e) **True/False** If matrix products $AB$ and $BA$ are both defined, then $A$ and $B$ are both square matrices of the same size.

(f) **True/False** If $\mathbf{x}_1$ is a solution for $A\mathbf{x} = \mathbf{b}_1$ (i.e., that $A\mathbf{x}_1 = \mathbf{b}_1$) and $\mathbf{x}_2$ is a solution for $B\mathbf{x} = \mathbf{b}_2$, then $\mathbf{x}_1 + \mathbf{x}_2$ is a solution for $(A + B)\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_2$.

(g) **True/False** If $B$ is an $m \times n$ matrix with two equal columns, then the matrix $AB$ has two equal columns for every $k \times m$ matrix.

(h) **True/False** If $A^2 = I_2$, then $A = -I_2$ or $A = I_2$.

## Project: Strassen's Algorithm and Partitioned Matrices

Strassen's algorithm is an algorithm for matrix multiplication that can be more efficient than the standard row-column method. To understand this method, we begin with the $2 \times 2$ case which will highlight the essential ideas.

**Project Activity 8.1.** We first work with the $2 \times 2$ case.

(a) Let $A = [a_{ij}] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = [b_{ij}] = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

i. Calculate the matrix product $AB$.

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ii. Rather than using eight multiplications to calculate $AB$, Strassen came up with the idea of using the following seven products:

$$\begin{aligned} h_1 &= (a_{11}+a_{22})(b_{11}+b_{22}) \\ h_2 &= (a_{21}+a_{22})b_{11} \\ h_3 &= a_{11}(b_{12}-b_{22}) \\ h_4 &= a_{22}(b_{21}-b_{11}) \\ h_5 &= (a_{11}+a_{12})b_{22} \\ h_6 &= (a_{21}-a_{11})(b_{11}+b_{12}) \\ h_7 &= (a_{12}-a_{22})(b_{21}+b_{22}). \end{aligned}$$

Calculate $h_1$ through $h_7$ for the given matrices $A$ and $B$. Then calculate the quantities

$$h_1+h_4-h_5+h_7,\ h_3+h_5,\ h_2+h_4,\ \text{ and } h_1+h_3-h_2+h_6.$$

What do you notice?

(b) Now we repeat part (a) in general. Suppose we want to calculate the matrix product $AB$ for arbitrary $2\times 2$ matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Let

$$\begin{aligned} h_1 &= (a_{11}+a_{22})(b_{11}+b_{22}) \\ h_2 &= (a_{21}+a_{22})b_{11} \\ h_3 &= a_{11}(b_{12}-b_{22}) \\ h_4 &= a_{22}(b_{21}-b_{11}) \\ h_5 &= (a_{11}+a_{12})b_{22} \\ h_6 &= (a_{21}-a_{11})(b_{11}+b_{12}) \\ h_7 &= (a_{12}-a_{22})(b_{21}+b_{22}). \end{aligned}$$

Show that

$$AB = \begin{bmatrix} h_1+h_4-h_5+h_7 & h_3+h_5 \\ h_2+h_4 & h_1+h_3-h_2+h_6 \end{bmatrix}.$$

The next step is to understand how Strassen's algorithm can be applied to larger matrices. This involves the idea of partitioned (or block) matrices. Recall that the matrix-matrix product of the $k\times m$ matrix $A$ and the $m\times n$ matrix $B = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_n]$ is defined as

$$AB = [A\mathbf{b}_1\ A\mathbf{b}_2\ \cdots\ A\mathbf{b}_n].$$

In this process, we think of $B$ as being partitioned into $n$ columns. We can expand on this idea to partition both $A$ and $B$ when calculating a matrix-matrix product.

**Project Activity 8.2.** We illustrate the idea of partitioned matrices with an example. Let $A = \begin{bmatrix} 1 & -2 & 3 & -6 & 4 \\ 7 & 5 & 2 & -1 & 0 \\ 3 & -8 & 1 & 0 & 9 \end{bmatrix}$.We can partition $A$ into smaller matrices

$$A = \left[\begin{array}{ccc|cc} 1 & -2 & 3 & -6 & 4 \\ 7 & 5 & 2 & -1 & 0 \\ \hline 3 & -8 & 1 & 0 & 9 \end{array}\right],$$

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which are indicated by the vertical and horizontal lines. As a shorthand, we can describe this partition of $A$ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} = \begin{bmatrix} 1 & -2 & 3 \\ 7 & 5 & 2 \end{bmatrix}$, $A_{12} = \begin{bmatrix} -6 & 4 \\ -1 & 0 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 3 & -8 & 1 \end{bmatrix}$, and $A_{22} = [0\ 9]$. The submatrices $A_{ij}$ are called *blocks*. If $B$ is a matrix such that $AB$ is defined, then $B$ must have five rows. As an example, $AB$ is defined if $B = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 4 & 1 \\ 6 & 5 \\ 4 & 2 \end{bmatrix}$. The partition of $A$ breaks $A$ up into blocks with three and two columns, respectively. So if we partition $B$ into blocks with three and two rows, then we can use the blocks to calculate the matrix product $AB$. For example, partition $B$ as

$$B = \left[\begin{array}{cc} 1 & 3 \\ 2 & 0 \\ 4 & 1 \\ \hline 6 & 5 \\ 4 & 2 \end{array}\right] = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}.$$

Show that

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}.$$

An advantage to using partitioned matrices is that computations with them can be done in parallel, which lessens the time it takes to do the work. In general, we can multiply partitioned matrices as though the submatrices are scalars. That is,

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{im} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{km} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mj} & \cdots & B_{mn} \end{bmatrix} = [P_{ij}],$$

where

$$P_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{im}B_{mj} = \sum_{t=1}^{m} A_{it}B_{tj},$$

provided that all the submatrix products are defined.

Now we can apply Strassen's algorithm to larger matrices using partitions. This method is sometimes referred to as divide and conquer.

**Project Activity 8.3.** Let $A$ and $B$ be two $r \times r$ matrices. If $r$ is not a power of 2, then pad the rows and columns of $A$ and $B$ with zeros to make them of size $2^m \times 2^m$ for some integer $m$. (From a

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practical perspective, we might instead just use unequal block sizes.) Let $n = 2^m$. Partition $A$ and $B$ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where each submatrix is of size $\frac{n}{2} \times \frac{n}{2}$. Now we use the Strassen algorithm just as in the $2 \times 2$ case, treating the submatrices as if they were scalars (with the additional constraints of making sure that the dimensions match up so that products are defined, and ensuring we multiply in the correct order). Letting

$$\begin{aligned}
M_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
M_2 &= (A_{21} + A_{22})B_{11} \\
M_3 &= A_{11}(B_{12} - B_{22}) \\
M_4 &= A_{22}(B_{21} - B_{11}) \\
M_5 &= (A_{11} + A_{12})B_{22} \\
M_6 &= (A_{21} - A_{11})(B_{11} + B_{12}) \\
M_7 &= (A_{12} - A_{22})(B_{21} + B_{22}),
\end{aligned}$$

then the same algebra as in Project Activity 8.1 shows that

$$AB = \begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 + M_3 - M_2 + M_6 \end{bmatrix}.$$

Apply Strassen's algorithm to calculate the matrix product $AB$, where

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 6 \\ 7 & -2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 5 & 3 \\ 2 & -4 & 1 \\ 1 & 6 & 4 \end{bmatrix}.$$

While Strassen's algorithm can be more efficient, it does not always speed up the process. We investigate this in the next activity.

**Project Activity 8.4.** We introduce a little notation to help us describe the efficiency of our calculations. We won't be formal with this notation, rather work with it in an informal way. Big O (the letter "O") notation is used to describe the complexity of an algorithm. Generally speaking, in computer science big O notation can be used to describe the run time of an algorithm, the space used by the algorithm, or the number of computations required. The letter "O" is used because the behavior described is also called the order. Big O measures the asymptotic time of an algorithm, not its exact time. For example, if it takes $6n^2 - n + 8$ steps to complete an algorithm, then we say that the algorithm grows at the order of $n^2$ (we ignore the constants and the smaller power terms, since they become insignificant as $n$ increases) and we describe its growth as $O(n^2)$. To measure the efficiency of an algorithm to determine a matrix product, we will measure the number of operations it takes to calculate the product.

(a) Suppose $A$ and $B$ are $n \times n$ matrices. Explain why the operation of addition (that is, calculating $A + B$) is $O(n^2)$.

(b) Suppose $A$ and $B$ are $n \times n$ matrices. How many multiplications are required to calculate the matrix product $AB$? Explain.

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(c) The standard algorithm for calculating a matrix product of two $n \times n$ matrices requires $n^3$ multiplications and a number of additions. Since additions are much less costly in terms of operations, the standard matrix product is $O(n^3)$. We won't show it here, but using Strassen's algorithm on a product of $2^m \times 2^m$ matrices is $O(n^{\log_2(7)})$, where $n = 2^m$. That means that Strassen's algorithm applied to an $n \times n$ matrix (where $n$ is a power of 2) requires approximately $n^{\log_2(7)}$ multiplications. We use this to analyze situations to determine when Strassen's algorithm is computationally more efficient than the standard algorithm.

i. Suppose $A$ and $B$ are $5 \times 5$ matrices. Determine the number of multiplications required to calculate the matrix product $AB$ using the standard matrix product. Then determine the approximate number of multiplications required to calculate the matrix product $AB$ using Strassen's algorithm. Which is more efficient? (Remember, we can only apply Strassen's algorithm to square matrices whose sizes are powers of 2.)

ii. Repeat part i. with $125 \times 125$ matrices. Which method is more efficient?

As a final note, Strassen's algorithm is approximately $O(n^{2.81})$. As of 2018, the best algorithm for matrix multiplication, developed by Virginia Williams at Stanford University, is approximately $O(n^{2.373})$.[3]

[3] V. V. Williams, Multiplying matrices in $O(n^{2.373})$ time, Stanford University, (2014).

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# Section 9

# Introduction to Eigenvalues and Eigenvectors

**Focus Questions**

*By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.*

- What is an eigenvalue of a matrix?
- What is an eigenvector of a matrix?
- How do we find eigenvectors of a matrix corresponding to an eigenvalue?
- How can the action of a matrix on an eigenvector be visualized?
- Why do we study eigenvalues and eigenvectors?
- What are discrete dynamical systems and how do we analyze the long-term behavior in them?

## Application: The Google PageRank Algorithm

The World Wide Web is a vast collection of information, searchable via search engines. A search engine looks for pages that are of interest to the user. In order to be effective, a search engine needs to be able to identify those pages that are relevant to the search criteria provided by the user. This involves determining the relative importance of different web pages by ranking the results of thousands or millions of pages fitting the search criteria. For Google, the PageRank algorithm is their method and is "the heart of our software" as they say. It is this PageRank algorithm that we will learn about later in this section. Eigenvalues and eigenvectors play an important role in this algorithm.

## Introduction

Given a matrix $A$, for some special non-zero vectors $\mathbf{v}$ the action of $A$ on $\mathbf{v}$ will be same as scalar multiplication, i.e., $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar $\lambda$. Geometrically, this means that the transformation $T$ defined by $T(\mathbf{x}) = A\mathbf{x}$ simply stretches or contracts the vector $\mathbf{v}$ but does not change its direction. Such a nonzero vector is called an *eigenvector* of $A$, while the scalar $\lambda$ is called the corresponding *eigenvalue* of $A$. The eigenvectors of a matrix tell us quite a bit about the transformation the matrix defines.

Eigenvalues and eigenvectors are used in many applications. Social media like Facebook and Google use eigenvalues to determine the influence of individual members on the network (which can affect advertising) or to rank the importance of web pages. Eigenvalues and eigenvectors appear in quantum physics, where atomic and molecular orbitals can be defined by the eigenvectors of a certain operator. They appear in principal component analysis, used to study large data sets, to diagonalize certain matrices and determine the long term behavior of systems as a result, and in the important singular value decomposition of a matrix. Matrices with real entries can have real or complex eigenvalues, and complex eigenvalues reveal a rotation that is encoded in every real matrix with complex eigenvalues which allows us to better understand certain matrix transformations.

**Definition 9.1.** Let $A$ be an $n \times n$ matrix. A non-zero vector $\mathbf{x}$ is an **eigenvector** (or **characteristic vector**) of $A$ if there is a scalar $\lambda$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The scalar $\lambda$ is an **eigenvalue** (or **characteristic value**) of $A$.

For example, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$ because $A\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, which is equal to $3\mathbf{v}$. On the other hand, $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector of $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ because $A\mathbf{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, which is not a multiple of $\mathbf{w}$.

**Preview Activity 9.1.**

(1) For each of the following parts, use the definition of an eigenvector to determine whether the given vector $\mathbf{v}$ is an eigenvector for the given matrix $A$. If it is, determine the corresponding eigenvalue.

(a) $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(2) We now consider how we can find the eigenvectors corresponding to an eigenvalue using the definition. Suppose $A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$. We consider whether we can find eigenvectors

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corresponding to eigenvalues 3, and 5. Effectively, this will help us determine whether 3 and/or 5 are eigenvalues of $A$.

(a) Rewrite the vector equation $A\mathbf{v} = 5\mathbf{v}$ where $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ as a vector equation.

(b) After writing $5\mathbf{v}$ as $5I\mathbf{v}$ where $I$ is the identity matrix, rearrange the variables to turn this vector equation into the homogeneous matrix equation $B\mathbf{v} = \mathbf{0}$ where $B = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$. If possible, find a non-zero (i.e. a non-trivial) solution to $B\mathbf{v} = \mathbf{0}$. Explain what this means about 5 being an eigenvalue of $A$ or not.

(c) Similarly, determine whether the vector equation $A\mathbf{v} = 3\mathbf{v}$ has non-zero solutions. Using your result, determine whether 3 is an eigenvalue of $A$ or not.

## Eigenvalues and Eigenvectors

Eigenvectors are especially useful in understanding the long-term behavior of dynamical systems, an example of which we will see shortly. The long-term behavior of a dynamical system is quite simple when the initial state vector is an eigenvector and this fact helps us analyze the system in general.

To find eigenvectors, we are interested in determining the vectors $\mathbf{x}$ for which $A\mathbf{x}$ has the same direction as $\mathbf{x}$. This will happen when

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar $\lambda$. Of course, $A\mathbf{x} = \lambda\mathbf{x}$ when $\mathbf{x} = \mathbf{0}$ for every $A$ and every $\lambda$, but that is uninteresting. So we really want to consider when there is a non-zero vector $\mathbf{x}$ so that $A\mathbf{x} = \lambda\mathbf{x}$. This prompts the definition of eigenvectors and eigenvalues as in Definition 9.1

In order for a matrix $A$ to have an eigenvector, one condition $A$ must satisfy is that $A$ has to be a square matrix, i.e. an $n \times n$ matrix. We will find that each $n \times n$ matrix has only finitely many eigenvalues.

The terms eigenvalue and eigenvector seem to come from Hilbert, using the German “eigen” (roughly translated as “own”, “proper”, or “characteristic”) to emphasize how eigenvectors and eigenvalues are connected to their matrices. To find the eigenvalues and eigenvectors of an $n \times n$ matrix $A$, we need to find the solutions to the equation

$$A\mathbf{x} = \lambda\mathbf{x}\,.$$

In Preview Activity 9.1, we considered this equation for $A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ and $\lambda = 5$. The homogeneous matrix equation we came up with was

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \mathbf{x} = \mathbf{0}\,.$$

To see the relationship between this homogeneous matrix equation and the eigenvalue-eigenvector

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equation better, let us consider the eigenvector equation using matrix algebra:

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ A\mathbf{x} - \lambda I_n\mathbf{x} &= \mathbf{0} \\ (A - \lambda I_n)\mathbf{x} &= \mathbf{0}, \end{aligned}$$

where $I_n$ is the $n \times n$ identity matrix. Notice that this description matches the homogenous equation matrix example above since we simply subtracted 5 from the diagonal terms of the matrix $A$. Hence, to find eigenvalues, we need to find the values of $\lambda$ so that the homogeneous equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ has non-trivial solutions.

**Activity 9.1.**

(a) Under what conditions on $A - \lambda I_n$ will the matrix equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ have non-trivial solutions? Describe at least two different but equivalent conditions.

(b) The real number 0 is an eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Check that your criteria in the previous part agrees with this result.

(c) Determine if 5 is an eigenvalue of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ using your criterion above.

(d) What are the two eigenvalues of the matrix $A = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$?

Since an eigenvector of $A$ corresponding to eigenvalue $\lambda$ is a non-trivial solution to the homogeneous equation $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$, the eigenvalues $\lambda$ which work are those for which the matrix $A - \lambda I_n$ has linearly dependent columns, or for which the row echelon form of the matrix $A - \lambda I_n$ does not have a pivot in every column. When we need to test if a specific $\lambda$ is an eigenvalue, this method works fine. However, finding which $\lambda$'s will work in general involves row reducing a matrix with $\lambda$'s subtracted on the diagonal algebraically. For certain types of matrices, this method still provides us the eigenvalues quickly. For general matrices though, row reducing algebraically is not efficient. We will later see an algebraic method which uses the determinants to find the eigenvalues.

**Activity 9.2.**

(a) For $\lambda$ to be an eigenvalue of $A$, we noted that $A - \lambda I_n$ must have a non-pivot column. Use this criterion to explain why $A = \begin{bmatrix} -2 & 2 \\ 0 & 4 \end{bmatrix}$ has eigenvalues $\lambda = -2$ and $\lambda = 4$.

(b) Determine the eigenvalues of $A = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) Generalize your results from the above parts in the form of a theorem in the most general $n \times n$ case.

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## Dynamical Systems

One real-life application of eigenvalues and eigenvectors is in analyzing the long-term behavior of *discrete dynamical systems*. A *dynamical system* is a system of variables whose values change with time. In discrete systems, the change is described by defining the values of the variables at time $t+1$ in terms of the values at time $t$. For example, the discrete dynamical system

$$y_{t+1} = y_t + t$$

relates the value of $y$ at time $t+1$ to the value of $y$ at time $t$. This is in contrast with a differential equation[1] such as

$$\frac{dy}{dt} = y + t,$$

which describes the instantaneous rate of change of $y(t)$ in terms of $y$ and $t$.

Discrete dynamical systems can be used in population modeling to provide a simplified model of predator-prey interactions in biology (see Preview Activity 9.2). Other applications include Markov chains (see Exercise 5), age structured population growth models, distillation of a binary ideal mixture of two liquids, cobweb model in economics concerning the interaction of supply and demand for a single good, queuing theory and traffic flow.

Eigenvectors can be used to analyze the long-term behavior of dynamical systems.

**Preview Activity 9.2.**

(1) Consider a discrete dynamical system providing a simplified model of predator-prey interactions in biology, such as the system describing the populations of rabbits and foxes in a certain area.

Suppose, for example, for a specific area the model is given by the following equations:

$$\begin{aligned} r_{k+1} &= 1.14r_k - 0.12f_k \\ f_{k+1} &= 0.08r_k + 0.86f_k \end{aligned} \tag{9.1}$$

where $r_i$ represents the number of rabbits in the area $i$ years after a starting time value, and $f_i$ represents the number of foxes in year $i$. We use $r_0, f_0$ to denote the initial population values.

(a) Suppose $r_k = 300$ and $f_k = 100$ for one year. Calculate rabbit and fox population values for the next year. In other words, find $r_{k+1}, f_{k+1}$ values.

(b) Consider the coefficients of the variables $r_k, f_k$ in the the system of equations in (9.1). Can you explain the reasoning behind the signs and absolute sizes of the coefficients from the story that it models?

(c) Let $\mathbf{x}_k = \begin{bmatrix} r_k \\ f_k \end{bmatrix}$. The vector $\mathbf{x}_k$ is called the *state vector* of the system at time $k$, because it describes the state of the whole system at time $k$. We can rewrite the system of equations in (9.1) as a matrix-vector equation in terms of the state vectors at time $k$ and $k+1$. More specifically, the equation will be of the form

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \tag{9.2}$$

[1] A differential equation is an equation that involves one or more derivatives of a function.

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where $A = \begin{bmatrix} 1.14 & -0.12 \\ 0.08 & 0.86 \end{bmatrix}$. We will call this matrix the *transition matrix* of the system. Check that $A \begin{bmatrix} 300 \\ 100 \end{bmatrix}$ gives us the population values you calculated in the first part above.

(d) The transition matrix will help us simplify calculations of the population values. Note that equation (9.2) implies that $\mathbf{x}_1 = A\mathbf{x}_0$, $\mathbf{x}_2 = A\mathbf{x}_1$, $\mathbf{x}_3 = A\mathbf{x}_2$, and so on. This is a recursive method to find the population values as each year's population values depend on the previous year's population values. Using this approach, calculate $\mathbf{x}_k$ for $k$ values up to 5 corresponding to the following three different initial rabbit-fox population values (all in thousands):

$$r_0 = 300\ ,\ f_0 = 100$$

$$r_0 = 100\ ,\ f_0 = 200$$

$$r_0 = 1200\ ,\ f_0 = 750$$

Can you guess the long-term behavior of the population values in each case? Are they both increasing? Decreasing? One increasing, one decreasing? How do the rabbit and fox populations compare?

A dynamical system is a system of variables whose values change with time. In Preview Activity 9.2, we considered the discrete dynamical system modeling the rabbit and fox population in an area, which is an example of a predator-prey system. The system was given by the equations from (9.1), where $r_i$ represented the number of rabbits in the area $i$ years after a starting time value, and $f_i$ represented the number of foxes in year $i$. In this notation, $r_0, f_0$ corresponded to the initial population values.

As we saw in Preview Activity 9.2, if we define the state vector as $\mathbf{x}_k = \begin{bmatrix} r_k \\ f_k \end{bmatrix}$, the system of equations representing the dynamical system can be expressed as

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \tag{9.3}$$

where $A = \begin{bmatrix} 1.14 & -0.12 \\ 0.08 & 0.86 \end{bmatrix}$ represents the transition matrix. Note that equation (9.3) encodes infinitely many equations including $\mathbf{x}_1 = A\mathbf{x}_0$, $\mathbf{x}_2 = A\mathbf{x}_1$, $\mathbf{x}_3 = A\mathbf{x}_2$, and so on. This is a recursive formula for the population values as each year's population values are expressed in terms of the previous year's population values. If we want to calculate $\mathbf{x}_{10}$, this formula requires first finding the population values for years 1-9. However, we can obtain a non-recursive formula using matrix algebra. If we substitute $\mathbf{x}_1 = A\mathbf{x}_0$ into $\mathbf{x}_2 = A\mathbf{x}_1$ and simplify, we find that

$$\mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0\ .$$

Similarly, substituting $\mathbf{x}_2 = A^2\mathbf{x}_1$ into the formula for $\mathbf{x}_3$ gives

$$\mathbf{x}_3 = A\mathbf{x}_2 = A(A^2\mathbf{x}_0) = A^3\mathbf{x}_0\ .$$

This process can be continued inductively to show that

$$\mathbf{x}_k = A^k\mathbf{x}_0 \tag{9.4}$$

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for every $k$ value. So to find the population values at any year $k$, we only need to know the initial state vector $\mathbf{x}_0$.

**Activity 9.3.** In this activity the matrix $A$ is the transition matrix for the rabbit and fox population model,

$$A = \begin{bmatrix} 1.14 & -0.12 \\ 0.08 & 0.86 \end{bmatrix}.$$

(a) Suppose that the initial state vector $\mathbf{x}_0$ is an eigenvector of $A$ corresponding to eigenvalue $\lambda$. In this case, explain why $\mathbf{x}_1 = \lambda\mathbf{x}_0$ and $\mathbf{x}_2 = \lambda^2\mathbf{x}_0$. Find the formula for $\mathbf{x}_k$ in terms of $\lambda, k$ and $\mathbf{x}_0$ by applying equation (9.3) iteratively.

(b) The initial state vector $\mathbf{x}_0 = \begin{bmatrix} 300 \\ 100 \end{bmatrix}$ is an eigenvector of $A$. Find the corresponding eigenvalue and, using your formula from (a) for $\mathbf{x}_k$ in terms of $\lambda, k$ and $\mathbf{x}_0$, find the state vector $\mathbf{x}_k$ in this case.

(c) The initial state vector $\mathbf{x}_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ is an eigenvector of $A$. Find the corresponding eigenvalue and, using your formula from (a) for $\mathbf{x}_k$ in terms of $\lambda, k$ and $\mathbf{x}_0$, find the state vector $\mathbf{x}_k$ in this case.

(d) Consider now an initial state vector of the form $\mathbf{x}_0 = a\mathbf{v}_0 + b\mathbf{w}_0$ where $a, b$ are constants, $\mathbf{v}_0$ is an eigenvector corresponding to eigenvalue $\lambda_1$ and $\mathbf{w}_0$ corresponding to eigenvalue $\lambda_2$ ($\mathbf{v}_0$ and $\mathbf{w}_0$ are not necessarily the eigenvectors from parts (b) and (c)). Use matrix algebra and equation (9.4) to explain why $\mathbf{x}_k = a\lambda_1^k\mathbf{v}_0 + b\lambda_2^k\mathbf{w}_0$.

(e) Express the initial state vector $\mathbf{x}_0 = \begin{bmatrix} 1200 \\ 750 \end{bmatrix}$ as a linear combination of the eigenvectors $\mathbf{v}_0 = \begin{bmatrix} 300 \\ 100 \end{bmatrix}, \mathbf{w}_0 = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$ and use your result from the previous part to find a formula for $\mathbf{x}_k$. What happens to the population values as $k \to \infty$?

As you discovered in Activity 9.3, we can use linearly independent eigenvectors of the transition matrix to find a closed formula for the state vector of a dynamical system, as long as the initial state vector can be expressed as a linear combination of the eigenvectors.

## Examples

What follows are worked examples that use the concepts from this section.

**Example 9.2.** Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

(a) Find all of the eigenvalues of $A$.

(b) Find a corresponding eigenvector for each eigenvalue found in part (a).

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**Example Solution.**

(a) Recall that a scalar $\lambda$ is a eigenvalue for $A$ if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ or $(A - \lambda I_2)\mathbf{x} = \mathbf{0}$. For this matrix $A$, we have

$$A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}.$$

To solve the homogeneous system $(A - \lambda I_2)\mathbf{x} = \mathbf{0}$, we row reduce $A - \lambda I_2$. To do this, we first interchange rows to get the following matrix that is row equivalent to $A - \lambda I_2$ (we do this to ensure that we have a nonzero entry in the first row and column)

$$\begin{bmatrix} 2 & 4-\lambda \\ 1-\lambda & 2 \end{bmatrix}.$$

Next we replace row two with $\frac{1}{2}(1-\lambda)$ times row one minus row two to obtain the row equivalent matrix

$$\begin{bmatrix} 2 & 4-\lambda \\ 0 & \frac{1}{2}(4-\lambda)(1-\lambda) - 2 \end{bmatrix}.$$

There will be a nontrivial solution to $(A - \lambda I_2)\mathbf{x} = \mathbf{0}$ if there is a row of zeros in this row echelon form. Thus, we look for values of $\lambda$ that make

$$\frac{1}{2}(4-\lambda)(1-\lambda) - 2 = 0.$$

Applying a little algebra shows that

$$\begin{aligned} \frac{1}{2}(4-\lambda)(1-\lambda) - 2 &= 0 \\ (4-\lambda)(1-\lambda) - 4 &= 0 \\ \lambda^2 - 5\lambda &= 0 \\ \lambda(\lambda - 5) &= 0. \end{aligned}$$

So the eigenvalues of $A$ are $\lambda = 0$ and $\lambda = 5$.

(b) Recall that an eigenvector for the eigenvalue $\lambda$ is a nonzero vector $\mathbf{x}$ such that $(A - \lambda I_2)\mathbf{x} = \mathbf{0}$. We consider each eigenvalue in turn.

- When $\lambda = 0$,

  $$A - 0I_2 = A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

  Technology shows that the reduced row echelon form of $A$ is

  $$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

  If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{0}$ implies that $x_2$ is free and $x_1 = -2x_2$. Choosing $x_2 = 1$ gives us the eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. As a check, note that

  $$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

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- When $\lambda = 5$,
$$A - 5I_2 = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}.$$
Technology shows that the reduced row echelon form of $A - 5I_2$ is
$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}.$$
If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $(A - 5I_2)\mathbf{x} = \mathbf{0}$ implies that $x_2$ is free and $x_1 = \frac{1}{2}x_2$. Choosing $x_2 = 2$ gives us the eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. As a check, note that
$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

**Example 9.3.** Accurately predicting the weather has long been an important task. Meteorologists use science, mathematics, and technology to construct models that help us understand weather patterns. These models are very sophisticated, but we will consider only a simple model. Suppose, for example, we want to learn something about whether it will be wet or dry in Grand Rapids, Michigan. To do this, we might begin by collecting some data about weather conditions in Grand Rapids and then use that to make predictions. Information taken over the course of 2017 from the National Weather Service Climate Data shows that if it was dry (meaning no measurable precipitation, either rain or snow) on a given day in Grand Rapids, it would be dry the next day with a probability of 64% and wet with a probability of 36%. Similarly, if it was wet on a given day it would be dry the next day with a probability of 47% and dry with a probability of 53%. Assuming that this pattern is one that continues in the long run, we can develop a mathematical model to make predictions about the weather.

This data tells us how the weather transitions from one day to the next, and we can succinctly represent this data in a *transition matrix*:
$$T = \begin{bmatrix} 0.64 & 0.47 \\ 0.36 & 0.53 \end{bmatrix}. \tag{9.5}$$
Whether it is dry or wet on a given day is called the *state* of that day. So our transition matrix tells us about the transition between states. Notice that if $T = [t_{ij}]$, then the probability of moving from state $j$ to state $i$ is given by $t_{ij}$. We can represent a state by a vector: the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represents the dry state and the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represents the wet state.

(a) Calculate $T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Interpret the meaning of this output.

(b) Calculate $T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Interpret the meaning of this output.

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(c) Calculate $T \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$. Interpret the meaning of this output.

(d) We can use the transition matrix to build a chain of probability vectors. We begin with an initial state, say it is dry on a given day. This initial state is represented by the initial state vector $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The probabilities that it will be dry or wet the following day are given by the vector

$$\mathbf{x}_1 = T\mathbf{x}_0 = \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix}.$$

This output vector tells us that the next day will be dry with a 64% probability and wet with a 36% probability.

For each $k \geq 1$, we let

$$\mathbf{x}_k = T\mathbf{x}_{k-1}. \tag{9.6}$$

Thus we create a sequence of vectors that tell us the probabilities of it being dry or wet on subsequent days. The vector $\mathbf{x}_k$ is called the *state vector* of the system at time $k$, because it describes the state of the whole system at time $k$. We can rewrite the system of equations in (9.1) as a matrix-vector equation in terms of the state vectors at time $k$ and $k + 1$. More specifically, the equation will be of the form

$$\mathbf{x}_{k+1} = T\mathbf{x}_k \tag{9.7}$$

for $k \geq 0$.

i. Starting with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, use appropriate technology to calculate $\mathbf{x}_k$ for $k$ values up to 10. Round to three decimal places. What do you notice about the entries?

ii. What does the result of the previous part tell us about eigenvalues of $T$? Explain.

iii. Rewrite $T$ as

$$T = \begin{bmatrix} \frac{64}{100} & \frac{47}{100} \\ \frac{36}{100} & \frac{53}{100} \end{bmatrix}.$$

We do this so we can use exact arithmetic. Let $\mathbf{x} = \begin{bmatrix} \frac{47}{83} \\ \frac{36}{83} \end{bmatrix}$. What is $T\mathbf{x}$? (Use exact arithmetic, no decimals.) Explain how $\mathbf{x}$ is related to the previous two parts of this problem. What does the vector $\mathbf{x}$ tells us about weather in Grand Rapids?

**Example Solution.**

(a) Here we have

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.47 \\ 0.36 & 0.53 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.64 \\ 0.36 \end{bmatrix}.$$

This output tells us the different probabilities of whether it will be dry or wet the day following a dry day.

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(b) Here we have

$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.47 \\ 0.36 & 0.53 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.47 \\ 0.53 \end{bmatrix}.$$

This output tells us the different probabilities of whether it will be dry or wet the day following a wet day.

(c) Here we have

$$T\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.47 \\ 0.36 & 0.53 \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \approx \begin{bmatrix} 0.52 \\ 0.48 \end{bmatrix}.$$

This output tells us there is a 52% chance of it being dry and a 48% chance of it being wet following a day when there is a 30% chance of it being dry and a 70% chance of it being wet.

(d) i. Technology shows that

$$\begin{aligned} \mathbf{x}_1 &= \begin{bmatrix} 0.640 \\ 0.360 \end{bmatrix} & \mathbf{x}_2 &= \begin{bmatrix} 0.579 \\ 0.421 \end{bmatrix} \\ \mathbf{x}_3 &= \begin{bmatrix} 0.568 \\ 0.432 \end{bmatrix} & \mathbf{x}_4 &= \begin{bmatrix} 0.567 \\ 0.433 \end{bmatrix} \\ \mathbf{x}_5 &= \begin{bmatrix} 0.566 \\ 0.434 \end{bmatrix} & \mathbf{x}_6 &= \begin{bmatrix} 0.566 \\ 0.434 \end{bmatrix} \\ \mathbf{x}_7 &= \begin{bmatrix} 0.566 \\ 0.434 \end{bmatrix} & \mathbf{x}_8 &= \begin{bmatrix} 0.566 \\ 0.434 \end{bmatrix} \\ \mathbf{x}_9 &= \begin{bmatrix} 0.566 \\ 0.434 \end{bmatrix} & \mathbf{x}_{10} &= \begin{bmatrix} 0.566 \\ 0.434 \end{bmatrix}. \end{aligned}$$

We can see that our vectors $\mathbf{x}_k$ are essentially the same as we let $k$ increase.

ii. Since our sequence seems to be converging to a vector $\mathbf{x}$ satisfying $T\mathbf{x} = \mathbf{x}$, we conclude that 1 is an eigenvalue of $T$.

iii. A matrix vector multiplication shows that

$$T\mathbf{x} = \begin{bmatrix} \frac{64}{100} & \frac{47}{100} \\ \frac{36}{100} & \frac{53}{100} \end{bmatrix}\begin{bmatrix} \frac{47}{83} \\ \frac{36}{83} \end{bmatrix} = \begin{bmatrix} \frac{47}{83} \\ \frac{36}{83} \end{bmatrix}.$$

In other words, $\mathbf{x}$ is an eigenvector for $T$ with eigenvalue 1. Notice that

$$\frac{47}{83} \approx 0.566 \text{ and } \frac{36}{83} \approx 0.434,$$

so these fractions give the same results we obtained with our sequence of vectors $\mathbf{x}_k$. These vectors provide a steady-state vector for Grand Rapids weather. In other words, if there is a $56.6\%$ chance of it being dry on a given day in Grand Rapids, then there is a $56.6\%$ chance it will be dry again the next day.

This is an example of a Markov process. Markov processes (named after Andrei Andreevich Markov) are widely used to model phenomena in biology, chemistry, business, physics, engineering, the social sciences, and much more. More specifically,

**Definition 9.4.** A **Markov process** is a process in which the probability of the system being in a given state depends only on the previous state.

If $\mathbf{x}_0$ is a vector which represents the initial state of a Markov process, then there is a matrix $T$ (the *transition matrix*) such that the state of the system after one iteration is given by the vector $T\mathbf{x}_0$. This produces a chain of state vectors $T\mathbf{x}_0$, $T^2\mathbf{x}_0$, $T^3\mathbf{x}_0$, etc., where the state of the system after $n$ iterations is given by $T^n\mathbf{x}_0$. Such a chain of vectors is called a *Markov chain*. A Markov process is characterized by two properties:

- the total number of observations remains fixed (this is reflected in the fact that the sum of the entries in each column of the matrix $T$ is 1), and
- no observation is lost (this means the entries in the matrix $T$ cannot be negative).

## Summary

We learned about eigenvalues and eigenvectors of a matrix in this section.

- A scalar $\lambda$ is an eigenvalue (or characteristic value) of a square matrix $A$ if there is a non-zero vector $\mathbf{x}$ so that $A\mathbf{x} = \lambda\mathbf{x}$.
- A non-zero vector $\mathbf{x}$ is an eigenvector (or characteristic vector) of a square matrix $A$ if there is a scalar $\lambda$ so that $A\mathbf{x} = \lambda\mathbf{x}$.
- To find the eigenvectors of an $n \times n$ matrix $A$ corresponding to an eigenvalue $\lambda$, we determine the non-trivial solutions to $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ where $I_n$ is the $n \times n$ identity matrix.
- We study eigenvectors and eigenvalues because the eigenvectors tell us quite a bit about the transformation corresponding to the matrix. These eigenvectors arise in many applications in physics, chemistry, statistics, economics, biology, sociology and other areas, and help understand the long-term behavior of dynamical systems.
- A dynamical system is a system of variables whose values change with time. In linear dynamical systems, the change in the state vector from one time period to the next is expressed by matrix multiplication by the transition matrix $A$. The eigenvectors of $A$ provide us a simple method to express the state vector at any given time period in terms of the initial state vector. Specifically, if the initial state vector is $\mathbf{x}_0 = a\mathbf{v}_0 + b\mathbf{w}_0$ where $\mathbf{v}_0, \mathbf{w}_0$ are eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2$, we have
$$\mathbf{x}_k = A^k\mathbf{x}_0 = \lambda_1^k a\mathbf{v}_0 + \lambda_2^k b\mathbf{w}_0 \,.$$

## Exercises

(1) For each of the following matrix-vector pairs, determine whether the given vector is an eigenvector of the matrix.

(a) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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(b) $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(2) For each of the following matrix-eigenvalue pairs, determine an eigenvector of $A$ for the given eigenvalue.

(a) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, \lambda = 3$ (b) $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}, \lambda = 3$

(c) $A = \begin{bmatrix} -1 & 4 & 1 \\ 3 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \lambda = 5$ (d) $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \lambda = 4$

(3) For each of the following matrix-$\lambda$ pairs, determine whether the given $\lambda$ will work as an eigenvalue. You do not need to find an eigenvector as long you can justify if $\lambda$ is a valid eigenvalue or not.

(a) $A = \begin{bmatrix} 4 & 3 \\ 4 & 8 \end{bmatrix}, \lambda = 2$ (b) $A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}, \lambda = 0$

(c) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, \lambda = -1$ (d) $A = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}, \lambda = -2$

(4) For a matrix $A$ with eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda_1 = 2$, and eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with eigenvalue $\lambda_2 = -1$, determine the value of the following expressions using matrix-vector product properties:

(a) $A(2\mathbf{v}_1 + 3\mathbf{v}_2)$

(b) $A(A(\mathbf{v}_1 + 2\mathbf{v}_2))$

(c) $A^{20}(4\mathbf{v}_1 - 2\mathbf{v}_2)$

(5) In this problem we consider a discrete dynamical system that forms what is called a *Markov chain* (see Definition 9.4) which models the number of students attending and skipping a linear algebra class in a semester. Assume the course starts with 1,000,000 students on day 0. For any given class day, 90% of the students who attend a class attend the next class (and 10% of these students skip next class) while only 30% of those absent are there the next time (and 70% of these students continue skipping class).

(a) We know that there will be 900,000 students in class on the second day and 100,000 students skipping class. On the third day, 90% of the 900,000 students (attenders) and 30% of the 100,000 students (skippers) will come back to class. Therefore, 840,000 students will attend class on the third day. On the other hand, 10% of 900,000 students and 70% of 100,000 students skip class on the third day, for a total of 160,000 students skipping class.

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We can use variables to represent these numbers. Let $a_n$ represent the number of students attending class $n$ days after first day. So $a_0 = 1,000,000, a_1 = 900,000, a_2 = 840,000$. Let $s_n$ represent the students skipping class. So $s_0 = 0, s_1 = 100,000, s_2 = 160,000$. Find $a_3, s_3, a_4, s_4$.

(b) Find a linear expression for $a_{k+1}$ in terms of the previous day values, $a_k$ and $s_k$, using the story given in the problem. Similarly, express $s_{k+1}$ in terms of $a_k$ and $s_k$.

(c) Let $\mathbf{x}_k$ represent the state vector: $\mathbf{x}_k = \begin{bmatrix} a_k \\ s_k \end{bmatrix}$. It describes the state of the whole system (students attending class and skipping class) in one vector. For example, $\mathbf{x}_0 = \begin{bmatrix} 1,000,000 \\ 0 \end{bmatrix}$ is the initial state. The state next day is $\mathbf{x}_1 = \begin{bmatrix} 900,000 \\ 100,000 \end{bmatrix}$. Using your answer to the previous part, find a matrix $A$ which describes how the system changes from one day to the other so that $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

(6) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** The number 0 cannot be an eigenvalue.

(b) **True/False** The **0** vector cannot be an eigenvector.

(c) **True/False** If $\mathbf{v}$ is an eigenvector of $A$, then so is $2\mathbf{v}$.

(d) **True/False** If $\mathbf{v}$ is an eigenvector of $A$, then it is also an eigenvector of $A^2$.

(e) **True/False** If $\mathbf{v}$ and $\mathbf{u}$ are eigenvectors of $A$ with the same eigenvalue, then $\mathbf{v} + \mathbf{u}$ is also an eigenvector with the same eigenvalue.

(f) **True/False** If $\lambda$ is an eigenvalue of $A$, then $\lambda^2$ is an eigenvalue of $A^2$.

(g) **True/False** A projection matrix satisfies $P^2 = P$. If $P$ is a projection matrix, then the eigenvalues of $P$ can only be 0 and 1.

(h) **True/False** If $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$, then $1 + \lambda$ is an eigenvalue of $I_n + A$.

(i) **True/False** If $\lambda$ is an eigenvalue of two matrices $A$ and $B$ of the same size, then $\lambda$ is an eigenvalue of $A + B$.

(j) **True/False** If $\mathbf{v}$ is an eigenvector of two matrices $A$ and $B$ of the same size, then it is also an eigenvector of $A + B$.

(k) **True/False** A matrix $A$ has 0 as an eigenvalue if and only if $A$ has linearly dependent columns.

## Project: Understanding the PageRank Algorithm

Sergey Brin and Lawrence Page, the founders of Google, decided that the importance of a web page can be judged by the number of links to it as well as the importance of those pages. It is this

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idea that leads to the PageRank algorithm.[2] Google uses this algorithm (and others) to order search engine results. According to Google:[3]

> PageRank works by counting the number and quality of links to a page to determine a rough estimate of how important the website is. The underlying assumption is that more important websites are likely to receive more links from other websites.

To rank how "important" a website is, we need to make some assumptions. We assume that a person visits a page and then surfs the web by selecting a link from that page – all links on a given page are assigned the same probability of being chosen. As an example, assume a small set of seven pages 1, 2, 3, 4, 5, 6, and 7 with links between the pages given by the arrows as shown in Figure 9.1.[4] So, for example, there is a hyperlink from page 4 to page 3, but no hyperlink in the opposite direction. If a web surfer starts on page 5, then there is probability of $\frac{1}{2}$ that this person will surf to page 6 and a probability of $\frac{1}{2}$ that the surfer will move to page 4. If there is no link leaving a page, as in the case of page 3, then the probability of remaining there is 1.

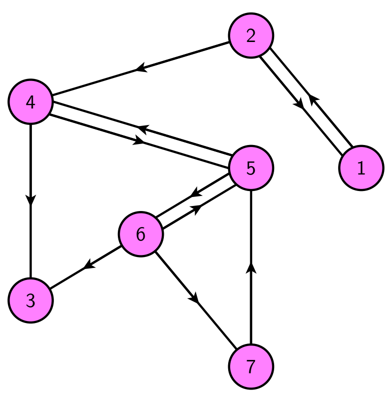


Figure 9.1: A seven page internet

To rank pages, we need to know how likely it is that a surfer will land on a given page. In our seven page example, a person can land on page 3 from page 4 with a probability of $\frac{1}{2}$ or from page 6 with a probability of $\frac{1}{3}$. If there is a link from a page we assume that the surfer leaves the page, and if there are no links from a page then the surfer stays on that page. We also assume that the surfer does not use the "Back" key. This information for our seven page internet example can be summarized in a *transition matrix* $T$ whose $i, j$th entry is the probability that a surfer lands on page

[2]Information for this project was taken from the websites http://www.ams.org/samplings/feature-column/fcarc-pagerank and http://faculty.winthrop.edu/polaskit/spring11/math550/chapter.pdf.

[3]http://web.archive.org/web/20111104131332/http://www.google.com/competition/howgooglesearchworks.html

[4]The Internet is very large and has upwards of 25 billion pages. This would leave us with an enormous transition matrix, even though most of its entries are 0. In fact, studies show that web pages have an average of about 10 links, so on average all but 10 entries of each column are 0. Working with such a large matrix is beyond what we want to do in this project, so we will just amuse ourselves with small examples that illustrate the general points.

$i$ from page $j$.

$$T = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

Let us assume in our seven page internet that a user starts on page 6. That is, the probability that the user is initially on page 6 is 1, and so the probability that the user is on some other page is 0. This information can be encapsulated in a *state vector*

$$\mathbf{x}_0 = [0\,0\,0\,0\,0\,1\,0]^{\mathsf{T}}.$$

Since there are links from page 6 to pages 3, 5, and 7, there is a $\frac{1}{3}$ probability that the surfer will next move to one of these pages. That means that at the next step, the state vector $\mathbf{x}_1$ for this user will be

$$\mathbf{x}_1 = \left[0\,0\,\frac{1}{3}\,0\,\frac{1}{3}\,0\,\frac{1}{3}\right]^{\mathsf{T}}.$$

Note that

$$\mathbf{x}_1 = T\mathbf{x}_0.$$

As the user continues to surf the internet, the probabilities that the surfer is on a given page after the second, third, and fourth steps are given in the state vectors

$$\mathbf{x}_2 = T\mathbf{x}_1 = T^2\mathbf{x}_0, \;\; \mathbf{x}_3 = T\mathbf{x}_2 = T^3\mathbf{x}_0, \;\; \mathbf{x}_4 = T\mathbf{x}_3 = T^4\mathbf{x}_0.$$

In general, the probabilities that the surfer is on a given page after the $n$th step is given by the state vector

$$\mathbf{x}_n = T\mathbf{x}_{n-1} = T^n\mathbf{x}_0.$$

This example illustrates the general nature of what is called a *Markov process* (see Definition 9.4). The two properties of the transition matrix $T$ make $T$ a special kind of matrix.

**Definition 9.5.** A **stochastic** matrix is a matrix in which entries are nonnegative and the sum of the entries in every column is one.

In a Markov process, each generation depends only on the preceding generation and there may be a limiting value as we let the process continue indefinitely. We can test to see if that happens for this Markov process defined by $T$ by doing some experimentation.

**Project Activity 9.1.** Use appropriate technology to do the following. Choose several different initial state vectors $\mathbf{x}_0$ and calculate the vectors in the sequence $\{T^n\mathbf{x}_0\}$ for large values of $n$. (Note that, as state vectors, the entries of $\mathbf{x}_0$ cannot be negative and the sum of the entries of $\mathbf{x}_0$ must be 1.) Explain the behavior of the sequence $\{\mathbf{x}_n\}$ as $n$ gets large. Do you notice anything strange? What aspects of our seven page internet do you think explain this behavior? Clearly communicate all of the experimentation that you do. You may use the GeoGebra applet at https://www.geogebra.org/m/b3dybnux.

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If there is a limit of the sequence $\{T^n \mathbf{x}_0\}$ (in other words, if there is a vector $\mathbf{v}$ such that $\mathbf{v} = \lim_{n\to\infty} T^n \mathbf{x}_0$), we call this limit a *steady-state* or *equilibrium* vector. Such a steady-state vector has another important property. Since $T$ is independent of $n$ we have

$$T\mathbf{v} = T\left(\lim_{n\to\infty} T^n \mathbf{x}_0\right) = \lim_{n\to\infty} T^{n+1}\mathbf{x}_0 = \mathbf{v}. \tag{9.8}$$

Equation (9.8) shows that a steady state vector $\mathbf{v}$ is an eigenvector for $T$ with eigenvalue 1. We can interpret the steady-state vector for $T$ in an important way. Let $t_j$ be the fraction of time we spend on page $j$ and let $l_j$ be the number of links on page $j$. Then the fraction of the time that we end up on page $i$ coming from page $j$ is $\frac{t_j}{l_j}$. If we sum over all the pages linked to page $i$ we have that

$$t_i = \sum \frac{t_j}{l_j}.$$

Notice that this is essentially the same process we used to obtain $\mathbf{x}_n$ from $\mathbf{x}_{n-1}$, and so we can interpret the steady-state vector $\mathbf{v}$ as telling us what fraction of a random web surfer's time was spent at each web page. If we assume that the time spent at a web page is a measure of its importance, then the steady-state vector tells us the relative importance of each web page. So this steady-state vector provides the page rankings for us. In other words,

> The importance of a webpage may be measured by the relative size of the corresponding entry in the steady-state vector for an appropriately chosen Markov chain.

**Project Activity 9.2.** Show that the limiting vector you found in Project Activity 9.1 is an eigenvector of $T$ with eigenvalue 1.

Project Activity 9.1 illustrates one problem with our seven page internet. The steady-state vector shows that page 3 is the only important page, but that hardly seems reasonable in the example since there are other pages that must have some importance. The problem is that page 3 is a "dangling" page and does not lead anywhere. Once a surfer reaches that page, they are stuck there, overemphasizing its importance. So this dangling page acts like a sink, ultimately drawing all surfers to it. To adjust for dangling pages, we make the assumption that if a surfer reaches a dangling page (one with no links emanating from it), the surfer will jump to any page on the web with equal probability. So in our seven page example, once a surfer reaches page 3 the surfer will jump to any page on the web with probability $\frac{1}{7}$.

**Project Activity 9.3.**

(a) Determine the transition matrix for our seven page internet with this adjustment.

(b) Approximate the steady-state vector for this adjusted matrix so that the entries are accurate to four decimal places. Use any appropriate technology to row reduce matrices.

(c) According to this adjusted model, which web page is now the most important? Why? Does this seem reasonable? Why?

There is one more issue to address before we can consider ourselves ready to rank web pages. Consider the example of the five page internet shown in Figure 9.2.

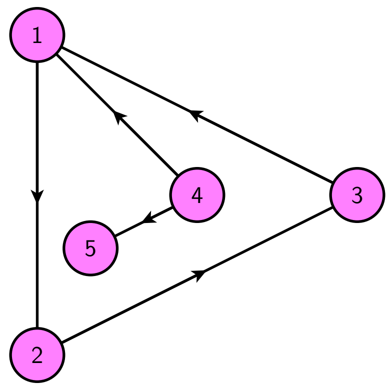


Figure 9.2: A five page internet

**Project Activity 9.4.**

(a) Explain why

$$\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} & \frac{1}{5} \\ 1 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{5} \end{bmatrix}.$$

is the transition matrix for this five page internet. (Keep in mind the adjustment we made for dangling pages.)

(b) Start with different initial state vectors $\mathbf{x}_0$ and determine if there is a limit to the Markov chain. Explain. You may use the GeoGebra applet at https://www.geogebra.org/m/b3dybnux.

Project Activity 9.4 shows that it is possible to construct an internet so that the corresponding Markov chain does not have a limit, even after adjusting for dangling pages. This is a significant problem if we want to provide a relative ranking of all web pages regardless of where a surfer starts. To fix this problem we need to make one final adjustment to arrive at a type of transition matrix that always provides a limit for our Markov chain.

**Definition 9.6.** A stochastic matrix is **regular** if its transition matrix $T$ has the property that for some power $k$, all the entries of $T^k$ are positive.

Note that the transition matrix from Project Activity 9.4 is not regular. Regular matrices have some especially nice properties, as the following theorem describes. We will not prove this theorem, but use it in the remainder of this project. The theorem shows that if we have a regular transition matrix, then there will a limit of the state vectors $\mathbf{x}_n$, and that this limit has a very interesting property.

**Theorem 9.7.** *Assume $n \geq 2$ and that $T$ is a regular $n \times n$ stochastic matrix.*

*(1)* $\lim_{k\to\infty} T^k$ *exists and is a stochastic matrix.*

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*(2) For any vector* $\mathbf{x}$*,*

$$\lim_{k\to\infty} T^k\mathbf{x} = \mathbf{c}$$

*for the same vector* $\mathbf{c}$*.*

*(3) The columns of* $\lim_{k\to\infty} T^k$ *are the same vector* $\mathbf{c}$*.*

*(4) The vector* $\mathbf{c}$ *is the unique eigenvector of* $T$ *whose entries sum to 1.*

*(5) If* $\lambda$ *is an eigenvalue of* $T$ *not equal to 1, then* $|\lambda| < 1$*.*

Having a regular transition matrix $T$ ensures that there is always the same limit $\mathbf{v}$ to the sequence $T^k\mathbf{x}_0$ for any starting vector $\mathbf{x}_0$. As mentioned before, the entries in $\mathbf{v} = \lim_{n\to\infty} T^n\mathbf{x}_0$ can be interpreted as telling us what fraction of the random surfers time was spent at each webpage. If we interpret the amount of time a surfer spends at a page as a measure of the page's importance, then this steady-state vector $\mathbf{v}$ provides a ranking of the relative importance of each page in the web. This is the essence of Google's PageRank.

To make our final adjustment in the transition matrix to be sure that we obtain a regular matrix, we need to deal with the problems of "loops" in our internet. Loops, as illustrated in Project Activity 9.4, can act as sinks just like the dangling pages we saw earlier and condemn a user that enters such a loop to spend his/her time only on those pages in the loop. Quite boring! To account for this problem, we make a second adjustment.

Let $p$ be a number between 0 and 1 (Google supposedly uses $p = 0.85$). Suppose a surfer is on page $i$. We assume with probability $p$ that the surfer will chose any link on page $i$ with equal probability. We make the additional assumption with probability $1 - p$ that the surfer will select with equal probability any page on the web.

If $T$ is a transition matrix, incorporating the method we used to deal with dangling pages, then the adjusted transition matrix $G$ (the Google matrix) is

$$G = pT + (1 - p)Q,$$

where $Q$ is the matrix all of whose entries are $\frac{1}{n}$, where $n$ is the number of pages in the internet ($n = 7$ in our seven page example). Since all of the entries of $G$ are positive, $G$ is a regular stochastic matrix.

**Project Activity 9.5.** Return to the seven page internet in Figure 9.1.

(a) Find the Google matrix $G$ for this internet.

(b) Approximate, to four decimal places, the steady-state vector for this internet.

(c) What is the relative rank of each page in this internet, and approximately what percentage of time does a random user spend on each page.

We conclude with two observations. Consider the role of the parameter $p$ in our final adjustment. Notice that if $p = 1$, then $G = T$ and we have the original hyperlink structure of the web. However, if $p = 0$, then $G = \frac{1}{n}I_n$, where $I_n$ is the $n \times n$ identity matrix with $n$ as the number of pages in the web. In this case, every page is linked to every other page and a random surfer spends equal time

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on any page. Here we have lost all of the character of the linked structure of the web. Choosing $p$ close to 1 retains much of the original hyperlink structure of the web.

Finally, the matrices that model the web are HUGE, and so the methods we used in this project to approximate the steady-state vectors are not practical. There are many methods for approximating eigenvectors that are often used in these situations, some of which we discuss in a later section.

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# Section 10

# The Inverse of a Matrix

**Focus Questions**

*By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.*

- What does it mean for a matrix $A$ to be invertible?
- How can we tell when an $n \times n$ matrix $A$ is invertible?
- If an $n \times n$ matrix $A$ is invertible, how do we find the inverse of $A$?
- If $A$ and $B$ are invertible $n \times n$ matrices, why is $AB$ invertible and what is $(AB)^{-1}$?
- How can we use the inverse of a matrix in solving matrix equations?

## Application: Modeling an Arms Race

Lewis Fry Richardson was a Quaker by conviction who was deeply troubled by the major wars that had been fought in his lifetime. Richardson's training as a physicist led him to believe that the causes of war were phenomena that could be quantified, studied, explained, and thus controlled. He collected considerable data on wars and constructed a model to represent an arms race. The equations in his model caused him concern about the future as indicated by the following statement:

> But it worried him that the equations also showed that the unilateral disarmament of Germany after 1918, enforced by the Allied Powers, combined with the persistent level of armaments of the victor countries would lead to the level of Germanys armaments growing again. In other words, the post-1918 situation was not stable. From the model he concluded that great statesmanship would be needed to prevent an unstable situation from developing, which could only be prevented by a change of policies.[1]

[1] *Nature* **135**, 830-831 (18 May 1935) "Mathematical Psychology of War" (3420).

Analyzing Richardson's arms race model utilizes matrix operations, including matrix inverses. We explore the basic ideas in Richardson's model later in this section.

## Introduction

To this point we have solved systems of linear equations with matrix forms $A\mathbf{x} = \mathbf{b}$ by row reducing the augmented matrices $[A \mid \mathbf{b}]$. These linear matrix-vector equations should remind us of linear algebraic equations of the form $ax = b$, where $a$ and $b$ are real numbers. Recall that we solved an equation of the form $ax = b$ by dividing both sides by $a$ (provided $a \neq 0$), giving the solution $x = \frac{b}{a}$, or equivalently $x = a^{-1}b$. The important property that the number $a^{-1}$ has that allows us to solve a linear equation in this way is that $a^{-1}a = 1$, so that $a^{-1}$ is the multiplicative inverse of $a$. We can solve certain types of matrix equations $A\mathbf{x} = \mathbf{b}$ in the same way, provided we can find a matrix $A^{-1}$ with similar properties. We investigate this situation in this section.

**Preview Activity 10.1.**

(1) Before we define the inverse matrix, recall that the identity matrix $I_n$ (with 1's along the diagonal and 0's everywhere else) is a multiplicative identity in the set of $n \times n$ matrices (just like the real number 1 is the multiplicative identity in the set of real number). In particular, $I_nA = AI_n = A$ for any $n \times n$ matrix $A$.

Now we can generalize the inverse operation to matrices. For an $n \times n$ matrix $A$, we define $A^{-1}$ to be the matrix which when multiplied by $A$ gives us the identity matrix. In other words, $AA^{-1} = A^{-1}A = I_n$. We can find the inverse of a matrix in a calculator by using the $x^{-1}$ button.

For each of the following matrices, determine if the inverse exists using your calculator or other appropriate technology. If the inverse does exist, write down the inverse and check that it satisfies the defining property of the inverse matrix, that is $AA^{-1} = A^{-1}A = I_n$. If the inverse doesn't exist, write down any error you received from the technology. Can you guess why the inverse does not exist for these matrices?

(a) $A = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 2 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (f) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

(2) Now we turn to the question of how to find the inverse of a matrix in general. With this approach, we will be able to determine which matrices have inverses as well.

We will consider the $2 \times 2$ case to make the calculations easier. Suppose $A$ is a $2 \times 2$ matrix. Our goal is to find a matrix $B$ so that $AB = I_2$ and $BA = I_2$. If such a matrix exists, we will call $B$ the inverse, $A^{-1}$, of $A$.

(a) What does the equation $AB = I_2$ tell us about the size of the matrix $B$?

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(b) Now let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. We want to find a matrix $B$ so that $AB = I_2$. Suppose $B$ has columns $\mathbf{b}_1$ and $\mathbf{b}_2$, i.e. $B = [\mathbf{b}_1\ \mathbf{b}_2]$. Our definition of matrix multiplication shows that

$$AB = [A\mathbf{b}_1\ A\mathbf{b}_2].$$

i. If $AB = I_2$, what must $A\mathbf{b}_1$ and $A\mathbf{b}_2$ equal?

ii. Use the result from part (a) to set up two matrix equations to solve to find $\mathbf{b}_1$ and $\mathbf{b}_2$. Then find $\mathbf{b}_1$ and $\mathbf{b}_2$. As a result, find the matrix $B$.

iii. When we solve the two systems we have found a matrix $B$ so that $AB = I_2$. Is this enough to say that $B$ is the inverse of $A$? If not, what else do we need to know to verify that $B$ is in fact $A^{-1}$? Verify that $B$ is $A^{-1}$.

(3) A matrix inverse is extremely useful in solving matrix equations and can help us in solving systems of equations. Suppose that $A$ is an invertible matrix, i.e., there exists $A^{-1}$ such that $AA^{-1} = A^{-1}A = I_n$.

(a) Consider the system $A\mathbf{x} = \mathbf{b}$. Use the inverse of $A$ to show that this system has a solution for every $\mathbf{b}$ and find an expression for this solution in terms of $\mathbf{b}$ and $A^{-1}$. (Note that since matrix multiplication is not commutative, we have to pay attention to the order in which we multiply matrices. For example, $A^{-1}AB = B$ while we cannot simplify $ABA^{-1}$ to $B$ unless $A$ and $B$ commute.)

(b) If $A$, $B$, and $C$ are matrices and $A + C = B + C$, then we can subtract the matrix $C$ from both sides to see that $A = B$. We saw in Section 8 that there is no corresponding general cancellation property for matrix multiplication when we found that $AB = AC$ could hold while $B \neq C$. However, we can cancel $A$ from this equation in certain circumstances. Suppose that $AB = AC$ and that $A$ is an invertible matrix. Show that we can cancel $A$ in this case and conclude that $B = C$. (Note: When simplifying the product of matrices, again keep in mind that matrix multiplication is not commutative.)

## Invertible Matrices

We now have an algebra of matrices in that we can add, subtract, and multiply matrices of the correct sizes. But what about division? In our early mathematics education we learned about *multiplicative inverses* (or reciprocals) of real numbers. The multiplicative inverse of a number $a$ is the real number which when multiplied by $a$ produces 1, the multiplicative identity of real numbers. This inverse is denoted $a^{-1}$. For example, the multiplicative inverse of 2 is $2^{-1} = \frac{1}{2}$ because

$$2 \cdot \frac{1}{2} = 1 = \frac{1}{2} \cdot 2.$$

Of course, we didn't have to write both products because multiplication of real numbers is a commutative operation. There are a couple of important things to note about multiplicative inverses – we can use the inverses of the number $a$ to solve the simple linear equation $ax + b = c$ for $x$ ($x = a^{-1}(c - b)$), and not every real number has an inverse. The latter means that the inverse is not

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defined on the entire set of real numbers. We can extend the idea of inverses to matrices, although we will see that there are many more matrices than just the zero matrix that do not have inverses.

To define matrix inverses[2] we make an analogy with the property of inverses in the real numbers: $x \cdot x^{-1} = 1 = x^{-1} \cdot x$.

**Definition 10.1.** Let $A$ be an $n \times n$ matrix.

(1) $A$ is **invertible** if there is an $n \times n$ matrix $B$ so that $AB = BA = I_n$.

(2) If $A$ is invertible, an **inverse** of $A$ is a matrix $B$ such that $AB = BA = I_n$.

If an $n \times n$ matrix $A$ is invertible, its inverse will be unique (see Exercise 1), and we denote the inverse of $A$ as $A^{-1}$. We also call an invertible matrix a *non-singular* matrix (with *singular* meaning non-invertible).

**Activity 10.1.**

(a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Calculate $AB$ where $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Using your result, explain why it is not possible to have $AB = I_2$, showing that $A$ is non-invertible.

(b) Calculate $AB$ where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Using your result, explain why the inverse of $A$ doesn't exist.

We saw in Activity 10.1 why the inverse does not exist for two specific matrices. We will find in the next section an easy criterion for determining when a matrix has an inverse. In short, when the RREF of the matrix has a pivot in every column and row, then the matrix will be invertible. We know that this condition relates to quite a few other linear algebra concepts we have seen so far, such as linear independence of columns and the columns spanning $\mathbb{R}^n$. We will put these criteria together in one big theorem in the next section.

**Activity 10.2.** Suppose that $A$ is an invertible $n \times n$ matrix. Hence we have an inverse matrix $A^{-1}$ for which $AA^{-1} = A^{-1}A = I_n$. We will see how the inverse is useful in solving matrix equations involving $A$.

(a) Explain why the matrix expressions

$$A^{-1}(AB),\ A^{-1}(A(BA)A^{-1}) \text{ and } BA^{-1}BAA^{-1}B^{-1}A$$

can all be simplified to $B$. (Hint: Use the associative property of matrix multiplication.)

(b) Suppose the system $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why then $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$. What does this equation simplify to?

(c) Since we found one single expression for the solution $\mathbf{x}$ in equation $A\mathbf{x} = \mathbf{b}$, this implies that the equation has a unique solution. What does this imply about the matrix $A$?

[2] We usually refer to a multiplicative inverse as just an inverse. Since every matrix has an additive inverse, there is no need to consider the existence of additive inverses.

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As we saw in Preview Activity 10.1, if the $n \times n$ matrix $A$ is invertible, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^n$ and has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. This means that $A$ has a pivot in every row and column, which is equivalent to the criterion that $A$ reduces to $I_n$, as we noted above.

Even though $\mathbf{x} = A^{-1}\mathbf{b}$ is an explicit expression for the solution of the system $A\mathbf{x} = \mathbf{b}$, using the inverse of a matrix is usually not a computationally efficient way to solve a matrix equation. Finding the RREF of a matrix computationally takes fewer steps to solve the matrix equation.

## Finding the Inverse of a Matrix

The next questions for us to address are how to tell when a matrix is invertible and how to find the inverse of an invertible matrix. Consider a $2 \times 2$ matrix $A$. To find the inverse matrix $B = [\mathbf{b}_1\ \mathbf{b}_2]$ of $A$, we have to solve the two matrix-vector equations $A\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to find the columns of $B$. Since $A$ is the coefficient matrix for both systems, we apply the same row operations on both systems to reduce $A$ to RREF. Thus, instead of solving the two matrix-vector equations separately, we could simply have found the RREF of

$$\left[\begin{array}{c|cc} A & 1 & 0 \\ & 0 & 1 \end{array}\right]$$

and done all of the work in one pass. Note that the right hand side of the augmented matrix is now $I_2$. So we row reduce $[A \mid I_2]$, and if the systems are consistent, the reduced row echelon form of $[A \mid I_2]$ must be $[I_2 \mid A^{-1}]$. You should be able to see that this same process works in any dimension.

**How to find the inverse of an $n \times n$ matrix $A$:**

- Augment $A$ with the identity matrix $I_n$.
- Apply row operations to reduce the augmented matrix $[A \mid I_n]$. If the system is consistent, then the reduced row echelon form of $[A \mid I_n]$ will have the form $[I_n \mid B]$ (by Activity 10.1 (d)). If the reduced row echelon form of $A$ is not $I_n$, then this step fails and $A$ is not invertible.
- If $A$ is row equivalent to $I_n$, then the matrix $B$ in the second step has the property that $AB = I_n$. We will show later that the matrix $B$ also satisfies $BA = I_n$ and so $B$ is the inverse of $A$.

**Activity 10.3.** Find the inverse of each matrix using the method above, if it exists. Compare the result with the inverse that you get from using appropriate technology to directly calculate the inverse.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

We can use this method of finding the inverse of a matrix to derive a concrete formula for the inverse of a $2 \times 2$ matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \tag{10.1}$$

provided that $ad - bc \neq 0$ (see Exercise 2). Hence, any $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse if and only if $ad - bc \neq 0$. We call this quantity *determinant of* $A$, $\det(A)$. We will see that the determinant of a general $n \times n$ matrix will be essential in determining invertibility of the matrix.

## Properties of the Matrix Inverse

As we have done with every new operation, we ask what properties the inverse of a matrix has.

**Activity 10.4.** Consider the following questions about matrix inverses. If two $n \times n$ matrices $A$ and $B$ are invertible, is the product $AB$ invertible? If so, what is the inverse of $AB$? We answer these questions in this activity.

(a) Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}.$$

i. Use formula (10.1) to find the inverses of $A$ and $B$.

ii. Find the matrix product $AB$. Is $AB$ invertible? If so, use formula (10.1) to find the inverse of $AB$.

iii. Calculate the products $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$. What do you notice?

(b) In part (a) we saw that the matrix product $B^{-1}A^{-1}$ was the inverse of the matrix product $AB$. Now we address the question of whether this is true in general. Suppose now that $C$ and $D$ are invertible $n \times n$ matrices so that the matrix inverses $C^{-1}$ and $D^{-1}$ exist.

i. Use matrix algebra to simplify the matrix product $(CD)\left(D^{-1}C^{-1}\right)$. (Hint: What do you know about $DD^{-1}$ and $CC^{-1}$?)

ii. Simplify the matrix product $\left(D^{-1}C^{-1}\right)(CD)$ in a manner similar to part i.

iii. What conclusion can we draw from parts i and ii? Explain. What property of matrix multiplication requires us to reverse the order of the product when we create the inverse of $CD$?

Activity 10.4 gives us one important property of matrix inverses. The other properties given in the next theorem can be verified similarly.

**Theorem 10.2.** *Let $A$ and $B$ be invertible $n \times n$ matrices. Then*

*(1)* $\left(A^{-1}\right)^{-1} = A$.

*(2) The product $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

*(3) The matrix $A^{\mathsf{T}}$ is invertible and $\left(A^{\mathsf{T}}\right)^{-1} = \left(A^{-1}\right)^{\mathsf{T}}$.*

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## Examples

What follows are worked examples that use the concepts from this section.

**Example 10.3.** For each of the following matrices $A$,

- Use appropriate technology to find the reduced row echelon form of $[A \mid I_3]$.
- Based on the result of part (a), is $A$ invertible? If yes, what is $A^{-1}$? If no, explain why.
- Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$. If $A$ is invertible, solve the matrix equation $A\mathbf{x} = \mathbf{b}$ using the inverse of $A$. If $A$ is not invertible, find all solutions, if any, to the equation $A\mathbf{x} = \mathbf{b}$ using whatever method you choose.

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

**Example Solution.**

(a) With $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, we have the following.

- The reduced row echelon form of $[A \mid I_3]$ is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -\frac{1}{2} & -1 & \frac{3}{2} \end{array}\right].$$

- Since $A$ is row equivalent to $I_3$, we conclude that $A$ is invertible. The reduced row echelon form of $[A \mid I_3]$ tells us that

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & -4 \\ -1 & -2 & 3 \end{bmatrix}.$$

- The solution to $A\mathbf{x} = \mathbf{b}$ is given by

$$vx = A^{-1}\mathbf{b} = \frac{1}{2}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & -4 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ -5 \end{bmatrix}.$$

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(b) With $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$, we have the following.

- The reduced row echelon form of $[A \mid I_3]$ is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -3 \end{array}\right].$$

- Since $A$ is not row equivalent to $I_3$, we conclude that $A$ is not invertible.
- The reduced row echelon form of $[A \mid \mathbf{b}]$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The fact that the augmented column is a pivot column means that the equation $A\mathbf{x} = \mathbf{b}$ has no solutions.

**Example 10.4.**

(a) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

i. Show that $A^2 \neq 0$ but $A^3 = 0$.

ii. Show that $I - A$ is invertible and find its inverse. Compare the inverse of $I - A$ to $I + A + A^2$.

(b) Let $M$ be an arbitrary square matrix such that $M^3 = 0$. Show that $M$ is invertible and find an inverse for $M$.

**Example Solution.**

(a) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

i. Using technology to calculate $A^2$ and $A^3$ we find that $A^3 = 0$ while $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

ii. For this matrix $A$ we have $I - A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. The reduced row echelon form of $I - A$ is

$$\left[\begin{array}{ccc|ccc} 1 & -0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right],$$

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so $I - A$ is invertible and $(I-A)^{-1} = \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$.

A straightforward matrix calculation also shows that

$$(I-A)^{-1} = I + A + A^2.$$

(b) We can try to emulate the result of part (a) here. Expanding using matrix operations gives us

$$\begin{aligned}(I-M)(I+M+M^2) &= (I+M+M^2) - (M+M^2+M^3) \\ &= (I+M+M^2) - (M+M^2+0) \\ &= I\end{aligned}$$

and

$$\begin{aligned}(I+M+M^2)(I-M) &= (I+M+M^2) - (M+M^2+M^3) \\ &= (I+M+M^2) - (M+M^2+0) \\ &= I.\end{aligned}$$

So $I - M$ is invertible and $(I-M)^{-1} = I + M + M^2$.

This argument can be generalized to show that if $M$ is a square matrix and $M^n = 0$ for some positive integer $n$, then $I - M$ is invertible and

$$(I-M)^{-1} = I + M + M^2 + \cdots + M^{n-1}.$$

## Summary

- If $A$ is an $n \times n$ matrix, then $A$ is invertible if there is a matrix $B$ so that $AB = BA = I_n$. The matrix $B$ is called the inverse of $A$ and is denoted $A^{-1}$.

- An $n \times n$ matrix $A$ is invertible if and only if $A$ the reduced row echelon form of $A$ is the $n \times n$ identity matrix $I_n$.

- To find the inverse of an invertible $n \times n$ matrix $A$, augment $A$ with the identity and row reduce. If $[A \mid I_n] \sim [I_n \mid B]$, then $B = A^{-1}$.

- If $A$ and $B$ are invertible $n \times n$ matrices, then $(AB)^{-1} = B^{-1}A^{-1}$. Since the inverse of $AB$ exists, the product of two invertible matrices is an invertible matrix.

- We can use the algebraic tools we have developed for matrix operations to solve equations much like we solve equations with real variables. We must be careful, though, to only multiply by inverses of invertible matrices, and remember that matrix multiplication is not commutative.

## Exercises

(1) Let $A$ be an invertible $n \times n$ matrix. In this exercise we will prove that the inverse of $A$ is unique. To do so, we assume that both $B$ and $C$ are inverses of $A$, that is $AB = BA = I_n$ and $AC = CA = I_n$. By considering the product $BAC$ simplified in two different ways, show that $B = C$, implying that the inverse of $A$ is unique.

(2) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary $2 \times 2$ matrix.

(a) If $A$ is invertible, perform row operations to determine a row echelon form of $A$. (Hint: You may need to consider different cases, e.g., when $a = 0$ and when $a \neq 0$.)

(b) Under certain conditions, we can row reduce $[A \mid I_2]$ to $[I_2 \mid B]$, where

$$B = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Use the row echelon form of $A$ from part (a) to find conditions under which the $2 \times 2$ matrix $A$ is invertible. Then derive the formula for the inverse $B$ of $A$.

(3) (a) For a few different $k$ values, find the inverse of $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. From these results, make a conjecture as to what $A^{-1}$ is in general.

(b) Prove your conjecture using the definition of inverse matrix.

(c) Find the inverse of $A = \begin{bmatrix} 1 & k & \ell \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}$.

(Note: You can combine the first two parts above by applying the inverse finding algorithm directly on $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.)

(4) Solve for the matrix $A$ in terms of the others in the following equation:

$$P^{-1}(D + CA)P = B$$

If you need to use an inverse, assume it exists.

(5) For which $c$ is the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 5 & c \end{bmatrix}$ invertible?

(6) For which $c$ is the matrix $A = \begin{bmatrix} c & 2 \\ 3 & c \end{bmatrix}$ invertible?

(7) Let $A$ and $B$ be invertible $n \times n$ matrices. Verify the remaining properties of Theorem 10.2. That is, show that

(a) $\left(A^{-1}\right)^{-1} = A$.

(b) The matrix $A^{\mathsf{T}}$ is invertible and $\left(A^{\mathsf{T}}\right)^{-1} = \left(A^{-1}\right)^{\mathsf{T}}$.

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(8) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** If $A$ is an invertible matrix, then for any two matrices $B, C$, $AB = AC$ implies $B = C$.

(b) **True/False** If $A$ is invertible, then so is $AB$ for any matrix $B$.

(c) **True/False** If $A$ and $B$ are invertible $n \times n$ matrices, then so is $AB$.

(d) **True/False** If $A$ is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b}$ in $\mathbb{R}^n$.

(e) **True/False** If $A$ is an invertible $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution when it is consistent.

(f) **True/False** If $A$ is invertible, then so is $A^2$.

(g) **True/False** If $A$ is invertible, then it reduces to the identity matrix.

(h) **True/False** If a matrix is invertible, then so is its transpose.

(i) **True/False** If $A$ and $B$ are invertible $n \times n$ matrices, then $A + B$ is invertible.

(j) **True/False** If $A^2 = 0$, then $I + A$ is invertible.

## Project: The Richardson Arms Race Model

How and why a nation arms itself for defense depends on many factors. Among these factors are the offensive military capabilities a nation deems its enemies have, the resources available for creating military forces and equipment, and many others. To begin to analyze such a situation, we will need some notation and background. In this section we will consider a two nation scenario, but the methods can be extended to any number of nations. In fact, after World War I, Richardson collected data and created a model for the countries Czechoslovakia, China, France, Germany, England, Italy, Japan, Poland, the USA, and the USSR.[3]

Let $N_1$ and $N_2$ represent 2 different nations. Each nation has some military capability (we will call this the *armament* of the nation) at time $n$ (think of $n$ as representing the year). Let $a_1(n)$ represent the armament of nation $N_1$ at time $n$, and $a_2(n)$ the armament of nation $N_2$ at time $n$. We could measure $a_i(n)$ in weaponry or dollars or whatever units make sense for armaments. The Richardson arms race model provides connections between the armaments of the two nations.

**Project Activity 10.1.** We continue to analyze a two nation scenario. Let us suppose that our two nations are Iran (nation $N_1$) and Iraq (nation $N_2$). In 1980, Iraq invaded Iran resulting in a long and brutal 8 year war. Richardson was interested in analyzing data to see if such wars could be predicted by the changes in armaments of each nation. We construct the two nation model in this activity.

During each time period every nation adds or subtracts from its armaments. In our model, we will consider three main effects on the changes in armaments: the defense effect, fatigue effect and

[3]The Union of Soviet Socialist Republics (USSR), headed by Russia, was a confederation of socialist republics in Eurasia. The USSR disbanded in 1991. Czechoslovakia was a sovereign state in central Europe that peacefully split into the Czech Republic and Slovakia in 1993.

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the grievance effect. In this activity we will discuss each effect in turn and then create a model to represent a two nation arms race.

- We first consider the defense effect. In a two nation scenario, each nation may react to the potential threat implied by an arms buildup of the other nation. For example, if nation $N_1$ feels threatened by nation $N_2$ (think of South and North Korea, or Ukraine and Russia, for example), then nation $N_2$'s level of armament might cause nation $N_1$ to increase its armament in response. We will let $\delta_{12}$ represent this effect of nation $N_2$'s armament on the armament of nation $N_1$. Nation $N_1$ will then increase (or decrease) its armament in time period $n$ by the amount $\delta_{12}a_2(n-1)$ based on the armament of nation $N_2$ in time period $n-1$. We will call $\delta_{12}$ a *defense coefficient*.[4]

- Next we discuss the fatigue effect. Keeping a strong defense is an expensive and taxing enterprise, often exacting a heavy toll on the resources of a nation. For example, consider the fatigue that the U.S. experienced fighting wars in Iraq and Afghanistan, losing much hardware and manpower in these conflicts. Let $\delta_{ii}$ represent this *fatigue factor* on nation $i$. Think of $\delta_{ii}$ as a measure of how much the nation has to replace each year, so a positive fatigue factor means that the nation is adding to its armament. The fatigue factor produces an effect of $\delta_{ii}a_i(n-1)$ on the armament of nation $i$ at time $t=n$ that is the effect of the armament at time $t=n-1$.

- The last factor we consider is what we will call a grievance factor. This can be thought of as the set of ambitions and/or grievances against other nations (such as the acquisition or reacquisition of territory currently belonging to another country). As an example, Argentina and Great Britain both claim the Falkland Islands as territory. In 1982 Argentina invaded the disputed Falkland Islands which resulted in a two-month long undeclared Falkland Islands war, which returned control to the British. It seems reasonable that one nation might want to have sufficient armament in place to support its claim if force becomes necessary. Assuming that these grievances and ambitions have a constant impact on the armament of a nation from year to year, let $g_i$ be this "grievance" constant for nation $i$.[5] The effect a grievance factor $g_i$ would have on the armament of nation $i$ in year $n$ would be to add $g_i$ directly to $a_i(n-1)$, since the factor $g_i$ is constant from year to year (paying for arms and soldier's wages, for example) and does not depend on the amount of existing armament.

(a) Taking the three effects discussed above into consideration, explain why

$$a_1(n) = \delta_{11}a_1(n-1) + \delta_{12}a_2(n-1) + a_1(n-1) + g_1.$$

Then explain why

$$a_1(n) = (\delta_{11}+1)\,a_1(n-1) + \delta_{12}a_2(n-1) + g_1. \tag{10.2}$$

[4]Of course, there are many other factors that have not been taken into account in the analysis. A nation may have heavily armed allies (like the U.S.) which may provide enough perceived security that this analysis is not relevant. Also, a nation might be a neutral state, such as Switzerland, and this analysis might not apply to such nations.

[5]It might be possible for $g_i$ to be negative if, for example, a nation feels that such disputes can and should only be settled by negotiation.

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(b) Write an equation similar to equation (10.2) that describes $a_2(n)$ in terms of the three effects.

(c) Let $\mathbf{a}_n = \begin{bmatrix} a_1(n) \\ a_2(n) \end{bmatrix}$. Explain why

$$\mathbf{a}_n = (D + I_2)\mathbf{a}_{n-1} + \mathbf{g},$$

where $D = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$ and $\mathbf{g} = [g_1\ g_2]^{\mathsf{T}}$.

| Year | Iran | Iraq |
|---|---|---|
| 1966 | 662 | 391 |
| 1967 | 903 | 378 |
| 1968 | 1090 | 495 |
| 1969 | 1320 | 615 |
| 1970 | 1470 | 600 |
| 1971 | 1970 | 618 |
| 1972 | 2500 | 589 |
| 1973 | 2970 | 785 |
| 1974 | 5970 | 2990 |
| 1975 | 7100 | 1690 |

Table 10.1: Military Expenditures of Iran and Iraq 1966-1975.

**Project Activity 10.2.** In order to analyze a specific arms race between nations, we need some data to determine values of the $\delta_{ij}$ and the $g_i$. Table 10.1 shows the military expenditures of Iran and Iraq in the years leading up to their war in 1975. (The data is in millions of US dollars, adjusted for inflation and is taken from "World Military Expenditures and Arms Transfers 1966-1975" by the U.S. Arms Control and Disarmament Agency.) We can perform regression (we will see how in a later section) on this data to obtain the following linear approximations:

$$a_1(n) = 2.0780a_1(n-1) - 1.7081a_2(n-1) - 126.9954 \quad (10.3)$$

$$a_2(n) = 0.9419a_1(n-1) - 1.3283a_2(n-1) - 101.2980. \quad (10.4)$$

(Of course, the data does not restrict itself to only factors between the two countries, so our model will not be as precise as we might like. However, it is a reasonable place to start.) Use the regression equations (10.3) and (10.4) to explain why

$$D = \begin{bmatrix} 1.0780 & -1.7081 \\ 0.94194 & -2.3283 \end{bmatrix} \text{ and } \mathbf{g} = [-126.9954\ \ -101.2980]^{\mathsf{T}}$$

for our Iran-Iraq arms race.

Activities 10.1 and 10.2 provide the basics to describe the general arms race model due to Richardson. If we have an $m$ nation arms race with $D = [\delta_{ij}]$ and $\mathbf{g} = [g_i]$ , then

$$\mathbf{a}_n = (D + I_m)\mathbf{a}_{n-1} + \mathbf{g}. \quad (10.5)$$

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**Project Activity 10.3.** The idea of an arms race, theoretically, is to reach a point at which all parties feel secure and no additional money needs to be spent on armament. If such a situation ever arises, then the armament of all nations is stable, or in equilibrium. If we have an equilibrium solution, then for large values of $n$ we will have $\mathbf{a}_n = \mathbf{a}_{n-1}$. So to find an equilibrium solution, if it exists, we need to find a vector $\mathbf{a}_E$ so that

$$\mathbf{a}_E = (D + I)\mathbf{a}_E + \mathbf{g} \tag{10.6}$$

where $I$ is the appropriate size identity matrix. If $\mathbf{a}_E$ exists, we call $\mathbf{a}_E$ an *equilibrium state*.

We can apply matrix algebra to find the equilibrium state vector $\mathbf{a}_E$ under certain conditions.

(a) Assuming that $\mathbf{a}_E$ exists, use matrix algebra and Equation 10.6 to show that

$$D\mathbf{a}_E + \mathbf{g} = 0. \tag{10.7}$$

(b) Under what conditions can we be assured that there will always be a unique equilibrium state $\mathbf{a}_E$? Explain. Under these conditions, how can we find this unique equilibrium state? Write this equilibrium state vector $\mathbf{a}_E$ as a matrix-vector product.

(c) Does the arms race model for Iran and Iraq have an equilibrium solution? If so, find it. If not, explain why not. Use technology as appropriate.

(d) Assuming an equilibrium exists and that both nations behave in a way that supports the equilibrium, explain what the appropriate entry of the equilibrium state vector $\mathbf{a}_E$ suggests about what Iran and Iraq's policies should be. What does this model say about why there might have been war between these two nations?

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# Section 11

# The Invertible Matrix Theorem

**Focus Questions**

*By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.*

- What does it mean for two statements to be equivalent?
- How can we efficiently prove that a string of statements are all equivalent?
- What is the Invertible Matrix Theorem and why is it important?
- What are the equivalent conditions to a matrix being invertible?

## Introduction

This section is different than others in this book in that it contains only one long proof of the equivalence of statements that we have already discussed. As such, this is a theoretical section and there is no application connected to it.

The Invertible Matrix Theorem is a theorem that provides many different statements that are equivalent to having a matrix be invertible. To understand the Invertible Matrix Theorem, we need to know what it means for two statements to be *equivalent*. By equivalent, we mean that if one of the statements is true, then so is the other. We examine this idea in this preview activity.

**Preview Activity 11.1.** Let $A$ be an $n \times n$ matrix. In this activity we endeavor to understand why the two statements

**I.** The matrix $A$ is invertible.

**II.** The matrix $A^{\mathsf{T}}$ is invertible.

are equivalent. To demonstrate that statements I and II are equivalent, we need to argue that if statement I is true, then so is statement II, and if statement II is true then so is statement I.

(1) Let's first show that if statement I is true, then so is statement II. So we assume statement I. That is, we assume that $A$ is an invertible matrix. So we know that there is an $n \times n$ matrix $B$ such that $AB = BA = I_n$, where $I_n$ is the $n \times n$ identity matrix. To demonstrate that statement II must also be true, we need to verify that $A^{\mathsf{T}}$ is an invertible matrix.

   (a) What is $I_n^{\mathsf{T}}$?

   (b) Take the transpose of both sides of the equation $AB = I_n$ and use the properties of the transpose to write $(AB)^{\mathsf{T}}$ in terms of $A^{\mathsf{T}}$ and $B^{\mathsf{T}}$.

   (c) Take the transpose of both sides of the equation $BA = I_n$ and use the properties of the transpose to write $(BA)^{\mathsf{T}}$ in terms of $A^{\mathsf{T}}$ and $B^{\mathsf{T}}$.

   (d) Explain how the previous two parts show that $B^{\mathsf{T}}$ is the inverse of $A^{\mathsf{T}}$, so that $A^{\mathsf{T}}$ is invertible. So we have shown that if statement I is true, so is statement II.[1]

(2) Now we prove that if statement II is true, then so is statement I. So we assume statement II. That is, we assume that the matrix $A^{\mathsf{T}}$ is invertible. We could do this in the same manner as part (a), or we could be a bit clever. Let's try to be clever.

   (a) What matrix is $\left(A^{\mathsf{T}}\right)^{\mathsf{T}}$?

   (b) Why can we use the result of part (a) with $A^{\mathsf{T}}$ in place of $A$ to conclude that $A$ is invertible? As a consequence, we have demonstrated that $A$ is invertible if $A^{\mathsf{T}}$ is invertible. This concludes our argument that statements I and II are equivalent.

## The Invertible Matrix Theorem

We have been introduced to many statements about existence and uniqueness of solutions to systems of linear equations, linear independence of columns of coefficient matrices, onto linear transformations, and many other items. In this section we will analyze these statements in light of how they are related to invertible matrices, with the main goal to understand the Invertible Matrix Theorem.

Recall that an $n \times n$ matrix $A$ is invertible if there is an $n \times n$ matrix $B$ such that $AB = BA = I_n$, where $I_n$ is the $n \times n$ identity matrix. The Invertible Matrix Theorem is an important theorem in that it provides us with a wealth of statements that are all equivalent to the statement that an $n \times n$ matrix $A$ is invertible, and connects many of the topics we have been discussing so far this semester into one big picture.

**Theorem 11.1** (The Invertible Matrix Theorem)**.** *Let $A$ be an $n \times n$ matrix. The following statements are equivalent:*

*(1) $A$ is an invertible matrix.*

*(2) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

*(3) $A$ has $n$ pivot columns.*

*(4) The columns of $A$ span $\mathbb{R}^n$.*

[1]Note that statement I does not have to be true. We are only assuming that IF statement I is true, then statement II must also be true.

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*(5) $A$ is row equivalent to the identity matrix $I_n$.*

*(6) The columns of $A$ are linearly independent.*

*(7) The columns of $A$ form a basis for $\mathbb{R}^n$.*

*(8) The matrix transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.*

*(9) The matrix equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each vector $\mathbf{b}$ in $\mathbb{R}^n$.*

*(10) The matrix transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is onto.*

*(11) There is an $n \times n$ matrix $C$ so that $AC = I_n$.*

*(12) There is an $n \times n$ matrix $D$ so that $DA = I_n$.*

*(13) The scalar 0 is not an eigenvalue of $A$.*

*(14) $A^{\mathsf{T}}$ is invertible.*

The Invertible Matrix Theorem is a theorem that provides many different statements that are equivalent to a matrix being invertible. As discussed in Preview Activity 11.1, two statements are said to be **equivalent** if, whenever one of the statements is true, then the other is also true. So to demonstrate, say, statements I and II are equivalent, we need to argue that

- if statement I is true, then so is statement II, and
- if statement II is true then so is statement I.

The Invertible Matrix Theorem, however, provides a long list of statements that are equivalent. It would be inefficient to prove, one by one, that each pair of statements is equivalent. (There are $\binom{14}{2} = 91$ such pairs.) Fortunately, there is a shorter method that we can use.

**Activity 11.1.** In this activity, we will consider certain parts of the Invertible Matrix Theorem and show that one implies another in a specific order. For all parts in this activity, we assume $A$ is an $n \times n$ matrix.

(a) Consider the following implication:
$(2) \implies (6)$:[2] If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the columns of $A$ are linearly independent. This shows that part 2 of the IMT implies part 6 of the IMT. Justify this implication as if it is a T/F problem.

(b) Justify the following implication:
$(6) \implies (9)$: If the columns of $A$ are linearly independent, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each vector $\mathbf{b}$ in $\mathbb{R}^n$.

(c) Justify the following implication:
$(9) \implies (4)$: If the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in $\mathbb{R}^n$, then the columns of $A$ span $\mathbb{R}^n$.

[2] The symbol $\implies$ is the implication symbol, so $(1) \implies (12)$ is read to mean that statement $(1)$ of the theorem implies statement $(12)$.

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(d) Justify the following implication:
$(4) \implies (2)$: If the columns of $A$ span $\mathbb{R}^n$, then the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) Using the above implications you proved, explain why we can conclude the following implication must also be true:
$(2) \implies (9)$: If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each vector $\mathbf{b}$ in $\mathbb{R}^n$.

(f) Using the above implications you proved, explain why any one of the implications $(2)$, $(6)$, $(9)$, and $(4)$ implies any of the others.

Using a similar ordering of circular implications as in Activity 11.1, we can prove the Invertible Matrix Theorem by showing that each statement in the list implies the next statement, and that the last statement implies the first.

*Proof of the Invertible Matrix Theorem*

**Statement (1) implies Statement (2).** This follows from work done in Activity 11.1.

**Statement (2) implies Statement (3).** This was done in Activity 11.1.

**Statement (3) implies Statement (4).** Suppose that every column of $A$ is a pivot column. The fact that $A$ is square means that every row of $A$ contains a pivot, and hence the columns of $A$ span $\mathbb{R}^n$.

**Statement (4) implies Statement (5).** Since the columns of $A$ span $\mathbb{R}^n$, it must be the case that every row of $A$ contains a pivot. This means that $A$ must be row equivalent to $I_n$.

**Statement (5) implies Statement (6).** If $A$ is row equivalent to $I_n$, there must be a pivot in every column, which means that the columns of $A$ are linearly independent.

**Statement (6) implies Statement (7).** If the columns of $A$ are linearly independent, then there is a pivot in every column. Since $A$ is a square matrix, there is a pivot in every row as well. So the columns of $A$ span $\mathbb{R}^n$. Since they are also linearly independent, the columns form a minimal spanning set, which is a basis of $\mathbb{R}^n$.

**Statement (7) implies Statement (8).** If the columns form a basis of $\mathbb{R}^n$, then the columns are linearly independent. This means that each column is a pivot column, which also implies $A\mathbf{x} = \mathbf{0}$ has a unique solution and that $T$ is one-to-one.

**Statement (8) implies Statement (9).** If $T$ is one-to-one, then $A$ has a pivot in every column. Since $A$ is square, every row of $A$ contains a pivot. Therefore, the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$ and has a unique solution.

**Statement (9) implies Statement (10).** If $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$, then the transformation $T$ is onto since $T(\mathbf{x}) = \mathbf{b}$ has a solution for every $\mathbf{b}$.

**Statement (10) implies Statement (11).** Assume that $T$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is onto. For each $i$, let $\mathbf{e}_i$ be the $i$th column of the $n \times n$ identity matrix $I_n$. That is, $\mathbf{e}_i$ is the vector in $\mathbb{R}^n$ with 1 in the $i$th component and 0 everywhere else. Since $T$ is onto, for each $i$ there is a vector $\mathbf{c}_i$ such that $T(\mathbf{c}_i) = A\mathbf{c}_i = \mathbf{e}_i$. Let $C = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n]$. Then

$$AC = A[\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n] = [A\mathbf{c}_1\ A\mathbf{c}_2\ \cdots\ A\mathbf{c}_n] = [\mathbf{e}_1\ \mathbf{e}_2\ \cdots\ \mathbf{e}_n] = I_n.$$

**Statement (11) implies Statement (12).** Assume $C$ is an $n \times n$ matrix so that $AC = I_n$. First we show that the matrix equation $C\mathbf{x} = \mathbf{0}$ has only the trivial solution. Suppose $C\mathbf{x} = \mathbf{0}$. Then multiplying both sides on the left by $A$ gives us

$$A(C\mathbf{x}) = A\mathbf{0}.$$

Simplifying this equation using $AC = I_n$, we find $\mathbf{x} = \mathbf{0}$.

Since $C\mathbf{x} = \mathbf{0}$ has only the trivial solution, every column of $C$ must be a pivot column. Since $C$ is an $n \times n$ matrix, it follows that every row of $C$ contains a pivot position. Thus, the matrix equation $C\mathbf{x} = \mathbf{b}$ is consistent and has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$. Let $\mathbf{v}_i$ be the vector in $\mathbb{R}^n$ satisfying $C\mathbf{v}_i = \mathbf{e}_i$ for each $i$ between 1 and $n$ and let $M = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$. Then $CM = I_n$.

Now we show that $CA = I_n$. Since

$$AC = I_n$$

we can multiply both sides on the left by $C$ to see that

$$C(AC) = CI_n.$$

Now we multiply both sides on the right by $M$ and obtain

$$(C(AC))M = CM.$$

Using the associative property of matrix multiplication and the fact that $CM = I_n$ shows that

$$\begin{aligned}(CA)(CM) &= CM \\ CA &= I_n.\end{aligned}$$

Thus, if $A$ and $C$ are $n \times n$ matrices and $AC = I_n$, then $CA = I_n$. So we have proved our implication with $D = C$

**Statement (12) implies Statement (13).** Assume that there is an $n \times n$ matrix $D$ so that $DA = I_n$. Suppose $A\mathbf{x} = \mathbf{0}$. Then multiplying both sides by $A$ on the left, we find that

$$\begin{aligned}D(A\mathbf{x}) &= D\mathbf{0} \\ (DA)\mathbf{x} &= \mathbf{0} \\ \mathbf{x} &= \mathbf{0}.\end{aligned}$$

So the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and 0 is not an eigenvalue for $A$.

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**Statement (13) implies Statement (14).** If 0 is not an eigenvalue of $A$, then the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Since statement 2 implies statement 11, there is an $n \times n$ matrix $C$ such that $AC = I_n$. The proof that statement 11 implies statement 12 shows that $CA = I_n$ as well. So $A$ is invertible. By taking the transpose of both sides of the equation $AA^{-1} = A^{-1}A = I_n$ (remembering $(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}$) we find

$$(A^{-1})^{\mathsf{T}}A^{\mathsf{T}} = A^{\mathsf{T}}(A^{-1})^{\mathsf{T}} = I_n^{\mathsf{T}} = I_n\,.$$

Therefore, $(A^{-1})^{\mathsf{T}}$ is the inverse of $A^{\mathsf{T}}$ by definition of the inverse.

**Statement (14) implies Statement (1).** Since statement (1) implies statement (14), we proved 'If $A$ is invertible, then $A^{\mathsf{T}}$ is invertible." Using this implication with $A^{\mathsf{T}}$ replaced by $A$, we find that 'If $A^{\mathsf{T}}$ is invertible, then $(A^{\mathsf{T}})^{\mathsf{T}}$ is invertible." But $(A^{\mathsf{T}})^{\mathsf{T}} = A$, which proves that statement (14) implies statement (1).

This concludes our proof of the Invertible Matrix Theorem.

## Examples

What follows are worked examples that use the concepts from this section.

**Example 11.2.** Let $M = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 3 & 2 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

(a) Without doing any calculations, is $M$ invertible? Explain your response.

(b) Is the equation $M\mathbf{x} = \mathbf{b}$ consistent for every $\mathbf{b}$ in $\mathbb{R}^4$? Explain.

(c) Is the equation $M\mathbf{x} = \mathbf{0}$ consistent? If so, how many solutions does this equation have? Explain.

(d) Is it possible to find a $4 \times 4$ matrix $P$ such that $PM = I_4$? Explain.

**Example Solution.**

(a) The third column of $M$ is twice the first, so the columns of $M$ are not linearly independent. We conclude that $M$ is not invertible.

(b) The equation $M\mathbf{x} = \mathbf{b}$ is not consistent for every $\mathbf{b}$ in $\mathbb{R}^4$. If it was, then the columns of $M$ would span $\mathbb{R}^4$ and, since there are exactly four columns, the columns of $M$ would be a basis for $\mathbb{R}^4$. Thus, $M$ would have to be invertible, which it is not.

(c) The homogeneous system is always consistent. Since the columns of $M$ are linearly dependent, the equation $M\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

(d) It is not possible to find a $4 \times 4$ matrix $P$ such that $PM = I_4$. Otherwise $M$ would have to be invertible.

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**Example 11.3.** Let $M$ be an $n \times n$ matrix whose eigenvalues are all nonzero.

(a) Let $\mathbf{b} \in \mathbb{R}^n$. Is the equation $M\mathbf{x} = \mathbf{b}$ consistent? If yes, explain why and find all solutions in terms of $M$ and $\mathbf{b}$. If no, explain why.

(b) Let $S$ be the matrix transformation defined by $S(\mathbf{x}) = M\mathbf{x}$. Suppose $S(\mathbf{a}) = S(\mathbf{b})$ for some vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$. Must there be any relationship between $\mathbf{a}$ and $\mathbf{b}$? If yes, explain the relationship. If no, explain why.

(c) Let $\mathbf{m}_1$, $\mathbf{m}_2$, ..., $\mathbf{m}_n$ be the columns of $M$. In how many ways can we write the zero vector as a linear combination of $\mathbf{m}_1$, $\mathbf{m}_2$, ..., $\mathbf{m}_n$? Explain.

**Example Solution.**

(a) Since 0 is not an eigenvalue of $M$, we know that $M$ is invertible. Therefore, the equation $M\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = M^{-1}\mathbf{b}$.

(b) The fact that $M$ is invertible implies that $S$ is one-to-one. So if $S(\mathbf{a}) = S(\mathbf{b})$, then it must be the case that $\mathbf{a} = \mathbf{b}$.

(c) Because $M$ is invertible, the columns of $M$ are linearly independent. Therefore, there is only the trivial solution to the equation

$$x_1\mathbf{m}_1 + x_2\mathbf{m}_2 + \cdots + x_n\mathbf{m}_n = \mathbf{0}.$$

## Summary

- Two statements are equivalent if, whenever one of the statements is true, then the other must also be true.
- To efficiently prove that a string of statements are all equivalent, we can prove that each statement in the list implies the next statement, and that the last statement implies the first.
- The Invertible Matrix Theorem gives us many conditions that are equivalent to an $n \times n$ matrix being invertible. This theorem is important because it connects many of the concepts we have been studying.

## Exercises

(1) Consider the matrix $A = \begin{bmatrix} 1 & 2 & a \\ -1 & 1 & b \\ 1 & 1 & c \end{bmatrix}$. Use the Invertible Matrix Theorem to geometrically describe the vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ which make $A$ invertible without doing any calculations.

(2) Suppose $A$ is an invertible $n \times n$ matrix. Let $T$ be the matrix transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in $\mathbb{R}^n$. Show that the matrix transformation $S$ defined by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the inverse of the transformation $T$ (i.e., $S$ is the inverse function to $T$ when the transformations are considered as functions).

(3) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** If $A^2$ is invertible, then $A$ is invertible.

(b) **True/False** If $A$ and $B$ are square matrices with $AB$ invertible, then $A$ and $B$ are invertible.

(c) **True/False** If the columns of an $n \times n$ matrix $A$ span $\mathbb{R}^n$, then the equation $A^{-1}\mathbf{x} = \mathbf{0}$ has a unique solution.

(d) **True/False** If the columns of $A$ and columns of $B$ form a basis of $\mathbb{R}^n$, then so do the columns of $AB$.

(e) **True/False** If the columns of a matrix $A$ form a basis of $\mathbb{R}^n$, then so do the rows of $A$.

(f) **True/False** If the matrix transformation $T$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one for an $n \times n$ matrix $A$, then the columns of $A^{-1}$ are linearly independent.

(g) **True/False** If the columns of an $n \times n$ matrix $A$ span $\mathbb{R}^n$, then so do the rows of $A$.

(h) **True/False** If there are two $n \times n$ matrices $A$ and $B$ such that $AB = I_n$, then the matrix transformation defined by $T(\mathbf{x}) = A^T\mathbf{x}$ is one-to-one.

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