Section 20

Products of Topological Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is the product of a finite number of topological spaces?
- How do we define a topology on the product of a finite number of topological spaces?
- What is a projection map from a product of a finite number of topological spaces?
- How can we use projection maps to determine the continuity of a function to a product of a finite number of topological spaces?
- What is a subbasis of a topological space?
- What properties do product spaces inherit from their factors?

Introduction

In Section 11 we saw how we can make a Cartesian product of two metric spaces into a metric space. This is exactly the construction that allows us to work with the Cartesian plane $\mathbb{R}^2$ as a metric space with the usual metric. As we discussed in Section 12, every metric space is a topological space, but not every topological space is metrizable. So knowing how to make a product of metric spaces into a metric space still leaves open the question of how we can make the product of topological spaces into a topological space. If we have two topological spaces (X, τ_X) and (Y, τ_Y), a natural approach to this problem might be to take as the open sets in $X \times Y$ the sets of the form $U \times V$ where $U \in \tau_X$ and $V \in \tau_Y$. We investigate this idea in Preview Activity 20.1.

Preview Activity 20.1. Let $X = \{a, b, c\}$ with $\tau_X = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$, and let $Y = \{1, 2\}$ with $\tau_Y = \{\emptyset, \{1\}, Y\}$.

(1) Let
$$\mathcal{B} = \{U \times V \mid U \in \tau_X \text{ and } V \in \tau_Y\}. \tag{20.1}$$
List all of the sets in $\mathcal{B}$ along with their elements.

(2) Assume that all of the sets in $\mathcal{B}$ are open sets in $X \times Y$. Should the set $A = \{(a,1),(a,2),(b,1)\}$ be an open set in $X \times Y$? Is the set A of the form $U \times V$ for some open sets U in X and V in Y? Explain. Is $\mathcal{B}$ a topology on $X \times Y$?

(3) If $\mathcal{B}$ is not a topology on $X \times Y$, what is the smallest collection of sets would we need to add to $\mathcal{B}$ to make a topology on $X \times Y$? Explain your process.

The Topology on a Product of Topological Spaces

In our preview activity we learned that we cannot make a topology on a product $X \times Y$ of topological spaces (X, τ_X) and (Y, τ_Y) with just the sets of the form $U \times V$ where $U \in \tau_X$ and $V \in \tau_Y$ as the open sets since the collection of these sets is not closed under arbitrary unions. What we can do instead is consider these unions of all of the sets of the form $U \times V$, where U is open in X and V is open in Y. In other words, consider these sets to be a basis for the topology on $X \times Y$.

Activity 20.1. Let (X, τ) and (Y, τ_Y) be topological spaces, and let $\mathcal{B}$ be as defined in (20.1). Prove that $\mathcal{B}$ is a basis for a topology on $X \times Y$.

The argument from Activity 20.1 can be extended to a product of any finite number of topological spaces. Let n be a positive integer and let (X_i, τ_i) be topological spaces for i from 1 to n. Let
$$\mathcal{B} = \left\{\Pi_{i=1}^n O_i \mid O_i \text{ is open in } X_i\right\}.$$
Since $X_i \in \tau_i$ for every i, every point in $\Pi_{i=1}^n X_i$ is in a set in $\mathcal{B}$. So $\mathcal{B}$ satisfies condition 1 of a basis. Now we show that $\mathcal{B}$ satisfies the second condition of a basis. Let $B_1 = \Pi_{i=1}^n U_i$ and $B_2 = \Pi_{i=1}^n V_i$ for some open sets U_i, V_i in X_i. Suppose $(x_i) \in (B_1 \cap B_2)$. Then for each j we have $x_j \in U_j \cap V_j$ and so
$$(x_i) \in \Pi_{i=1}^n (U_i \cap V_i).$$
Since $U_i \cap V_i$ is an open set in X_i, it follows that $\Pi_{i=1}^n (U_i \cap V_i)$ is in $\mathcal{B}$. Thus, $\mathcal{B}$ is a basis for a topology on $X \times Y$.

This topology generated by products of open sets is called the *box* or *product* topology.

Definition 20.1. Let (X_α, τ_α) be a collection of topological spaces for α in some finite indexing set I. The **box topology** or **product topology** on the product $\Pi_{\alpha \in I} X_\alpha$ is the topology with basis
$$\mathcal{B} = \left\{\Pi_{\alpha \in I} U_\alpha \mid U_\alpha \in \tau_\alpha \text{ for each } \alpha \in I\right\}.$$

So we can always make the product of topological spaces into a topological space using the box topology.

Three Examples

In this section we consider three specific examples of a product of topological spaces.

Activity 20.2. Let $X = [1, 2]$ and $Y = [3, 4]$ as subspaces of $\mathbb{R}^2$.

(a) Explain in detail what the product space $X \times Y$ looks like.

(b) Find, if possible, an open subset of $X \times Y$ that is not of the form $U \times V$ where U is open in X and V is open in Y.

Activity 20.3. Let $X = \mathbb{R}$ and $Y = S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$, the unit circle as a subset of $\mathbb{R}^2$.

(a) Draw a picture of $\mathbb{R}$. For each $x \in \mathbb{R}$, the set $\mathbb{R}_x = \{(x, y) \mid y \in S^1\}$ is a subset of $\mathbb{R} \times S^1$. On your graph of $\mathbb{R}$, draw pictures of $\mathbb{R}_x$ for x equal to -1, 0, and 1. Explain in detail what the product space $\mathbb{R} \times S^1$ looks like.

(b) Consider the sets of the form $B \cap S^1$, where B is an open ball in $\mathbb{R}^2$ (relatively open sets in S^1). What do these sets look like?

(c) Describe the shape of the basis elements for the product topology on $\mathbb{R} \times S^1$ that result from products of the form $U \times V$, where U is an open interval in $\mathbb{R}$ and V is the intersections of S^1 with an open ball in $\mathbb{R}^2$.

Activity 20.4. Let $2S^1 = \{(x, y) \mid x^2 + y^2 = 4\}$ be the circle of radius 2 centered at the origin as a subset of $\mathbb{R}^2$. In this activity we investigate the space $2S^1 \times S^1$.

(a) Draw a picture of $2S^1$ in the xy-plane. For each $p \in S^1$, the set $S^1_p = \{(p, y) \mid y \in S^1\}$ is a subset of $S^1 \times S^1$. On your graph of S^1, draw pictures of S^1_p for p equal to $(1, 0)$, $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, and $(0, 1)$. Orient the graphs so that the copies of S^1 are perpendicular to $2S^1$. Explain in detail what the product space $2S^1 \times S^1$ looks like.

(b) Consider the sets of the form $B \cap S^1$, where B is an open ball in $\mathbb{R}^2$. What do these sets look like?

(c) Describe the shape of the basis elements for the product topology on $2S^1 \times S^1$ that result from products of the form $U \times V$, where U and V are intersections of S^1 with open balls in $\mathbb{R}^2$.

Projections and Continuous Functions on Products

Given topological spaces (X_1, τ_1) and (X_2, τ_2), we define $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ by $\pi_1((x, y)) = x$ and $\pi_2((x, y)) = y$. These functions π_1 and π_2 are the *projections* of $X_1 \times X_2$ onto X_1 and X_2, respectively. These projection functions can help us determine when a function f from a topological space Y to $X_1 \times X_2$ is continuous.

Activity 20.5. Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let O_1 be an open set in X_1.

(a) Determine which set is $\pi_1^{-1}(O_1)$. Verify your conjecture.

(b) Explain why π_1 is continuous.

The same argument as in Activity 20.5 shows that π_2 is also a continuous function. In general, if $X = \Pi_{i=1}^n X_i$ is a finite product of topological spaces, then the projection $\pi_k : X \to X_k$ is a continuous function for each k, where $\pi_k((x_1, x_2, \ldots, x_n)) = x_k$.

Let $O = \Pi_{i=1}^n O_i$ be a basic open set in $X = \Pi_{i=1}^n X_i$, where X_i is a topological space for each i. We can extend the result of Activity 20.5 to see that

$$\pi_i^{-1}(O_i) = X_1 \times X_2 \times \cdots \times X_{i-1} \times O_i \times X_{i+1} \times \cdots \times X_n.$$

So

$$\Pi_{i=1}^n O_i = \bigcap_{i=1}^n \pi_i^{-1}(O_i).$$

So each basic open set is a finite intersection of sets of the form $\pi_i^{-1}(O_i)$ where O_i is open in X_i. When this happens, we call the collection of sets of the form $\pi_i^{-1}(O_i)$ a *subbasis* of the topology.

Definition 20.2. Let (X, τ) be a topological space. A subset $\mathcal{S}$ of τ is a **subbasis** or **subbase** for τ if the set of all finite intersections of elements of $\mathcal{S}$ is a basis for τ.

As an example, since finite intersections of intervals of the form $(-\infty, b)$ and (a, ∞) give all intervals of the form (a, b), the collection $\mathcal{S} = \{(-\infty, b), (a, \infty) \mid a, b \in \mathbb{R}\}$ is a subbasis for the standard topology on $\mathbb{R}$. Note that this collection itself is not a basis for the standard topology on $\mathbb{R}$. If $X = \Pi_{i=1}^n X_i$ is a product of topological space, then another example of a subbasis is the collection

$$\mathcal{S} = \bigcup_{i=1}^n \{\pi_i^{-1}(O_i) \mid O_i \text{ is open in } X_i\}.$$

This set is a subbasis for the product topology on X (the verification of this is left to Exercise (1)).

We note here that there is another topology, called the *product topology*, on X with subbasis $S = \bigcup_{\alpha \in I} S_\alpha$, where

$$S_\alpha = \{\pi_\alpha^{-1}(U_\alpha) \mid U_\alpha \text{ is open in } X_\alpha\}.$$

For reasons we won't go into, the product topology is preferred to the box topology for infinite products (many important theorems that hold for finite products will not hold for infinite products using the box topology, but will hold using the product topology). However, the product topology and the box topology are the same for finite products, and since we won't consider infinite products here we will not worry about the distinction. For our purposes we will use the terms "box topology" and "product topology" interchangeably.

As we have discussed before, it can often be easier to define a topology using a basis or subbasis than it is to describe all of the sets in the topology. As we might expect, since the continuity of a function can be determined by the inverse image of basis elements, the continuity of a function can also be determined by the inverse image of subbasis elements.

Activity 20.6. Prove Theorem 20.3. (Hint: Recall that f is continuous if $f^{-1}(B)$ is open in X for each basic open set B.)

Theorem 20.3. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, let $\mathcal{S}$ be a subbasis for τ_Y, and let $f : X \to Y$ be a function. If $f^{-1}(S)$ is open in X for each $S \in \mathcal{S}$, then f is continuous.*

Now suppose that X_1, X_2, and Y are topological spaces, and that $f : Y \to X_1 \times X_2$ is a function. Then $\pi_1 \circ f$ maps Y to X_1 and $\pi_2 \circ f$ maps Y to X_2. Since the composition of continuous functions is continuous, we can see that if f is continuous so are $\pi_1 \circ f$ and $\pi_2 \circ f$. To determine if f is a continuous function, it would be useful to know if the converse is true. A key idea in the proof is the result of Exercise (9) on page 26 that if R, S, and T are sets, and $g : R \to S$ and $h : S \to T$ are functions, then $(h \circ g)^{-1}(O) = g^{-1}(h^{-1}(O)$ for any subset O of T.

Now we can use projections to determine when functions to product spaces are continuous.

Theorem 20.4. *Let X_i for i from 1 to n and Y be topological spaces, and let $f : Y \to \Pi_{i=1}^{n} X_i$ be a function. Then f is continuous if and only if $\pi_i \circ f$ is continuous for each i.*

Proof. Let X_i for i from 1 to n and Y be topological spaces, and let $f : Y \to \Pi X_i$ be a function. If f is continuous, the facts that each π_i is continuous and that composites of continuous functions are continuous show that $\pi_i \circ f$ is continuous for each i.

Now suppose that $\pi_i \circ f$ is continuous for each i. Recall that

$$\mathcal{S} = \{\pi_i^{-1}(O_i) \mid O_i \text{ is open in } X_i\}$$

is a subbasis for the product topology on $\Pi_{i=1}^{n} X_i$. To prove that f is continuous, Theorem 20.3 tells us that it is enough to show that $f^{-1}(S)$ is open for each S in $\mathcal{S}$. Let O_i be an open set in X_i. Exercise (9) on page 26 tells us that

$$f^{-1}(\pi_i^{-1}(O_i)) = (\pi_i \circ f)^{-1}(O_i),$$

which is open in Y because $\pi_i \circ f$ is continuous. Therefore, f is continuous. ■

Properties of Products of Topological Spaces

It is natural to ask what topological properties of the topological spaces (X, τ_X) and (Y, τ_Y) are inherited by the product $X \times Y$. We have studied Hausdorff, connected, and compact spaces, and we now consider those properties.

Activity 20.7. Let (X, τ_X) and (Y, τ_Y) be Hausdorff spaces.

(a) What will it take to prove that the space $X \times Y$ with the product topology is Hausdorff?

(b) Suppose that $(x_1, y_1), (x_2, y_2) \in X \times Y$. What does the fact that X is Hausdorff tell us about x_1 and x_2? What can we say about y_1 and y_2?

(c) Complete the proof of the following theorem.

Theorem 20.5. *If (X, τ_X) and (Y, τ_Y) are Hausdorff spaces, then $X \times Y$ with the product topology is a Hausdorff space.*

The proofs that a product of connected spaces is connected, that a product of path connected spaces is path connected, and that a product of compact spaces is compact are a bit more complicated. To prove that a product of two connected spaces is connected, we will use the result of Activity 18.6 in Section 18 that the union of connected subsets is connected if the intersection of the subsets is nonempty. A consequence of this result is the following.

Lemma 20.6. *Let X be a topological space, and let A_α be a connected subset of X for all α in some indexing set I. Let B be a connected subset of X such that $A_\alpha \cap B \neq \emptyset$ for every $\alpha \in I$. Then $B \cup \left(\bigcup_{\alpha \in I} A_\alpha\right)$ is connected.*

Proof. Let X be a topological space, and let A_α be a connected subset of X for all α in some indexing set I. Let B be a connected subset of X such that $A_\alpha \cap B \neq \emptyset$ for every $\alpha \in I$. For each $\alpha \in I$ let $B_\alpha = B \cup A_\alpha$. Let $\beta \in I$. Since $B \cap A_\beta \neq \emptyset$, Lemma 20.6 shows that B_β is connected. Given that B is not empty, and $B \subseteq \bigcap_{\alpha \in I} B_\alpha$, we see that $\bigcap_{\alpha \in I} B_\alpha \neq \emptyset$. Lemma 20.6 allows us to conclude that $\bigcup_{\alpha \in I} B_\alpha$ is connected. But

$$\bigcup_{\alpha \in I} B_\alpha = \bigcup_{\alpha \in I} (B \cup A_\alpha) = B \cup \left(\bigcup_{\alpha \in I} A_\alpha \right),$$

and so $B \cup \left(\bigcup_{\alpha \in I} A_\alpha\right)$ is connected. ■

We will use Lemma 20.6 to show that a product of connected spaces is connected.

Theorem 20.7. *If (X, τ_X) and (Y, τ_Y) are connected topological spaces, then $X \times Y$ with the product topology is a connected topological space.*

Proof. Assume (X, τ_X) and (Y, τ_Y) are connected topological spaces. Our approach to proving that $X \times Y$ is connected is to write $X \times Y$ as a union of two connected subspaces whose intersection is not empty. Let $a \in X$. The space $X_a = \{a\} \times Y$ is homeomorphic to Y via the inclusion map i which sends $(a, t) \in \{a\} \times Y$ to the point $t \in Y$. Since Y is connected, so is X'. Let $b \in Y$. The space $Y_b = X \times \{b\}$ is homeomorphic to X via the inclusion map i which sends $(s, b) \in X \times \{b\}$ to the point $s \in X$. Since X is connected, so is Y_b. (The verification of these homeomorphisms is left to the reader.) The point (a, b) is in $X_a \cap Y_b$, so $X_a \cap Y_b \neq \emptyset$ for every $b \in Y$. It follows that $X_a \cup \left(\bigcup_{t \in Y} Y_t\right)$ is connected by Lemma 20.6. All that remains is to prove that $X_a \cup \left(\bigcup_{t \in Y} Y_t\right) = X \times Y$ and we will have demonstrated that $X \times Y$ is connected. The fact that $X_a \subseteq X \times Y$ and $Y_t \subseteq X \times Y$ for every $t \in Y$ implies that $X_a \cup \left(\bigcup_{t \in Y} Y_t\right) \subseteq X \times Y$. It then remains to show that $X \times Y \subseteq X_a \cup \left(\bigcup_{t \in Y} Y_t\right)$. Let $(u, v) \in X \times Y$. Then $u \in X$ and $v \in Y$ and $(u, v) \in Y_v$. Thus, $X \times Y \subseteq X_a \cup \left(\bigcup_{t \in Y} Y_t\right)$ and so $X \times Y = X_a \cup \left(\bigcup_{t \in Y} Y_t\right)$. Therefore, $X \times Y$ is connected. ■

Once we know that a product of connected topological spaces is connected, we can extend that result to any finite number of connected spaces by induction.

Corollary 20.8. *Let X_k be a connected topological space for k from 1 to n. Then the product $\Pi_{k=1}^n X_k$ is connected.*

The proof is left to Exercise (6).

We conclude this section by demonstrating that a product of compact topological spaces is compact. It is also true that finite products of path connected and compact spaces are path connected and compact. The proofs are left to Exercises (7) and (8).

Theorem 20.9. *If X and Y are compact topological spaces, then $X \times Y$ is a compact topological space under the product topology.*

Proof. Let (X, τ_X) and (Y, τ_Y) be compact topological spaces. Let $\mathcal{C} = \{O_\alpha\}$ be an open cover of $X \times Y$ for α in some indexing set I. Let $a \in X$ and let $Y_a = \{a\} \times Y$. Since Y_a is homeomorphic to Y, we know that Y_a is compact. The collection $\{O_\alpha \cap Y_a\}$ is an open cover of Y_a, and so has a finite sub-cover $\{O_{\alpha_i}\}_{1 \leq i \leq n}$. The set $N_a = \bigcup_{1 \leq i \leq n} O_{\alpha_i}$ is an open set that contains Y_a. We will show that there is a neighborhood W_a of a that N_a contains the entire set $W_a \times Y$.

Cover the set Y_a with open sets that are contained in N_a (since N_a is open, we can intersect any open set with N_a and still have an open set). Each open set is a union of basis elements, so we can cover Y_a with basis elements $U \times V$ that are contained in N_a. Since Y_a is compact, there is a finite collection $U_1 \times V_1$, $U_2 \times V_2$, . . ., $U_m \times V_m$ of basis elements contained in N_a that cover Y_a. Assume that each $U_i \times V_i$ intersects Y_a (otherwise, we can remove that set and still have a cover). Let $W_a = U_1 \cap U_2 \cap \cdots \cap U_m$. Since $a \in U_i$ for each i, we know that W_a is not empty. Each U_i is open in X and so W_a is open in X. Thus, W_a is a neighborhood of a in X. Now we demonstrate that $W_a \times Y \subseteq \bigcup_{1 \leq i \leq m} U_i \times V_i$. Let $(x, y) \in W_a \times Y$. Since the collection $\{U_i \times V_i\}_{1 \leq i \leq m}$ covers Y_a, the point (a, y) is in $U_k \times V_k$ for some k between 1 and m. So $y \in Y_k$. But $x \in W_a = \bigcap_{1 \leq i \leq m} U_i$, so $x \in U_k$. Thus, $(x, y) \in U_k \times V_k$ and we conclude that $W_a \times Y \subseteq \bigcup_{1 \leq i \leq m} U_i \times V_i$.

So for each $a \in X$, the set N_a contains a set of the form $W_a \times Y$, where W_a is a neighborhood of a in X. So $W_a \times Y$ is covered by a finite sub-cover of our open cover $\mathcal{C}$ of $X \times Y$. The collection $\{W_a \times Y\}_{a \in X}$ is an open cover of $X \times Y$. Since X is compact, the is a finite sub-cover W_1, W_2, . . ., W_r of the open cover $\{W_a\}_{a \in X}$ of X. is an open cover of X. It follows that the sets $W_1 \times Y$, $W_2 \times Y$, . . ., $W_r \times Y$ is a cover of $X \times Y$. For each i, the set $W_i \times Y$ is covered by finitely many of the sets in $\mathcal{C}$, and so the collection of these sets forms a finite sub-cover of $X \times Y$ in $\mathcal{C}$. Therefore, $X \times Y$ is compact. ∎

Summary

Important ideas that we discussed in this section include the following. Throughout, let (X_i, τ_i) be topological spaces for i from 1 to some integer n

- The product of the X_i is the Cartesian product $\Pi_{i=1}^n X_i$.
- The set
$$\mathcal{B} = \{\Pi_{i=1}^n O_i \mid O_i \text{ is open in } X_i\}$$
is a basis for the box topology on $\Pi_{i=1}^n X_i$.
- The mapping $\pi_j : \Pi_{i=1}^n X_i \to X_j$ defined by $\pi_j((x_i)) = x_j$ is the projection map onto X_j for j from 1 to n.
- A function f mapping a topological space Y to $\Pi_{i=1}^n X_i$ is continuous if and only if $\pi_j \circ f$ is continuous for every j from 1 to n.
- Let (X, τ) be a topological space. A subset $\mathcal{S}$ of τ is a subbasis for τ if the set $\mathcal{S}$ of all finite intersections of elements of $\mathcal{S}$ is a basis for τ.
- If each X_i is (a) connected, (b) path connected, (c) compact, then $\Pi_{i=1}^n X_i$ is (a) connected, (b) path connected, (c) compact with respect to the product topology.

Exercises

(1) (a) Let (Y_1, τ_1) and (Y_2, τ_2) be topological spaces, where $Y_1 = \{a, b, c\}$ with $\tau_1 = \{\emptyset, \{a\}, \{b, c\}, Y_1\}$ and $Y_2 = \{1, 2\}$ with $\tau_2 = \{\emptyset, \{1\}, Y_2\}$. Find all of the sets of the form

$$\mathcal{S} = \{\pi_1^{-1}(O_1) \mid O_1 \text{ is open in } Y_1\} \cup \{\pi_2^{-1}(O_2) \mid O_2 \text{ is open in } Y_2\}$$

and verify that these sets generate the product topology on $Y_1 \times Y_2$.

(b) Let X_i for i from 1 to n be topological spaces and let $X = \Pi_{i=1}^{n} X_i$. Show that the collection

$$\mathcal{S} = \bigcup_{i=1}^{n} \{\pi_i^{-1}(O_i) \mid O_i \text{ is open in } X_i\}$$

is a subbasis for the box topology on X.

(2) Let X be the set of real numbers with the standard Euclidean metric topology and let Y be the real numbers with the discrete topology.

(a) Explain why the set of all "horizontal intervals" of the form

$$I = (a, b) \times \{c\} = \{(x, c) \mid a < x < b\}$$

is a base for the product topology on $X \times Y$.

(b) Find the interior and closure of each of the following subsets of $X \times Y$.

(a) $A = \{(x, 0) \mid 0 \leq x < 1\}$
(b) $B = \{(0, y) \mid 0 \leq y < 1\}$
(c) $C = \{(x, y) \mid 0 \leq x < 1, 0 \leq y < 1\}$

(3) Let X_1 and X_2 be topological space and let A_1 be a subset of X_1 and A_2 a subset of X_2. Assume the product topology on $X_1 \times X_2$. Prove each of the following.

(a) $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$

(b) $\text{Int}(A_1 \times A_2) \subseteq \text{Int}A_1 \times \text{Int}A_2$

(4) Let X and Y be topological spaces and let $\{U_\alpha\}_{\alpha \in I}$ and $\{V_\beta\}_{\beta \in J}$ be collections of open sets in X and Y, respectively, for some indexing sets I and J. Show that

$$\bigcup_{\substack{\alpha \in I \\ \beta \in J}} (U_\alpha \times V_\beta) = \left(\bigcup_{\alpha \in I} U_\alpha\right) \times \left(\bigcup_{\beta \in J} V_\beta\right).$$

(5) (a) If S_1, S_2, T_1, and T_2 are sets, show that

$$(S_1 \times T_1) \cap (S_2 \times T_2) = (S_1 \cap S_2) \times (T_1 \cap T_2).$$

(b) If $\mathcal{B}_X$ is a base for a topology τ_X on a space X and $\mathcal{B}_Y$ is a base for a topology τ_Y on a space Y, show that $\mathcal{B}_X \times \mathcal{B}_Y$ is a base for the product topology on $X \times Y$.

(6) Prove that the product of any finite number of connected spaces is connected.

(7) Prove that the product of any finite number of compact spaces is compact.

(8) (a) Prove that if (X_1, τ_1) and (X_2, τ_2) are path connected topological spaces, then $X_1 \times X_2$ with the product topology is a path connected topological space.

(b) Prove that the product of any finite number of path connected spaces is path connected.

(9) Let X_1 and X_2 be topological spaces and $X = X_1 \times X_2$. Is it true that if X is compact, then both X_1 and X_2 are compact. Prove your answer.

(10) Let X_1 and X_2 be topological spaces and let $\pi_i : X_1 \times X_2 \to X_i$ be the projection mapping. We have shown that π_i is continuous. Now show that π_i is an open map for i equal 1 and 2. Assume the standard product topology.

(11) Let X_1 and X_2 be topological spaces with cardinalities at least 2. Let $X = X_1 \times X_2$. Prove that the product space topology on $X = X_1 \times X_2$ is the discrete topology if and only if the topologies on X_1 and X_2 are the discrete topologies.

(12) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why. Throughout, let X_1 and X_2 be topological spaces and $X = X_1 \times X_2$ with the product topology.

(a) If $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$, then $(X_1 \times X_2) \setminus (A_1 \times A_2) = (X_1 \setminus A_1) \times (X_2 \setminus A_2)$.

(b) If $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$, then $\text{Bdry}(A_1 \times A_2) \subseteq \text{Bdry}A_1 \times \text{Bdry}A_2$.

(c) If $A_1 \subseteq X_1$ and $A_2 \subseteq X_2$, then $\text{Bdry}A_1 \times \text{Bdry}A_2 \subseteq \text{Bdry}(A_1 \times A_2)$.

(d) If O_1 is an open subset of X_1 and O_2 is an open subset of X_2, then $O_1 \times O_2$ is an open subset of $X_1 \times X_2$.

(e) If O_1 is a subset of X_1 and O_2 is a subset of X_2 and $O_1 \times O_2$ is an open subset of $X_1 \times X_2$, then O_1 is an open subset of X_1 and O_2 is an open subset of X_2.

(f) If C_1 is a closed subset of X_1 and C_2 is a closed subset of X_2, then $C_1 \times C_2$ is a closed subset of $X_1 \times X_2$.

(g) If C_1 is a subset of X_1 and C_2 is a subset of X_2 and $C_1 \times C_2$ is a closed subset of $X_1 \times X_2$, then C_1 is a closed subset of X_1 and C_2 is a closed subset of X_2.

Applications of Products of Topological Spaces

Computers represent information from the real world digitally. That is, a computer screen consists of discrete pixels that are used to mimic the continuous information from the real world. So we exist in $\mathbb{R}^3$, but a computer screen represents information in $\mathbb{Z}^2$ as illustrated in Figure 20.1. It is important to be able to accurately mimic the continuous information from digital data. One of the key ideas is to have a digital version of the Jordan curve theorem which states that a Jordan curve (a continuous loop that does not intersect itself) in the Euclidean plane separates the remainder of

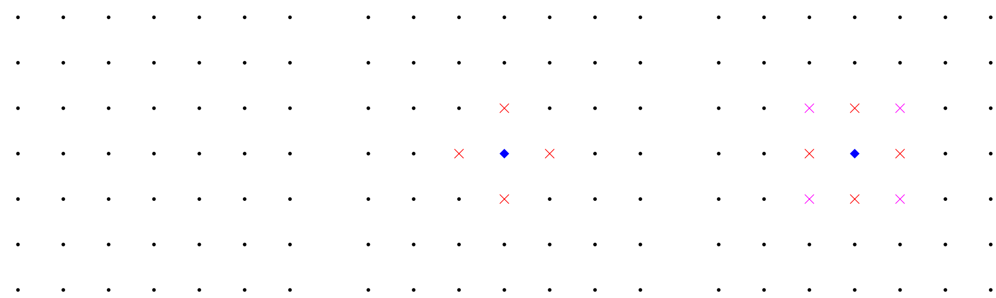

Figure 20.1: Left: The digital plane. Middle: 4-neighbors of a point. Right: 8-neighbors of a point.

the plane into two connected components (the inside and the outside of the curve). Additionally, if a single point is removed from a Jordan curve, the remainder of the plane becomes connected. The reason a digital Jordan curve theorem is important is that it is only necessary to save the Jordan curves which determine regions, along with the associated colors of the regions, rather than having to save the color of every single pixel in an image.

A natural start to building a digital topology might be to identify neighborhoods. The idea of a neighborhood is to consider elements that are close to a point, and in the digital world there are different ways to do this. Given a point (x, y) in $\mathbb{Z}^2$, the 4-neighbors of (x, y) are the points vertically or horizontally adjacent to (x, y): that is, the points $(x \pm 1, y)$ and $(x, y \pm 1)$. The 8-neighbors of (x, y) are the 4-neighbors along with the points diagonally adjacent to (x, y): that is, $(x \pm 1, y)$, $(x, y \pm 1)$, $(x \pm 1, y \pm 1)$. These neighbors are illustrated in Figure 20.1, with the crosses indicating the neighbors of the highlighted point.

In the continuous case, we define a path between points to be a continuous function from $[0, 1]$ to the space. However, we cannot have continuity in the digital world. So we define paths by moving through neighbor points. That is, if k is either 4 or 8, a k-path is a finite sequence p_0, p_1, $\ldots$, p_m in $\mathbb{Z}^2$ such that p_1 is a k-neighbor of p_2, p_2 is a k-neighbor of p_3, $\ldots$, and p_{m-1} is a k-neighbor of p_m.

Activity 20.8.

(a) Show that there is a 4-path connecting any two points in $\mathbb{Z}^2$. Then explain why there is an 8-path connecting any two points in $\mathbb{Z}^2$.

(b) In the continuous case, every Jordan curve separates $\mathbb{R}^2$ into two connected regions. To have a similar theorem in the discrete case, we need a notion of connectedness in $\mathbb{Z}^2$. Every image is made up of a finite number of pixels, and so we can think of a digital image as existing in a finite subspace of $\mathbb{Z}^2$. Since connectedness and path connectedness are equivalent in finite topological spaces, we us the idea of k-paths to define connectedness in $\mathbb{Z}^2$. We say that a subset S of $\mathbb{Z}^2$ is $k - connected$ if if any two of its points can be joined by a k-path in S.

Figure 20.2 show two sets (curves) in the digital plane indicated by the points that connect the line segments (examples taken from A Topological Approach to Digital Topology, T. Yung Kong, R. Kopperman, and P. Meyer, *American Mathematical Monthly*, 98 (1991), no. 10, 901-917). Let S_1 be the set illustrated at left in Figure 20.2 and S_2 the set at right.

Is S_1 4 connected? Is S_1 8 connected? Verify your answer. Repeat with S_2.

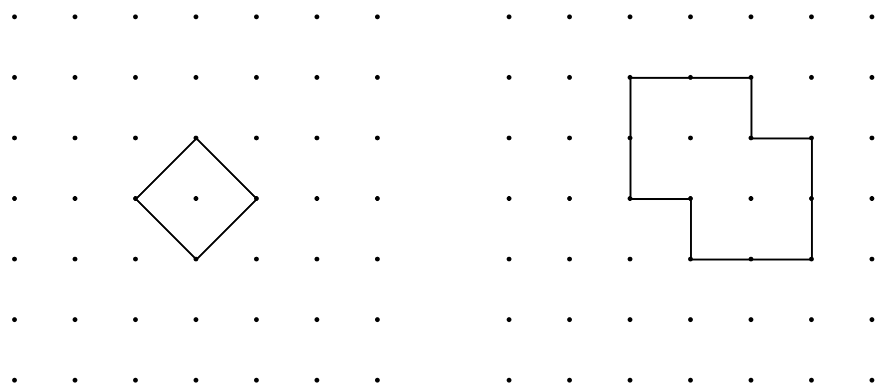

Figure 20.2: Sets S_1 (left) and S_2 (right) in the digital plane.

(c) We can now define a Jordan k-curve to be a finite k-connected set which contains exactly two k-neighbors for each of its points.

Is S_1 a Jordan 4-curve? Is S_1 a Jordan 8-curve? Verify your answer. Repeat with S_2.

(d) As usual, we define a component to be a maximal connected set. Explain why S_1 is a Jordan 8-curve whose complement is connected and why S_2 is a Jordan 4-curve whose complement consists of three connected 4-components. This example shows that there is no Jordan curve theorem in digital topology using the standard notions of k-connectedness with k either 4 or 8. So neither 4-adjacency nor 8-adjacency provides an analogue of the Jordan curve theorem and it is necessary to use a combination of both. That is, a Jordan 4-curve with at least five points separates $\mathbb{Z}^2$ into exactly two 8-components, and a Jordan 8-curve with at least five points separates $\mathbb{Z}^2$ into exactly two 4-components.

In Activity 20.8 we discussed the importance of a digital Jordan curve theorem. In the next activity we describe a topology in which such a theorem exists.

Activity 20.9. Consider Z with the topology τ_1 with basis $\{B(n)\}$, where

$$B(n) = \begin{cases} \{n\} & \text{if } n \text{ is odd,} \\ \{n-1, n, n+1\} & \text{if } n \text{ is even.} \end{cases}$$

This topology is called the *digital line topology* or the *Khalimsky topology* on $\mathbb{Z}$. Notice that all sets of the form $\{n\}$ are open when n is odd.

(a) Show that any set of the form $\{n\}$ where n is even is closed in the digital line topology.

(b) To define a *Khalimsky topology* on $\mathbb{Z}^2$ we use the product topology. Explain why the collection of sets $\{B(m,n)\}$ where

$$B(m,n) = \begin{cases} \{(m,n)\} & m \text{ and } n \text{ odd,} \\ \{(m-i, n-j) \mid -1 \leq i \leq 1, -1 \leq j \leq 1\} & m \text{ and } n \text{ even,} \\ \{(m, n-1), (m,n), (m, n+1)\} & m \text{ odd and } n \text{ even,} \\ \{(m-1, n), (m,n), (m+1, n)\} & m \text{ even and } n \text{ odd} \end{cases}$$

is a basis for the Khalimsky topology τ_2 on $\mathbb{Z}^2$. (This topology was originally published by E. Khalimsky in *Applications of connected ordered topological spaces in topology*, Conference of math. departments of Povolsia, 1970.)

(c) New we want to define a digital Jordan curve. Our first step is to define a *digital path*. Recall that a path in a topological space is a homeomorphism from the interval $[0, 1]$ into the space. So we need the concept of a digital interval. If $z_1 < z_2$ in $(\mathbb{Z}, \tau_1)$, the *digital interval* $[z_1, z_2]$ is the set

$$[z_1, z_2] = \{z \in \mathbb{Z} \mid z_1 \leq z \leq z_2\}.$$

The integers z_1 and z_2 are called the *endpoints* of the digital interval $[z_1, z_2]$.

Definition 20.10. Let X be a topological space.

- A **digital path** in X is the range of a continuous function from a digital interval to X.
- A **digital arc** in X is the range of a homeomorphism from a digital interval to X.

Let

$$S_1 = \{(1, -1), (1, 1), (-1, 1), (-1, -1)\},$$
$$S_2 = \{(0, 0), (1, -1), (2, 0), (1, 1)\}, \text{ and}$$
$$S_3 = \{(1, -1), (1, 0), (1, 1), (0, 1), (-1, 1), (-1, 0), (-1, -1), (0, -1)\}.$$

Show that S_1 is not a digital path but S_2 and S_3 are digital paths.

(d) To produce a digital Jordan Curve Theorem, we need a definition of a digital Jordan curve.

Definition 20.11. A **digital Jordan curve** is a finite connected set J with $|J| \geq 4$ such that $J \setminus \{j\}$ is a digital arc for each $j \in J$.

So every digital Jordan curve is a connected set. Show that any finite digital path in $\mathbb{Z}^2$ is a connected set. (Hint: Is every digital interval connected?)

(e) The upshot of all of this is the following theorem (a proof can be found in A Topological Approach to Digital Topology, T. Yung Kong, R. Kopperman, and P. Meyer, *American Mathematical Monthly*, 98 (1991), no. 10, 901-917).

Theorem 20.12. *If J is a digital Jordan curve in the digital plane $\mathbb{Z}^2$, then $\mathbb{Z}^2 \setminus J$ has exactly two components.*

The two components in Theorem 20.12 split the digital plane into an infinite region (the outside) and a finite region (the inside).

Show that S_2 is a digital Jordan curve (and thus splits $\mathbb{Z}^2$ into two connected components).

Digital Jordan curves, as described in Activity 20.9, are important in order to have a digital Jordan curve theorem. Christer O. Kiselman presents the following theorem to characterize digital Jordan curves in *Discrete Geometry for Computer Imagery*, Springer-Verlag, 2000, p. 46-56.

Theorem 20.13. *A subset J of $\mathbb{Z}^2$ equipped with the Khalimsky topology is a digital Jordan curve if and only if $J = \{P_1, P_2, \ldots, P_m\}$ for some even integer $m \geq 4$ and for all j, $P_{j?1}$ and P_{j+1} and no other points are adjacent to P_j; moreover each path consisting of three consecutive points $P_{i?1}$, P_i, P_{i+1} turns at P_i by 45° or 90° or not at all if P_i is a pure point, and goes straight ahead if P_i is mixed.*

We investigate this theorem in the next activity.

Activity 20.10.

(a) We need to first define the appropriate terms. Let X be a topological space. Two points x and y in X are *adjacent* if $x \neq y$ and the set $\{x, y\}$ is connected. Then let $N(x)$ to be the intersection of all neighborhoods of x.

Show that distinct elements x and y in a topological space X are adjacent if and only if $x \in N(y)$ or $y \in N(x)$.

(b) A point (x_1, x_2) in $\mathbb{Z}^2$ is called *pure* if x_1 and x_2 have the same parity. Otherwise, the point is *mixed*. Find $N(P)$ if P is a pure point or a mixed point.

(c) In Activity 20.9 we show that the set $S_1 = \{(1, -1), (1, 1), (-1, 1), (-1, -1)\}$ is not a digital path and so not a digital Jordan curve. Which part of Theorem 20.13 does S_1 violate?

(d) In Activity 20.9 we show that the set $S_2 = \{(0, 0), (1, -1), (2, 0), (1, 1)\}$ is a digital Jordan curve. Show that in S_2, the property from Theorem 20.13 that x_{j-1} and x_{j+1} and no other points are adjacent to x_j is satisfied for each j.