Chapter 3

Using Derivatives

3.1 Using derivatives to identify extreme values

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What are the critical numbers of a function f and how are they connected to identifying the most extreme values the function achieves?

- How does the first derivative of a function reveal important information about the behavior of the function, including the function's extreme values?

- How can the second derivative of a function be used to help identify extreme values of the function?

Introduction

In many different settings, we are interested in knowing where a function achieves its least and greatest values. These can be important in applications – say to identify a point at which maximum profit or minimum cost occurs – or in theory to understand how to characterize the behavior of a function or a family of related functions. Consider the simple and familiar example of a parabolic function such as $s(t) = -16t^2 + 32t + 48$ (shown at left in Figure 3.1) that represents the height of an object tossed vertically: its maximum value occurs at the vertex of the parabola and represents the highest value that the object reaches. Moreover, this maximum value identifies an especially important point on the graph, the point at which the curve changes from increasing to decreasing.

   More generally, for any function we consider, we can investigate where its lowest
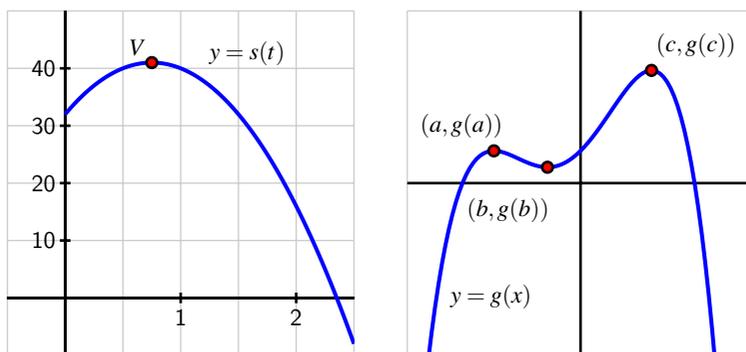
Figure 3.1: At left, $s(t) = -16t^2 + 24t + 32$ whose vertex is $(\frac{3}{4}, 41)$; at right, a function g that demonstrates several high and low points.

and highest points occur in comparison to points nearby or to all possible points on the graph. Given a function f, we say that $f(c)$ is a *global* or *absolute maximum* provided that $f(c) \geq f(x)$ for all x in the domain of f, and similarly call $f(c)$ a *global* or *absolute minimum* whenever $f(c) \leq f(x)$ for all x in the domain of f. For instance, for the function g given at right in Figure 3.1, g has a global maximum of $g(c)$, but g does not appear to have a global minimum, as the graph of g seems to decrease without bound. We note that the point $(c, g(c))$ marks a fundamental change in the behavior of g, where g changes from increasing to decreasing; similar things happen at both $(a, g(a))$ and $(b, g(b))$, although these points are not global mins or maxes.

For any function f, we say that $f(c)$ is a *local maximum* or *relative maximum* provided that $f(c) \geq f(x)$ for all x near c, while $f(c)$ is called a *local* or *relative minimum* whenever $f(c) \leq f(x)$ for all x near c. Any maximum or minimum may be called an *extreme value* of f. For example, in Figure 3.1, g has a relative minimum of $g(b)$ at the point $(b, g(b))$ and a relative maximum of $g(a)$ at $(a, g(a))$. We have already identified the global maximum of g as $g(c)$; this global maximum can also be considered a relative maximum.

We would like to use fundamental calculus ideas to help us identify and classify key function behavior, including the location of relative extremes. Of course, if we are given a graph of a function, it is often straightforward to locate these important behaviors visually. We investigate this situation in the following preview activity.

Preview Activity 3.1. Consider the function h given by the graph in Figure 3.2. Use the graph to answer each of the following questions.

(a) Identify all of the values of c for which $h(c)$ is a local maximum of h.

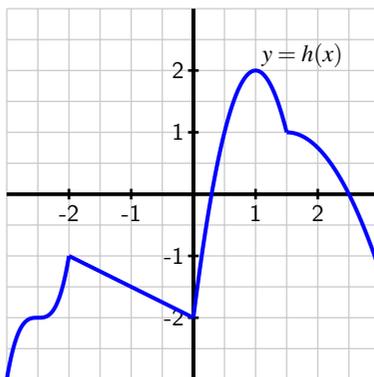(b) Identify all of the values of c for which $h(c)$ is a local minimum of h.

Figure 3.2: The graph of a function h on the interval $[-3, 3]$.

(c) Does h have a global maximum on the interval $[-3, 3]$? If so, what is the value of this global maximum?

(d) Does h have a global minimum on the interval $[-3, 3]$? If so, what is its value?

(e) Identify all values of c for which $h'(c) = 0$.

(f) Identify all values of c for which $h'(c)$ does not exist.

(g) True or false: every relative maximum and minimum of h occurs at a point where $h'(c)$ is either zero or does not exist.

(h) True or false: at every point where $h'(c)$ is zero or does not exist, h has a relative maximum or minimum.

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Critical numbers and the first derivative test

If a function has a relative extreme value at a point $(c, f(c))$, the function must change its behavior at c regarding whether it is increasing or decreasing before or after the point.

For example, if a continuous function has a relative maximum at c, such as those pictured in the two leftmost functions in Figure 3.3, then it is both necessary and sufficient that the function change from being increasing just before c to decreasing just after c. In the same way, a continuous function has a relative minimum at c if and only if the function changes from decreasing to increasing at c. See, for instance, the two functions pictured at right in Figure 3.3. There are only two possible ways for these changes in behavior to occur: either $f'(c) = 0$ or $f'(c)$ is undefined.
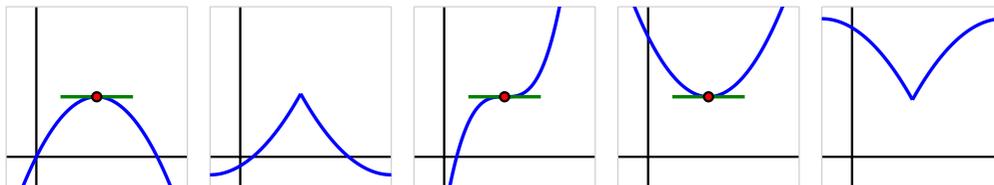
Figure 3.3: From left to right, a function with a relative maximum where its derivative is zero; a function with a relative maximum where its derivative is undefined; a function with neither a maximum nor a minimum at a point where its derivative is zero; a function with a relative minimum where its derivative is zero; and a function with a relative minimum where its derivative is undefined.

Because these values of c are so important, we call them *critical numbers*. More specifically, we say that a function f has a *critical number* at $x = c$ provided that c is in the domain of f, and $f'(c) = 0$ or $f'(c)$ is undefined. Critical numbers provide us with the only possible locations where the function f may have relative extremes. Note that not every critical number produces a maximum or minimum; in the middle graph of Figure 3.3, the function pictured there has a horizontal tangent line at the noted point, but the function is increasing before and increasing after, so the critical number does not yield a location where the function is greater than every value nearby, nor less than every value nearby.

We also sometimes use the terminology that, when c is a critical number, that $(c, f(c))$ is a *critical point* of the function, or that $f(c)$ is a *critical value* .

The *first derivative test* summarizes how sign changes in the first derivative indicate the presence of a local maximum or minimum for a given function.

> **First Derivative Test**: If p is a critical number of a continuous function f that is differentiable near p (except possibly at $x = p$), then f has a relative maximum at p if and only if f' changes sign from positive to negative at p, and f has a relative minimum at p if and only if f' changes sign from negative to positive at p.

We consider an example to show one way the first derivative test can be used to identify the relative extreme values of a function.

Example 3.1. Let f be a function whose derivative is given by the formula $f'(x) = e^{-2x}(3 - x)(x + 1)^2$. Determine all critical numbers of f and decide whether a relative maximum, relative minimum, or neither occurs at each.

Solution. Since we already have $f'(x)$ written in factored form, it is straightforward to find the critical numbers of f. Since $f'(x)$ is defined for all values of x, we need only

determine where $f'(x) = 0$. From the equation

$$e^{-2x}(3 - x)(x + 1)^2 = 0$$

and the zero product property, it follows that $x = 3$ and $x = -1$ are critical numbers of f. (Note particularly that there is no value of x that makes $e^{-2x} = 0$.)

Next, to apply the first derivative test, we'd like to know the sign of $f'(x)$ at inputs near the critical numbers. Because the critical numbers are the only locations at which f' can change sign, it follows that the sign of the derivative is the same on each of the intervals created by the critical numbers: for instance, the sign of f' must be the same for every $x < -1$. We create a first derivative sign chart to summarize the sign of f' on the relevant intervals along with the corresponding behavior of f.
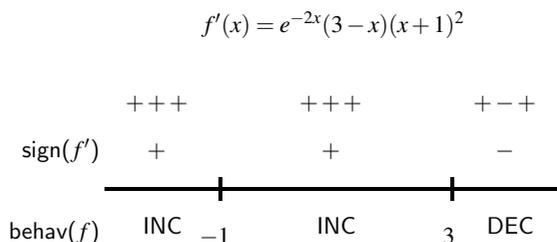
$$f'(x) = e^{-2x}(3 - x)(x + 1)^2$$

Figure 3.4: The first derivative sign chart for a function f whose derivative is given by the formula $f'(x) = e^{-2x}(3 - x)(x + 1)^2$.

The first derivative sign chart in Figure 3.4 comes from thinking about the sign of each of the terms in the factored form of $f'(x)$ at one selected point in the interval under consideration. For instance, for $x < -1$, we could consider $x = -2$ and determine the sign of e^{-2x}, $(3 - x)$, and $(x + 1)^2$ at the value $x = -2$. We note that both e^{-2x} and $(x + 1)^2$ are positive regardless of the value of x, while $(3 - x)$ is also positive at $x = -2$. Hence, each of the three terms in f' is positive, which we indicate by writing "$+ + +$." Taking the product of three positive terms obviously results in a value that is positive, which we denote by the "$+$" in the interval to the left of $x = -1$ indicating the overall sign of f'. And, since f' is positive on that interval, we further know that f is increasing, which we summarize by writing "INC" to represent the corresponding behavior of f. In a similar way, we find that f' is positive and f is increasing on $-1 < x < 3$, and f' is negative and f is decreasing for $x > 3$.

Now, by the first derivative test, to find relative extremes of f we look for critical numbers at which f' changes sign. In this example, f' only changes sign at $x = 3$, where f' changes from positive to negative, and thus f has a relative maximum at $x = 3$. While f has a critical number at $x = -1$, since f is increasing both before and after $x = -1$, f

has neither a minimum nor a maximum at $x = -1$.

Activity 3.1.

Suppose that $g(x)$ is a function continuous for every value of $x \neq 2$ whose first derivative is $g'(x) = \dfrac{(x+4)(x-1)^2}{x-2}$. Further, assume that it is known that g has a vertical asymptote at $x = 2$.

    (a) Determine all critical numbers of g.

    (b) By developing a carefully labeled first derivative sign chart, decide whether g has as a local maximum, local minimum, or neither at each critical number.

    (c) Does g have a global maximum? global minimum? Justify your claims.

    (d) What is the value of $\lim\limits_{x \to \infty} g'(x)$? What does the value of this limit tell you about the long-term behavior of g?

    (e) Sketch a possible graph of $y = g(x)$.

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The second derivative test

Recall that the second derivative of a function tells us several important things about the behavior of the function itself. For instance, if f'' is positive on an interval, then we know that f' is increasing on that interval and, consequently, that f is concave up, which also tells us that throughout the interval the tangent line to $y = f(x)$ lies below the curve at every point. In this situation where we know that $f'(p) = 0$, it turns out that the sign of the second derivative determines whether f has a local minimum or local maximum at the critical number p.

In Figure 3.5, we see the four possibilities for a function f that has a critical number p at which $f'(p) = 0$, provided $f''(p)$ is not zero on an interval including p (except possibly at p). On either side of the critical number, f'' can be either positive or negative, and hence f can be either concave up or concave down. In the first two graphs, f does not change concavity at p, and in those situations, f has either a local minimum or local maximum. In particular, if $f'(p) = 0$ and $f''(p) < 0$, then we know f is concave down at p with a horizontal tangent line, and this guarantees f has a local maximum there. This fact, along with the corresponding statement for when $f''(p)$ is positive, is stated in the
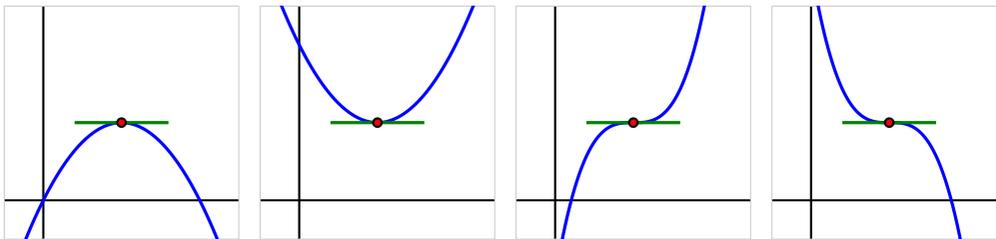
Figure 3.5: Four possible graphs of a function f with a horizontal tangent line at a critical point.

second derivative test.

> **Second Derivative Test**: If p is a critical number of a continuous function f such that $f'(p) = 0$ and $f''(p) \neq 0$, then f has a relative maximum at p if and only if $f''(p) < 0$, and f has a relative minimum at p if and only if $f''(p) > 0$.

In the event that $f''(p) = 0$, the second derivative test is inconclusive. That is, the test doesn't provide us any information. This is because if $f''(p) = 0$, it is possible that f has a local minimum, local maximum, or neither.[1]

Just as a first derivative sign chart reveals all of the increasing and decreasing behavior of a function, we can construct a second derivative sign chart that demonstrates all of the important information involving concavity.

Example 3.2. Let $f(x)$ be a function whose first derivative is $f'(x) = 3x^4 - 9x^2$. Construct both first and second derivative sign charts for f, fully discuss where f is increasing and decreasing and concave up and concave down, identify all relative extreme values, and sketch a possible graph of f.

Solution. Since we know $f'(x) = 3x^4 - 9x^2$, we can find the critical numbers of f by solving $3x^4 - 9x^2 = 0$. Factoring, we observe that

$$0 = 3x^2(x^2 - 3) = 3x^2(x + \sqrt{3})(x - \sqrt{3}),$$

so that $x = 0, \pm\sqrt{3}$ are the three critical numbers of f. It then follows that the first derivative sign chart for f is given in Figure 3.6. Thus, f is increasing on the intervals $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$, while f is decreasing on $(-\sqrt{3}, 0)$ and $(0, \sqrt{3})$. Note particularly that by the first derivative test, this information tells us that f has a local maximum at

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[1]Consider the functions $f(x) = x^4$, $g(x) = -x^4$, and $h(x) = x^3$ at the critical point $p = 0$.

$$f'(x) = 3x^2(x + \sqrt{3})(x - \sqrt{3})$$

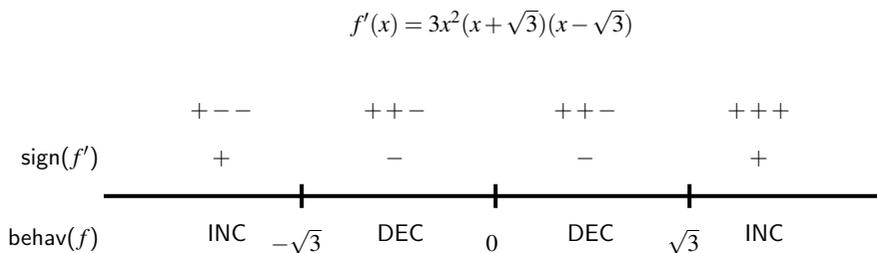|	$+--$	$++-$	$++-$	$+++$
sign(f')	$+$	$-$	$-$	$+$
behav(f)	INC $\quad -\sqrt{3}$	DEC $\quad 0$	DEC $\quad \sqrt{3}$	INC

Figure 3.6: The first derivative sign chart for f when $f'(x) = 3x^4 - 9x^2 = 3x^2(x^2 - 3)$.

$x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$. While f also has a critical number at $x = 0$, neither a maximum nor minimum occurs there since f' does not change sign at $x = 0$.

Next, we move on to investigate concavity. Differentiating $f'(x) = 3x^4 - 9x^2$, we see that $f''(x) = 12x^3 - 18x$. Since we are interested in knowing the intervals on which f'' is positive and negative, we first find where $f''(x) = 0$. Observe that

$$0 = 12x^3 - 18x = 12x\left(x^2 - \frac{3}{2}\right) = 12x\left(x + \sqrt{\frac{3}{2}}\right)\left(x - \sqrt{\frac{3}{2}}\right),$$

which implies that $x = 0, \pm\sqrt{\frac{3}{2}}$. Building a sign chart for f'' in the exact same way we do for f', we see the result shown in Figure 3.7. Therefore, f is concave down on the
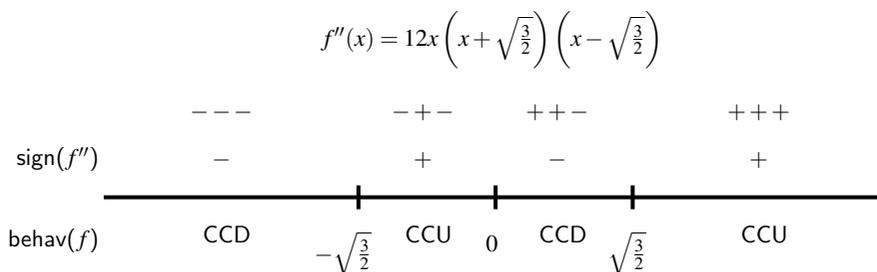
$$f''(x) = 12x\left(x + \sqrt{\frac{3}{2}}\right)\left(x - \sqrt{\frac{3}{2}}\right)$$

	$---$	$-+-$	$++-$	$+++$
sign(f'')	$-$	$+$	$-$	$+$
behav(f)	CCD $\quad -\sqrt{\frac{3}{2}}$	CCU $\quad 0$	CCD $\quad \sqrt{\frac{3}{2}}$	CCU

Figure 3.7: The second derivative sign chart for f when $f''(x) = 12x^3 - 18x = 12x^2\left(x^2 - \sqrt{\frac{3}{2}}\right)$.

intervals $(-\infty, -\sqrt{\frac{3}{2}})$ and $(0, \sqrt{\frac{3}{2}})$, and concave up on $(-\sqrt{\frac{3}{2}}, 0)$ and $(\sqrt{\frac{3}{2}}, \infty)$.

Putting all of the above information together, we now see a complete and accurate

possible graph of f in Figure 3.8. The point $A = (-\sqrt{3}, f(-\sqrt{3}))$ is a local maximum, as
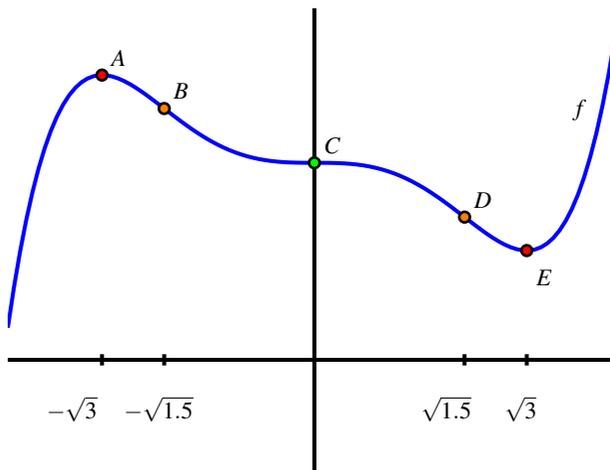

Figure 3.8: A possible graph of the function f in Example 3.2.

f is increasing prior to A and decreasing after; similarly, the point $E = (\sqrt{3}, f(\sqrt{3}))$ is a local minimum. Note, too, that f is concave down at A and concave up at B, which is consistent both with our second derivative sign chart and the second derivative test. At points B and D, concavity changes, as we saw in the results of the second derivative sign chart in Figure 3.7. Finally, at point C, f has a critical point with a horizontal tangent line, but neither a maximum nor a minimum occurs there since f is decreasing both before and after C. It is also the case that concavity changes at C.

While we completely understand where f is increasing and decreasing, where f is concave up and concave down, and where f has relative extremes, we do not know any specific information about the y-coordinates of points on the curve. For instance, while we know that f has a local maximum at $x = -\sqrt{3}$, we don't know the value of that maximum because we do not know $f(-\sqrt{3})$. Any vertical translation of our sketch of f in Figure 3.8 would satisfy the given criteria for f.

Points B, C, and D in Figure 3.8 are locations at which the concavity of f changes. We give a special name to any such point: if p is a value in the domain of a continuous function f at which f changes concavity, then we say that $(p, f(p))$ is an *inflection point* of f. Just as we look for locations where f changes from increasing to decreasing at points where $f'(p) = 0$ or $f'(p)$ is undefined, so too we find where $f''(p) = 0$ or $f''(p)$ is undefined to see if there are points of inflection at these locations.

It is important at this point in our study to remind ourselves of the big picture that derivatives help to paint: the sign of the first derivative f' tells us *whether* the function

f is increasing or decreasing, while the sign of the second derivative f'' tells us *how* the function f is increasing or decreasing.

Activity 3.2.

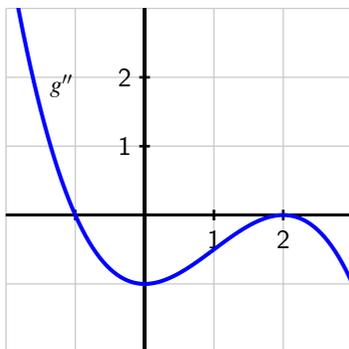Suppose that g is a function whose second derivative, g'', is given by the following graph.

Figure 3.9: The graph of $y = g''(x)$.

(a) Find the x-coordinates of all points of inflection of g.

(b) Fully describe the concavity of g by making an appropriate sign chart.

(c) Suppose you are given that $g'(-1.67857351) = 0$. Is there is a local maximum, local minimum, or neither (for the function g) at this critical number of g, or is it impossible to say? Why?

(d) Assuming that $g''(x)$ is a polynomial (and that all important behavior of g'' is seen in the graph above), what degree polynomial do you think $g(x)$ is? Why?

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As we will see in more detail in the following section, derivatives also help us to understand families of functions that differ only by changing one or more parameters. For instance, we might be interested in understanding the behavior of all functions of the form $f(x) = a(x - h)^2 + k$ where a, h, and k are numbers that may vary. In the following activity, we investigate a particular example where the value of a single parameter has considerable impact on how the graph appears.

Activity 3.3.

Consider the family of functions given by $h(x) = x^2 + \cos(kx)$, where k is an arbitrary positive real number.

(a) Use a graphing utility to sketch the graph of h for several different k-values, including $k = 1, 3, 5, 10$. Plot $h(x) = x^2 + \cos(3x)$ on the axes provided below.

What is the smallest value of k at which you think you can see (just by looking
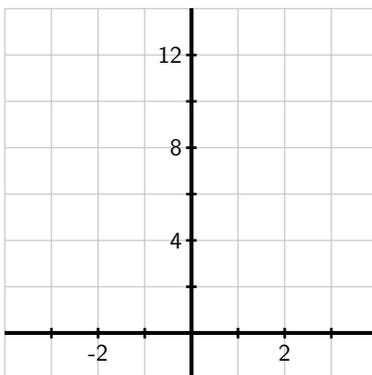

Figure 3.10: Axes for plotting $y = h(x)$.

at the graph) at least one inflection point on the graph of h?

(b) Explain why the graph of h has no inflection points if $k \leq \sqrt{2}$, but infinitely many inflection points if $k > \sqrt{2}$.

(c) Explain why, no matter the value of k, h can only have finitely many critical numbers.

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Summary

In this section, we encountered the following important ideas:

- The critical numbers of a continuous function f are the values of p for which $f'(p) = 0$ or $f'(p)$ does not exist. These values are important because they identify horizontal tangent lines or corner points on the graph, which are the only possible locations at which a local maximum or local minimum can occur.

- Given a differentiable function f, whenever f' is positive, f is increasing; whenever f' is negative, f is decreasing. The first derivative test tells us that at any point where f changes from increasing to decreasing, f has a local maximum, while conversely at any point where f changes from decreasing to increasing f has a local minimum.

- Given a twice differentiable function f, if we have a horizontal tangent line at $x = p$ and $f''(p)$ is nonzero, then the fact that f'' tells us the concavity of f will determine whether f has a maximum or minimum at $x = p$. In particular, if $f'(p) = 0$ and $f''(p) < 0$, then f is concave down at p and f has a local maximum there, while if $f'(p) = 0$ and $f''(p) > 0$, then f has a local minimum at p. If $f'(p) = 0$ and $f''(p) = 0$, then the second derivative does not tell us whether f has a local extreme at p or not.

Exercises

1. This problem concerns a function about which the following information is known:

- f is a differentiable function defined at every real number x
- $f(0) = -1/2$
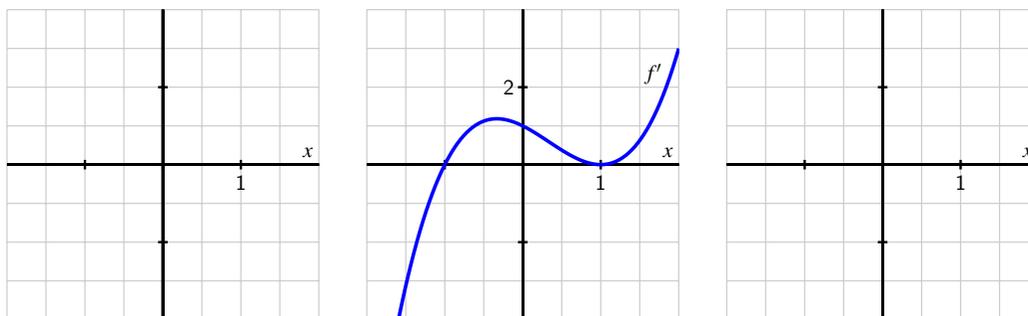- $y = f'(x)$ has its graph given at center in Figure 3.11

Figure 3.11: At center, a graph of $y = f'(x)$; at left, axes for plotting $y = f(x)$; at right, axes for plotting $y = f''(x)$.

(a) Construct a first derivative sign chart for f. Clearly identify all critical numbers of f, where f is increasing and decreasing, and where f has local extrema.

(b) On the right-hand axes, sketch an approximate graph of $y = f''(x)$.

(c) Construct a second derivative sign chart for f. Clearly identify where f is concave up and concave down, as well as all inflection points.

(d) On the left-hand axes, sketch a possible graph of $y = f(x)$.

2. Suppose that g is a differentiable function and $g'(2) = 0$. In addition, suppose that on $1 < x < 2$ and $2 < x < 3$ it is known that $g'(x)$ is positive.

(a) Does g have a local maximum, local minimum, or neither at $x = 2$? Why?

(b) Suppose that $g''(x)$ exists for every x such that $1 < x < 3$. Reasoning graphically, describe the behavior of $g''(x)$ for x-values near 2.

(c) Besides being a critical number of g, what is special about the value $x = 2$ in terms of the behavior of the graph of g?

3. Suppose that h is a differentiable function whose first derivative is given by the graph in Figure 3.12.
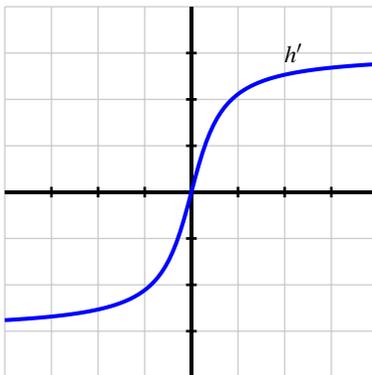
Figure 3.12: The graph of $y = h'(x)$.

(a) How many real number solutions can the equation $h(x) = 0$ have? Why?

(b) If $h(x) = 0$ has two distinct real solutions, what can you say about the signs of the two solutions? Why?

(c) Assume that $\lim_{x \to \infty} h'(x) = 3$, as appears to be indicated in Figure 3.12. How will the graph of $y = h(x)$ appear as $x \to \infty$? Why?

(d) Describe the concavity of $y = h(x)$ as fully as you can from the provided information.

4. Let p be a function whose second derivative is $p''(x) = (x + 1)(x - 2)e^{-x}$.

(a) Construct a second derivative sign chart for p and determine all inflection points of p.

(b) Suppose you also know that $x = \frac{\sqrt{5}-1}{2}$ is a critical number of p. Does p have a local minimum, local maximum, or neither at $x = \frac{\sqrt{5}-1}{2}$? Why?

(c) If the point $(2, \frac{12}{e^2})$ lies on the graph of $y = p(x)$ and $p'(2) = -\frac{5}{e^2}$, find the equation of the tangent line to $y = p(x)$ at the point where $x = 2$. Does the tangent line lie above the curve, below the curve, or neither at this value? Why?

3.2 Using derivatives to describe families of functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- Given a family of functions that depends on one or more parameters, how does the shape of the graph of a typical function in the family depend on the value of the parameters?

- How can we construct first and second derivative sign charts of functions that depend on one or more parameters while allowing those parameters to remain arbitrary constants?

Introduction

Mathematicians are often interested in making general observations, say by describing patterns that hold in a large number of cases. For example, think about the Pythagorean Theorem: it doesn't tell us something about a single right triangle, but rather a fact about *every* right triangle, thus providing key information about every member of the right triangle family. In the next part of our studies, we would like to use calculus to help us make general observations about families of functions that depend on one or more parameters. People who use applied mathematics, such as engineers and economists, often encounter the same types of functions in various settings where only small changes to certain constants occur. These constants are called *parameters*.

We are already familiar with certain families of functions. For example, $f(t) = a \sin(b(t - c)) + d$ is a stretched and shifted version of the sine function with amplitude a, period $\frac{2\pi}{b}$, phase shift c, and vertical shift d. We understand from experience with trigonometric functions that a affects the size of the oscillation, b the rapidity of oscillation, and c where the oscillation starts, as shown in Figure 3.13, while d affects the vertical positioning of the graph.

In addition, there are several basic situations that we already understand completely. For instance, every function of the form $y = mx + b$ is a line with slope m and y-intercept $(0, b)$. Note that the form $y = mx + b$ allows us to consider every possible line by using two parameters (except for vertical lines which are of the form $x = a$). Further, we understand that the value of m affects the line's steepness and whether the line rises or falls from left to right, while the value of b situates the line vertically on the coordinate axes.

For other less familiar families of functions, we would like to use calculus to understand and classify where key behavior occurs: where members of the family are increasing or decreasing, concave up or concave down, where relative extremes occur, and more, all
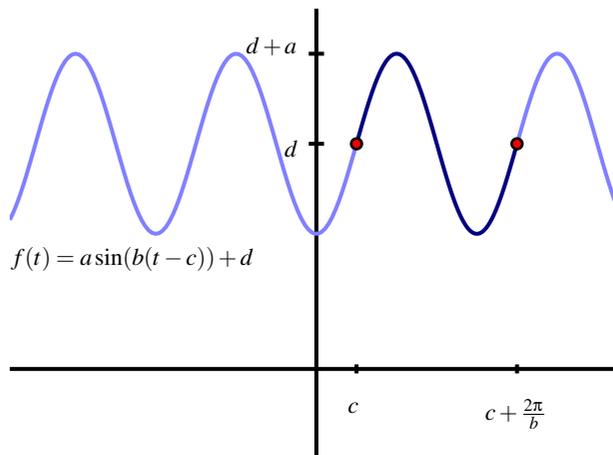
Figure 3.13: The graph of $f(t) = a\sin(b(t - c)) + d$ based on parameters a, b, c, and d.

in terms of the parameters involved. To get started, we revisit a common collection of functions to see how calculus confirms things we already know.

Preview Activity 3.2. Let a, h, and k be arbitrary real numbers with $a \neq 0$, and let f be the function given by the rule $f(x) = a(x - h)^2 + k$.

(a) What familiar type of function is f? What information do you know about f just by looking at its form? (Think about the roles of a, h, and k.)

(b) Next we use some calculus to develop familiar ideas from a different perspective. To start, treat a, h, and k as constants and compute $f'(x)$.

(c) Find all critical numbers of f. (These will depend on at least one of a, h, and k.)

(d) Assume that $a < 0$. Construct a first derivative sign chart for f.

(e) Based on the information you've found above, classify the critical values of f as maxima or minima.

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Describing families of functions in terms of parameters

Given a family of functions that depends on one or more parameters, our goal is to describe the key characteristics of the overall behavior of each member of the familiy in terms of those parameters. By finding the first and second derivatives and constructing first and second derivative sign charts (each of which may depend on one or more of the

parameters), we can often make broad conclusions about how each member of the family will appear. The fundamental steps for this analysis are essentially identical to the work we did in Section 3.1, as we demonstrate through the following example.

Example 3.3. Consider the two-parameter family of functions given by $g(x) = axe^{-bx}$, where a and b are positive real numbers. Fully describe the behavior of a typical member of the family in terms of a and b, including the location of all critical numbers, where g is increasing, decreasing, concave up, and concave down, and the long term behavior of g.

Solution. We begin by computing $g'(x)$. By the product rule,

$$g'(x) = ax \frac{d}{dx}\left[e^{-bx}\right] + e^{-bx} \frac{d}{dx}[ax],$$

and thus by applying the chain rule and constant multiple rule, we find that

$$g'(x) = axe^{-bx}(-b) + e^{-bx}(a).$$

To find the critical numbers of g, we solve the equation $g'(x) = 0$. Here, it is especially helpful to factor $g'(x)$. We thus observe that setting the derivative equal to zero implies

$$0 = ae^{-bx}(-bx + 1).$$

Since we are given that $a \neq 0$ and we know that $e^{-bx} \neq 0$ for all values of x, the only way the preceding equation can hold is when $-bx + 1 = 0$. Solving for x, we find that $x = \frac{1}{b}$, and this is therefore the only critical number of g.

Now, recall that we have shown $g'(x) = ae^{-bx}(1 - bx)$ and that the only critical number of g is $x = \frac{1}{b}$. This enables us to construct the first derivative sign chart for g that is shown in Figure 3.14.

$$g'(x) = ae^{-bx}(1 - bx)$$

$$
\begin{array}{ccc}
 & ++ & +- \\
\text{sign}(g') & + & - \\
\hline
\text{behav}(g) & \text{INC} & \text{DEC} \\
 & \frac{1}{b} &
\end{array}
$$

Figure 3.14: The first derivative sign chart for $g(x) = axe^{-bx}$.

Note particularly that in $g'(x) = ae^{-bx}(1 - bx)$, the term ae^{-bx} is always positive, so the sign depends on the linear term $(1 - bx)$, which is zero when $x = \frac{1}{b}$. Note that this line has negative slope $(-b)$, so $(1 - bx)$ is positive for $x < \frac{1}{b}$ and negative for $x > \frac{1}{b}$. Hence we can not only conclude that g is always increasing for $x < \frac{1}{b}$ and decreasing for $x > \frac{1}{b}$, but also that g has a global maximum at $(\frac{1}{b}, g(\frac{1}{b}))$ and no local minimum.

We turn next to analyzing the concavity of g. With $g'(x) = -abxe^{-bx} + ae^{-bx}$, we differentiate to find that

$$g''(x) = -abxe^{-bx}(-b) + e^{-bx}(-ab) + ae^{-bx}(-b).$$

Combining like terms and factoring, we now have

$$g''(x) = ab^2xe^{-bx} - 2abe^{-bx} = abe^{-bx}(bx - 2).$$

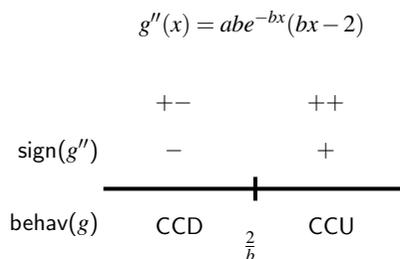Similar to our work with the first derivative, we observe that abe^{-bx} is always positive,

$$g''(x) = abe^{-bx}(bx - 2)$$

Figure 3.15: The second derivative sign chart for $g(x) = axe^{-bx}$.

and thus the sign of g'' depends on the sign of $(bx - 2)$, which is zero when $x = \frac{2}{b}$. Since $(bx - 2)$ represents a line with positive slope (b), the value of $(bx - 2)$ is negative for $x < \frac{2}{b}$ and positive for $x > \frac{2}{b}$, and thus the sign chart for g'' is given by the one shown in Figure 3.15. Thus, g is concave down for all $x < \frac{2}{b}$ and concave up for all $x > \frac{2}{b}$.

Finally, we analyze the long term behavior of g by considering two limits. First, we note that

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} axe^{-bx} = \lim_{x \to \infty} \frac{ax}{e^{bx}}.$$

Since this limit has indeterminate form $\frac{\infty}{\infty}$, we can apply L'Hopital's Rule and thus find that $\lim_{x \to \infty} g(x) = 0$. In the other direction,

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} axe^{-bx} = -\infty,$$

since $ax \to -\infty$ and $e^{-bx} \to \infty$ as $x \to -\infty$. Hence, as we move left on its graph, g decreases without bound, while as we move to the right, $g(x) \to 0$.

All of the above information now allows us to produce the graph of a typical member of this family of functions without using a graphing utility (and without choosing particular values for a and b), as shown in Figure 3.16.

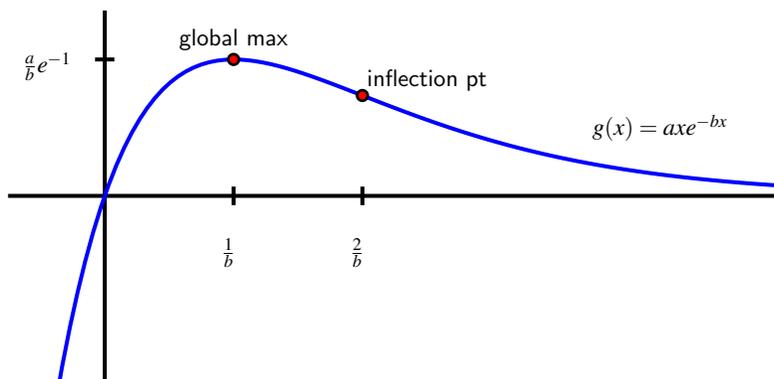Figure 3.16: The graph of $g(x) = axe^{-bx}$.

We note that the value of b controls the horizontal location of the global maximum and the inflection point, as neither depends on a. The value of a affects the vertical stretch of the graph. For example, the global maximum occurs at the point $(\frac{1}{b}, g(\frac{1}{b})) = (\frac{1}{b}, \frac{a}{b}e^{-1})$, so the larger the value of a, the greater the value of the global maximum.

The kind of work we've completed in Example 3.3 can often be replicated for other families of functions that depend on parameters. Normally we are most interested in determining all critical numbers, a first derivative sign chart, a second derivative sign chart, and some analysis of the limit of the function as $x \to \infty$. Throughout, we strive to work with the parameters as arbitrary constants. If stuck, it is always possible to experiment with some particular values of the parameters present to reduce the algebraic complexity of our work. The following sequence of activities offers several key examples where we see that the values of different parameters substantially affect the behavior of individual functions within a given family.

Activity 3.4.

Consider the family of functions defined by $p(x) = x^3 - ax$, where $a \neq 0$ is an arbitrary constant.

(a) Find $p'(x)$ and determine the critical numbers of p. How many critical numbers does p have?

(b) Construct a first derivative sign chart for p. What can you say about the overall

behavior of p if the constant a is positive? Why? What if the constant a is negative? In each case, describe the relative extremes of p.

(c) Find $p''(x)$ and construct a second derivative sign chart for p. What does this tell you about the concavity of p? What role does a play in determining the concavity of p?

(d) Without using a graphing utility, sketch and label typical graphs of $p(x)$ for the cases where $a > 0$ and $a < 0$. Label all inflection points and local extrema.

(e) Finally, use a graphing utility to test your observations above by entering and plotting the function $p(x) = x^3 - ax$ for at least four different values of a. Write several sentences to describe your overall conclusions about how the behavior of p depends on a.

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Activity 3.5.

Consider the two-parameter family of functions of the form $h(x) = a(1 - e^{-bx})$, where a and b are positive real numbers.

(a) Find the first derivative and the critical numbers of h. Use these to construct a first derivative sign chart and determine for which values of x the function h is increasing and decreasing.

(b) Find the second derivative and build a second derivative sign chart. For which values of x is a function in this family concave up? concave down?

(c) What is the value of $\lim\limits_{x \to \infty} a(1 - e^{-bx})$? $\lim\limits_{x \to -\infty} a(1 - e^{-bx})$?

(d) How does changing the value of b affect the shape of the curve?

(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function h and how this behavior depends on a and b.

◁

Activity 3.6.

Let $L(t) = \dfrac{A}{1 + ce^{-kt}}$, where A, c, and k are all positive real numbers.

(a) Observe that we can equivalently write $L(t) = A(1 + ce^{-kt})^{-1}$. Find $L'(t)$ and explain why L has no critical numbers. Is L always increasing or always decreasing? Why?

(b) Given the fact that

$$L''(t) = Ack^2 e^{-kt} \frac{ce^{-kt} - 1}{(1 + ce^{-kt})^3},$$

find all values of t such that $L''(t) = 0$ and hence construct a second derivative sign chart. For which values of t is a function in this family concave up? concave down?

(c) What is the value of $\lim\limits_{t \to \infty} \dfrac{A}{1 + ce^{-kt}}$? $\lim\limits_{t \to -\infty} \dfrac{A}{1 + ce^{-kt}}$?

(d) Find the value of $L(x)$ at the inflection point found in (b).

(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function L and how this behavior depends on A, c, and k number.

(f) Explain why it is reasonable to think that the function $L(t)$ models the growth of a population over time in a setting where the largest possible population the surrounding environment can support is A.

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Summary

In this section, we encountered the following important ideas:

- Given a family of functions that depends on one or more parameters, by investigating how critical numbers and locations where the second derivative is zero depend on the values of these parameters, we can often accurately describe the shape of the function in terms of the parameters.

- In particular, just as we can created first and second derivative sign charts for a single function, we often can do so for entire families of functions where critical numbers and possible inflection points depend on arbitrary constants. These sign charts then reveal where members of the family are increasing or decreasing, concave up or concave down, and help us to identify relative extremes and inflection points.

Exercises

1. Consider the one-parameter family of functions given by $p(x) = x^3 - ax^2$, where $a > 0$.

(a) Sketch a plot of a typical member of the family, using the fact that each is a cubic polynomial with a repeated zero at $x = 0$ and another zero at $x = a$.

(b) Find all critical numbers of p.

(c) Compute p'' and find all values for which $p''(x) = 0$. Hence construct a second derivative sign chart for p.

(d) Describe how the location of the critical numbers and the inflection point of p change as a changes. That is, if the value of a is increased, what happens to the critical numbers and inflection point?

2. Let $q(x) = \dfrac{e^{-x}}{x - c}$ be a one-parameter family of functions where $c > 0$.

   (a) Explain why q has a vertical asymptote at $x = c$.

   (b) Determine $\lim\limits_{x \to \infty} q(x)$ and $\lim\limits_{x \to -\infty} q(x)$.

   (c) Compute $q'(x)$ and find all critical numbers of q.

   (d) Construct a first derivative sign chart for q and determine whether each critical number leads to a local minimum, local maximum, or neither for the function q.

   (e) Sketch a typical member of this family of functions with important behaviors clearly labeled.

3. Let $E(x) = e^{-\frac{(x-m)^2}{2s^2}}$, where m is any real number and s is a positive real number.

   (a) Compute $E'(x)$ and hence find all critical numbers of E.

   (b) Construct a first derivative sign chart for E and classify each critical number of the function as a local minimum, local maximum, or neither.

   (c) It can be shown that $E''(x)$ is given by the formula

$$E''(x) = e^{-\frac{(x-m)^2}{2s^2}} \left(\frac{(x - m)^2 - s^2}{s^4} \right).$$

     Find all values of x for which $E''(x) = 0$.

   (d) Determine $\lim\limits_{x \to \infty} E(x)$ and $\lim\limits_{x \to -\infty} E(x)$.

   (e) Construct a labeled graph of a typical function E that clearly shows how important points on the graph of $y = E(x)$ depend on m and s.

3.3 Global Optimization

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What are the differences between finding relative extreme values and global extreme values of a function?

- How is the process of finding the global maximum or minimum of a function over the function's entire domain different from determining the global maximum or minimum on a restricted domain?

- For a function that is guaranteed to have both a global maximum and global minimum on a closed, bounded interval, what are the possible points at which these extreme values occur?

Introduction

We have seen that we can use the first derivative of a function to determine where the function is increasing or decreasing, and the second derivative to know where the function is concave up or concave down. Each of these approaches provides us with key information that helps us determine the overall shape and behavior of the graph, as well as whether the function has a relative minimum or relative maximum at a given critical number. Remember that the difference between a relative maximum and a global maximum is that there is a relative maximum of f at $x = p$ if $f(p) \geq f(x)$ for all x near p, while there is a global maximum at p if $f(p) \geq f(x)$ for all x in the domain of f. For instance,
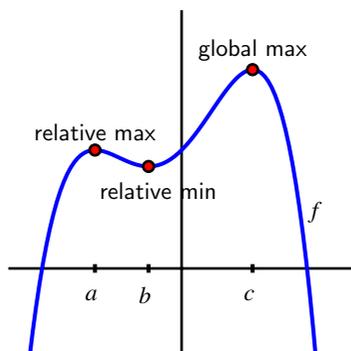
Figure 3.17: A function f with a global maximum, but no global minimum.

in Figure 3.17, we see a function f that has a global maximum at $x = c$ and a relative maximum at $x = a$, since $f(c)$ is greater than $f(x)$ for every value of x, while $f(a)$ is only greater than the value of $f(x)$ for x near a. Since the function appears to decrease without bound, f has no global minimum, though clearly f has a relative minimum at $x = b$.

Our emphasis in this section is on finding the global extreme values of a function (if they exist). In so doing, we will either be interested in the behavior of the function over its entire domain or on some restricted portion. The former situation is familiar and similar to work that we did in the two preceding sections of the text. We explore this through a particular example in the following preview activity.

Preview Activity 3.3. Let $f(x) = 2 + \dfrac{3}{1 + (x + 1)^2}$.

    (a) Determine all of the critical numbers of f.

    (b) Construct a first derivative sign chart for f and thus determine all intervals on which f is increasing or decreasing.

    (c) Does f have a global maximum? If so, why, and what is its value and where is the maximum attained? If not, explain why.

    (d) Determine $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$.

    (e) Explain why $f(x) > 2$ for every value of x.

    (f) Does f have a global minimum? If so, why, and what is its value and where is the minimum attained? If not, explain why.

⋈

Global Optimization

For the functions in Figure 3.17 and Preview Activity 3.3, we were interested in finding the global minimum and global maximum on the entire domain, which turned out to be $(-\infty, \infty)$ for each. At other times, our perspective on a function might be more focused due to some restriction on its domain. For example, rather than considering $f(x) = 2 + \frac{3}{1+(x+1)^2}$ for every value of x, perhaps instead we are only interested in those x for which $0 \le x \le 4$, and we would like to know which values of x in the interval $[0, 4]$ produce the largest possible and smallest possible values of f. We are accustomed to critical numbers playing a key role in determining the location of extreme values of a function; now, by restricting the domain to an interval, it makes sense that the endpoints of the interval will also be important to consider, as we see in the following activity. When limiting ourselves to a particular interval, we will often refer to the *absolute* maximum or minimum value, rather than the *global* maximum or minimum.

Activity 3.7.

Let $g(x) = \frac{1}{3}x^3 - 2x + 2$.

  (a) Find all critical numbers of g that lie in the interval $-2 \le x \le 3$.

  (b) Use a graphing utility to construct the graph of g on the interval $-2 \le x \le 3$.

  (c) From the graph, determine the x-values at which the absolute minimum and absolute maximum of g occur on the interval $[-2, 3]$.

  (d) How do your answers change if we instead consider the interval $-2 \le x \le 2$?

  (e) What if we instead consider the interval $-2 \le x \le 1$?

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In Activity 3.7, we saw how the absolute maximum and absolute minimum of a function on a closed, bounded interval $[a, b]$, depend not only on the critical numbers of the function, but also on the selected values of a and b. These observations demonstrate several important facts that hold much more generally. First, we state an important result called the Extreme Value Theorem.

The Extreme Value Theorem: If f is a continuous function on a closed interval $[a, b]$, then f attains both an absolute minimum and absolute maximum on $[a, b]$. That is, for some value x_m such that $a \le x_m \le b$, it follows that $f(x_m) \le f(x)$ for all x in $[a, b]$. Similarly, there is a value x_M in $[a, b]$ such that $f(x_M) \ge f(x)$ for all x in $[a, b]$. Letting $m = f(x_m)$ and $M = f(x_M)$, it follows that $m \le f(x) \le M$ for all x in $[a, b]$.

The Extreme Value Theorem tells us that provided a function is continuous, on any closed interval $[a, b]$ the function has to achieve both an absolute minimum and an absolute maximum. Note, however, that this result does not tell us where these extreme values occur, but rather only that they must exist. As seen in the examples of Activity 3.7, it is apparent that the only possible locations for relative extremes are either the endpoints of the interval or at a critical number (the latter being where a relative minimum or maximum could occur, which is a potential location for an absolute extreme). Thus, we have the following approach to finding the absolute maximum and minimum of a continuous function f on the interval $[a, b]$:

- find all critical numbers of f that lie in the interval;

- evaluate the function f at each critical number in the interval and at each endpoint of the interval;

- from among the noted function values, the smallest is the absolute minimum of f on the interval, while the largest is the absolute maximum.

Activity 3.8.

Find the *exact* absolute maximum and minimum of each function on the stated interval.

   (a) $h(x) = xe^{-x}$, $[0, 3]$

   (b) $p(t) = \sin(t) + \cos(t)$, $[-\frac{\pi}{2}, \frac{\pi}{2}]$

   (c) $q(x) = \frac{x^2}{x-2}$, $[3, 7]$

   (d) $f(x) = 4 - e^{-(x-2)^2}$, $(-\infty, \infty)$

   (e) $h(x) = xe^{-ax}$, $[0, \frac{2}{a}]$ $(a > 0)$

   (f) $f(x) = b - e^{-(x-a)^2}$, $(-\infty, \infty)$, $a, b > 0$

◁

One of the big lessons in finding absolute extreme values is the realization that the interval we choose has nearly the same impact on the problem as the function under consideration. Consider, for instance, the function pictured in Figure 3.18. In sequence,
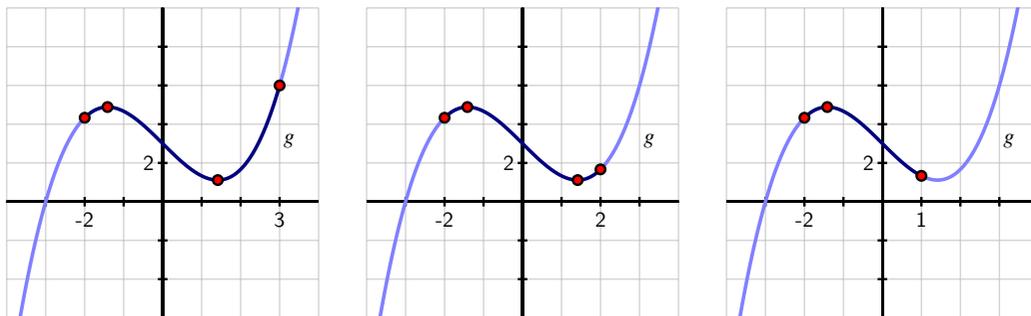

Figure 3.18: A function g considered on three different intervals.

from left to right, as we see the interval under consideration change from $[-2, 3]$ to $[-2, 2]$ to $[-2, 1]$, we move from having two critical numbers in the interval with the absolute minimum at one critical number and the absolute maximum at the right endpoint, to still having both critical numbers in the interval but then with the absolute minimum and maximum at the two critical numbers, to finally having just one critical number in the interval with the absolute maximum at one critical number and the absolute minimum at one endpoint. It is particularly essential to always remember to only consider the critical numbers that lie within the interval.

Moving towards applications

In Section 3.4, we will focus almost exclusively on applied optimization problems: problems where we seek to find the absolute maximum or minimum value of a function that represents some physical situation. We conclude this current section with an example of one such problem because it highlights the role that a closed, bounded domain can play in finding absolute extrema. In addition, these problems often involve considerable preliminary work to develop the function which is to be optimized, and this example demonstrates that process.

Example 3.4. A 20 cm piece of wire is cut into two pieces. One piece is used to form a square and the other an equilateral triangle. How should the wire be cut to maximize the total area enclosed by the square and triangle? to minimize the area?

Solution. We begin by constructing a picture that exemplifies the given situation. The primary variable in the problem is where we decide to cut the wire. We thus label that point x, and note that the remaining portion of the wire then has length $20 - x$ As shown
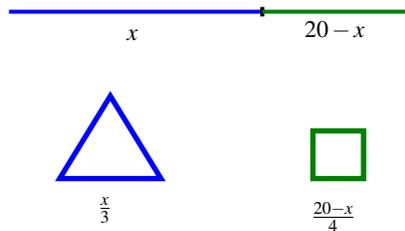

Figure 3.19: A 20 cm piece of wire cut into two pieces, one of which forms an equilateral triangle, the other which yields a square.

in Figure 3.19, we see that the x cm of the wire that are used to form the equilateral triangle result in a triangle with three sides of length $\frac{x}{3}$. For the remaining $20 - x$ cm of wire, the square that results will have each side of length $\frac{20-x}{4}$.

At this point, we note that there are obvious restrictions on x: in particular, $0 \leq x \leq 20$. In the extreme cases, all of the wire is being used to make just one figure. For instance, if $x = 0$, then all 20 cm of wire are used to make a square that is 5×5.

Now, our overall goal is to find the absolute minimum and absolute maximum areas that can be enclosed. We note that the area of the triangle is $A_\triangle = \frac{1}{2}bh = \frac{1}{2} \cdot \frac{x}{3} \cdot \frac{x\sqrt{3}}{6}$, since the height of an equilateral triangle is $\sqrt{3}$ times half the length of the base. Further, the

area of the square is $A_\square = s^2 = \left(\frac{20-x}{4}\right)^2$. Therefore, the total area function is

$$A(x) = \frac{\sqrt{3}x^2}{36} + \left(\frac{20-x}{4}\right)^2.$$

Again, note that we are only considering this function on the restricted domain $[0, 20]$ and we seek its absolute minimum and absolute maximum.

Differentiating $A(x)$, we have

$$A'(x) = \frac{\sqrt{3}x}{18} + 2\left(\frac{20-x}{4}\right)\left(-\frac{1}{4}\right) = \frac{\sqrt{3}}{18}x + \frac{1}{8}x - \frac{5}{2}.$$

Setting $A'(x) = 0$, it follows that $x = \frac{180}{4\sqrt{3}+9} \approx 11.3007$ is the only critical number of A, and we note that this lies within the interval $[0, 20]$.

Evaluating A at the critical number and endpoints, we see that

- $A\left(\frac{180}{4\sqrt{3}+9}\right) = \frac{\sqrt{3}(\frac{180}{4\sqrt{3}+9})^2}{4} + \left(\frac{20 - \frac{180}{4\sqrt{3}+9}}{4}\right)^2 \approx 10.8741$

- $A(0) = 25$

- $A(20) = \frac{\sqrt{3}}{36}(400) = \frac{100}{9}\sqrt{3} \approx 19.2450$

Thus, the absolute minimum occurs when $x \approx 11.3007$ and results in the minimum area of approximately 10.8741 square centimeters, while the absolute maximum occurs when we invest all of the wire in the square (and none in the triangle), resulting in 25 square centimeters of area. These results are confirmed by a plot of $y = A(x)$ on the interval $[0, 20]$, as shown in Figure .

Activity 3.9.

A piece of cardboard that is 10×15 (each measured in inches) is being made into a box without a top. To do so, squares are cut from each corner of the box and the remaining sides are folded up. If the box needs to be at least 1 inch deep and no more than 3 inches deep, what is the maximum possible volume of the box? what is the minimum volume? Justify your answers using calculus.

(a) Draw a labeled diagram that shows the given information. What variable should we introduce to represent the choice we make in creating the box? Label the diagram appropriately with the variable, and write a sentence to state what the variable represents.
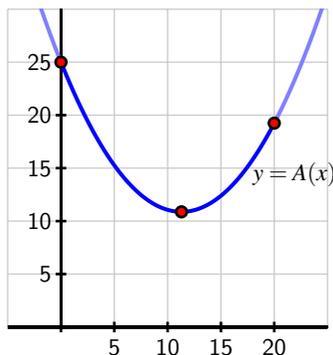
Figure 3.20: A plot of the area function from Example 3.4.

(b) Determine a formula for the function V (that depends on the variable in (a)) that tells us the volume of the box.

(c) What is the domain of the function V? That is, what values of x make sense for input? Are there additional restrictions provided in the problem?

(d) Determine all critical numbers of the function V.

(e) Evaluate V at each of the endpoints of the domain and at any critical numbers that lie in the domain.

(f) What is the maximum possible volume of the box? the minimum?

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The approaches shown in Example 3.4 and experienced in Activity 3.9 include standard steps that we undertake in almost every applied optimization problem: we draw a picture to demonstrate the situation, introduce one or more variables to represent quantities that are changing, work to find a function that models the quantity to be optimized, and then decide an appropriate domain for that function. Once that work is done, we are in the familiar situation of finding the absolute minimum and maximum of a function over a particular domain, at which time we apply the calculus ideas that we have been studying to this point in Chapter 3.

Summary

In this section, we encountered the following important ideas:

- To find relative extreme values of a function, we normally use a first derivative sign chart and classify all of the function's critical numbers. If instead we are interested in absolute extreme values, we first decide whether we are considering the entire domain

of the function or a particular interval.

- In the case of finding global extremes over the function's entire domain, we again use a first or second derivative sign chart in an effort to make overall conclusions about whether or not the function can have a absolute maximum or minimum. If we are working to find absolute extremes on a restricted interval, then we first identify all critical numbers of the function that lie in the interval.

- For a continuous function on a closed, bounded interval, the only possible points at which absolute extreme values occur are the critical numbers and the endpoints. Thus, to find said absolute extremes, we simply evaluate the function at each endpoint and each critical number in the interval, and then we compare the results to decide which is largest (the absolute maximum) and which is smallest (the absolute minimum).

Exercises

1. Based on the given information about each function, decide whether the function has global maximum, a global minimum, neither, both, or that it is not possible to say without more information. Assume that each function is twice differentiable and defined for all real numbers, unless noted otherwise. In each case, write one sentence to explain your conclusion.

    (a) f is a function such that $f''(x) < 0$ for every x.

    (b) g is a function with two critical numbers a and b (where $a < b$), and $g'(x) < 0$ for $x < a$, $g'(x) < 0$ for $a < x < b$, and $g'(x) > 0$ for $x > b$.

    (c) h is a function with two critical numbers a and b (where $a < b$), and $h'(x) < 0$ for $x < a$, $h'(x) > 0$ for $a < x < b$, and $h'(x) < 0$ for $x > b$. In addition, $\lim_{x \to \infty} h(x) = 0$ and $\lim_{x \to -\infty} h(x) = 0$.

    (d) p is a function differentiable everywhere except at $x = a$ and $p''(x) > 0$ for $x < a$ and $p''(x) < 0$ for $x > a$.

2. For each family of functions that depends on one or more parameters, determine the function's absolute maximum and absolute minimum on the given interval.

    (a) $p(x) = x^3 - a^2 x$, $[0, a]$ $(a > 0)$

    (b) $r(x) = axe^{-bx}$, $[\frac{1}{2b}, b]$ $(a, b > 0)$

    (c) $w(x) = a(1 - e^{-bx})$, $[b, 3b]$ $(a, b > 0)$

    (d) $s(x) = \sin(kx)$, $[\frac{\pi}{3k}, \frac{5\pi}{6k}]$

3. For each of the functions described below (each continuous on $[a, b]$), state the location of the function's absolute maximum and absolute minimum on the interval $[a, b]$, or say there is not enough information provided to make a conclusion. Assume that

any critical numbers mentioned in the problem statement represent all of the critical numbers the function has in $[a, b]$. In each case, write one sentence to explain your answer.

(a) $f'(x) \leq 0$ for all x in $[a, b]$

(b) g has a critical number at c such that $a < c < b$ and $g'(x) > 0$ for $x < c$ and $g'(x) < 0$ for $x > c$

(c) $h(a) = h(b)$ and $h''(x) < 0$ for all x in $[a, b]$

(d) $p(a) > 0$, $p(b) < 0$, and for the critical number c such that $a < c < b$, $p'(x) < 0$ for $x < c$ and $p'(x) > 0$ for $x > c$

4. Let $s(t) = 3 \sin(2(t - \frac{\pi}{6})) + 5$. Find the exact absolute maximum and minimum of s on the provided intervals by testing the endpoints and finding and evaluating all relevant critical numbers of s.

(a) $[\frac{\pi}{6}, \frac{7\pi}{6}]$

(b) $[0, \frac{\pi}{2}]$

(c) $[0, 2\pi]$

(d) $[\frac{\pi}{3}, \frac{5\pi}{6}]$

critical number

3.4   Applied Optimization

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- In a setting where a situation is described for which optimal parameters are sought, how do we develop a function that models the situation and use calculus to find the desired maximum or minimum?

Introduction

Near the conclusion of Section 3.3, we considered two examples of optimization problems where determining the function to be optimized was part of a broader question. In Example 3.4, we sought to use a single piece of wire to build two geometric figures (an equilateral triangle and square) and to understand how various choices for how to cut the wire led to different values of the area enclosed. One of our conclusions was that in order to maximize the total combined area enclosed by the triangle and square, all of the wire must be used to make a square. In the subsequent Activity 3.9, we investigated how the volume of a box constructed from a piece of cardboard by removing squares from each corner and folding up the sides depends on the size of the squares removed.

Both of these problems exemplify situations where there is not a function explicitly provided to optimize. Rather, we first worked to understand the given information in the problem, drawing a figure and introducing variables, and then sought to develop a formula for a function that models the quantity (area or volume, in the two examples, respectively) to be optimized. Once the function was established, we then considered what domain was appropriate on which to pursue the desired absolute minimum or maximum (or both). At this point in the problem, we are finally ready to apply the ideas of calculus to determine and justify the absolute minimum or maximum. Thus, what is primarily different about problems of this type is that the problem-solver must do considerable work to introduce variables and develop the correct function and domain to represent the described situation.

Throughout what follows in the current section, the primary emphasis is on the reader solving problems. Initially, some substantial guidance is provided, with the problems progressing to require greater independence as we move along.

Preview Activity 3.4. According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where by "girth" we mean the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square

end that can be sent by mail? What are the dimensions of the package of largest volume?
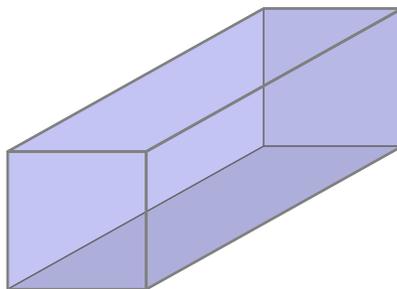

Figure 3.21: A rectangular parcel with a square end.

(a) Let x represent the length of one side of the square end and y the length of the longer side. Label these quantities appropriately on the image shown in Figure .

(b) What is the quantity to be optimized in this problem? Find a formula for this quantity in terms of x and y.

(c) The problem statement tells us that the parcel's girth plus length may not exceed 108 inches. In order to maximize volume, we assume that we will actually need the girth plus length to equal 108 inches. What equation does this produce involving x and y?

(d) Solve the equation you found in (c) for one of x or y (whichever is easier).

(e) Now use your work in (b) and (d) to determine a formula for the volume of the parcel so that this formula is a function of a single variable.

(f) Over what domain should we consider this function? Note that both x and y must be positive; how does the constraint that girth plus length is 108 inches produce intervals of possible values for x and y?

(g) Find the absolute maximum of the volume of the parcel on the domain you established in (f) and hence also determine the dimensions of the box of greatest volume. Justify that you've found the maximum using calculus.

More applied optimization problems

Many of the steps in Preview Activity 3.4 are ones that we will execute in any applied optimization problem. We briefly summarize those here to provide an overview of our approach in subsequent questions.

- Draw a picture and introduce variables. It is essential to first understand what quantities are allowed to vary in the problem and then to represent those values with variables. Constructing a figure with the variables labeled is almost always an essential first step. Sometimes drawing several diagrams can be especially helpful to get a sense of the situation. A nice example of this can be seen at http://gvsu.edu/s/99, where the choice of where to bend a piece of wire into the shape of a rectangle determines both the rectangle's shape and area.

- Identify the quantity to be optimized as well as any key relationships among the variable quantities. Essentially this step involves writing equations that involve the variables that have been introduced: one to represent the quantity whose minimum or maximum is sought, and possibly others that show how multiple variables in the problem may be interrelated.

- Determine a function of a single variable that models the quantity to be optimized; this may involve using other relationships among variables to eliminate one or more variables in the function formula. For example, in Preview Activity 3.4, we initially found that $V = x^2 y$, but then the additional relationship that $4x + y = 108$ (girth plus length equals 108 inches) allows us to relate x and y and thus observe equivalently that $y = 108 - 4x$. Substituting for y in the volume equation yields $V(x) = x^2(108 - 4x)$, and thus we have written the volume as a function of the single variable x.

- Decide the domain on which to consider the function being optimized. Often the physical constraints of the problem will limit the possible values that the independent variable can take on. Thinking back to the diagram describing the overall situation and any relationships among variables in the problem often helps identify the smallest and largest values of the input variable.

- Use calculus to identify the absolute maximum and/or minimum of the quantity being optimized. This always involves finding the critical numbers of the function first. Then, depending on the domain, we either construct a first derivative sign chart (for an open or unbounded interval) or evaluate the function at the endpoints and critical numbers (for a closed, bounded interval), using ideas we've studied so far in Chapter 3.

- Finally, we make certain we have answered the question: does the question seek the absolute maximum of a quantity, or the values of the variables that produce the

maximum? That is, finding the absolute maximum volume of a parcel is different from finding the dimensions of the parcel that produce the maximum.

Activity 3.10.

A soup can in the shape of a right circular cylinder is to be made from two materials. The material for the side of the can costs \$0.015 per square inch and the material for the lids costs \$0.027 per square inch. Suppose that we desire to construct a can that has a volume of 16 cubic inches. What dimensions minimize the cost of the can?

(a) Draw a picture of the can and label its dimensions with appropriate variables.

(b) Use your variables to determine expressions for the volume, surface area, and cost of the can.

(c) Determine the total cost function as a function of a single variable. What is the domain on which you should consider this function?

(d) Find the absolute minimum cost and the dimensions that produce this value.

◁

Familiarity with common geometric formulas is particularly helpful in problems like the one in Activity 3.10. Sometimes those involve perimeter, area, volume, or surface area. At other times, the constraints of a problem introduce right triangles (where the Pythagorean Theorem applies) or other functions whose formulas provide relationships among variables present.

Activity 3.11.

A hiker starting at a point P on a straight road walks east towards point Q, which is on the road and 3 kilometers from point P. Two kilometers due north of point Q is a cabin. The hiker will walk down the road for a while, at a pace of 8 kilometers per hour. At some point Z between P and Q, the hiker leaves the road and makes a straight line towards the cabin through the woods, hiking at a pace of 3 kph, as pictured in Figure 3.22. In order to minimize the time to go from P to Z to the cabin, where should the hiker turn into the forest?

◁

In more geometric problems, we often use curves or functions to provide natural constraints. For instance, we could investigate which isosceles triangle that circumscribes a unit circle has the smallest area, which you can explore for yourself at http://gvsu.edu/s/9b. Or similarly, for a region bounded by a parabola, we might seek the rectangle of largest area that fits beneath the curve, as shown at http://gvsu.edu/s/9c. The next activity is similar to the latter problem.
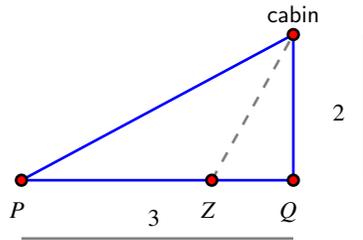
critical number

Activity 3.12.

Figure 3.22: A hiker walks from P to Z to the cabin, as pictured.

Consider the region in the x-y plane that is bounded by the x-axis and the function $f(x) = 25 - x^2$. Construct a rectangle whose base lies on the x-axis and is centered at the origin, and whose sides extend vertically until they intersect the curve $y = 25 - x^2$. Which such rectangle has the maximum possible area? Which such rectangle has the greatest perimeter? Which has the greatest combined perimeter and area? (Challenge: answer the same questions in terms of positive parameters a and b for the function $f(x) = b - ax^2$.)

◁

Activity 3.13.

A trough is being constructed by bending a 4×24 (measured in feet) rectangular piece of sheet metal. Two symmetric folds 2 feet apart will be made parallel to the longest side of the rectangle so that the trough has cross-sections in the shape of a trapezoid, as pictured in Figure 3.23. At what angle should the folds be made to produce the trough of maximum volume?
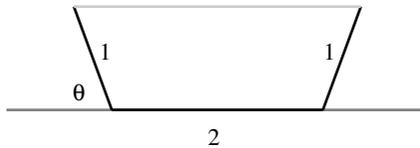

Figure 3.23: A cross-section of the trough formed by folding to an angle of θ.

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Summary

In this section, we encountered the following important ideas:

- While there is no single algorithm that works in every situation where optimization is used, in most of the problems we consider, the following steps are helpful: draw a picture and introduce variables; identify the quantity to be optimized and find relationships among the variables; determine a function of a single variable that models the quantity to be optimized; decide the domain on which to consider the function being optimized; use calculus to identify the absolute maximum and/or minimum of the quantity being optimized.

Exercises

1. A rectangular box with a square bottom and closed top is to be made from two materials. The material for the side costs $1.50 per square foot and the material for the bottom costs $3.00 per square foot. If you are willing to spend $15 on the box, what is the largest volume it can contain? Justify your answer completely using calculus.

2. A farmer wants to start raising cows, horses, goats, and sheep, and desires to have a rectangular pasture for the animals to graze in. However, no two different kinds of animals can graze together. In order to minimize the amount of fencing she will need, she has decided to enclose a large rectangular area and then divide it into four equally sized pens by adding three segments of fence inside the large rectangle that are parallel to two existing sides. She has decided to purchase 7500 ft of fencing. What is the maximum possible area that each of the four pens will enclose?

3. Two vertical poles of heights 60 ft and 80 ft stand on level ground, with their bases 100 ft apart. A cable that is stretched from the top of one pole to some point on the ground between the poles, and then to the top of the other pole. What is the minimum possible length of cable required? Justify your answer completely using calculus.

4. A company is designing propane tanks that are cylindrical with hemispherical ends. Assume that the company wants tanks that will hold 1000 cubic feet of gas, and that the ends are more expensive to make, costing $5 per square foot, while the cylindrical barrel between the ends costs $2 per square foot. Use calculus to determine the minimum cost to construct such a tank.

3.5 Related Rates

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- If two quantities that are related, such as the radius and volume of a spherical balloon, are both changing as implicit functions of time, how are their rates of change related? That is, how does the relationship between the values of the quantities affect the relationship between their respective derivatives with respect to time?

Introduction

In most of our applications of the derivative so far, we have worked in settings where one quantity (often called y) depends explicitly on another (say x), and in some way we have been interested in the instantaneous rate at which y changes with respect to x, leading us to compute $\frac{dy}{dx}$. These settings emphasize how the derivative enables us to quantify how the quantity y is changing as x changes at a given x-value.

We are next going to consider situations where multiple quantities are related to one another and changing, but where each quantity can be considered an implicit function of the variable t, which represents time. Through knowing how the quantities are related, we will be interested in determining how their respective rates of change with respect to time are related. For example, suppose that air is being pumped into a spherical balloon in such a way that its volume increases at a constant rate of 20 cubic inches per second. It makes sense that since the balloon's volume and radius are related, by knowing how fast the volume is changing, we ought to be able to relate this rate to how fast the radius is changing. More specifically, can we find how fast the radius of the balloon is increasing at the moment the balloon's diameter is 12 inches?

The following preview activity leads you through the steps to answer this question.

Preview Activity 3.5. A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon's diameter is 12 inches? Is the radius changing more rapidly when $d = 12$ or when $d = 16$? Why?

(a) Draw several spheres with different radii, and observe that as volume changes, the radius, diameter, and surface area of the balloon also change.

(b) Recall that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Note well that in the setting of this problem, *both* V and r are changing as time t changes, and thus

both V and r may be viewed as implicit functions of t, with respective derivatives $\frac{dV}{dt}$ and $\frac{dr}{dt}$.

Differentiate both sides of the equation $V = \frac{4}{3}\pi r^3$ with respect to t (using the chain rule on the right) to find a formula for $\frac{dV}{dt}$ that depends on both r and $\frac{dr}{dt}$.

(c) At this point in the problem, by differentiating we have "related the rates" of change of V and r. Recall that we are given in the problem that the balloon is being inflated at a constant *rate* of 20 cubic inches per second. Is this rate the value of $\frac{dr}{dt}$ or $\frac{dV}{dt}$? Why?

(d) From part (c), we know the value of $\frac{dV}{dt}$ at every value of t. Next, observe that when the diameter of the balloon is 12, we know the value of the radius. In the equation $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is $\frac{dr}{dt}$. How fast is the radius changing at the instant $d = 12$?

(e) How is the situation different when $d = 16$? When is the radius changing more rapidly, when $d = 12$ or when $d = 16$?

⋈

Related Rates Problems

In problems where two or more quantities can be related to one another, and all of the variables involved can be viewed as implicit functions of time, t, we are often interested in how the rates of change of the individual quantities with respect to time are themselves related; we call these *related rates* problems. Often these problems involve identifying one or more key underlying geometric relationships to relate the variables involved. Once we have an equation establishing the fundamental relationship among variables, we differentiate implicitly with respect to time to find connections among the rates of change.

For example, consider the situation where sand is being dumped by a conveyor belt on a pile so that the sand forms a right circular cone, as pictured in Figure 3.24. As sand falls from the conveyor belt onto the top of the pile, obviously several features of the sand pile will change: the volume of the pile will grow, the height will increase, and the radius will get bigger, too. All of these quantities are related to one another, and the rate at which each is changing is related to the rate at which sand falls from the conveyor.

The first key steps in any related rates problem involve identifying which variables are changing and how they are related. In the current problem involving a conical pile of sand, we observe that the radius and height of the pile are related to the volume of the pile by the standard equation for the volume of a cone,
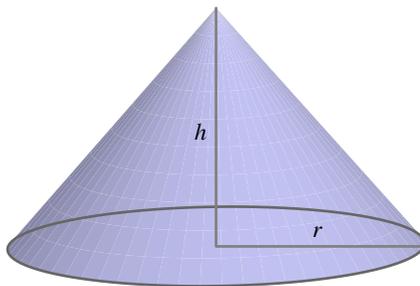
$$V = \frac{1}{3}\pi r^2 h.$$

Figure 3.24: A conical pile of sand.

Viewing each of V, r, and h as functions of t, we can differentiate implicitly to determine an equation that relates their respective rates of change. Taking the derivative of each side of the equation with respect to t,

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right].$$

On the left, $\frac{d}{dt}[V]$ is simply $\frac{dV}{dt}$. On the right, the situation is more complicated, as both r and h are implicit functions of t, hence we have to use the product and chain rules. Doing so, we find that

$$\begin{aligned}
\frac{dV}{dt} &= \frac{d}{dt}\left[\frac{1}{3}\pi r^2 h\right] \\
&= \frac{1}{3}\pi r^2 \frac{d}{dt}[h] + \frac{1}{3}\pi h \frac{d}{dt}[r^2] \\
&= \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{1}{3}\pi h 2r \frac{dr}{dt}
\end{aligned}$$

Note particularly how we are using ideas from Section 2.7 on implicit differentiation. There we found that when y is an implicit function of x, $\frac{d}{dx}[y^2] = 2y\frac{dy}{dx}$. The exact same thing is occurring here when we compute $\frac{d}{dt}[r^2] = 2r\frac{dr}{dt}$.

With our arrival at the equation

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt},$$

we have now related the rates of change of V, h, and r. If we are given sufficient information, we may then find the value of one or more of these rates of change at one or more points in time. Say, for instance, that we know the following: (a) sand falls from the conveyor in such a way that the height of the pile is always half the radius, and (b) sand falls from the conveyor belt at a constant rate of 10 cubic feet per minute. With this information given,

we can answer questions such as: how fast is the height of the sandpile changing at the moment the radius is 4 feet?

The information that the height is always half the radius tells us that for all values of t, $h = \frac{1}{2}r$. Differentiating with respect to t, it follows that $\frac{dh}{dt} = \frac{1}{2}\frac{dr}{dt}$. These relationships enable us to relate $\frac{dV}{dt}$ exclusively to just one of r or h. Substituting the expressions involving r and $\frac{dr}{dt}$ for h and $\frac{dh}{dt}$, we now have that

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \cdot \frac{1}{2}\frac{dr}{dt} + \frac{2}{3}\pi r \cdot \frac{1}{2}r \cdot \frac{dr}{dt}. \tag{3.1}$$

Since sand falls from the conveyor at the constant rate of 10 cubic feet per minute, this tells us the value of $\frac{dV}{dt}$, the rate at which the volume of the sand pile changes. In particular, $\frac{dV}{dt} = 10$ ft^3/min. Furthermore, since we are interested in how fast the height of the pile is changing at the instant $r = 4$, we use the value $r = 4$ along with $\frac{dV}{dt} = 10$ in Equation (3.1), and hence find that

$$10 = \frac{1}{3}\pi 4^2 \cdot \frac{1}{2}\frac{dr}{dt}\Big|_{r=4} + \frac{2}{3}\pi 4 \cdot \frac{1}{2}4 \cdot \frac{dr}{dt}\Big|_{r=4} = \frac{8}{3}\pi \frac{dr}{dt}\Big|_{r=4} + \frac{16}{3}\pi \frac{dr}{dt}\Big|_{r=4}.$$

With only the value of $\frac{dr}{dt}\big|_{r=4}$ remaining unknown, we solve for $\frac{dr}{dt}\big|_{r=4}$ and find that $10 = 8\pi \frac{dr}{dt}\big|_{r=4}$, so that

$$\frac{dr}{dt}\Big|_{r=4} = \frac{10}{8\pi} \approx 0.39789$$

feet per second. Because we were interested in how fast the height of the pile was changing at this instant, we want to know $\frac{dh}{dt}$ when $r = 4$. Since $\frac{dh}{dt} = \frac{1}{2}\frac{dr}{dt}$ for all values of t, it follows

$$\frac{dh}{dt}\Big|_{r=4} = \frac{5}{8\pi} \approx 0.19894 \text{ ft/min.}$$

Note particularly how we distinguish between the notations $\frac{dr}{dt}$ and $\frac{dr}{dt}\big|_{r=4}$. The former represents the rate of change of r with respect to t at an arbitrary value of t, while the latter is the rate of change of r with respect to t at a particular moment, in fact the moment $r = 4$. While we don't know the exact value of t, because information is provided about the value of r, it is important to distinguish that we are using this more specific data.

The relationship between h and r, with $h = \frac{1}{2}r$ for all values of t, enables us to transition easily between questions involving r and h. Indeed, had we known this information at the problem's outset, we could have immediately simplified our work. Using $h = \frac{1}{2}r$, it follows that since $V = \frac{1}{3}\pi r^2 h$, we can write V solely in terms of r to have

$$V = \frac{1}{3}\pi r^2 \left(\frac{1}{2}h\right) = \frac{1}{6}\pi r^3.$$

From this last equation, differentiating with respect to t implies

$$\frac{dV}{dt} = \frac{1}{2}\pi r^2 \frac{dr}{dt},$$

from which the same conclusions made earlier about $\frac{dr}{dt}$ and $\frac{dh}{dt}$ can be made.

Our work with the sandpile problem above is similar in many ways to our approach in Preview Activity 3.5, and these steps are typical of most related rates problems. In certain ways, they also resemble work we do in applied optimization problems, and here we summarize the main approach for consideration in subsequent problems.

- Identify the quantities in the problem that are changing and choose clearly defined variable names for them. Draw one or more figures that clearly represent the situation.

- Determine all rates of change that are known or given and identify the rate(s) of change to be found.

- Find an equation that relates the variables whose rates of change are known to those variables whose rates of change are to be found.

- Differentiate implicitly with respect to t to relate the rates of change of the involved quantities.

- Evaluate the derivatives and variables at the information relevant to the instant at which a certain rate of change is sought. Use proper notation to identify when a derivative is being evaluated at a particular instant, such as $\frac{dr}{dt}\big|_{r=4}$.

In the first step of identifying changing quantities and drawing a picture, it is important to think about the dynamic ways in which the involved quantities change. Sometimes a sequence of pictures can be helpful; for some already-drawn pictures that can be easily modified as applets built in Geogebra, see the following links[2] which represent

- how a circular oil slick's area grows as its radius increases `http://gvsu.edu/s/9n`;

- how the location of the base of a ladder and its height along a wall change as the ladder slides `http://gvsu.edu/s/9o`;

- how the water level changes in a conical tank as it fills with water at a constant rate `http://gvsu.edu/s/9p` (compare the problem in Activity 3.14);

- how a skateboarder's shadow changes as he moves past a lamppost `http://gvsu.edu/s/9q`.

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[2]We again refer to the work of Prof. Marc Renault of Shippensburg University, found at `http://gvsu.edu/s/5p`.

Drawing well-labeled diagrams and envisioning how different parts of the figure change is a key part of understanding related rates problems and being successful at solving them.

Activity 3.14.

A water tank has the shape of an inverted circular cone (point down) with a base of radius 6 feet and a depth of 8 feet. Suppose that water is being pumped into the tank at a constant instantaneous rate of 4 cubic feet per minute.

(a) Draw a picture of the conical tank, including a sketch of the water level at a point in time when the tank is not yet full. Introduce variables that measure the radius of the water's surface and the water's depth in the tank, and label them on your figure.

(b) Say that r is the radius and h the depth of the water at a given time, t. What equation relates the radius and height of the water, and why?

(c) Determine an equation that relates the volume of water in the tank at time t to the depth h of the water at that time.

(d) Through differentiation, find an equation that relates the instantaneous rate of change of water volume with respect to time to the instantaneous rate of change of water depth at time t.

(e) Find the instantaneous rate at which the water level is rising when the water in the tank is 3 feet deep.

(f) When is the water rising most rapidly: at $h = 3$, $h = 4$, or $h = 5$?

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Recognizing familiar geometric configurations is one way that we relate the changing quantities in a given problem. For instance, while the problem in Activity 3.14 is centered on a conical tank, one of the most important observations is that there are two key right triangles present. In another setting, a right triangle might be indicative of an opportunity to take advantage of the Pythagorean Theorem to relate the legs of the triangle. But in the conical tank, the fact that the water at any time fills a portion of the tank in such a way that the ratio of radius to depth is constant turns out to be the most important relationship with which to work. That enables us to write r in terms of h and reduce the overall problem to one that involves only one variable, where the volume of water depends simply on h, and hence to subsequently relate $\frac{dV}{dt}$ and $\frac{dh}{dt}$. In other situations where a changing angle is involved, a right triangle may offer the opportunity to find relationships among various parts of the triangle using trigonometric functions.

Activity 3.15.

A television camera is positioned 4000 feet from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. In addition, the auto-focus of the camera has to take into

account the increasing distance between the camera and the rocket. We assume that the rocket rises vertically. (A similar problem is discussed and pictured dynamically at http://gvsu.edu/s/9t. Exploring the applet at the link will be helpful to you in answering the questions that follow.)

(a) Draw a figure that summarizes the given situation. What parts of the picture are changing? What parts are constant? Introduce appropriate variables to represent the quantities that are changing.

(b) Find an equation that relates the camera's angle of elevation to the height of the rocket, and then find an equation that relates the instantaneous rate of change of the camera's elevation angle to the instantaneous rate of change of the rocket's height (where all rates of change are with respect to time).

(c) Find an equation that relates the distance from the camera to the rocket to the rocket's height, as well as an equation that relates the instantaneous rate of change of distance from the camera to the rocket to the instantaneous rate of change of the rocket's height (where all rates of change are with respect to time).

(d) Suppose that the rocket's speed is 600 ft/sec at the instant it has risen 3000 feet. How fast is the distance from the television camera to the rocket changing at that moment? If the camera is following the rocket, how fast is the camera's angle of elevation changing at that same moment?

(e) If from an elevation of 3000 feet onward the rocket continues to rise at 600 feet/sec, will the rate of change of distance with respect to time be greater when the elevation is 4000 feet than it was at 3000 feet, or less? Why?

◁

In addition to being able to find instantaneous rates of change at particular points in time, we are often able to make more general observations about how particular rates themselves will change over time. For instance, when a conical tank (point down) is filling with water at a constant rate, we naturally intuit that the depth of the water should increase more slowly over time. Note how carefully we need to speak: we mean to say that while the depth, h, of the water is increasing, its rate of change $\frac{dh}{dt}$ is decreasing (both as a function of t and as a function of h). These observations may often be made by taking the general equation that relates the various rates and solving for one of them, and doing this without substituting any particular values for known variables or rates. For instance, in the conical tank problem in Activity 3.14, we established that

$$\frac{dV}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt},$$

and hence

$$\frac{dh}{dt} = \frac{16}{\pi h^2}\frac{dV}{dt}.$$

Provided that $\frac{dV}{dt}$ is constant, it is immediately apparent that as h gets larger, $\frac{dh}{dt}$ will get smaller, while always remaining positive. Hence, the depth of the water is increasing at a decreasing rate.

Activity 3.16.

As pictured in the applet at http://gvsu.edu/s/9q, a skateboarder who is 6 feet tall rides under a 15 foot tall lamppost at a constant rate of 3 feet per second. We are interested in understanding how fast his shadow is changing at various points in time.

(a) Draw an appropriate right triangle that represents a snapshot in time of the skateboarder, lamppost, and his shadow. Let x denote the horizontal distance from the base of the lamppost to the skateboarder and s represent the length of his shadow. Label these quantities, as well as the skateboarder's height and the lamppost's height on the diagram.

(b) Observe that the skateboarder and the lamppost represent parallel line segments in the diagram, and thus similar triangles are present. Use similar triangles to establish an equation that relates x and s.

(c) Use your work in (b) to find an equation that relates $\frac{dx}{dt}$ and $\frac{ds}{dt}$.

(d) At what rate is the length of the skateboarder's shadow increasing at the instant the skateboarder is 8 feet from the lamppost?

(e) As the skateboarder's distance from the lamppost increases, is his shadow's length increasing at an increasing rate, increasing at a decreasing rate, or increasing at a constant rate?

(f) Which is moving more rapidly: the skateboarder or the tip of his shadow? Explain, and justify your answer.

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As we progress further into related rates problems, less direction will be provided. In the first three activities of this section, we have been provided with guided instruction to build a solution in a step by step way. For the closing activity and the following exercises, most of the detailed work is left to the reader.

Activity 3.17.

A baseball diamond is $90'$ square. A batter hits a ball along the third base line and runs to first base. At what rate is the distance between the ball and first base changing when the ball is halfway to third base, if at that instant the ball is traveling 100 feet/sec? At what rate is the distance between the ball and the runner changing at the same instant, if at the same instant the runner is 1/8 of the way to first base running at 30 feet/sec?

◁

Summary

In this section, we encountered the following important ideas:

- When two or more related quantities are changing as implicit functions of time, their rates of change can be related by implicitly differentiating the equation that relates the quantities themselves. For instance, if the sides of a right triangle are all changing as functions of time, say having lengths x, y, and z, then these quantities are related by the Pythagorean Theorem: $x^2 + y^2 = z^2$. It follows by implicitly differentiating with respect to t that their rates are related by the equation

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt},$$

so that if we know the values of x, y, and z at a particular time, as well as two of the three rates, we can deduce the value of the third.

Exercises

1. A sailboat is sitting at rest near its dock. A rope attached to the bow of the boat is drawn in over a pulley that stands on a post on the end of the dock that is 5 feet higher than the bow. If the rope is being pulled in at a rate of 2 feet per second, how fast is the boat approaching the dock when the length of rope from bow to pulley is 13 feet?

2. A swimming pool is 60 feet long and 25 feet wide. Its depth varies uniformly from 3 feet at the shallow end to 15 feet at the deep end, as shown in the Figure 3.25. Suppose
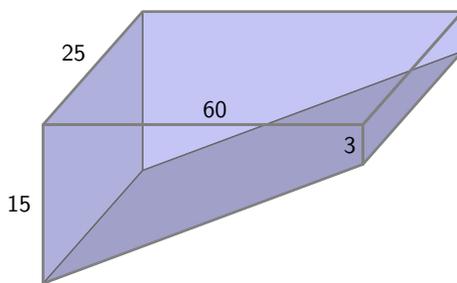

Figure 3.25: The swimming pool described in Exercise 2.

the pool has been emptied and is now being filled with water at a rate of 800 cubic feet per minute. At what rate is the depth of water (measured at the deepest point of the pool) increasing when it is 5 feet deep at that end? Over time, describe how the depth of the water will increase: at an increasing rate, at a decreasing rate, or at a constant rate. Explain.

3. A baseball diamond is a square with sides 90 feet long. Suppose a baseball player is advancing from second to third base at the rate of 24 feet per second, and an umpire is standing on home plate. Let θ be the angle between the third baseline and the line of sight from the umpire to the runner. How fast is θ changing when the runner is 30 feet from third base?

4. Sand is being dumped off a conveyor belt onto a pile in such a way that the pile forms in the shape of a cone whose radius is always equal to its height. Assuming that the sand is being dumped at a rate of 10 cubic feet per minute, how fast is the height of the pile changing when there are 1000 cubic feet on the pile?