Section 2

Functions

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a function?
- What is the domain of a function?
- What is the difference between the range and codomain of a function?
- What does it mean for a function to be an injection? A surjection?
- When and how is the composite of two functions defined?
- When and how is the inverse of a function defined?
- What do we mean by the image and inverse image of a set under a function?
- What properties relate images and inverse images of sets and set unions?

Introduction

Many topological properties are defined using continuous functions. We will focus on continuity later – for now we review some important concepts related to functions. Much of this should be familiar, but some might be new.

First we present the basic definitions. Much of our previous work has probably been with functions that map from the reals to the reals, but we will be considering functions form a more general perspective. We start with a formal definition of a function.

Definition 2.1. A **function** f from a nonempty set A to a set B is a collection of ordered pairs (a, b) so that

- for each $a \in A$ there is a pair (a, b) in f, and
- if (a, b) and (a, b') are in f, then $b = b'$.

Note that the first property is an existence property – that if $a \in A$ then there is an element b in B that matches up with a. This first property also says that every element in A is used, or that every element in A is paired with an element in B, and the element in B depends on the element in A that is chosen. The second property is a uniqueness one – that there is only one element b in B that is paired with a given element a in A.

We generally use an alternate notation for a function. If (a, b) is an element of a function f, we write

$$f(a) = b,$$

and in this way we think of f as a mapping from the set A to the set B. We indicate that f is a mapping from set A to set B with the notation

$$f : A \to B.$$

If f maps the element $a \in A$ to the element $b \in B$ we also use the notation

$$f : a \mapsto b.$$

There is some familiar terminology and notation associated with functions. Let f be a function from a set A to a set B.

- The set A is called the **domain** of f, and we write $\text{dom}(f) = A$.
- The set B is called the **codomain** of f, and we write $\text{codom}(f) = B$.
- The subset $\{f(a) \mid a \in A\}$ of B is called the **range** of f, which we denote by $\text{range}(f)$.
- If $a \in A$, then $f(a)$ is the **image** of a under f. Since each a in A is paired with a unique $b \in B$, there is only one image of a under f. That is why it is appropriate to use the work "the" when referring to the image of an element.
- If $b \in B$ and $b = f(a)$ for some $a \in A$, then a is called a **preimage** of b. For a given $b \in B$, there may be many different preimages of b, no preimages of b, or just one preimage of b. It can be instructive to construct examples of each situation. The fact that a preimage of an element b may not be unique is the reason we use the word "a" when referring to a preimage.

Knowing the domains and codomains is very important when working with functions, and we will pay a lot of attention to these sets.

We have likely been exposed to one-to-one and onto function in our past mathematical experiences. One-to-one functions (or injections) and onto functions (or surjections) are special types of functions and we present their definitions here.

Definition 2.2. Let f be a function from a set A to a set B.

(1) The function f is an **injection** if, whenever (a, b) and (a', b) are in f, then $a = a'$. Alternatively, using the function notation, f is an injection if $f(a) = f(a')$ implies $a = a'$.

(2) The function f is a **surjection** if, whenever $b \in B$, then there is an $a \in A$ so that (a, b) is in f. Alternatively, using the function notation, f is a surjection if for each $b \in B$ there exists an $a \in A$ so that $f(a) = b$.

(3) The function f is a **bijection** if f is both an injection and a surjection.

Preview Activity 2.1. We often define functions with rules, but functions can also be defined by tables or graphs. We will work with functions defined by rules in this activity. The goal of this activity is to illustrate why the domain and the codomain are just as important as the rule defining the outputs when want to determine if a function is one-to-one and/or onto. As an example, let $f(x) = x^2 + 1$. (Note that f is the function and $f(x)$ is the image of x under f.) Notice that

$$f(2) = 5 \text{ and } f(-2) = 5.$$

This observation is enough to prove that the function f is not an injection since we can see that there exist two different inputs that produce the same output.

Since $f(x) = x^2 + 1$, we know that $f(x) \geq 1$ for all $x \in \mathbb{R}$. This implies that the function f is not a surjection. For example, -2 is in the codomain of f and $f(x) \neq -2$ for all x in the domain of f.

(1) We can change the domain of a function so that the function is defined on a subset of the original domain. Such a function is called a restriction.

Definition 2.3. Let f be a function from a set A to a set B and let C be a subset of A. The **restriction** of f to C is the function $F : C \to B$ satisfying

$$F(c) = f(c) \text{ for all } c \in C.$$

A notation used for the restriction is also $F = f\mid_C$. We also call f an *extension* of F.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$, and let $h = f\mid_{\mathbb{R}^+}$, where $\mathbb{R}^+$ is the set of positive real numbers. So h has the same codomain as f, but a different domain.

(a) Show that h is an injection.

(b) Is h a surjection? Justify your conclusion.

(2) Let $T = \{y \in \mathbb{R} \mid y \geq 1\}$, and let $F : \mathbb{R} \to T$ be defined by $F(x) = f(x)$. Notice that the function F uses the same formula as the function f and has the same domain as f, but has a different codomain than f.

(a) Explain why F is not an injection.

(b) Is F a surjection? Justify your conclusion.

(3) Let $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \geq 0\}$. Define $g : \mathbb{R}^* \to T$ by $g(x) = x^2 + 1$.

(a) Prove or disprove: the function g is an injection.

(b) Prove or disprove: the function g is a surjection.

In our preview activity, the same mathematical formula was used to determine the outputs for the functions. However:

- One of the functions was neither an injection nor a surjection.
- One of the functions was not an injection but was a surjection.
- One of the functions was an injection but was not a surjection.
- One of the functions was both an injection and a surjection.

This illustrates the important fact that whether a function is injective or surjective not only depends on the formula that defines the output of the function but also on the domain and codomain of the function.

An important special function that is always an injection and surjection is the *identity* function on a set. If A is a set, the identity function on A is denoted as i_A, and $i_A(a) = a$ for every $a \in A$.

Composites of Functions

In our past mathematical experiences, we have often added and multiplied functions together (e.g., if $f(x) = x^2$ and $g(x) = x + 1$ map from $\mathbb{R}$ to $\mathbb{R}$, then $(fg)(x) = x^2(x + 1)$ and $(f + g)(x) = x^2 + (x + 1)$). In topology, we generally don't care about any algebraic structure a set might have, so we will move away from sums and products, and focus on compositions of functions.

The basic idea of function composition is that, when possible, the output of a function f is used as the input of a function g. The resulting function can be referred to as "f followed by g" and is called the composite of f with g. The notation we use is $g \circ f$ (note the order – f is applied first). For example, if

$$f(x) = 3x^2 + 2 \text{ and } g(x) = \sin(x),$$

both mapping $\mathbb{R}$ to $\mathbb{R}$, then we can compute $(g \circ f)(x)$ as follows:

$$(g \circ f)(x) = g(f(x)) = g(3x^2 + 2) = \sin\left(3x^2 + 2\right).$$

In this case, $f(x)$, the output of the function f, was used as the input for the function g. This idea motivates the formal definition of the composition of two functions.

Definition 2.4. Let A, B, and C be nonempty sets, and let $f : A \to B$ and $g : B \to C$ be functions. The **composite** of f and g is the function $g \circ f : A \to C$ defined by

$$(g \circ f)(x) = g(f(x))$$

for all $x \in A$

We refer to the function $g \circ f$ as a composite function, and we read $(g \circ f)(x)$ as "g of f" of x.

Activity 2.1. Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, and $C = \{\alpha, \beta, \gamma\}$. Define $f : A \to B$, $g : A \to B$, and $h : B \to C$ by

$$f(1) = b,\ f(2) = c,\ f(3) = a,$$

$$g(1) = d,\ g(2) = c,\ g(3) = d, \text{ and}$$

$$h(a) = \gamma,\ h(b) = \alpha,\ h(c) = \beta,\ h(d) = \alpha.$$

(a) Find the images of the elements in A under the function $h \circ f$.

(b) Find the images of the elements in A under the function $h \circ g$.

(c) Are any of f, g, and h injections? Are any of f, g, and h surjections?

(d) Is $h \circ f$ an injection? Is $h \circ f$ a surjection? Explain.

(e) Is $h \circ g$ an injection? Is $h \circ g$ a surjection? Explain.

In Activity 2.1, we asked questions about whether certain composite functions were injections and/or surjections. In mathematics, it is typical to explore whether certain properties of an object transfer to related objects. In particular, we might want to know whether or not the composite of two injective functions is also an injection. (Of course, we could ask a similar question for surjections.) These questions are explored in the next activity.

Activity 2.2. Let the sets A, B, C, and D be as follows:

$$A = \{a, b, c\}, \quad B = \{p, q, r\}, \quad C = \{u, v, w, x\}, \quad \text{and} \quad D = \{u, v\}.$$

(a) Construct a function $f : A \to B$ that is an injection and a function $g : B \to C$ that is an injection. In this case, is the composite function $g \circ f : A \to C$ an injection? Explain.

(b) Construct a function $f : A \to B$ that is a surjection and a function $g : B \to D$ that is a surjection. In this case, is the composite function $g \circ f : A \to D$ a surjection? Explain.

(c) Construct a function $f : A \to B$ that is a bijection and a function $g : B \to A$ that is a bijection. In this case, is the composite function $g \circ f : A \to A$ a bijection? Explain.

In Activity 2.2, we explored some properties of composite functions related to injections, surjections, and bijections. The following theorem summarizes the results that these explorations were intended to illustrate.

Theorem 2.5. *Let A, B, and C be nonempty sets, and assume that $f : A \to B$ and $g : B \to C$.*

(1) If f and g are both injections, then $(g \circ f) : A \to C$ is an injection.

(2) If f and g are both surjections, then $(g \circ f) : A \to C$ is a surjection.

(3) If f and g are both bijections, then $(g \circ f) : A \to C$ is a bijection.

Activity 2.3.

(a) Prove part (1) of Theorem 2.5.

(b) Prove part (2) of Theorem 2.5.

(c) Why is the proof of part (3) of Theorem 2.5 a direct consequence of parts (1) and (2)?

Inverse Functions

Now that we have studied composite functions, we will move on to consider another important idea: the inverse of a function. In previous mathematics courses, you probably learned that the exponential function (with base e) and the natural logarithm functions are inverses of each other. You may have seen this relationship expressed as follows:

$$\text{For each } x \in \mathbb{R} \text{ with } x > 0 \text{ and for each } y \in \mathbb{R},$$
$$y = \ln(x) \text{ if and only if } x = e^y.$$

Notice that x is the input and y is the output for the natural logarithm function if and only if y is the input and x is the output for the exponential function. In essence, the inverse function (in this case, the exponential function) reverses the action of the original function (in this case, the natural logarithm function). In terms of ordered pairs (input-output pairs), this means that if (x, y) is an ordered pair for a function, then (y, x) is an ordered pair for its inverse. The idea of reversing the roles of the first and second coordinates is the basis for our definition of the inverse of a function.

Definition 2.6. Let $f : A \to B$ be a function. The **inverse** of f, denoted by f^{-1}, is the set of ordered pairs

$$f^{-1} = \{(b, a) \in B \times A \mid (a, b) \in f\}.$$

Notice that this definition does not state that f^{-1} is a function. Rather, f^{-1} is simply a subset of $B \times A$. In Activity 2.4, we will explore the conditions under which the inverse of a function $f : A \to B$ is itself a function from B to A.

Activity 2.4. Let $A = \{a, b, c\}$, $B = \{a, b, c, d\}$, and $C = \{p, q, r\}$. Define

$f : A \to C$ by	$g : A \to C$ by	$h : B \to C$ by
$f(a) = r$	$g(a) = p$	$h(a) = p$
$f(b) = p$	$g(b) = q$	$h(b) = q$
$f(c) = q$	$g(c) = p$	$h(c) = r$
		$h(d) = q$

(a) Determine the inverse of each function as a set of ordered pairs.

(b) i. Is f^{-1} a function from C to A? Explain.

ii. Is g^{-1} a function from C to A? Explain.

iii. Is h^{-1} a function from C to B? Explain.

(c) Make a conjecture about what conditions on a function $F : S \to T$ will ensure that its inverse is a function from T to S.

The result of the Activity 2.4 should have been the following theorem.

Theorem 2.7. *Let A and B be nonempty sets, and let $f : A \to B$. The inverse of f is a function from B to A if and only if f is a bijection.*

The proof of Theorem 2.7 is outlined in the following activity.

Activity 2.5. Theorem 2.7 is a biconditional statement, so we need to prove both directions. Let A and B be nonempty sets, and let $f : A \to B$.

(a) Assume that f is a bijection. We will prove that f^{-1} is a function, that is that f^{-1} satisfies the conditions of Definition 2.1.

i. Let $b \in B$. What property does f have that ensures that $(b, a) \in f^{-1}$ for some $a \in A$? What conclusion can we draw about f^{-1}?

ii. Now let $b \in B$, $a_1, a_2 \in A$ and assume that

$$(b, a_1) \in f^{-1} \text{ and } (b, a_2) \in f^{-1}.$$

What does this tell us about elements that must be in f? What property of f ensures that $a_1 = a_2$? What conclusion can we draw about f^{-1}?

(b) Now assume that f^{-1} is a function from B to A. We will prove that f is a bijection.

i. What does it take to prove that f is an injection? Use the fact that f^{-1} is a function to prove that f is an injection.

ii. What does it take to prove that f is a surjection? Use the fact that f^{-1} is a function to prove that f is a surjection.

In the situation where $f : A \to B$ is a bijection and f^{-1} is a function from B to A, we can write $f^{-1} : B \to A$. In this case, we frequently say that f is an **invertible function**, and we usually do not use the ordered pair representation for either f or f^{-1}. Instead of writing $(a, b) \in f$, we write $f(a) = b$, and instead of writing $(b, a) \in f^{-1}$, we write $f^{-1}(b) = a$. Using the fact that $(a, b) \in f$ if and only if $(b, a) \in f^{-1}$, we can now write $f(a) = b$ if and only if $f^{-1}(b) = a$. Theorem 2.8 formalizes this observation.

Theorem 2.8. *Let A and B be nonempty sets, and let $f : A \to B$ be a bijection. Then $f^{-1} : B \to A$ is a function, and for every $a \in A$ and $b \in B$,*

$$f(a) = b \text{ if and only if } f^{-1}(b) = a.$$

The next result provide useful information about inverse functions. The proofs are left for Exercise (8).

Corollary 2.9. *Let A and B be nonempty sets, and let $f : A \to B$ be a bijection. Then*

(1) For every x in A, $\left(f^{-1} \circ f\right)(x) = x$.

(2) For every y in B, $\left(f \circ f^{-1}\right)(y) = y$.

The next question to address is what we can say about a composition of bijections. In particular, if $f : A \to B$ and $g : B \to C$ are both bijections, then $f^{-1} : B \to A$ and $g^{-1} : C \to B$ are both functions. Must it be the case that $g \circ f$ is invertible and, if so, what is $(g \circ f)^{-1}$?

Activity 2.6. Let $f : A \to B$ and $g : B \to C$ both be bijections.

(a) Why do we know that $g \circ f$ is invertible?

(b) Now we determine the inverse of $g \circ f$. We might be tempted to think that $(g \circ f)^{-1}$ is $g^{-1} \circ f^{-1}$, but this composite is not defined because g^{-1} maps B to C and f^{-1} maps B to A. However, $f^1 \circ g^{-1}$ is defined. To prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, we need to prove that two functions are equal. How do we prove that two functions are equal?

(c) Suppose $c \in C$.

i. What tells us that there is a $b \in B$ so that $g(b) = c$?

ii. What tells us that there is an $a \in A$ so that $f(a) = b$?

iii. What element is $(g \circ f)^{-1}(c)$? Why?

iv. What element is $f^{-1}(b)$? Why? What element is $g^{-1}(c)$? Why?

v. What element is $(f^{-1} \circ g^{-1})(c)$? Why? What can we conclude about $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$? Explain.

The result of Activity 2.6 is contained in the next theorem.

Theorem 2.10. *Let $f : A \to B$ and $g : B \to C$ be bijections. Then $g \circ f$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Functions and Sets

We conclude this section with a connection between subsets and functions. A bit of notation first. If f is a function from a set X to a set Y, and if A is a subset of X and B is a subset of Y, we define $f(A)$ and $f^{-1}(B)$ as

$$f(A) = \{f(a) \mid a \in C\},$$

and

$$f^{-1}(B) = \{a \in A \mid f(a) \in B\}.$$

We call $f(A)$ the image of the set A under f and $f^{-1}(B)$ is the preimage of the set B under f. Note that $f^{-1}(B)$ is defined for any function, not just invertible functions. So it is important to recognize that the use of the notation $f^{-1}(B)$ does not imply that f is invertible.

When we work with continuous functions in later sections, we will need to understand how a function behaves with respect to subsets. One result is in the following lemma.

Lemma 2.11. *Let $f : X \to Y$ be a function and let $\{A_\alpha\}$ be a collection of subsets of X for α in some indexing set I, and $\{B_\beta\}$ be a collection of subsets of Y for β in some indexing set J. Then*

(1) $f\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} f(A_\alpha)$ *and*

(2) $f^{-1}\left(\bigcup_{\beta \in J} B_\beta\right) = \bigcup_{\beta \in J} f^{-1}(B_\beta)$.

Proof. Let $f : X \to Y$ be a function and let $\{A_\alpha\}$ be a collection of subsets of X for α in some indexing set I. To prove part 1, we demonstrate the containment in both directions.

Let $b \in f\left(\bigcup_{\alpha \in I} A_\alpha\right)$. Then $b = f(a)$ for some $a \in \bigcup_{\alpha \in I} A_\alpha$. It follows that $a \in A_\rho$ for some $\rho \in I$. Thus, $b \in f(A_\rho) \subseteq \bigcup_{\alpha \in I} f(A_\alpha)$. We conclude that $f\left(\bigcup_{\alpha \in I} A_\alpha\right) \subseteq \bigcup_{\alpha \in I} f(A_\alpha)$.

Now let $b \in \bigcup_{\alpha \in I} f(A_\alpha)$. Then $b \in f(A_\rho)$ for some $\rho \in I$. Since $A_\rho \subseteq \bigcup_{\alpha \in I} A_\alpha$, it follows that $b \in f\left(\bigcup_{\alpha \in I} A_\alpha\right)$. Thus, $\bigcup_{\alpha \in I} f(A_\alpha) \subseteq f\left(\bigcup_{\alpha \in I} A_\alpha\right)$. The two containments prove part 1.

For part 2, we again demonstrate the containments in both directions. Let $a \in f^{-1}\left(\bigcup_{\beta \in J} B_\beta\right)$. Then $f(a) \in \bigcup_{\beta \in J} B_\beta$. So there exists $\mu \in J$ such that $f(a) \in B_\mu$. This implies that $a \in f^{-1}(B_\mu) \subseteq \bigcup_{\beta \in J} f^{-1}(B_\beta)$. We conclude that $f^{-1}\left(\bigcup_{\beta \in J} B_\beta\right) \subseteq \bigcup_{\beta \in J} f^{-1}(B_\beta)$.

For the reverse containment, let $a \in \bigcup_{\beta \in J} f^{-1}(B_\beta)$. Then $a \in f^{-1}(B_\mu)$ for some $\mu \in J$. Thus, $f(a) \in B_\mu \subseteq \bigcup_{\beta \in J} B_\beta$. So $a \in f^{-1}\left(\bigcup_{\beta \in J} B_\beta\right)$. Thus, $\bigcup_{\beta \in J} f^{-1}(B_\beta) \subseteq f^{-1}\left(\bigcup_{\beta \in J} B_\beta\right)$. The two containments verify part 2. ■

At this point it is reasonable to ask if Lemma 2.11 would still hold if we replace unions with intersections. We leave that question for Exercise (7).

Another result is contained in the next activity.

Activity 2.7. Let X, Y, and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions. Let C be a subset of Z. There is a relationship between $(g \circ f)^{-1}(C)$ and $f^{-1}(g^{-1}(C))$. Find and prove this relationship.

The Cardinality of a Set

How big is a set? When a set is finite, we can count the number of elements in the set and answer the question directly. When a set is infinite, the question is a little more complicated. For example, how big is $\mathbb{Z}$? How big is $\mathbb{Q}$? Since $\mathbb{Z}$ is a subset of $\mathbb{Q}$, we might think that $\mathbb{Q}$ contains more elements than $\mathbb{Z}$. But $\mathbb{Z}$ is infinite and how many more elements can we have than infinity? We won't answer that question in this section, but it is an interesting one to consider.

If two finite sets have the same number of elements, then it should seem natural to say that the sets are of the same size. How do we extend this to infinite sets? If two finite sets have the same number of elements, then we can pair each element in one set with exactly one element in the other. This is exactly what a bijection does. So a set with n elements can be paired with the set $\{1, 2, \ldots, n\}$, where n is a positive integer. This is how we can define a finite set.

Definition 2.12. A set A is a **finite** set if $A = \emptyset$ or there is a bijection f mapping A to the set $\{1, 2, 3, \ldots, n\}$ for some positive integer n.

In the case that $A = \emptyset$, we say that A has *cardinality* 0, and if there is a bijection from A to the set $\{1, 2, \ldots, n\}$, we say that A has cardinality n. If there is no positive integer n such that there is a bijection from set A to $\{1, 2, \ldots, n\}$ we say that A is an *infinite* set and say that A has infinite cardinality. We use the word *cardinality* instead of number of elements because we can't actually count the number of elements in an infinite set. We denote the cardinality of the set (the number of elements in the set) A by $|A|$. It is left to the homework to show that if A and B are sets with $|A| = n$ and $|B| = m$, then $n = m$ if and only if there is a bijection $f : A \to B$. This tells us that cardinality is well defined. Since composites of bijections are bijections with inverses that are bijections, if there is a bijection from set A to $\{1, 2, \ldots, n\}$ and a bijection from a set B

to $\{1, 2, \ldots, n\}$ for some positive integer n, then there is a bijection between A and B. Using this idea, we say that two sets (either finite or infinite) have the same cardinality if there is a bijection between the sets. We will discuss cardinality in more detail a bit later.

Summary

Important ideas that we discussed in this section include the following.

- A function f from a nonempty set A to a set B is a collection of ordered pairs (a, b) so that for each $a \in A$ there is a pair (a, b) in f, and if (a, b) and (a, b') are in f, then $b = b'$. If f is a function we use the notation $f(a) = b$ to indicate that $(a, b) \in f$.

- If f is a function from A to B, the set A is the domain of the function.

- If f is a function from A to B, the set B is the codomain of the function. The set

$$\{f(a) \mid a \in A\}$$

  is the range of the function. So the range of a function is a subset of the codomain.

- A function f from a set A to a set B is an injection if, whenever $f(a) = f(a')$ for a, $a' \in A$, then $a = a'$. The function f is a surjection if, whenever $b \in B$, then there is an $a \in A$ so that $f(a) = b$.

- If f is a function from a set A to a set B and if g is a function from B to a set C, then the composite $g \circ f$ is a function from A to C defined by $(g \circ f)(a) = g(f(a))$ for every $a \in A$.

- A function f from a set A to a set B is a bijection if f is both a surjection and injection. When f is a bijection from A to B, then f has an inverse f^{-1} defined by $f^{-1}(b) = a$ when $f(a) = b$.

- If f is a function from a set A to a set B, and if C is a subset of A, then image of C under f is the set

$$f(C) = \{f(c) \mid c \in C\},$$

  and if D is a subset of Y, the inverse image of D is the set

$$f^{-1}(D) = \{a \in A \mid f(a) \in D\}.$$

- Important properties that relate images and inverse images of sets and set unions are the following. If f is a function from a set X to a set Y, and if $\{A_\alpha\}$ is a collection of subsets of X for α in some indexing set I, and $\{B_\beta\}$ be a collection of subsets of Y for β in some indexing set J, then

  i. $f\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} f(A_\alpha)$ and
  ii. $f^{-1}\left(\bigcup_{\beta \in J} B_\beta\right) = \bigcup_{\beta \in J} f^{-1}(B_\beta)$.

Exercises

(1) (a) Find a function $f : \mathbb{R} \to \mathbb{R}$ such that each element in the codomain has exactly one preimage.

(b) Find a function $f : \mathbb{R} \to \mathbb{R}$ such that each element in the codomain has at least two preimages.

(c) Find a function $f : \mathbb{R} \to \mathbb{R}$ such that each element has exactly two preimages.

(d) Find a function $f : \mathbb{R} \to \mathbb{R}$ such that there is an element in the codomain that has exactly three preimages and another element in the codomain that as exactly two preminages.

(e) Find a function $f : \mathbb{R} \to \mathbb{R}$ such that there is an element in the codomain that has infinitely many preimages.

(2) For each of the following functions, determine if the function is an injection, a surjection, a bijection, or none of these. Remember to be careful about the domain and range in each case. Justify all of your conclusions.

(a) $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 5x + 3$, for all $x \in \mathbb{R}$

(b) $G : \mathbb{Z} \to \mathbb{Z}$ defined by $G(x) = 5x + 3$, for all $x \in \mathbb{Z}$

(c) $f : (\mathbb{R} \setminus \{4\}) \to \mathbb{R}$ defined by $f(x) = \dfrac{3x}{x-4}$, for all $x \in (\mathbb{R} \setminus \{4\})$

(d) $g : (\mathbb{R} \setminus \{4\}) \to (\mathbb{R} \setminus \{3\})$ defined by $g(x) = \dfrac{3x}{x-4}$, for all $x \in (\mathbb{R} \setminus \{4\})$

(e) $h : \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined by $h(x) = x^2$ for every $x \in \mathbb{R}$, where $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$

(f) $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $k(x) = x^2$ for every $x \in \mathbb{R}_{\geq 0}$

(3) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and let $B = \{a, b, c, d, e, g\}$, and define $f : A \to B$ as given in Table 2.1.

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	c	d	a	g	a	c	e	d	c	a

Table 2.1: A function from A to B.

(a) If f an injection? Is f a surjection? Explain.

(b) Find a largest subset C of A (largest in the number of elements of C) such that $f|_C$ is an injection.

(c) Find a subset D of B such that f is a surjection.

(d) Find subsets X of A and Y of B such that $f|_X : X \to Y$ is a bijection.

(4) Let A and B be sets, both of which have at least two distinct members.

(a) Illustrate a subset $X \subset A \times B$ that is the Cartesian product of a subset of A with a subset of B.

(b) Show that there is a subset $W \subset A \times B$ that is not the Cartesian product of a subset of A with a subset of B. [Thus, not every subset of a Cartesian product is the Cartesian product of a pair of subsets.]

(5) The cardinality of a finite set is defined to be the number of elements of that set. We denote the cardinality of a set A as $|A|$. Let A and B be sets with $|A| = n$ and $|B| = m$ for some positive integers m and n. Prove that there is a bijection $f : A \to B$ if and only if $n = m$.

(6) Let X and Y be sets and let $f : X \to Y$ be a function.

(a) Let A be a subset of X. Show that $A \subseteq f^{-1}(f(A))$. Make an example to show that in general, $A \neq f^{-1}(f(A))$. (Hint: To show that the sets are not equal, consider sets X and Y with two elements.)

(b) Let B be a subset of Y. Show that $f(f^{-1}(B)) \subseteq B$. Make an example to show that in general, $f(f^{-1}(B)) \neq B$. (Hint: To show that the sets are not equal, consider sets X and Y with two elements.)

(c) Prove that f is a surjection if and only if $f(f^{-1}(B)) = B$ for every subset B of Y.

(d) Prove that f is an injection if and only if $f^{-1}(f(A)) = A$ for every subset A of X.

(7) Let $f : X \to Y$ be a function and let $\{A_\alpha\}$ be a collection of subsets of X for α in some indexing set I, and $\{B_\beta\}$ be a collection of subsets of Y for β in some indexing set J. Prove or disprove each of the following. If a statement is not true, is either containment true? Prove your answers.

(a) $f\left(\bigcap_{\alpha\in I} A_\alpha\right) = \bigcap_{\alpha\in I} f(A_\alpha)$

(b) $f^{-1}\left(\bigcap_{\beta\in J} B_\beta\right) = \bigcap_{\beta\in J} f^{-1}(B_\beta)$

(8) Let A and B be nonempty sets, and let $f : A \to B$ be a bijection. Prove that

(a) For every x in A, $\left(f^{-1} \circ f\right)(x) = x$.

(b) For every y in B, $\left(f \circ f^{-1}\right)(y) = y$.

(9) Let R, S, and T be sets, and let $g : R \to S$ and $h : S \to T$ be functions. Let O be a subset of T. Show that $(h \circ g)^{-1}(O) = g^{-1}(h^{-1}(O)$.

(10) Let X_1 and X_2 be nonempty sets, and let $X = X_1 \times X_2$. Define $\pi_i : X \to X_i$ by $\pi_i(x) = x_i$, where $x = (x_1, x_2)$. We call π_i the *projection* of X onto X_i. Let Y_1 and Y_2 be nonempty sets, and let $Y = Y_1 \times Y_2$. Assume that for each i there is a function $f_i : X_i \to Y_i$. For example, let $X_i = \{i, i+1\}$ and $Y_i = \{-i, -i-1\}$. We could then define f_i by $f_i(x) = -x$ for i either 1 or 2.

(a) Prove that π_i is a surjection for each i.

(b) Prove that there is a unique function $f : X \to Y$ such that $\pi_i \circ f = f_i \circ \pi_i$ for each i. (Note that one of the π_i maps X to X_i and the other maps Y to Y_i.)

(c) The function f from part (b) is denoted as $f = f_1 \times f_2$. Let Z_1 and Z_2 be two nonempty sets, and let $Z = Z_1 \times Z_2$. Assume that there are functions $g_i : Y_i \to Z_i$ for each i. Show that

$$(g_1 \times g_2) \circ (f_1 \times f_2) = (g_1 \circ f_1) \times (g_2 \circ f_2).$$

(d) Suppose that each f_i has an inverse h_i. Show that $(f_1 \times f_2)^{-1} = h_1 \times h_1$.

(11) Let $\mathbb{N}$ be the set of positive integers. Define $f : \mathbb{N} \to \mathbb{Z}$ as follows: For each $n \in \mathbb{N}$, let

$$f(n) = \frac{1 + (-1)^n(2n-1)}{4}.$$

Is the function f an injection? Is the function f a surjection? Justify your conclusions. (Hint: Start by calculating several outputs for the function before you attempt to write a proof. In exploring whether or not the function is an injection, it might be a good idea to use cases based on whether the inputs are even or odd. In exploring whether f is a surjection, consider using cases based on whether the output is positive or less than or equal to zero.)

(12) An operation $*$ on a set S is a function from $S \times S$ to S that assigns to the pair $(x, y) \in S \times S$ the element $x * y$ in S. For example, addition of integers can be defined as a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ that maps the pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ to the integer $f(a, b) = a + b$.

(a) Is the function f an injection? Justify your conclusion.

(b) Is the function f a surjection? Justify your conclusion.

(13) Let A, B, and C be sets and let $f : A \to B$ and $g : B \to C$ be functions.

(a) Is it true that if $g \circ f$ is an injection, then both f and g are injections? If the answer is no, are there any conditions that f or g must satisfy if $g \circ f$ an injection? Prove your answers.

(b) Is it true that if $g \circ f$ is a surjection, then both f and g are surjections? If the answer is no, are there any conditions that f or g must satisfy if $f \circ g$ a surjection? Prove your answers.

(14) (a) Is composition of functions a commutative operation? Prove your answer.

(b) Is composition of functions an associative operation? Prove your answer.

(15) (a) Define $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $f([x]) = \left[x^2 + 4\right]$ for all $[x] \in \mathbb{Z}_5$. Write the inverse of f as a set of ordered pairs, and explain why f^{-1} is not a function.

(b) Define $g : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $g([x]) = \left[x^3 + 4\right]$ for all $[x] \in \mathbb{Z}_5$. Write the inverse of g as a set of ordered pairs, and explain why g^{-1} is a function.

(c) Is it possible to write a formula for $g^{-1}([y])$, where $[y] \in \mathbb{Z}_5$? The answer to this question depends on whether or not it is possible to define a cube root of elements of $\mathbb{Z}_5$. Recall that for a real number x, we define the cube root of x to be the real number y such that $y^3 = x$. That is,

$$y = \sqrt[3]{x} \text{ if and only if } y^3 = x.$$

Using this idea, is it possible to define the cube root of each element of $\mathbb{Z}_5$? If so, what is $\sqrt[3]{[0]}$, $\sqrt[3]{[1]}$, $\sqrt[3]{[2]}$, $\sqrt[3]{[3]}$, and $\sqrt[3]{[4]}$.

(d) Now answer the question posed at the beginning of part (c). If possible, determine a formula for $g^{-1}([y])$ where $g^{-1} : \mathbb{Z}_5 \to \mathbb{Z}_5$.

(16) Let A be the set of all functions $f : [a,b] \to \mathbb{R}$ that are continuous on $[a,b]$ (use your memory of continuous functions from calculus for this problem). Let B be the subset of A consisting of all functions possessing a continuous derivative on $[a,b]$. Let C be the subset of B consisting of all functions whose value at a is 0.

(a) i. Give an example of a function that is in A and not in B with $[a,b] = [-1,1]$.
ii. Give an example of a function that is in B but not in C with $[a,b] = [-1,1]$.
iii. Give an example of a function that is in C with $[a,b] = [-1,1]$.

(b) Let $d : B \to A$ be defined by
$$d(f) = f'.$$
Is the function d invertible? Justify your response.

(c) To each function $f \in A$, let $h(f)$ be the function defined by
$$(h(f))(x) = \int_a^x f(t)\,dt$$
for $x \in [a,b]$.
i. Verify that h maps A to C.
ii. Show that h is invertible by finding a function $g : C \to A$ such that g and h are inverse functions.

(17) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) If A is a subset of X, then $A \subseteq f^{-1}(f(A))$.

(b) If A is a subset of X, then $f^{-1}(f(A)) \subseteq A$.

(c) If B is a subset of Y, then $B \subseteq f(f^{-1}(B))$.

(d) If B is a subset of Y, then $f(f^{-1}(B)) \subseteq B$.

(e) If A_1 and A_2 are subsets of X with $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.

(f) If B_1 and B_2 are subsets of Y with $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

(g) If B_1 and B_2 are subsets of Y with $B_1 \subseteq B_2$, then $f^{-1}(B_2) \subseteq f^{-1}(B_1)$.

(h) If A_1 and A_2 are subsets of X, then $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(i) If B_1 and B_2 are subsets of Y, then $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

(j) If A_1 and A_2 are subsets of X, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

(k) If B_1 and B_2 are subsets of Y, then $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(l) If A_1 and A_2 are subsets of X, then $f(A_1 \setminus A_2) = f(A_1) \setminus f(A_2)$.

(m) If B_1 and B_2 are subsets of Y, then $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$.