Chapter 11

Multiple Integrals

11.1 Double Riemann Sums and Double Integrals over Rectangles

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a double Riemann sum?
- How is the double integral of a continuous function $f = f(x, y)$ defined?
- What are two things the double integral of a function can tell us?

Introduction

In single-variable calculus, recall that we approximated the area under the graph of a positive function f on an interval $[a, b]$ by adding areas of rectangles whose heights are determined by the curve. The general process involved subdividing the interval $[a, b]$ into smaller subintervals, constructing rectangles on each of these smaller intervals to approximate the region under the curve on that subinterval, then summing the areas of these rectangles to approximate the area under the curve. We will extend this process in this section to its three-dimensional analogs, double Riemann sums and double integrals over rectangles.

Preview Activity 11.1. In this activity we introduce the concept of a double Riemann sum.

(a) Review the concept of the Riemann sum from single-variable calculus. Then, explain how we define the definite integral $\int_a^b f(x)\, dx$ of a continuous function of a single variable x on an interval $[a, b]$. Include a sketch of a continuous function on an interval $[a, b]$ with appropriate labeling in order to illustrate your definition.

(b) In our upcoming study of integral calculus for multivariable functions, we will first extend the idea of the single-variable definite integral to functions of two variables over rectangular domains. To do so, we will need to understand how to partition a rectangle into subrectangles. Let R be rectangular domain $R = \{(x, y) : 0 \leq x \leq 6, 2 \leq y \leq 4\}$ (we can also represent this domain with the notation $[0, 6] \times [2, 4]$), as pictured in Figure 11.1.

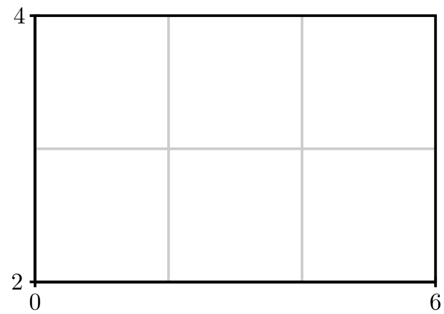

Figure 11.1: Rectangular domain R with subrectangles.

To form a partition of the full rectangular region, R, we will partition both intervals $[0, 6]$ and $[2, 4]$; in particular, we choose to partition the interval $[0, 6]$ into three uniformly sized subintervals and the interval $[2, 4]$ into two evenly sized subintervals as shown in Figure 11.1. In the following questions, we discuss how to identify the endpoints of each subinterval and the resulting subrectangles.

i. Let $0 = x_0 < x_1 < x_2 < x_3 = 6$ be the endpoints of the subintervals of $[0, 6]$ after partitioning. What is the length Δx of each subinterval $[x_{i-1}, x_i]$ for i from 1 to 3?
ii. Explicitly identify x_0, x_1, x_2, and x_3. On Figure 11.1 or your own version of the diagram, label these endpoints.
iii. Let $2 = y_0 < y_1 < y_2 = 4$ be the endpoints of the subintervals of $[2, 4]$ after partitioning. What is the length Δy of each subinterval $[y_{j-1}, y_j]$ for j from 1 to 2? Identify y_0, y_1, and y_2 and label these endpoints on Figure 11.1.
iv. Let R_{ij} denote the subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Appropriately label each subrectangle in your drawing of Figure 11.1. How does the total number of subrectangles depend on the partitions of the intervals $[0, 6]$ and $[2, 4]$?
v. What is area ΔA of each subrectangle?

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Double Riemann Sums over Rectangles

For the definite integral in single-variable calculus, we considered a continuous function over a closed, bounded interval $[a, b]$. In multivariable calculus, we will eventually develop the idea of a definite integral over a closed, bounded region (such as the interior of a circle). We begin with a simpler situation by thinking only about rectangular domains, and will address more complicated domains in the following section.

Let $f = f(x, y)$ be a continuous function defined on a rectangular domain $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. As we saw in Preview Activity 11.1, the domain is a rectangle R and we want to partition R into subrectangles. We do this by partitioning each of the intervals $[a, b]$ and $[c, d]$ into subintervals and using those subintervals to create a partition of R into subrectangles. In the first activity, we address the quantities and notations we will use in order to define double Riemann sums and double integrals.

Activity 11.1.

Let $f(x, y) = 100 - x^2 - y^2$ be defined on the rectangular domain $R = [a, b] \times [c, d]$. Partition the interval $[a, b]$ into four uniformly sized subintervals and the interval $[c, d]$ into three evenly sized subintervals as shown in Figure 11.2. As we did in Preview Activity 11.1, we will need a method for identifying the endpoints of each subinterval and the resulting subrectangles.

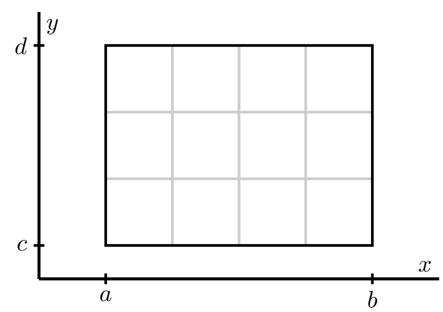

Figure 11.2: Rectangular domain with subrectangles.

(a) Let $a = x_0 < x_1 < x_2 < x_3 < x_4 = b$ be the endpoints of the subintervals of $[a, b]$ after partitioning. Label these endpoints in Figure 11.2.

(b) What is the length Δx of each subinterval $[x_{i-1}, x_i]$? Your answer should be in terms of a and b.

(c) Let $c = y_0 < y_1 < y_2 < y_3 = d$ be the endpoints of the subintervals of $[c, d]$ after partitioning. Label these endpoints in Figure 11.2.

(d) What is the length Δy of each subinterval $[y_{j-1}, y_j]$? Your answer should be in terms of c and d.

(e) The partitions of the intervals $[a, b]$ and $[c, d]$ partition the rectangle R into subrectangles. How many subrectangles are there?

(f) Let R_{ij} denote the subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Label each subrectangle in Figure 11.2.

(g) What is area ΔA of each subrectangle?

(h) Now let $[a, b] = [0, 8]$ and $[c, d] = [2, 6]$. Let (x^*_{11}, y^*_{11}) be the point in the upper right corner of the subrectangle R_{11}. Identify and correctly label this point in Figure 11.2. Calculate the product

$$f(x^*_{11}, y^*_{11})\Delta A.$$

Explain, geometrically, what this product represents.

(i) For each i and j, choose a point (x^*_{ij}, y^*_{ij}) in the subrectangle $R_{i,j}$. Identify and correctly label these points in Figure 11.2. Explain what the product

$$f(x^*_{ij}, y^*_{ij})\Delta A$$

represents.

(j) If we were to add all the values $f(x^*_{ij}, y^*_{ij})\Delta A$ for each i and j, what does the resulting number approximate about the surface defined by f on the domain R? (You don't actually need to add these values.)

(k) Write a double sum using summation notation that expresses the arbitrary sum from part (j).

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Double Riemann Sums and Double Integrals

Now we use the process from the most recent activity to formally define double Riemann sums and double integrals.

Definition 11.1. Let f be a continuous function on a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. A **double Riemann sum for f over R** is created as follows.

- Partition the interval $[a, b]$ into m subintervals of equal length $\Delta x = \frac{b-a}{m}$. Let x_0, x_1, …, x_m be the endpoints of these subintervals, where $a = x_0 < x_1 < x_2 < \cdots < x_m = b$.
- Partition the interval $[c, d]$ into n subintervals of equal length $\Delta y = \frac{d-c}{n}$. Let y_0, y_1, …, y_n be the endpoints of these subintervals, where $c = y_0 < y_1 < y_2 < \cdots < y_n = d$.
- These two partitions create a partition of the rectangle R into mn subrectangles R_{ij} with opposite vertices (x_{i-1}, y_{j-1}) and (x_i, y_j) for i between 1 and m and j between 1 and n. These rectangles all have equal area $\Delta A = \Delta x \cdot \Delta y$.
- Choose a point (x^*_{ij}, y^*_{ij}) in each rectangle R_{ij}. Then, a double Riemann sum for f over R is given by

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f(x^*_{ij}, y^*_{ij}) \cdot \Delta A.$$

If $f(x, y) \geq 0$ on the rectangle R, we may ask to find the volume of the solid bounded above by f over R, as illustrated on the left of Figure 11.3. This volume is approximated by a Riemann sum, which sums the volumes of the rectangular boxes shown on the right of Figure 11.3.

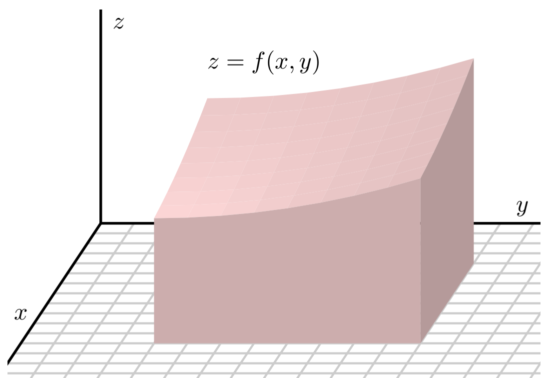

Figure 11.3: The volume under a graph approximated by a Riemann Sum

As we let the number of subrectangles increase without bound (in other words, as both m and n in a double Riemann sum go to infinity), as illustrated in Figure 11.4, the sum of the volumes of the rectangular boxes approaches the volume of the solid bounded above by f over R. The value of this limit, provided it exists, is the double integral.

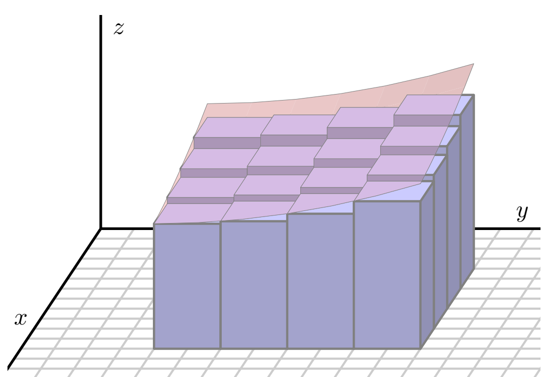

Figure 11.4: Finding better approximations by using smaller subrectangles.

Definition 11.2. Let R be a rectangular region in the x-y plane and f a continuous function over R. With terms defined as in a double Riemann sum, the **double integral of f over R** is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*)\cdot \Delta A.$$

Interpretation of Double Riemann Sums and Double integrals.

At the moment, there are two ways we can interpret the value of the double integral.

- Suppose that $f(x,y)$ assumes both positive and negatives values on the rectangle R, as shown on the left of Figure 11.5. When constructing a Riemann sum, for each i and j, the product $f(x_{ij}^*, y_{ij}^*)\cdot\Delta A$ can be interpreted as a "signed" volume of a box with base area ΔA and "signed" height $f(x_{ij}^*, y_{ij}^*)$. Since f can have negative values, this "height" could be negative. The sum

  $$\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*)\cdot\Delta A$$

  can then be interpreted as a sum of "signed" volumes of boxes, with a negative sign attached to those boxes whose heights are below the xy-plane. We can then realize the double integral

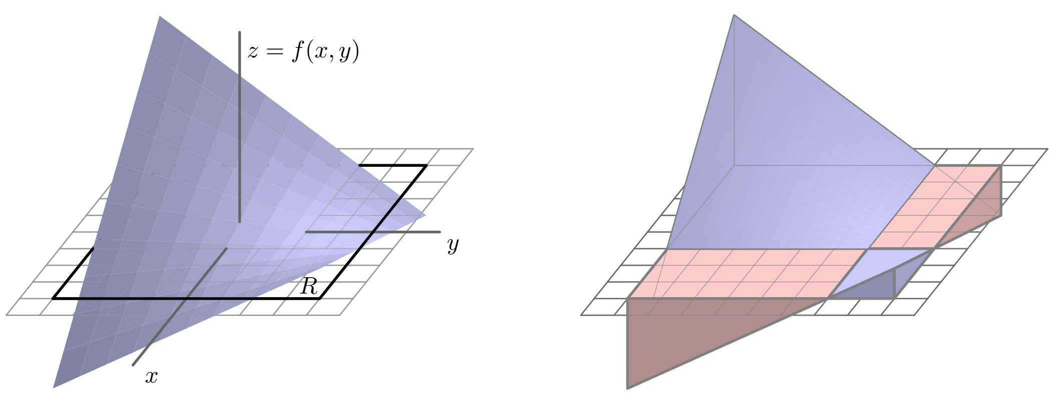

Figure 11.5: The integral measures signed volume.

  $\iint_R f(x,y)\,dA$ as a difference in volumes: $\iint_R f(x,y)\,dA$ tells us the volume of the solids the graph of f bounds above the xy-plane over the rectangle R minus the volume of the solids the graph of f bounds below the xy-plane under the rectangle R. This is shown on the right of Figure 11.5.

- The average of the finitely many mn values $f(x_{ij}^*, y_{ij}^*)$ that we take in a double Riemann sum is given by

$$\text{Avg}_{mn} = \frac{1}{mn} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*).$$

  If we take the limit as m and n go to infinity, we obtain what we define as the average value of f over the region R, which is connected to the value of the double integral. First, to view Avg_{mn} as a double Riemann sum, note that

$$\Delta x = \frac{b-a}{m} \quad \text{and} \quad \Delta y = \frac{d-c}{n}.$$

  Thus,

$$\frac{1}{mn} = \frac{\Delta x \cdot \Delta y}{(b-a)(d-c)} = \frac{\Delta A}{A(R)},$$

  where $A(R)$ denotes the area of the rectangle R. Then, the average value of the function f over R, $f_{\text{AVG(R)}}$, is given by

$$\begin{aligned} f_{\text{AVG(R)}} &= \lim_{m,n\to\infty} \frac{1}{mn} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*) \\ &= \lim_{m,n\to\infty} \frac{1}{A(R)} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A \\ &= \frac{1}{A(R)} \iint_R f(x,y)\, dA. \end{aligned}$$

  Therefore, the double integral of f over R divided by the area of R gives us the average value of the function f on R. Finally, if $f(x,y) \geq 0$ on R, we can interpret this average value of f on R as the height of the box with base R that has the same volume as the volume of the surface defined by f over R.

Activity 11.2.

Let $f(x,y) = x + 2y$ and let $R = [0,2] \times [1,3]$.

(a) Draw a picture of R. Partition $[0,2]$ into 2 subintervals of equal length and the interval $[1,3]$ into two subintervals of equal length. Draw these partitions on your picture of R and label the resulting subrectangles using the labeling scheme we established in the definition of a double Riemann sum.

(b) For each i and j, let (x_{ij}^*, y_{ij}^*) be the midpoint of the rectangle R_{ij}. Identify the coordinates of each (x_{ij}^*, y_{ij}^*). Draw these points on your picture of R.

(c) Calculate the Riemann sum

$$\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A$$

using the partitions we have described. If we let (x_{ij}^*, y_{ij}^*) be the midpoint of the rectangle R_{ij} for each i and j, then the resulting Riemann sum is called a *midpoint sum*.

(d) Give two interpretations for the meaning of the sum you just calculated.

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Activity 11.3.

Let $f(x,y) = \sqrt{4-y^2}$ on the rectangular domain $R = [1,7] \times [-2,2]$. Partition $[1,7]$ into 3 equal length subintervals and $[-2,2]$ into 2 equal length subintervals. A table of values of f at some points in R is given in Table 11.1, and a graph of f with the indicated partitions is shown in Figure 11.6.

	-2	-1	0	1	2
1	0	$\sqrt{3}$	2	$\sqrt{3}$	0
2	0	$\sqrt{3}$	2	$\sqrt{3}$	0
3	0	$\sqrt{3}$	2	$\sqrt{3}$	0
4	0	$\sqrt{3}$	2	$\sqrt{3}$	0
5	0	$\sqrt{3}$	2	$\sqrt{3}$	0
6	0	$\sqrt{3}$	2	$\sqrt{3}$	0
7	0	$\sqrt{3}$	2	$\sqrt{3}$	0

Table 11.1: Table of values of $f(x,y) = \sqrt{4-y^2}$.

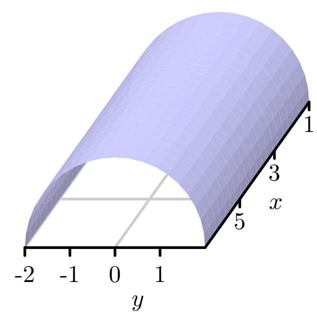

Figure 11.6: Graph of $f(x,y) = \sqrt{4-y^2}$ on R.

(a) Outline the partition of R into subrectangles on the table of values in Table 11.1.

(b) Calculate the double Riemann sum using the given partition of R and the values of f in the upper right corner of each subrectangle.

(c) Use geometry to calculate the exact value of $\iint_R f(x,y)\, dA$ and compare it to your approximation. How could we obtain a better approximation?

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We conclude this section with a list of properties of double integrals. Since similar properties are satisfied by single-variable integrals and the arguments for double integrals are essentially the same, we omit their justification.

Properties of Double Integrals. Let f and g be continuous functions on a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$, and let k be a constant. Then

1. $\iint_R (f(x, y) + g(x, y))\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$.
2. $\iint_R kf(x, y)\, dA = k \iint_R f(x, y)\, dA$.
3. If $f(x, y) \geq g(x, y)$ on R, then $\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$.

Summary

- Let f be a continuous function on a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. The double Riemann sum for f over R is created as follows.
  - Partition the interval $[a, b]$ into m subintervals of equal length $\Delta x = \frac{b-a}{m}$. Let x_0, x_1, ..., x_m be the endpoints of these subintervals, where $a = x_0 < x_1 < x_2 < \cdots < x_m = b$.
  - Partition the interval $[c, d]$ into n subintervals of equal length $\Delta y = \frac{d-c}{n}$. Let y_0, y_1, ..., y_n be the endpoints of these subintervals, where $c = y_0 < y_1 < y_2 < \cdots < y_n = d$.
  - These two partitions create a partition of the rectangle R into mn subrectangles R_{ij} with opposite vertices (x_{i-1}, y_{j-1}) and (x_i, y_j) for i between 1 and m and j between 1 and n. These rectangles all have equal area $\Delta A = \Delta x \cdot \Delta y$.
  - Choose a point (x^*_{ij}, y^*_{ij}) in each rectangle R_{ij}. Then a double Riemann sum for f over R is given by
  $$\sum_{j=1}^{n} \sum_{i=1}^{m} f(x^*_{ij}, y^*_{ij}) \cdot \Delta A.$$
- With terms defined as in the Double Riemann Sum, the double integral of f over R is
$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x^*_{ij}, y^*_{ij}) \cdot \Delta A.$$
- Two interpretations of the double integral $\displaystyle\iint_R f(x, y)\, dA$ are:
  - The volume of the solids the graph of f bounds above the xy-plane over the rectangle R minus the volume of the solids the graph of f bounds below the xy-plane under the rectangle R;
  - Dividing the double integral of f over R by the area of R gives us the average value of the function f on R. If $f(x, y) \geq 0$ on R, we can interpret this average value of f on R as the height of the box with base R that has the same volume as the volume of the surface defined by f over R.

Exercises

1. The temperature at any point on a metal plate in the xy plane is given by $T(x, y) = 100 - 4x^2 - y^2$, where x and y are measured in inches and T in degrees Celsius. Consider the portion of the plate that lies on the rectangular region $R = [1, 5] \times [3, 6]$.

(a) Estimate the value of $\iint_R T(x, y)\, dA$ by using a double Riemann sum with two subintervals in each direction and choosing (x_i^*, y_j^*) to be the point that lies in the upper right corner of each subrectangle.

(b) Determine the area of the rectangle R.

(c) Estimate the average temperature, $T_{\text{AVG(R)}}$, over the region R.

(d) Do you think your estimate in (c) is an over- or under-estimate of the true temperature? Why?

2. The wind chill, as frequently reported, is a measure of how cold it feels outside when the wind is blowing. In Table 11.2, the wind chill $w = w(v, T)$, measured in degrees Fahrenheit, is a function of the wind speed v, measured in miles per hour, and the ambient air temperature T, also measured in degrees Fahrenheit. Approximate the average wind chill on the rectangle $[5, 35] \times [-20, 20]$ using 3 subintervals in the v direction, 4 subintervals in the T direction, and the point in the lower left corner in each subrectangle.

$v \backslash T$	-20	-15	-10	-5	0	5	10	15	20
5	-34	-28	-22	-16	-11	-5	1	7	13
10	-41	-35	-28	-22	-16	-10	-4	3	9
15	-45	-39	-32	-26	-19	-13	-7	0	6
20	-48	-42	-35	-29	-22	-15	-9	-2	4
25	-51	-44	-37	-31	-24	-17	-11	-4	3
30	-53	-46	-39	-33	-26	-19	-12	-5	1
35	-55	-48	-41	-34	-27	-21	-14	-7	0

Table 11.2: Wind chill as a function of wind speed and temperature.

3. Consider the box with a sloped top that is given by the following description: the base is the rectangle $R = [0, 4] \times [0, 3]$, while the top is given by the plane $z = p(x, y) = 20 - 2x - 3y$.

(a) Estimate the value of $\iint_R p(x, y)\, dA$ by using a double Riemann sum with four subintervals in the x direction and three subintervals in the y direction, and choosing (x_i^*, y_j^*) to be the point that is the midpoint of each subrectangle.

(b) What important quantity does your double Riemann sum in (a) estimate?

(c) Suppose it can be determined that $\iint_R p(x,y)\,dA = 138$. What is the exact average value of p over R?

(d) If you wanted to build a rectangular box (with the same base) that has the same volume as the box with the sloped top described here, how tall would the rectangular box have to be?

11.2 Iterated Integrals

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How do we evaluate a double integral over a rectangle as an iterated integral, and why does this process work?

Introduction

Recall that we defined the double integral of a continuous function $f = f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$ as

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) \cdot \Delta A,$$

where the different variables and notation are as described in Section 11.1. Thus $\iint_R f(x,y)\,dA$ is a limit of double Riemann sums, but while this definition tells us exactly what a double integral is, it is not very helpful for determining the value of a double integral. Fortunately, there is a way to view a double integral as an *iterated integral*, which will make computations feasible in many cases.

The viewpoint of an iterated integral is closely connected to an important idea from single-variable calculus. When we studied solids of revolution, such as the one shown in Figure 11.7, we saw that in some circumstances we could slice the solid perpendicular to an axis and have each slice be approximately a circular disk. From there, we were able to find the volume of each disk, and then use an integral to add the volumes of the slices. In what follows, we are able to use single integrals to generalize this approach to handle even more general geometric shapes.

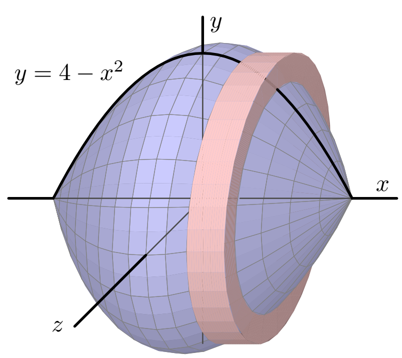

Figure 11.7: A solid of revolution.

Preview Activity 11.2. Let $f(x, y) = 25 - x^2 - y^2$ on the rectangular domain $R = [-3, 3] \times [-4, 4]$.

As with partial derivatives, we may treat one of the variables in f as constant and think of the resulting function as a function of a single variable. Now we investigate what happens if we integrate instead of differentiate.

(a) Choose a fixed value of x in the interior of $[-3, 3]$. Let

$$A(x) = \int_{-4}^{4} f(x, y)\, dy.$$

What is the geometric meaning of the value of $A(x)$ relative to the surface defined by f. (Hint: Think about the trace determined by the fixed value of x, and consider how $A(x)$ is related to Figure 11.8.)

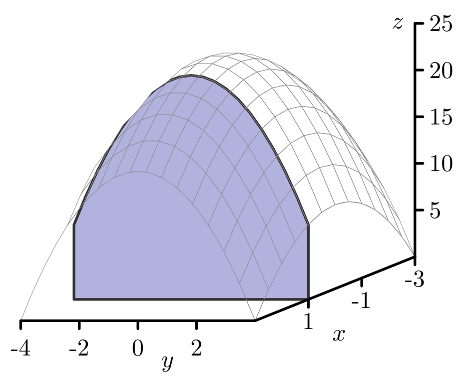

Figure 11.8: A cross section with fixed x.

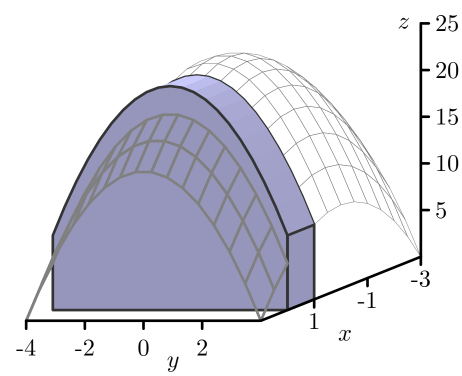

Figure 11.9: A cross section with fixed x and Δx.

(b) For a fixed value of x, say x_i^*, what is the geometric meaning of $A(x_i^*)\,\Delta x$? (Hint: Consider how $A(x_i^*)\Delta x$ is related to Figure 11.9.)

(c) Since f is continuous on R, we can define the function $A = A(x)$ at every value of x in $[-3, 3]$. Now think about subdividing the x-interval $[-3, 3]$ into m subintervals, and choosing a value x_i^* in each of those subintervals. What will be the meaning of the sum $\sum_{i=1}^{m} A(x_i^*)\,\Delta x$?

(d) Explain why $\int_{-3}^{3} A(x)\, dx$ will determine the exact value of the volume under the surface $z = f(x, y)$ over the rectangle R.

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Iterated Integrals

The ideas that we explored in Preview Activity 11.2 work more generally and lead to the idea of an iterated integral. Let f be a continuous function on a rectangular domain $R = [a, b] \times [c, d]$, and let

$$A(x) = \int_c^d f(x, y)\, dy.$$

The function $A = A(x)$ determines the value of the cross sectional area[1] in the y direction for the fixed value of x of the solid bounded between the surface defined by f and the xy-plane.

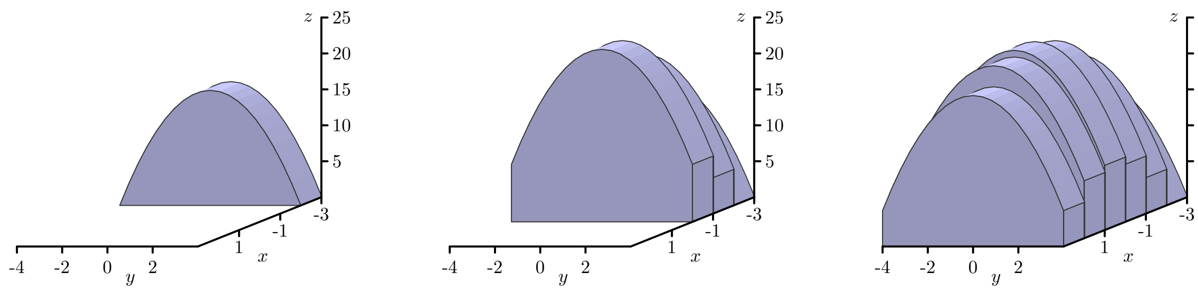

Figure 11.10: Summing cross section slices.

The value of this cross sectional area is determined by the input x in A. Since A is a function of x, it follows that we can integrate A with respect to x. In doing so, we use a partition of $[a, b]$ and make an approximation to the integral given by

$$\int_a^b A(x)\, dx \approx \sum_{i=1}^{m} A(x_i^*)\Delta x,$$

where x_i^* is any number in the subinterval $[x_{i-1}, x_i]$. Each term $A(x_i^*)\Delta x$ in the sum represents an approximation of a fixed cross sectional slice of the surface in the y direction with a fixed width of Δx as illustrated in Figure 11.9. We add the signed volumes of these slices as shown in the frames in Figure 11.10 to obtain an approximation of the total signed volume.

As we let the number of subintervals in the x direction approach infinity, we can see that the Riemann sum $\sum_{i=1}^{m} A(x_i^*)\Delta x$ approaches a limit and that limit is the sum of signed volumes bounded by the function f on R. Therefore, since $A(x)$ is itself determined by an integral, we have

$$\iint_R f(x, y)\, dA = \lim_{m \to \infty} \sum_{i=1}^{m} A(x_i^*)\Delta x = \int_a^b A(x)\, dx = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx.$$

[1]By area we mean "signed" area.

Hence, we can compute the double integral of f over R by first integrating f with respect to y on $[c, d]$, then integrating the resulting function of x with respect to x on $[a, b]$. The nested integral

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

is called an *iterated integral*, and we see that each double integral may be represented by two single integrals.

We made a choice to integrate first with respect to y. The same argument shows that we can also find the double integral as an iterated integral integrating with respect to x first, or

$$\iint_R f(x, y)\, dA = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

The fact that integrating in either order results in the same value is known as Fubini's Theorem.

Fubini's Theorem. If $f = f(x, y)$ is a continuous function on a rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Fubini's theorem enables us to evaluate iterated integrals without resorting to the limit definition. Instead, working with one integral at a time, we can use the Fundamental Theorem of Calculus from single-variable calculus to find the exact value of each integral, starting with the inner integral.

Activity 11.4.

Let $f(x, y) = 25 - x^2 - y^2$ on the rectangular domain $R = [-3, 3] \times [-4, 4]$.

(a) Viewing x as a fixed constant, use the Fundamental Theorem of Calculus to evaluate the integral

$$A(x) = \int_{-4}^{4} f(x, y)\, dy.$$

Note that you will be integrating with respect to y, and holding x constant. Your result should be a function of x only.

(b) Next, use your result from (a) along with the Fundamental Theorem of Calculus to determine the value of $\int_{-3}^{3} A(x)\, dx$.

(c) What is the value of $\iint_R f(x, y)\, dA$? What are two different ways we may interpret the meaning of this value?

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Activity 11.5.

Let $f(x,y) = x + y^2$ on the rectangle $R = [0,2] \times [0,3]$.

(a) Evaluate $\iint_R f(x,y)\,dA$ using an iterated integral. Choose an order for integration by deciding whether you want to integrate first with respect to x or y.

(b) Evaluate $\iint_R f(x,y)\,dA$ using the iterated integral whose order of integration is the opposite of the order you chose in (a).

◁

Summary

- We can evaluate the double integral $\iint_R f(x,y)\,dA$ over a rectangle $R = [a,b] \times [c,d]$ as an iterated integral in one of two ways:

  - $\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$, or
  - $\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$.

  This process works because each inner integral represents a cross-sectional (signed) area and the outer integral then sums all of the cross-sectional (signed) areas. Fubini's Theorem guarantees that the resulting value is the same, regardless of the order in which we integrate.

Exercises

1. Evaluate each of the following double or iterated integrals exactly.

(a) $\int_1^3 \left(\int_2^5 xy\,dy \right) dx$

(b) $\int_0^{\pi/4} \left(\int_0^{\pi/3} \sin(x)\cos(y)\,dx \right) dy$

(c) $\int_0^1 \left(\int_0^1 e^{-2x-3y}\,dy \right) dx$

(d) $\iint_R \sqrt{2x+5y}\,dA$, where $R = [0,2] \times [0,3]$.

2. The temperature at any point on a metal plate in the xy plane is given by $T(x,y) = 100-4x^2-y^2$, where x and y are measured in inches and T in degrees Celsius. Consider the portion of the plate that lies on the rectangular region $R = [1,5] \times [3,6]$.

(a) Write an iterated integral whose value represents the volume under the surface T over the rectangle R.

(b) Evaluate the iterated integral you determined in (a).

(c) Find the area of the rectangle, R.

(d) Determine the exact average temperature, $T_{\text{AVG(R)}}$, over the region R.

3. Consider the box with a sloped top that is given by the following description: the base is the rectangle $R = [1, 4] \times [2, 5]$, while the top is given by the plane $z = p(x, y) = 30 - x - 2y$.

(a) Write an iterated integral whose value represents the volume under p over the rectangle R.

(b) Evaluate the iterated integral you determined in (a).

(c) What is the exact average value of p over R?

(d) If you wanted to build a rectangular box (with an identical base) that has the same volume as the box with the sloped top described here, how tall would the rectangular box have to be?

11.3 Double Integrals over General Regions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How do we define a double integral over a non-rectangular region?
- What general form does an iterated integral over a non-rectangular region have?

Introduction

Recall that we defined the double integral of a continuous function $f = f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$ as

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*) \cdot \Delta A,$$

where the notation is as described in Section 11.1. Furthermore, we have seen that we can evaluate a double integral $\iint_R f(x, y)\, dA$ over R as an iterated integral of either of the forms

$$\int_a^b \int_c^d f(x, y)\, dy\, dx \quad \text{or} \quad \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

It is natural to wonder how we might define and evaluate a double integral over a non-rectangular region; we explore one such example in the following preview activity.

Preview Activity 11.3. A tetrahedron is a three-dimensional figure with four faces, each of which is a triangle. A picture of the tetrahedron T with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ is shown in Figure 11.11. If we place one vertex at the origin and let vectors **a**, **b**, and **c** be determined by the edges of the tetrahedron that have one end at the origin, then a formula that tells us the volume V of the tetrahedron is

$$V = \frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|. \tag{11.1}$$

(a) Use the formula (11.1) to find the volume of the tetrahedron T.

(b) Instead of memorizing or looking up the formula for the volume of a tetrahedron, we can use a double integral to calculate the volume of the tetrahedron T. To see how, notice that the top face of the tetrahedron T is the plane whose equation is

$$z = 1 - (x + y).$$

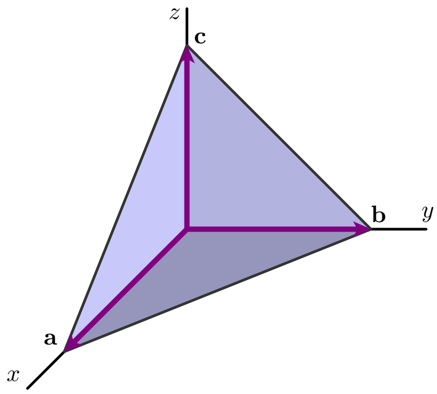

Figure 11.11: The tetrahedron T.

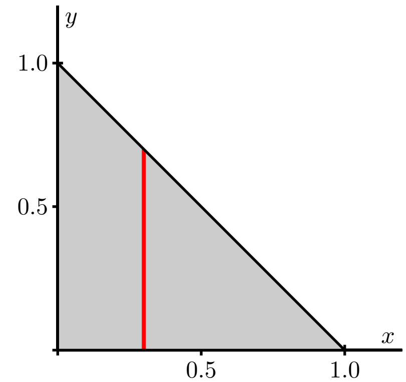

Figure 11.12: Projecting T onto the xy-plane.

Provided that we can use an iterated integral on a non-rectangular region, the volume of the tetrahedron will be given by an iterated integral of the form

$$\int_{x=?}^{x=?} \int_{y=?}^{y=?} 1-(x+y)\,dy\,dx.$$

The issue that is new here is how we find the limits on the integrals; note that the outer integral's limits are in x, while the inner ones are in y, since we have chosen $dA = dy\,dx$. To see the domain over which we need to integrate, think of standing way above the tetrahedron looking straight down on it, which means we are projecting the entire tetrahedron onto the xy-plane. The resulting domain is the triangular region shown in Figure 11.12.

Explain why we can represent the triangular region with the inequalities

$$0 \le y \le 1-x \quad \text{and} \quad 0 \le x \le 1.$$

(Hint: Consider the cross sectional slice shown in Figure 11.12.)

(c) Explain why it makes sense to now write the volume integral in the form

$$\int_{x=?}^{x=?} \int_{y=?}^{y=?} 1-(x+y)\,dy\,dx = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} 1-(x+y)\,dy\,dx.$$

(d) Use the Fundamental Theorem of Calculus to evaluate the iterated integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=1-x} 1-(x+y)\,dy\,dx$$

and compare to your result from part (a). (As with iterated integrals over rectangular regions, start with the inner integral.)

⋈

Double Integrals over General Regions

So far, we have learned that a double integral over a rectangular region may be interpreted in one of two ways:

- $\iint_R f(x,y)\,dA$ tells us the volume of the solids the graph of f bounds above the xy-plane over the rectangle R minus the volume of the solids the graph of f bounds below the xy-plane under the rectangle R;

- $\frac{1}{A(R)}\iint_R f(x,y)\,dA$, where $A(R)$ is the area of R tells us the average value of the function f on R. If $f(x,y) \geq 0$ on R, we can interpret this average value of f on R as the height of the box with base R that has the same volume as the volume of the surface defined by f over R.

As we saw in Preview Activity 11.1, a function $f = f(x,y)$ may be considered over regions other than rectangular ones, and thus we want to understand how to set up and evaluate double integrals over non-rectangular regions. Note that if we can, then the two interpretations of the double integral noted above will naturally extend to solid regions with non-rectangular bases.

So, suppose f is a continuous function on a closed, bounded domain D. For example, consider D as the circular domain shown in Figure 11.13. We can enclose D in a rectangular domain R as

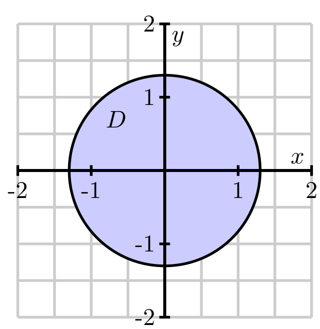

Figure 11.13: A non-rectangular domain.

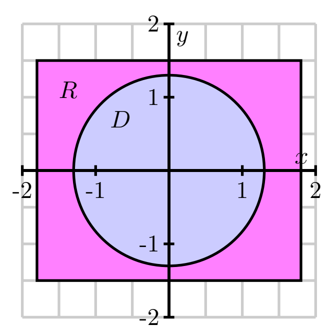

Figure 11.14: Enclosing this domain in a rectangle.

shown in Figure 11.14 and extend the function f to be defined over R in order to be able to use the definition of the double integral over a rectangle. We extend f in such a way that its values at the points in R that are not in D contribute 0 to the value of the integral. In other words, define a function $F = F(x, y)$ on R as

$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D, \\ 0, & \text{if } (x,y) \notin D \end{cases}.$$

We then say that the double integral of f over D is the same as the double integral of F over R, and thus

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA.$$

In practice, we just ignore everything that is in R but not in D, since these regions contribute 0 to the value of the integral.

Just as with double integrals over rectangles, a double integral over a domain D can be evaluated as an iterated integral. If the region D can be described by the inequalities $g_1(x) \le y \le g_2(x)$ and $a \le x \le b$, where $g_1 = g_1(x)$ and $g_2 = g_2(x)$ are functions of only x, then

$$\iint_D f(x,y)\,dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx.$$

Alternatively, if the region D is described by the inequalities $h_1(y) \le x \le h_2(y)$ and $c \le y \le d$, where $h_1 = h_1(y)$ and $h_2 = h_2(y)$ are functions of only y, we have

$$\iint_D f(x,y)\,dA = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy.$$

The structure of an iterated integral is of particular note:

In an iterated double integral:

- the limits on the outer integral must be constants;
- the limits on the inner integral must be constants or in terms of only the remaining variable – that is, if the inner integral is with respect to y, then its limits may only involve x and constants, and vice versa.

We next consider a detailed example.

Example 11.1. Let $f(x, y) = x^2y$ be defined on the triangle D with vertices $(0, 0)$, $(2, 0)$, and $(2, 3)$ as shown in Figure 11.15. To evaluate $\displaystyle\iint_D f(x,y)\,dA$, we must first describe the region D in terms of the variables x and y. We take two approaches.

Approach 1: Integrate first with respect to y. In this case we choose to evaluate the double integral as an iterated integral in the form

$$\iint_D x^2y\,dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} x^2y\,dy\,dx,$$

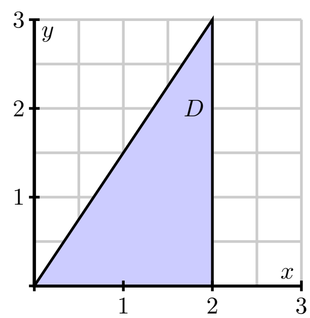

Figure 11.15: A triangular domain.

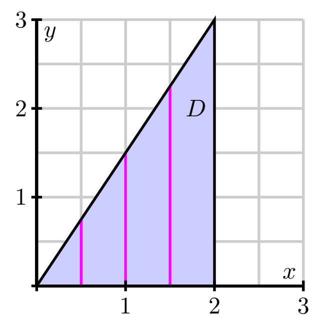

Figure 11.16: Slices in the y direction.

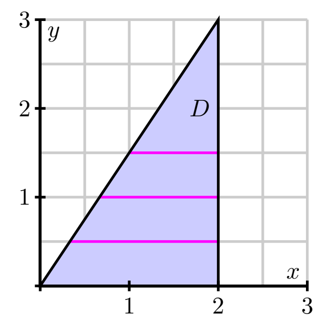

Figure 11.17: Slices in the x direction.

and therefore we need to describe D in terms of inequalities

$$g_1(x) \le y \le g_2(x) \quad \text{and} \quad a \le x \le b.$$

Since we are integrating with respect to y first, the iterated integral has the form

$$\iint_D x^2 y \, dA = \int_{x=a}^{x=b} A(x) \, dx,$$

where $A(x)$ is a cross sectional area in the y direction. So we are slicing the domain perpendicular to the x-axis and want to understand what a cross sectional area of the overall solid will look like. Several slices of the domain are shown in Figure 11.16. On a slice with fixed x value, the y values are bounded below by 0 and above by the y coordinate on the hypotenuse of the right triangle. Thus, $g_1(x) = 0$; to find $y = g_2(x)$, we need to write the hypotenuse as a function of x. The hypotenuse connects the points (0,0) and (2,3) and hence has equation $y = \frac{3}{2}x$. This gives the upper bound on y as $g_2(x) = \frac{3}{2}x$. The leftmost vertical cross section is at $x = 0$ and the rightmost one is at $x = 2$, so we have $a = 0$ and $b = 2$. Therefore,

$$\iint_D x^2 y \, dA = \int_{x=0}^{x=2} \int_{y=0}^{y=\frac{3}{2}x} x^2 y \, dy \, dx.$$

We evaluate the iterated integral by applying the Fundamental Theorem of Calculus first to the inner integral, and then to the outer one, and find that

$$\begin{aligned}\int_{x=0}^{x=2}\int_{y=0}^{y=\frac{3}{2}x} x^2y\,dy\,dx &= \int_{x=0}^{x=2}\left[x^2\cdot\frac{y^2}{2}\right]\Bigg|_{y=0}^{y=\frac{3}{2}x}\,dx\\ &= \int_{x=0}^{x=2}\frac{9}{8}x^4\,dx\\ &= \frac{9}{8}\frac{x^5}{5}\Bigg|_{x=0}^{x=2}\\ &= \left(\frac{9}{8}\right)\left(\frac{32}{5}\right)\\ &= \frac{36}{5}.\end{aligned}$$

Approach 2: Integrate first with respect to x. In this case, we choose to evaluate the double integral as an iterated integral in the form

$$\iint_D x^2y\,dA = \int_{y=c}^{y=d}\int_{x=h_1(y)}^{x=h_2(y)} x^2y\,dx\,dy$$

and thus need to describe D in terms of inequalities

$$h_1(y) \le x \le h_2(y) \quad \text{and} \quad c \le y \le d.$$

Since we are integrating with respect to x first, the iterated integral has the form

$$\iint_D x^2y\,dA = \int_c^d A(y)\,dy,$$

where $A(y)$ is a cross sectional area of the solid in the x direction. Several slices of the domain – perpendicular to the y-axis – are shown in Figure 11.17.

On a slice with fixed y value, the x values are bounded below by the x coordinate on the hypotenuse of the right triangle and above by 2. So $h_2(y) = 2$; to find $h_1(y)$, we need to write the hypotenuse as a function of y. Solving the earlier equation we have for the hypotenuse ($y = \frac{3}{2}x$) for x gives us $x = \frac{2}{3}y$. This makes $h_1(y) = \frac{2}{3}y$. The lowest horizontal cross section is at $y = 0$ and the uppermost one is at $y = 3$, so we have $c = 0$ and $d = 3$. Therefore,

$$\iint_D x^2y\,dA = \int_{y=0}^{y=3}\int_{x=(2/3)y}^{x=2} x^2y\,dx\,dy.$$

We evaluate the resulting iterated integral as before by twice applying the Fundamental

Theorem of Calculus, and find that

$$\begin{aligned}
\int_{y=0}^{y=3} \int_{x=\frac{2}{3}y}^{2} x^2 y \, dx \, dy &= \int_{y=0}^{y=3} \left[\frac{x^3}{3} \right] \Big|_{x=\frac{2}{3}y}^{x=2} y \, dx \\
&= \int_{y=0}^{y=3} \left[\frac{8}{3} y - \frac{8}{81} y^4 \right] dy \\
&= \left[\frac{8}{3} \frac{y^2}{2} - \frac{8}{81} \frac{y^5}{5} \right] \Big|_{y=0}^{y=3} \\
&= \left(\frac{8}{3} \right) \left(\frac{9}{2} \right) - \left(\frac{8}{81} \right) \left(\frac{243}{5} \right) \\
&= 12 - \frac{24}{5} \\
&= \frac{36}{5}.
\end{aligned}$$

We see, of course, that in the situation where D can be described in two different ways, the order in which we choose to set up and evaluate the double integral doesn't matter, and the same value results in either case.

The meaning of a double integral over a non-rectangular region, D, parallels the meaning over a rectangular region. In particular,

- $\iint_D f(x, y) \, dA$ tells us the volume of the solids the graph of f bounds above the xy-plane over the closed, bounded region D minus the volume of the solids the graph of f bounds below the xy-plane under the region D;
- $\frac{1}{A(D)} \iint_R f(x, y) \, dA$, where $A(D)$ is the area of D tells us the average value of the function f on D. If $f(x, y) \geq 0$ on D, we can interpret this average value of f on D as the height of the solid with base D and constant cross-sectional area D that has the same volume as the volume of the surface defined by f over D.

Activity 11.6.

Consider the double integral $\iint_D (4 - x - 2y) \, dA$, where D is the triangular region with vertices (0,0), (4,0), and (0,2).

(a) Write the given integral as an iterated integral of the form $\iint_D (4 - x - 2y) \, dy \, dx$. Draw a labeled picture of D with relevant cross sections.

(b) Write the given integral as an iterated integral of the form $\iint_D (4 - x - 2y) \, dx \, dy$. Draw a labeled picture of D with relevant cross sections.

(c) Evaluate the two iterated integrals from (a) and (b), and verify that they produce the same value. Give at least one interpretation of the meaning of your result.

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Activity 11.7.

Consider the iterated integral $\int_{x=3}^{x=5} \int_{y=-x}^{y=x^2} (4x + 10y)\, dy\, dx$.

(a) Sketch the region of integration, D, for which

$$\iint_D (4x + 10y)\, dA = \int_{x=3}^{x=5} \int_{y=-x}^{y=x^2} (4x + 10y)\, dy\, dx.$$

(b) Determine the equivalent iterated integral that results from integrating in the opposite order ($dx\, dy$, instead of $dy\, dx$). That is, determine the limits of integration for which

$$\iint_D (4x + 10y)\, dA = \int_{y=?}^{y=?} \int_{x=?}^{x=?} (4x + 10y)\, dx\, dy.$$

(c) Evaluate one of the two iterated integrals above. Explain what the value you obtained tells you.

(d) Set up and evaluate a single definite integral to determine the exact area of D, $A(D)$.

(e) Determine the exact average value of $f(x, y) = 4x + 10y$ over D.

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Activity 11.8.

Consider the iterated integral $\int_{x=0}^{x=4} \int_{y=x/2}^{y=2} e^{y^2}\, dy\, dx$.

(a) Explain why we cannot antidifferentiate e^{y^2} with respect to y, and thus are unable to evaluate the iterated integral $\int_{x=0}^{x=4} \int_{y=x/2}^{y=2} e^{y^2}\, dy\, dx$ using the Fundamental Theorem of Calculus.

(b) Sketch the region of integration, D, so that $\iint_D e^{y^2}\, dA = \int_{x=0}^{x=4} \int_{y=x/2}^{y=2} e^{y^2}\, dy\, dx$.

(c) Rewrite the given iterated integral in the opposite order, using $dA = dx\, dy$.

(d) Use the Fundamental Theorem of Calculus to evaluate the iterated integral you developed in (d). Write one sentence to explain the meaning of the value you found.

(e) What is the important lesson this activity offers regarding the order in which we set up an iterated integral?

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Summary

- For a double integral $\iint_D f(x,y)\,dA$ over a non-rectangular region D, we enclose D in a rectangle R and then extend integrand f to a function F so that $F(x,y) = 0$ at all points in R outside of D and $F(x,y) = f(x,y)$ for all points in D. We then define $\iint_D f(x,y)\,dA$ to be equal to $\iint_R F(x,y)\,dA$.
- In an iterated double integral, the limits on the outer integral must be constants while the limits on the inner integral must be constants or in terms of only the remaining variable. In other words, an iterated double integral has one of the following forms (which result in the same value):

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx,$$

where $g_1 = g_1(x)$ and $g_2 = g_2(x)$ are functions of x only and the region D is described by the inequalities $g_1(x) \le y \le g_2(x)$ and $a \le x \le b$ or

$$\int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy,$$

where $h_1 = h_1(y)$ and $h_2 = h_2(y)$ are functions of y only and the region D is described by the inequalities $h_1(y) \le x \le h_2(y)$ and $c \le y \le d$.

Exercises

1. For each of the following iterated integrals, (a) sketch the region of integration, (b) write an equivalent iterated integral expression in the opposite order of integration, and (c) choose one of the two orders and evaluate the integral.

(a) $\displaystyle\int_{x=0}^{x=1} \int_{y=x^2}^{y=x} xy\,dy\,dx$

(b) $\displaystyle\int_{y=0}^{y=2} \int_{x=-\sqrt{4-y^2}}^{x=0} xy\,dx\,dy$

(c) $\displaystyle\int_{x=0}^{x=1} \int_{y=x^4}^{y=x^{1/4}} x+y\,dy\,dx$

(d) $\displaystyle\int_{y=0}^{y=2} \int_{x=y/2}^{x=2y} x+y\,dx\,dy$

2. The temperature at any point on a metal plate in the xy plane is given by $T(x,y) = 100-4x^2-y^2$, where x and y are measured in inches and T in degrees Celsius. Consider the portion of the plate that lies on the region D that is the finite region that lies between the parabolas $x = y^2$ and $x = 3 - 2y^2$.

(a) Construct a labeled sketch of the region D.

(b) Set up an integrated integral whose value is $\iint_D T(x,y)\,dA$.

(c) Set up an integrated integral whose value is $\iint_D T(x,y)\,dA$.

(d) Use the Fundamental Theorem of Calculus to evaluate the integrals you determined in (b) and (c).

(e) Determine the exact average temperature, $T_{\text{AVG(D)}}$, over the region D.

3. Consider the solid that is given by the following description: the base is the given region D, while the top is given by the surface $z = p(x,y)$. In each setting below, set up, but do not evaluate, an iterated integral whose value is the exact volume of the solid. Include a labeled sketch of D in each case.

(a) D is the interior of the quarter circle of radius 2, centered at the origin, that lies in the second quadrant of the plane; $p(x,y) = 16 - x^2 - y^2$.

(b) D is the finite region between the line $y = x+1$ and the parabola $y = x^2$; $p(x,y) = 10 - x - 2y$.

(c) D is the triangular region with vertices $(1,1)$, $(2,2)$, and $(2,3)$; $p(x,y) = e^{-xy}$.

(d) D is the region bounded by the y-axis, $y = 4$ and $x = \sqrt{y}$; $p(x,y) = \sqrt{1+x^2+y^2}$.

4. Consider the iterated integral $I = \displaystyle\int_{x=0}^{x=4} \int_{y=\sqrt{x}}^{y=2} \cos(y^3)\,dy\,dx$.

(a) Sketch the region of integration.

(b) Write an equivalent iterated integral with the order of integration reversed.

(c) Choose one of the two orders of integration and evaluate the iterated integral you chose by hand. Explain the reasoning behind your choice.

(d) Determine the exact average value of $\cos(y^3)$ over the region D that is determined by the iterated integral I.

11.4 Applications of Double Integrals

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- If we have a mass density function for a lamina (thin plate), how does a double integral determine the mass of the lamina?
- How may a double integral be used to find the area between two curves?
- Given a mass density function on a lamina, how can we find the lamina's center of mass?
- What is a joint probability density function? How do we determine the probability of an event if we know a probability density function?

Introduction

So far, we have interpreted the double integral of a function f over a domain D in two different ways. First, $\iint_D f(x,y)\,dA$ tells us a difference of volumes – the volume the surface defined by f bounds above the xy-plane on D minus the volume the surface bounds below the xy-plane on D. In addition, $\frac{1}{A(D)}\iint_D f(x,y)\,dA$ determines the average value of f on D. In this section, we investigate several other applications of double integrals, using the integration process as seen in Preview Activity 11.4: we partition into small regions, approximate the desired quantity on each small region, then use the integral to sum these values exactly in the limit.

The following preview activity explores how a double integral can be used to determine the density of a thin plate with a mass density distribution. Recall that in single-variable calculus, we considered a similar problem and computed the mass of a one-dimensional rod with a mass-density distribution. There, as here, the key idea is that if density is constant, mass is the product of density and volume.

Preview Activity 11.4. Suppose that we have a flat, thin object (called a *lamina*) whose density varies across the object. We can think of the density on a lamina as a measure of mass per unit area. As an example, consider a circular plate D of radius 1 cm centered at the origin whose density δ varies depending on the distance from its center so that the density in grams per square centimeter at point (x,y) is

$$\delta(x,y) = 10 - 2(x^2 + y^2).$$

(a) Suppose that we partition the plate into subrectangles R_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq n$, of equal area ΔA, and select a point (x_{ij}^*, y_{ij}^*) in R_{ij} for each i and j.

What is the meaning of the quantity $\delta(x_{ij}^*, y_{ij}^*)\Delta A$?

(b) State a double Riemann sum that provides an approximation of the mass of the plate.

(c) Explain why the double integral

$$\iint_D \delta(x, y)\, dA$$

tells us the exact mass of the plate.

(d) Determine an iterated integral which, if evaluated, would give the exact mass of the plate. Do not actually evaluate the integral.[2]

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Mass

Density is a measure of some quantity per unit area or volume. For example, we can measure the human population density of some region as the number of humans in that region divided by the area of that region. In physics, the mass density of an object is the mass of the object per unit area or volume. As suggested by Preview Activity 11.4, the following holds in general.

If $\delta(x, y)$ describes the density of a lamina defined by a planar region D, then the **mass** of D is given by the double integral $\iint_D \delta(x, y)\, dA$.

Activity 11.9.

Let D be a half-disk lamina of radius 3 in quadrants IV and I, centered at the origin as shown in Figure 11.18. Assume the density at point (x, y) is given by $\delta(x, y) = x$. Find the exact mass of the lamina.

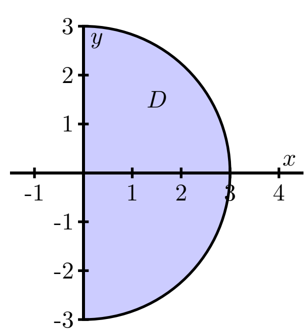

Figure 11.18: A half disk lamina.

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[2] This integral is considerably easier to evaluate in polar coordinates, which we will learn more about in Section 11.5.

Area

If we consider the situation where the mass-density distribution is constant, we can also see how a double integral may be used to determine the area of a region. Assuming that $\delta(x, y) = 1$ over a closed bounded region D, where the units of δ are "mass per unit of area," it follows that $\iint_D 1\, dA$ is the mass of the lamina. But since the density is constant, the numerical value of the integral is simply the area.

As the following activity demonstrates, we can also see this fact by considering a three-dimensional solid whose height is always 1.

Activity 11.10.

Suppose we want to find the area of the bounded region D between the curves

$$y = 1 - x^2 \quad \text{and} \quad y = x - 1.$$

A picture of this region is shown in Figure 11.19.

(a) We know that the volume of a solid with constant height is given by the area of the base times the height. Hence, we may interpret the area of the region D as the volume of a solid with base D and of uniform height 1. Determine a double integral whose value is the area of D.

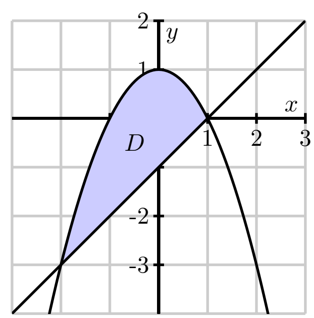

Figure 11.19: The graphs of $y = 1 - x^2$ and $y = x - 1$.

(b) Write an iterated integral whose value equals the double integral you found in (a).

(c) Use the Fundamental Theorem of Calculus to evaluate *only* the inner integral in the iterated integral in (b).

(d) After completing part (c), you should see a standard single area integral from calc II. Evaluate this remaining integral to find the exact area of D.

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We now formally state the conclusion from our earlier discussion and Activity 11.10.

Given a closed, bounded region D in the plane, the area of D, denoted $A(D)$, is given by the double integral

$$A(D) = \iint_D 1\,dA.$$

Center of Mass

The center of mass of an object is a point at which the object will balance perfectly. For example, the center of mass of a circular disk of uniform density is located at its center. For any object, if we throw it through the air, it will spin around its center of mass and behave as if all the mass is located at the center of mass.

In order to understand the role that integrals play in determining the center of a mass of an object with a nonuniform mass distribution, we start by finding the center of mass of a collection of N distinct point-masses in the plane.

Let m_1, m_2, ..., m_N be N masses located in the plane. Think of these masses as connected by rigid rods of negligible weight from some central point (x, y). A picture with four masses is shown in Figure 11.20. Now imagine balancing this system by placing it on a thin pole at the point (x, y) perpendicular to the plane containing the masses. Unless the masses are perfectly balanced, the system will fall off the pole. The point $(\overline{x}, \overline{y})$ at which the system will balance perfectly is called the *center of mass* of the system. Our goal is to determine the center of mass of a system of discrete masses, then extend this to a continuous lamina.

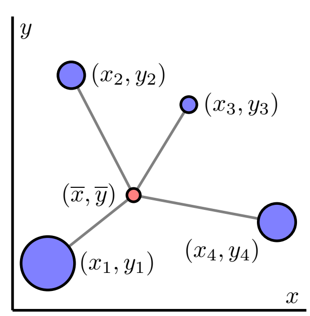

Figure 11.20: A center of mass $(\overline{x}, \overline{y})$ of four masses.

Each mass exerts a force (called a *moment*) around the lines $x = \overline{x}$ and $y = \overline{y}$ that causes the system to tilt in the direction of the mass. These moments are dependent on the mass and the distance from the given line. Let (x_1, y_1) be the location of mass m_1, (x_2, y_2) the location of mass m_2, etc. In order to balance perfectly, the moments in the x direction and in the y direction must be in equilibrium. We determine these moments and solve the resulting system to find the equilibrium point $(\overline{x}, \overline{y})$ at the center of mass.

The force that mass m_1 exerts to tilt the system from the line $y = \overline{y}$ is

$$m_1 g(\overline{y} - y_1),$$

where g is the gravitational constant. Similarly, the force mass m_2 exerts to tilt the system from the line $y = \overline{y}$ is

$$m_2 g(\overline{y} - y_2).$$

In general, the force that mass m_k exerts to tilt the system from the line $y = \overline{y}$ is

$$m_k g(\overline{y} - y_k).$$

For the system to balance, we need the forces to sum to 0, so that

$$\sum_{k=1}^{N} m_k g(\overline{y} - y_k) = 0.$$

Solving for $\overline{y}$, we find that

$$\overline{y} = \frac{\sum_{k=1}^{N} m_k y_k}{\sum_{k=1}^{N} m_k}.$$

A similar argument shows that

$$\overline{x} = \frac{\sum_{k=1}^{N} m_k x_k}{\sum_{k=1}^{N} m_k}.$$

The value $M_x = \sum_{k=1}^{N} m_k y_k$ is called the *total moment* with respect to the x-axis; $M_y = \sum_{k=1}^{N} m_k x_k$ is the *total moment* with respect to the y-axis. Hence, the respective quotients of the moments to the total mass, M, determines the center of mass of a point-mass system:

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right).$$

Now, suppose that rather than a point-mass system, we have a continuous lamina with a variable mass-density $\delta(x, y)$. We may estimate its center of mass by partitioning the lamina into mn subrectangles of equal area ΔA, and treating the resulting partitioned lamina as a point-mass system. In particular, we select a point (x_{ij}^*, y_{ij}^*) in the ijth subrectangle, and observe that the quanity

$$\delta(x_{ij}^*, y_{ij}^*)\Delta A$$

is density times area, so $\delta(x_{ij}^*, y_{ij}^*)\Delta A$ approximates the mass of the small portion of the lamina determined by the subrectangle R_{ij}.

We now treat $\delta(x_{ij}^*, y_{ij}^*)\Delta A$ as a point mass at the point (x_{ij}^*, y_{ij}^*). The coordinates $(\overline{x}, \overline{y})$ of the center of mass of these mn point masses are thus given by

$$\overline{x} = \frac{\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij}^* \delta(x_{ij}^*, y_{ij}^*)\Delta A}{\sum_{j=1}^{n} \sum_{i=1}^{m} \delta(x_{ij}^*, y_{ij}^*)\Delta A} \quad \text{and} \quad \overline{y} = \frac{\sum_{j=1}^{n} \sum_{i=1}^{m} y_{ij}^* \delta(x_{ij}^*, y_{ij}^*)\Delta A}{\sum_{j=1}^{n} \sum_{i=1}^{m} \delta(x_{ij}^*, y_{ij}^*)\Delta A}.$$

If we take the limit as m and n go to infinity, we obtain the exact center of mass $(\overline{x}, \overline{y})$ of the continuous lamina.

The coordinates $(\overline{x}, \overline{y})$ of the **center of mass of a lamina** D with density $\delta = \delta(x, y)$ are given by

$$\overline{x} = \frac{\iint_D x\delta(x,y)\,dA}{\iint_D \delta(x,y)\,dA} \quad \text{and} \quad \overline{y} = \frac{\iint_D y\delta(x,y)\,dA}{\iint_D \delta(x,y)\,dA}.$$

The numerators of $\overline{x}$ and $\overline{y}$ are called the respective *moments* of the lamina about the coordinate axes. Thus, the moment of a lamina D with density $\delta = \delta(x, y)$ about the y-axis is

$$M_y = \iint_D x\delta(x,y)\,dA$$

and the moment of D about the x-axis is

$$M_x = \iint_D y\delta(x,y)\,dA.$$

If M is the mass of the lamina, it follows that the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$.

Activity 11.11.

In this activity we determine integrals that represent the center of mass of a lamina D described by the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant if the density at point (x, y) is $\delta(x, y) = 6x + 6y + 6$. A picture of the lamina is shown in Figure 11.21.

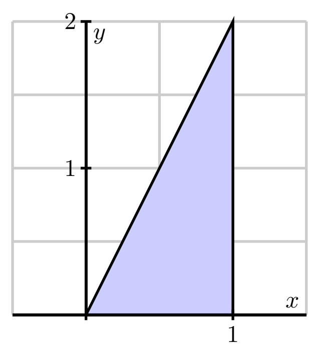

Figure 11.21: The lamina bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant.

(a) Set up an iterated integral that represents the mass of the lamina.

(b) Assume the mass of the lamina is 14. Set up two iterated integrals that represent the coordinates of the center of mass of the lamina.

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Probability

Calculating probabilities is a very important application of integration in the physical, social, and life sciences. To understand the basics, consider the game of darts in which a player throws a dart at a board and tries to hit a particular target. Let us suppose that a dart board is in the form of a disk D with radius 10 inches. If we assume that a player throws a dart at random, and is not aiming at any particular point, then it is equally probable that the dart will strike any single point on the board. For instance, the probability that the dart will strike a particular 1 square inch region is $\frac{1}{100\pi}$, or the ratio of the area of the desired target to the total area of D (assuming that the dart thrower always hits the board itself at some point). Similarly, the probability that the dart strikes a point in the disk D_3 of radius 3 inches is given by the area of D_3 divided by the area of D. In other words, the probability that the dart strikes the disk D_3 is

$$\frac{9\pi}{100\pi} = \iint_{D_3} \frac{1}{100\pi}\, dA.$$

The integrand, $\frac{1}{100\pi}$, may be thought of as a *distribution function*, describing how the dart strikes are distributed across the board. In this case the distribution function is constant since we are assuming a uniform distribution, but we can easily envision situations where the distribution function varies. For example, if the player is fairly good and is aiming for the bulls eye (the center of D), then the distribution function f could be skewed toward the center, say

$$f(x, y) = Ke^{-(x^2+y^2)}$$

for some constant positive K. If we assume that the player is consistent enough so that the dart always strikes the board, then the probability that the dart strikes the board somewhere is 1, and the distribution function f will have to satisfy[3]

$$\iint_D f(x, y)\, dA = 1.$$

For such a function f, the probability that the dart strikes in the disk D_1 of radius 1 would be

$$\iint_{D_1} f(x, y)\, dA.$$

Indeed, the probability that the dart strikes in any region R that lies within D is given by

$$\iint_R f(x, y)\, dA.$$

The preceding discussion highlights the general idea behind calculating probabilities. We assume we have a *joint probability density function* f, a function of two independent variables x and y defined on a domain D that satisfies the conditions

[3]This makes $K = \frac{1}{\pi\left(1-e^{-100}\right)}$, which you can check.

- $f(x, y) \geq 0$ for all x and y in D,
- the probability that x is between some values a and b while y is between some values c and d is given by

$$\int_a^b \int_c^d f(x, y)\, dy\, dx,$$

- The probability that the point (x, y) is in D is 1, that is

$$\iint_D f(x, y)\, dA = 1. \tag{11.2}$$

Note that it is possible that D could be an infinite region and the limits on the integral in Equation (11.2) could be infinite. When we have such a probability density function $f = f(x, y)$, the probability that the point (x, y) is in some region R contained in the domain D (the notation we use here is "$P((x, y) \in R)$") is determined by

$$P((x, y) \in R) = \iint_R f(x, y)\, dA.$$

Activity 11.12.

A firm manufactures smoke detectors. Two components for the detectors come from different suppliers – one in Michigan and one in Ohio. The company studies these components for their reliability and their data suggests that if x is the life span (in years) of a randomly chosen component from the Michigan supplier and y the life span (in years) of a randomly chosen component from the Ohio supplier, then the joint probability density function f might be given by

$$f(x, y) = e^{-x}e^{-y}.$$

(a) Theoretically, the components might last forever, so the domain D of the function f is the set D of all (x, y) such that $x \geq 0$ and $y \geq 0$. To show that f is a probability density function on D we need to demonstrate that

$$\int\int_D f(x, y)\, dA = 1,$$

or that

$$\int_0^\infty \int_0^\infty f(x, y)\, dy\, dx = 1.$$

Use your knowledge of improper integrals to verify that f is indeed a probability density function.

(b) Assume that the smoke detector fails only if both of the supplied components fail. To determine the probability that a randomly selected detector will fail within one year, we will need to determine the probability that the life span of each component is between 0 and 1 years. Set up an appropriate iterated integral, and evaluate the integral to determine the probability.

(c) What is the probability that a randomly chosen smoke detector will fail between years 3 and 7?

(d) Suppose that the manufacturer determines that one of the components is more likely to fail than the other, and hence conjectures that the probability density function is instead $f(x, y) = Ke^{-x}e^{-2y}$. What is the value of K?

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Summary

- The mass of a lamina D with a mass density function $\delta = \delta(x, y)$ is $\iint_D \delta(x, y)\, dA$.
- The area of a region D in the plane has the same numerical value as the volume of a solid of uniform height 1 and base D, so the area of D is given by $\iint_D 1\, dA$.
- The center of mass, $(\overline{x}, \overline{y})$, of a continuous lamina with a variable density $\delta(x, y)$ is given by

$$\overline{x} = \frac{\iint_D x\delta(x, y)\, dA}{\iint_D \delta(x, y)\, dA} \quad \text{and} \quad \overline{y} = \frac{\iint_D y\delta(x, y)\, dA}{\iint_D \delta(x, y)\, dA}.$$

- Given a joint probability density function f is a function of two independent variables x and y defined on a domain D, if R is some subregion of D, then the probability that (x, y) is in R is given by

$$\iint_R f(x, y)\, dA.$$

Exercises

1. A triangular plate is bounded by the graphs of the equations $y = 2x$, $y = 4x$, and $y = 4$. The plate's density at (x, y) is given by $\delta(x, y) = 4xy^2 + 1$, measured in grams per square centimeter (and x and y are measured in centimeters).

(a) Set up an iterated integral whose value is the mass of the plate. Include a labeled sketch of the region of integration. Why did you choose the order of integration you did?

(b) Determine the mass of the plate.

(c) Determine the exact center of mass of the plate. Draw and label the point you find on your sketch from (a).

(d) What is the average density of the plate? Include units on your answer.

2. Let D be a half-disk lamina of radius 3 in quadrants IV and I, centered at the origin as in Activity 11.9. Assume the density at point (x, y) is equal to x.

(a) Before doing any calculations, what do you expect the y-coordinate of the center of mass to be? Why?

(b) Set up iterated integral expressions which, if evaluated, will determine the exact center of mass of the lamina.

(c) Use appropriate technology to evaluate the integrals to find the center of mass numerically.

3. Let x denote the time (in minutes) that a person spends waiting in a checkout line at a grocery store and y the time (in minutes) that it takes to check out. Suppose the joint probability density for x and y is

$$f(x, y) = \frac{1}{8}e^{-x/4-y/2}.$$

(a) What is the exact probability that a person spends between 0 to 5 minutes waiting in line, and then 0 to 5 minutes waiting to check out?

(b) Set up, but do not evaluate, an iterated integral whose value determines the exact probability that a person spends at most 10 minutes total both waiting in line and checking out at this grocery store.

(c) Set up, but do not evaluate, an iterated integral expression whose value determines the exact probability that a person spends at least 10 minutes total both waiting in line and checking out, but not more than 20 minutes.

11.5 Double Integrals in Polar Coordinates

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What are the polar coordinates of a point in two-space?
- How do we convert between polar coordinates and rectangular coordinates?
- What is the area element in polar coordinates?
- How do we convert a double integral in rectangular coordinates to a double integral in polar coordinates?

Introduction

While we have naturally defined double integrals in the rectangular coordinate system, starting with domains that are rectangular regions, there are many of these integrals that are difficult, if not impossible, to evaluate. For example, consider the domain D that is the unit circle and $f(x,y) = e^{-x^2-y^2}$. To integrate f over D, we would use the iterated integral

$$\iint_D f(x,y)\,dA = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} e^{-x^2-y^2}\,dy\,dx.$$

For this particular integral, regardless of the order of integration, we are unable to find an antiderivative of the integrand; in addition, even if we were able to find an antiderivative, the inner limits of integration involve relatively complicated functions.

It is useful, therefore, to be able to translate to other coordinate systems where the limits of integration and evaluation of the involved integrals is simpler. In this section we provide a quick discussion of one such system – polar coordinates – and then introduce and investigate their ramifications for double integrals. The rectangular coordinate system allows us to consider domains and graphs relative to a rectangular grid. The polar coordinate system is an alternate coordinate system that allows us to consider domains less suited to rectangular coordinates, such as circles.

Preview Activity 11.5. The coordinates of a point determine its location. In particular, the rectangular coordinates of a point P are given by an ordered pair (x,y), where x is the (signed) distance the point lies from the y-axis to P and y is the (signed) distance the point lies from the x-axis to P. In polar coordinates, we locate the point by considering the distance the point lies from the origin, $(0,0)$, and the angle the line segment from the origin to P forms with the positive x-axis.

(a) Determine the rectangular coordinates of the following points:

i. The point P that lies 1 unit from the origin on the positive x-axis.

ii. The point Q that lies 2 units from the origin and such that $\overline{OQ}$ makes an angle of $\frac{\pi}{2}$ with the positive x-axis, where O is the origin, $(0,0)$.

iii. The point R that lies 3 units from the origin such that $\overline{OR}$ makes an angle of $\frac{2\pi}{3}$ with the positive x-axis.

(b) Part (a) indicates that the two pieces of information completely determine the location of a point: either the traditional (x, y) coordinates, or alternately, the distance r from the point to the origin along with the angle θ that the line through the origin and the point makes with the positive x-axis. We write "(r, θ)" to denote the point's location in its polar coordinate representation. Find polar coordinates for the points with the given rectangular coordinates.

i. $(0, -1)$ ii. $(-2, 0)$ iii. $(-1, 1)$

(c) For each of the following points whose coordinates are given in polar form, determine the rectangular coordinates of the point.

i. $(5, \frac{\pi}{4})$ ii. $(2, \frac{5\pi}{6})$ iii. $(\sqrt{3}, \frac{5\pi}{3})$

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A Quick Overview of Polar Coordinates

The rectangular coordinate system is best suited for graphs and regions that are naturally considered over a rectangular grid. The polar coordinate system is an alternative that offers good options for functions and domains that have more circular characteristics. A point P in rectangular coordinates that is described by an ordered pair (x, y), where x is the displacement from P to the y-axis and y is the displacement from P to the x-axis, as seen in Preview Activity 11.5, can also be described with polar coordinates (r, θ), where r is the distance from P to the origin and θ is the angle formed by the line segment $\overline{OP}$ and the positive x-axis, as shown in Figure 11.22. Trigonometry and the Pythagorean Theorem allow for straightforward conversion from rectangular to polar, and vice versa.

Converting from rectangular to polar. If we are given the rectangular coordinates (x, y) of a point P, then the polar coordinates (r, θ) of P satisfy

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan(\theta) = \frac{y}{x}, \text{ assuming } x \neq 0.$$

Converting from polar to rectangular. If we are given the polar coordinates (r, θ) of a point P, then the rectangular coordinates (x, y) of P satisfy

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

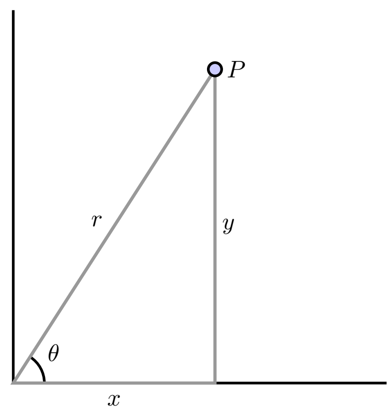

Figure 11.22: The polar coordinates of a point.

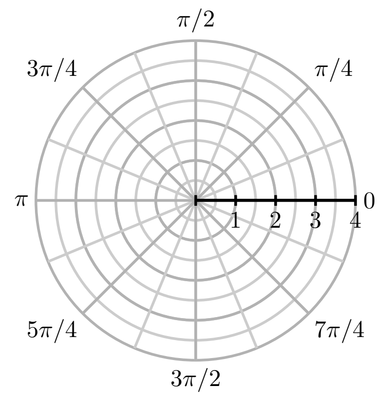

Figure 11.23: The polar coordinate grid.

We can draw graphs of curves in polar coordinates in a similar way to how we do in rectangular coordinates. However, when plotting in polar coordinates, we use a grid that considers changes in angles and changes in distance from the origin. In particular, the angles θ and distances r partition the plane into small wedges as shown in Figure 11.23.

Activity 11.13.

Most polar graphing devices[4] can plot curves in polar coordinates of the form $r = f(\theta)$. Use such a device to complete this activity.

(a) Before plotting the polar curve $r = 1$, think about what shape it should have, in light of how r is connected to x and y. Then use appropriate technology to draw the graph and test your intuition.

(b) The equation $\theta = 1$ does not define r as a function of θ, so we can't graph this equation on many polar plotters. What do you think the graph of the polar curve $\theta = 1$ looks like? Why?

(c) Before plotting the polar curve $r = \theta$, what do you think the graph looks like? Why? Use technology to plot the curve and compare your intuition.

(d) What about the curve $r = \sin(\theta)$? After plotting this curve, experiment with others of your choosing and think about why the curves look the way they do.

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[4]You can use your calculator in POL mode, or a web applet such as http://webspace.ship.edu/msrenault/ggb/polar_grapher.html

Integrating in Polar Coordinates

Consider the double integral

$$\iint_D e^{x^2+y^2}\, dA,$$

where D is the unit disk. While we cannot directly evaluate this integral in rectangular coordinates, a change to polar coordinates will convert it to one we can easily evaluate.

We have seen how to evaluate a double integral $\iint_D f(x,y)\, dA$ as an iterated integral of the form

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

in rectangular coordinates, because we know that $dA = dy\, dx$ in rectangular coordinates. To make the change to polar coordinates, we not only need to represent the variables x and y in polar coordinates, but we also must understand how to write the area element, dA, in polar coordinates. That is, we must determine how the area element dA can be written in terms of dr and $d\theta$ in the context of polar coordinates. We address this question in the following activity.

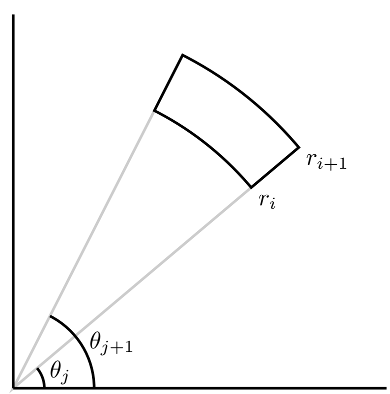

Figure 11.24: A polar rectangle.

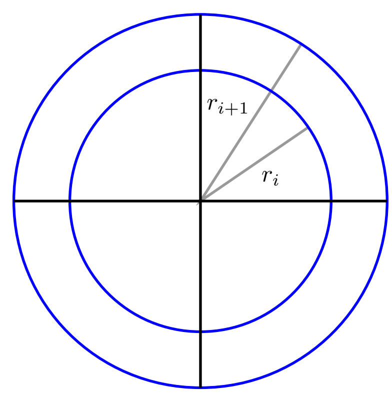

Figure 11.25: An annulus.

Activity 11.14.

Consider a polar rectangle R, with r between r_i and r_{i+1} and θ between θ_j and θ_{j+1} as shown in Figure 11.24. Let $\Delta r = r_{i+1} - r_i$ and $\Delta\theta = \theta_{j+1} - \theta_j$. Let ΔA be the area of this region.

(a) Explain why the area ΔA in polar coordinates is not $\Delta r\, \Delta\theta$.

(b) Now find ΔA by the following steps:

i. Find the area of the annulus (the washer-like region) between r_i and r_{i+1}, as shown at right in Figure 11.25. This area will be in terms of r_i and r_{i+1}.

ii. Observe that the region R is only a portion of the annulus, so the area ΔA of R is only a fraction of the area of the annulus. For instance, if $\theta_{i+1} - \theta_i$ were $\frac{\pi}{4}$, then the resulting wedge would be

$$\frac{\frac{\pi}{4}}{2\pi} = \frac{1}{8}$$

of the entire annulus. In this more general context, using the wedge between the two noted angles, what fraction of the area of the annulus is the area ΔA?

iii. Write an expression for ΔA in terms of r_i, r_{i+1}, θ_j, and θ_{j+1}.

iv. Finally, write the area ΔA in terms of r_i, r_{i+1}, Δr, and $\Delta\theta$, where each quantity appears only once in the expression. (Hint: Think about how to factor a difference of squares.)

(c) As we take the limit as Δr and $\Delta\theta$ go to 0, Δr becomes dr, $\Delta\theta$ becomes $d\theta$, and ΔA becomes dA, the area element. Using your work in (iv), write dA in terms of r, dr, and $d\theta$.

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From the result of Activity 11.14, we see when we convert an integral from rectangular coordinates to polar coordinates, we must not only convert x and y to being in terms of r and θ, but we also have to change the area element to $dA = r\,dr\,d\theta$ in polar coordinates. In other words, given a double integral $\iint_D f(x,y)\,dA$ in rectangular coordinates, to write a corresponding iterated integral in polar coordinates, we replace x with $r\cos(\theta)$, y with $r\sin(\theta)$ and dA with $r\,dr\,d\theta$. Of course, we need to describe the region D in polar coordinates as well. To summarize:

> The double integral $\iint_D f(x,y)\,dA$ in rectangular coordinates can be converted to a double integral in polar coordinates as $\iint_D f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$.

Example 11.2. Let $f(x,y) = e^{x^2+y^2}$ on the disk $D = \{(x,y) : x^2 + y^2 \leq 1\}$. We will evaluate $\iint_D f(x,y)\,dA$.

In rectangular coordinates the double integral $\displaystyle\iint_D f(x,y)\,dA$ can be written as the iterated integral

$$\iint_D f(x,y)\,dA = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx.$$

We cannot evaluate this iterated integral, because $e^{x^2+y^2}$ does not have an elementary antiderivative with respect to either x or y. However, since $r^2 = x^2 + y^2$ and the region D is circular, it is natural to wonder whether converting to polar coordinates will allow us to evaluate the new integral. To do so, we replace x with $r\cos(\theta)$, y with $r\sin(\theta)$, and $dy\,dx$ with $r\,dr\,d\theta$ to obtain

$$\iint_D f(x,y)\,dA = \iint_D e^{r^2}\,r\,dr\,d\theta.$$

The disc D is described in polar coordinates by the constraints $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Therefore, it follows that

$$\iint_D e^{r^2} \, r \, dr \, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} e^{r^2} \, r \, dr \, d\theta.$$

We can evaluate the resulting iterated polar integral as follows:

$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} e^{r^2} \, r \, dr \, d\theta &= \int_{\theta=0}^{2\pi} \left(\frac{1}{2} e^{r^2} \Big|_{r=0}^{r=1} \right) d\theta \\
&= \frac{1}{2} \int_{\theta=0}^{\theta=2\pi} (e-1) \, d\theta \\
&= \frac{1}{2}(e-1) \int_{\theta=0}^{\theta=2\pi} d\theta \\
&= \frac{1}{2}(e-1) \, [\theta] \Big|_{\theta=0}^{\theta=2\pi} \\
&= \pi(e-1).
\end{aligned}$$

While there is no firm rule for when polar coordinates can or should be used, they are a natural alternative anytime the domain of integration may be expressed simply in polar form, and/or when the integrand involves expressions such as $\sqrt{x^2+y^2}$.

Activity 11.15.

Let $f(x, y) = x + y$ and $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

(a) Write the double integral of f over D as an iterated integral in rectangular coordinates.

(b) Write the double integral of f over D as an iterated integral in polar coordinates.

(c) Evaluate one of the iterated integrals. Why is the final value you found not surprising?

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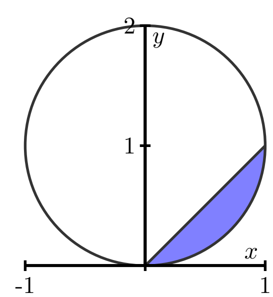

Figure 11.26: The graphs of $y = x$ and $x^2 + (y-1)^2 = 1$, for use in Activity 11.16.

Activity 11.16.

Consider the circle given by $x^2 + (y-1)^2 = 1$ as shown in Figure 11.26.

(a) Determine a polar curve in the form $r = f(\theta)$ that traces out the circle $x^2 + (y-1)^2 = 1$.

(b) Find the exact average value of $g(x, y) = \sqrt{x^2 + y^2}$ over the interior of the circle $x^2 + (y-1)^2 = 1$.

(c) Find the volume under the surface $h(x, y) = x$ over the region D, where D is the region bounded above by the line $y = x$ and below by the circle.

(d) Explain why in both (b) and (c) it is advantageous to use polar coordinates.

◁

Summary

- The polar representation of a point P is the ordered pair (r, θ) where r is the distance from the origin to P and θ is the angle the ray through the origin and P makes with the positive x-axis.
- The polar coordinates r and θ of a point (x, y) in rectangular coordinates satisfy

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan(\theta) = \frac{y}{x};$$

  the rectangular coordinates x and y of a point (r, θ) in polar coordinates satisfy

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

- The area element dA in polar coordinates is determined by the area of a slice of an annulus and is given by

$$dA = r\,dr\,d\theta.$$

- To convert the double integral $\iint_D f(x, y)\,dA$ to an iterated integral in polar coordinates, we substitute $r\cos(\theta)$ for x, $r\sin(\theta)$ for y, and $r\,dr\,d\theta$ for dA to obtain the iterated integral

$$\iint_D f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta.$$

Exercises

1. Consider the iterated integral $I = \int_{-3}^{0} \int_{-\sqrt{9-y^2}}^{0} \frac{y}{x^2 + y^2 + 1}\,dx\,dy$.

(a) Sketch (and label) the region of integration.

(b) Convert the given iterated integral to one in polar coordinates.

(c) Evaluate the iterated integral in (b).

(d) State one possible interpretation of the value you found in (c).

2. Let D be the region that lies inside the unit circle in the plane.

(a) Set up and evaluate an iterated integral in polar coordinates whose value is the area of D.

(b) Determine the exact average value of $f(x, y) = y$ over the upper half of D.

(c) Find the exact center of mass of the lamina over the portion of D that lies in the first quadrant and has its mass density distribution given by $\delta(x, y) = 1$. (Before making any calculations, where do you expect the center of mass to lie? Why?)

(d) Find the exact volume of the solid that lies under the surface $z = 8 - x^2 - y^2$ and over the unit disk, D.

3. For each of the following iterated integrals, (a) sketch and label the region of integration, (b) convert the integral to the other coordinate system (if given in polar, to rectangular; if given in rectangular, to polar), and (c) choose one of the two iterated integrals to evaluate exactly.

(a) $\displaystyle\int_{\pi}^{3\pi/2} \int_0^3 r^3 \, dr \, d\theta$

(b) $\displaystyle\int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} \sqrt{x^2 + y^2} \, dy \, dx$

(c) $\displaystyle\int_0^{\pi/2} \int_0^{\sin(\theta)} r\sqrt{1 - r^2} \, dr \, d\theta.$

(d) $\displaystyle\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} \cos(x^2 + y^2) \, dx \, dy.$

11.6 Surfaces Defined Parametrically and Surface Area

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a parameterization of a surface?
- How do we find the surface area of a parametrically defined surface?

Introduction

We have now studied at length how curves in space can be defined parametrically by functions of the form $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, and surfaces can be represented by functions $z = f(x, y)$. In what follows, we will see how we can also define surfaces parametrically. A one-dimensional curve in space results from a vector function that relies upon one parameter, so a two-dimensional surface naturally involves the use of two parameters. If $x = x(s, t)$, $y = y(s, t)$, and $z = z(s, t)$ are functions of independent parameters s and t, then the terminal points of all vectors of the form

$$\mathbf{r}(s, t) = x(s, t)\mathbf{i} + y(s, t)\mathbf{j} + z(s, t)\mathbf{k}$$

form a surface in space. The equations $x = x(s, t)$, $y = y(s, t)$, and $z = z(s, t)$ are the *parametric equations* for the surface, or a *parametrization* of the surface. In Preview Activity 11.6 we investigate how to parameterize a cylinder and a cone.

Preview Activity 11.6. Recall the standard parameterization of the unit circle that is given by

$$x(t) = \cos(t) \quad \text{and} \quad y(t) = \sin(t),$$

where $0 \leq t \leq 2\pi$.

(a) Determine a parameterization of the circle of radius 1 in $\mathbb{R}^3$ that has its center at $(0, 0, 1)$ and lies in the plane $z = 1$.

(b) Determine a parameterization of the circle of radius 1 in 3-space that has its center at $(0, 0, -1)$ and lies in the plane $z = -1$.

(c) Determine a parameterization of the circle of radius 1 in 3-space that has its center at $(0, 0, 5)$ and lies in the plane $z = 5$.

(d) Taking into account your responses in (a), (b), and (c), describe the graph that results from the set of parametric equations

$$x(s, t) = \cos(t), \quad y(s, t) = \sin(t), \quad \text{and} \quad z(s, t) = s,$$

where $0 \leq t \leq 2\pi$ and $-5 \leq s \leq 5$. Explain your thinking.

(e) Just as a cylinder can be viewed as a "stack" of circles of constant radius, a cone can be viewed as a stack of circles with varying radius. Modify the parametrizations of the circles above in order to construct the parameterization of a cone whose vertex lies at the origin, whose base radius is 4, and whose height is 3, where the base of the cone lies in the plane $z = 3$. Use appropriate technology[5] to plot the parametric equations you develop. (Hint: The cross sections parallel to the xz plane are circles, with the radii varying linearly as z increases.)

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Parametric Surfaces

In a single-variable setting, any function may have its graph expressed parametrically. For instance, given $y = g(x)$, by considering the parameterization $\langle t, g(t) \rangle$ (where t belongs to the domain of g), we generate the same curve. What is more important is that certain curves that are not functions may be represented parametrically; for instance, the circle (which cannot be represented by a single function) can be parameterized by $\langle \cos(t), \sin(t) \rangle$, where $0 \leq t \leq 2\pi$.

In the same way, in a two-variable setting, the surface $z = f(x, y)$ may be expressed parametrically by considering

$$\langle x(s,t), y(s,t), z(s,t) \rangle = \langle s, t, f(s,t) \rangle,$$

where (s, t) varies over the entire domain of f. Therefore, any familiar surface that we have studied so far can be generated as a parametric surface. But what is more powerful is that there are surfaces that cannot be generated by a single function $z = f(x, y)$ (such as the unit sphere), but that can be represented parametrically. We now consider an important example.

Example 11.3. Consider the torus (or doughnut) shown in Figure 11.27.

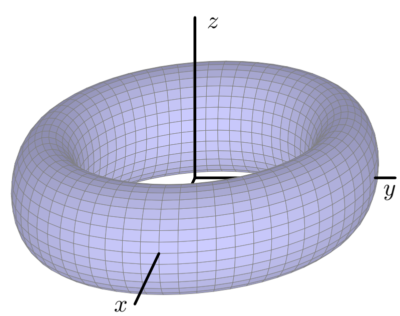

Figure 11.27: A torus

To find a parametrization of this torus, we recall our work in Preview Activity 11.6. There, we saw that a circle of radius r that has its center at the point $(0, 0, z_0)$ and is contained in the

[5] e.g., http://www.flashandmath.com/mathlets/multicalc/paramrec/surf_graph_rectan.html

horizontal plane $z = z_0$, as shown in Figure 11.28, can be parametrized using the vector-valued function $\mathbf{r}$ defined by

$$\mathbf{r}(t) = r\cos(t)\mathbf{i} + r\sin(t)\mathbf{j} + z_0\mathbf{k}$$

where $0 \le t \le 2\pi$.

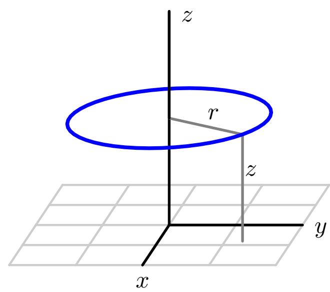

Figure 11.28: A circle in a horizontal plane centered at $(0, 0, z_0)$.

To obtain the torus in Figure 11.27, we begin with a circle of radius a in the xz-plane centered at $(b, 0)$, as shown on the left of Figure 11.29. We may parametrize the points on this circle, using the parameter s, by using the equations

$$x(s) = b + a\cos(s) \qquad \text{and} \qquad z(s) = a\sin(s),$$

where $0 \le s \le 2\pi$.

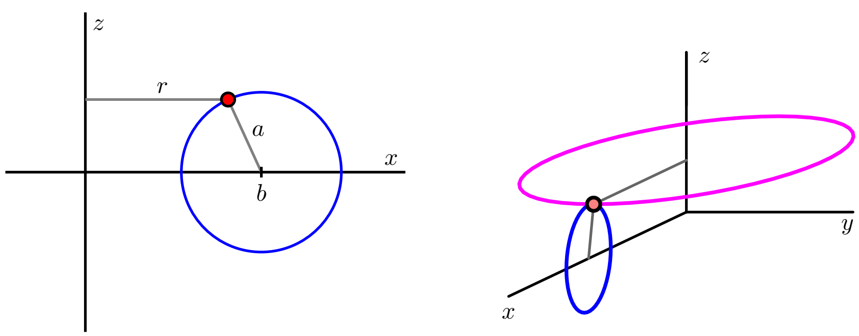

Figure 11.29: Revolving a circle to obtain a torus.

Let's focus our attention on one point on this circle, such as the indicated point, which has coordinates $(x(s), 0, z(s))$ for a fixed value of the parameter s. When this point is revolved about the z-axis, we obtain a circle contained in a horizontal plane centered at $(0, 0, z(s))$ and having radius $x(s)$, as shown on the right of Figure 11.29. If we let t be the new parameter that generates the circle for the rotation about the z-axis, this circle may be parametrized by

$$\mathbf{r}(s, t) = x(s)\cos(t)\mathbf{i} + x(s)\sin(t)\mathbf{j} + z(s)\mathbf{k}.$$

Now using our earlier parametric equations for $x(s)$ and $z(s)$ for the original smaller circle, we have an overall parameterization of the torus given by

$$\mathbf{r}(s,t) = (b + a\cos(s))\cos(t)\mathbf{i} + (b + a\cos(s))\sin(t)\mathbf{j} + a\sin(s)\mathbf{k}.$$

To trace out the entire torus, we require that the parameters vary through the values $0 \le s \le 2\pi$ and $0 \le t \le 2\pi$.

Activity 11.17.

In this activity, we seek a parametrization of the sphere of radius R centered at the origin, as shown on the left in Figure 11.30. Notice that this sphere may be obtained by revolving a half-circle contained in the xz-plane about the z-axis, as shown on the right.

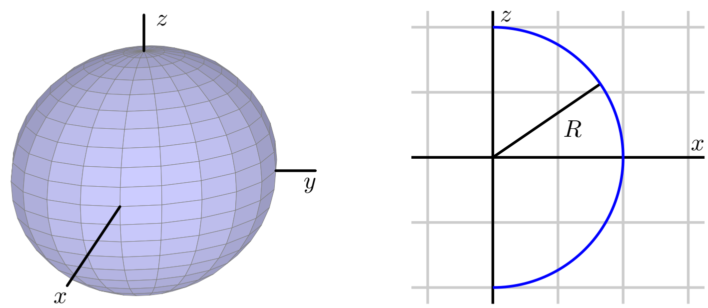

Figure 11.30: A sphere obtained by revolving a half-circle.

(a) Begin by writing a parametrization of this half-circle using the parameter s:

$$x(s) = \ldots, \qquad z(s) = \ldots.$$

Be sure to state the domain of the parameter s.

(b) By revolving the points on this half-circle about the z-axis, obtain a parametrization $\mathbf{r}(s,t)$ of the points on the sphere of radius R. Be sure to include the domain of both parameters s and t. (Hint: What is the radius of the circle obtained when revolving a point on the half-circle around the z axis?)

(c) Draw the surface defined by your parameterization with appropriate technology[6].

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The Surface Area of Parametrically Defined Surfaces

Recall that a differentiable function is locally linear – that is, if we zoom in on the surface around a point, the surface looks like its tangent plane. We now exploit this idea in order to determine

[6]e.g., http://web.monroecc.edu/manila/webfiles/calcNSF/JavaCode/CalcPlot3D.htm or http://www.flashandmath.com/mathlets/multicalc/paramrec/surf_graph_rectan.html

the surface area generated by a parametrization $\langle x(s,t), y(s,t), z(s,t)\rangle$. The basic idea is a familiar one: we will subdivide the surface into small pieces, in the approximate shape of small parallelograms, and thus estimate the entire the surface area by adding the areas of these approximation parallelograms. Ultimately, we use an integral to sum these approximations and determine the exact surface area.

Let

$$\mathbf{r}(s,t) = x(s,t)\mathbf{i} + y(s,t)\mathbf{j} + z(s,t)\mathbf{k}$$

define a surface over a rectangular domain $a \le s \le b$ and $c \le t \le d$. As a function of two variables, s and t, it is natural to consider the two partial derivatives of the vector-valued function $\mathbf{r}$, which we define by

$$\mathbf{r}_s(s,t) = x_s(s,t)\mathbf{i} + y_s(s,t)\mathbf{j} + z_s(s,t)\mathbf{k} \quad \text{and} \quad \mathbf{r}_t(s,t) = x_t(s,t)\mathbf{i} + y_t(s,t)\mathbf{j} + z_t(s,t)\mathbf{k}.$$

In the usual way, we slice the domain into small rectangles. In particular, we partition the interval $[a,b]$ into m subintervals of length $\Delta s = \frac{b-a}{n}$ and let s_0, s_1, ..., s_m be the endpoints of these subintervals, where $a = s_0 < s_1 < s_2 < \cdots < s_m = b$. Also partition the interval $[c,d]$ into n subintervals of equal length $\Delta t = \frac{d-c}{n}$ and let t_0, t_1, ..., t_n be the endpoints of these subintervals, where $c = t_0 < t_1 < t_2 < \cdots < t_n = d$. These two partitions create a partition of the rectangle $R = [a,b] \times [c,d]$ in st-coordinates into mn sub-rectangles R_{ij} with opposite vertices (s_{i-1}, t_{j-1}) and (s_i, t_j) for i between 1 and m and j between 1 and n. These rectangles all have equal area $\Delta A = \Delta s \cdot \Delta t$.

Now we want to think about the small piece of area on the surface itself that lies above one of these small rectangles in the domain. Observe that if we increase s by a small amount Δs from the point (s_{i-1}, t_{j-1}) in the domain, then $\mathbf{r}$ changes by approximately $\mathbf{r}_s(s_{i-1}, t_{j-1})\Delta s$. Similarly, if we increase t by a small amount Δt from the point (s_{i-1}, t_{j-1}), then $\mathbf{r}$ changes by approximately $\mathbf{r}_t(s_{i-1}, t_{j-1})\Delta t$. So we can approximate the surface defined by $\mathbf{r}$ on the st-rectangle $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ with the parallelogram determined by the vectors $\mathbf{r}_s(s_{i-1}, t_{j-1})\Delta s$ and $\mathbf{r}_t(s_{i-1}, t_{j-1})\Delta t$, as seen in Figure 11.31.

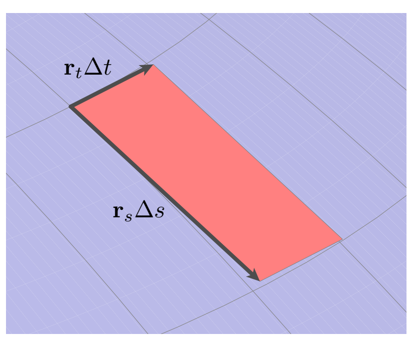

Figure 11.31: Approximation surface area with a parallelogram.

Say that the small parallelogram has area S_{ij}. If we can find its area, then all that remains is to sum the areas of all of the generated parallelograms and take a limit. Recall from our earlier work in the course that given two vectors $\mathbf{u}$ and $\mathbf{v}$, the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ is given by the magnitude of their cross product, $|\mathbf{u} \times \mathbf{v}|$. In the present context, it follows that the area, S_{ij}, of the parallelogram determined by the vectors $\mathbf{r}_s(s_{i-1}, t_{j-1})\Delta s$ and $\mathbf{r}_t(s_{i-1}, t_{j-1})\Delta t$ is

$$S_{ij} = |(\mathbf{r}_s(s_{i-1}, t_{j-1})\Delta s) \times (\mathbf{r}_t(s_{i-1}, t_{j-1})\Delta t)| = |\mathbf{r}_s(s_{i-1}, t_{j-1}) \times \mathbf{r}_t(s_{i-1}, t_{j-1})|\Delta s \Delta t, \qquad (11.3)$$

where the latter equality holds from standard properties of the cross product and length.

We sum the surface area approximations from Equation (11.3) over all sub-rectangles to obtain an estimate for the total surface area, S, given by

$$S \approx \sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{r}_s(s_{i-1}, t_{j-1}) \times \mathbf{r}_t(s_{i-1}, t_{j-1})|\Delta s \Delta t.$$

Taking the limit as $m, n \to \infty$ shows that the surface area of the surface defined by $\mathbf{r}$ over the domain D is given as follows.

Let $\mathbf{r}(s, t) = \langle x(s, t), y(s, t), z(s, t)\rangle$ be a parameterization of a smooth surface over a domain D. The **area of the surface** defined by $\mathbf{r}$ on D is given by

$$S = \iint_D |\mathbf{r}_s \times \mathbf{r}_t| \, dA. \qquad (11.4)$$

Activity 11.18.

Consider the cylinder with radius a and height h defined parametrically by

$$\mathbf{r}(s, t) = a\cos(s)\mathbf{i} + a\sin(s)\mathbf{j} + t\mathbf{k}$$

for $0 \leq s \leq 2\pi$ and $0 \leq t \leq h$, as shown in Figure 11.32.

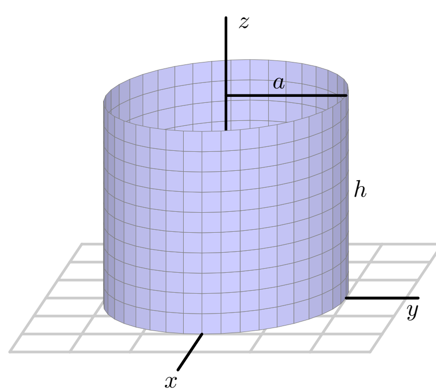

Figure 11.32: A cylinder.

(a) Set up an iterated integral to determine the surface area of this cylinder.

(b) Evaluate the iterated integral.

(c) Recall that one way to think about the surface area of a cylinder is to cut the cylinder horizontally and find the perimeter of the resulting cross sectional circle, then multiply by the height. Calculate the surface area of the given cylinder using this alternate approach, and compare your work in (b).

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As we noted earlier, we can take any surface $z = f(x, y)$ and generate a corresponding parameterization for the surface by writing $\langle s, t, f(s, t)\rangle$. Hence, we can use our recent work with parametrically defined surfaces to find the surface area that is generated by a function $f = f(x, y)$ over a given domain.

Activity 11.19.

Let $z = f(x, y)$ define a smooth surface, and consider the corresponding parameterization $\mathbf{r}(s, t) = \langle s, t, f(s, t)\rangle$.

(a) Let D be a region in the domain of f. Using Equation 11.4, show that the area, S, of the surface defined by the graph of f over D is

$$S = \iint_D \sqrt{(f_x(x, y))^2 + (f_y(x, y))^2 + 1}\, dA.$$

(b) Use the formula developed in (a) to calculate the area of the surface defined by $f(x, y) = \sqrt{4 - x^2}$ over the rectangle $D = [-2, 2] \times [0, 3]$.

(c) Observe that the surface of the solid describe in (b) is half of a circular cylinder. Use the standard formula for the surface area of a cylinder to calculate the surface area in a different way, and compare your result from (b).

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Summary

- A parameterization of a curve describes the coordinates of a point on the curve in terms of a single parameter t, while a parameterization of a surface describes the coordinates of points on the surface in terms of two independent parameters.
- If $\mathbf{r}(s, t) = \langle x(s, t), y(s, t), z(s, t)\rangle$ describes a smooth surface in 3-space on a domain D, then the area, S, of that surface is given by

$$S = \iint_D |\mathbf{r}_s \times \mathbf{r}_t|\, dA.$$

Exercises

1. Consider the ellipsoid given by the equation

$$\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{9} = 1.$$

In Activity 11.17, we found that a parameterization of the sphere S of radius R centered at the origin is

$$x(r,s) = R\cos(s)\cos(t),\ y(s,t) = R\cos(s)\sin(t),\ \text{ and }\ z(s,t) = R\sin(s)$$

for $-\frac{\pi}{2} \leq s \leq \frac{\pi}{2}$ and $0 \leq t \leq 2\pi$.

(a) Let (x,y,z) be a point on the ellipsoid and let $X = \frac{x}{4}$, $Y = \frac{y}{5}$, and $Z = \frac{z}{3}$. Show that (X,Y,Z) lies on the sphere S. Hence, find a parameterization of S in terms of X, Y, and Z as functions of s and t.

(b) Use the result of part (a) to find a parameterization of the ellipse in terms of x, y, and z as functions of s and t. Check your parametrization by substituting x, y, and z into the equation of the ellipsoid. Then check your work by plotting the surface defined by your parameterization with appropriate technology[7].

2. In this exercise, we explore how to use a parametrization and iterated integral to determine the surface area of a sphere.

(a) Set up an iterated integral whose value is the portion of the surface area of a sphere of radius R that lies in the first octant (see the parameterization you developed in Activity 11.17).

(b) Then, evaluate the integral to calculate the surface area of this portion of the sphere.

(c) By what constant must you multiply the value determined in (b) in order to find the total surface area of the entire sphere.

(d) Finally, compare your result to the standard formula for the surface area of sphere.

3. Consider the plane generated by $z = f(x,y) = 24 - 2x - 3y$ over the region $D = [0,2] \times [0,3]$.

(a) Sketch a picture of the overall solid generated by the plane over the given domain.

(b) Determine a parameterization $\mathbf{r}(s,t)$ for the plane over the domain D.

(c) Use Equation 11.4 to determine the surface area generated by f over the domain D.

(d) Observe that the vector $\mathbf{u} = \langle 2, 0, -4\rangle$ points from $(0,0,24)$ to $(2,0,20)$ along one side of the surface generated by the plane f over D. Find the vector $\mathbf{v}$ such that $\mathbf{u}$ and $\mathbf{v}$ together span the parallelogram that represents the surface defined by f over D, and hence compute $|\mathbf{u} \times \mathbf{v}|$. What do you observe about the value you find?

[7] e.g., http://web.monroecc.edu/manila/webfiles/calcNSF/JavaCode/CalcPlot3D.htm or http://www.flashandmath.com/mathlets/multicalc/paramrec/surf_graph_rectan.html

4. A cone with base radius a and height h can be realized as the surface defined by $z = \frac{h}{a}\sqrt{x^2 + y^2}$, where a and h are positive.

(a) Find a parameterization of the cone described by $z = \frac{h}{a}\sqrt{x^2 + y^2}$. (Hint: Compare to the parameterization of a cylinder as seen in Activity 11.18.)

(b) Set up an iterated integral to determine the surface area of this cone.

(c) Evaluate the iterated integral to find a formula for the lateral surface area of a cone of height h and base a.

11.7 Triple Integrals

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How are a triple Riemann sum and the corresponding triple integral of a continuous function $f = f(x, y, z)$ defined?
- What are two things the triple integral of a function can tell us?

Introduction

We have now learned that we define the double integral of a continuous function $f = f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$ as a limit of a double Riemann sum, and that these ideas parallel the single-variable integral of a function $g = g(x)$ on an interval $[a, b]$. Moreover, this double integral has natural interpretations and applications, and can even be considered over non-rectangular regions, D. For instance, given a continuous function f over a region D, the average value of f, $f_{\text{AVG(D)}}$, is given by

$$f_{\text{AVG(R)}} = \frac{1}{A(D)} \iint_D f(x, y)\, dA,$$

where $A(D)$ is the area of D. Likewise, if $\delta(x, y)$ describes a mass density function on a lamina over D, the mass, M, of the lamina is given by

$$M = \iint_D \delta(x, y)\, dA.$$

It is natural to wonder if it is possible to extend these ideas of double Riemann sums and double integrals for functions of two variables to triple Riemann sums and then triple integrals for functions of three variables. We begin investigating in Preview Activity 11.7.

Preview Activity 11.7. Consider a solid piece granite in the shape of a box $B = \{(x, y, z) : 0 \leq x \leq 4, 0 \leq y \leq 6, 0 \leq z \leq 8\}$, whose density varies from point to point. Let $\delta(x, y, z)$ represent the mass density of the piece of granite at point (x, y, z) in kilograms per cubic meter (so we are measuring x, y, and z in meters). Our goal is to find the mass of this solid.

Recall that if the density was constant, we could find the mass by multiplying the density and volume; since the density varies from point to point, we will use the approach we did with two-variable lamina problems, and slice the solid into small pieces on which the density is roughly constant.

Partition the interval $[0, 4]$ into 2 subintervals of equal length, the interval $[0, 6]$ into 3 subintervals of equal length, and the interval $[0, 8]$ into 2 subintervals of equal length. This partitions the box B into sub-boxes as shown in Figure 11.33.

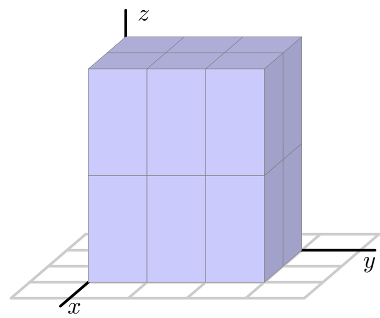


Figure 11.33: A partitioned three-dimensional domain.

(a) Let $0 = x_0 < x_1 < x_2 = 4$ be the endpoints of the subintervals of $[0, 4]$ after partitioning. Draw a picture of Figure 11.33 and label these endpoints on your drawing. Do likewise with $0 = y_0 < y_1 < y_2 < y_3 = 6$ and $0 = z_0 < z_1 < z_2 = 8$

What is the length Δx of each subinterval $[x_{i-1}, x_i]$ for i from 1 to 2? the length of Δy? of Δz?

(b) The partitions of the intervals $[0, 4]$, $[0, 6]$ and $[0, 8]$ partition the box B into sub-boxes. How many sub-boxes are there? What is volume ΔV of each sub-box?

(c) Let B_{ijk} denote the sub-box $[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$. Say that we choose a point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in the i, j, kth sub-box for each possible combination of i, j, k. What is the meaning of $\delta(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$? What physical quantity will $\delta(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$ approximate?

(d) What final step(s) would it take to determine the exact mass of the piece of granite?

⋈

Triple Riemann Sums and Triple Integrals

Through the application of a mass density distribution over a three-dimensional solid, Preview Activity 11.7 suggests that the generalization from double Riemann sums of functions of two variables to triple Riemann sums of functions of three variables is natural. In the same way, so is the generalization from double integrals to triple integrals. By simply adding a z-coordinate to our earlier work, we can define both a triple Riemann sum and the corresponding triple integral.

The Triple Riemann Sum.

Definition 11.3. Let $f = f(x, y, z)$ be a continuous function on a box $B = [a, b] \times [c, d] \times [r, s]$. The **triple Riemann sum of f over B** is created as follows.

- Partition the interval $[a, b]$ into m subintervals of equal length $\Delta x = \frac{b-a}{m}$. Let x_0, x_1, ..., x_m be the endpoints of these subintervals, where $a = x_0 < x_1 < x_2 < \cdots < x_m = b$. Do likewise with the interval $[c, d]$ using n subintervals of equal length $\Delta y = \frac{d-c}{n}$ to generate $c = y_0 < y_1 < y_2 < \cdots < y_n = d$, and with the interval $[r, s]$ using ℓ subintervals of equal length $\Delta z = \frac{s-r}{\ell}$ to have $r = z_0 < z_1 < z_2 < \cdots < z_l = s$.
- Let B_{ijk} be the sub-box of B with opposite vertices $(x_{i-1}, y_{j-1}, z_{k-1})$ and (x_i, y_j, z_k) for i between 1 and m, j between 1 and n, and k between 1 and ℓ. The volume of each B_{ijk} is $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$.
- Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be a point in box B_{ijk} for each i, j, and k. The resulting triple Riemann sum for f on B is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{\ell} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \cdot \Delta V.$$

If $f(x, y, z)$ represents the mass density of the box B, then, as we saw in Preview Activity 11.7, the triple Riemann sum approximates the total mass of the box B. In order to find the exact mass of the box, we need to let the number of sub-boxes increase without bound (in other words, let m, n, and ℓ go to infinity); in this case, the finite sum of the mass approximations becomes the actual mass of the solid B. More generally, we have the following definition of the triple integral.

The Triple Integral Over a Box.

Definition 11.4. With following notation defined as in a triple Riemann sum, the **triple integral of f over B** is

$$\iiint_B f(x, y, z)\, dV = \lim_{m,n,\ell \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{\ell} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \cdot \Delta V.$$

As we noted earlier, if $f(x, y, z)$ represents the density of the solid B at each point (x, y, z), then

$$M = \iiint_B f(x, y, z)\, dV$$

is the mass of B. Even more importantly, for any continuous function f over the solid B, we can use a triple integral to determine the average value of f over B, $f_{\text{AVG(B)}}$. We note this generalization of our work with functions of two variables along with several others in the following important boxed information. Note that each of these quantities may actually be considered over a general domain S in $\mathbb{R}^3$, not simply a box, B.

- The triple integral
$$V(S) = \iiint_S 1\, dV$$
represents the **volume** of the solid S.

- The **average value** of the function $f = f(x, y, x)$ over a solid domain S is given by
$$f_{\text{AVG(S)}} = \left(\frac{1}{V(S)}\right) \iiint_S f(x, y, z)\, dV,$$
where $V(S)$ is the volume of the solid S.

- The **center of mass** of the solid S with density $\delta = \delta(x, y, z)$ is $(\overline{x}, \overline{y}, \overline{z})$, where
$$\overline{x} = \frac{\iiint_S x\, \delta(x, y, z)\, dV}{M}, \quad \overline{y} = \frac{\iiint_S y\, \delta(x, y, z)\, dV}{M}, \quad \overline{z} = \frac{\iiint_S z\, \delta(x, y, z)\, dV}{M},$$
and $M = \iiint_S \delta(x, y, z)\, dV$ is the mass of the solid S.

In the Cartesian coordinate system, the volume element dV is $dz\, dy\, dx$, and, as a consequence, a triple integral of a function f over a box $B = [a, b] \times [c, d] \times [r, s]$ in Cartesian coordinates can be evaluated as an iterated integral of the form
$$\iiint_B f(x, y, z)\, dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx.$$

If we want to evaluate a triple integral as an iterated integral over a solid S that is not a box, then we need to describe the solid in terms of variable limits.

Activity 11.20.

(a) Set up and evaluate the triple integral of $f(x, y, z) = x - y + 2z$ over the box $B = [-2, 3] \times [1, 4] \times [0, 2]$.

(b) Let S be the solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$. A picture of S is shown at right in Figure 11.34. Our goal in what follows is to set up an iterated integral of the form
$$\int_{x=?}^{x=?} \int_{y=?}^{y=?} \int_{z=?}^{z=?} \delta(x, y, z)\, dz\, dy\, dx \tag{11.5}$$
to represent the mass of S in the setting where $\delta(x, y, z)$ tells us the density of S at the point (x, y, z). Our particular task is to find the limits on each of the three integrals.

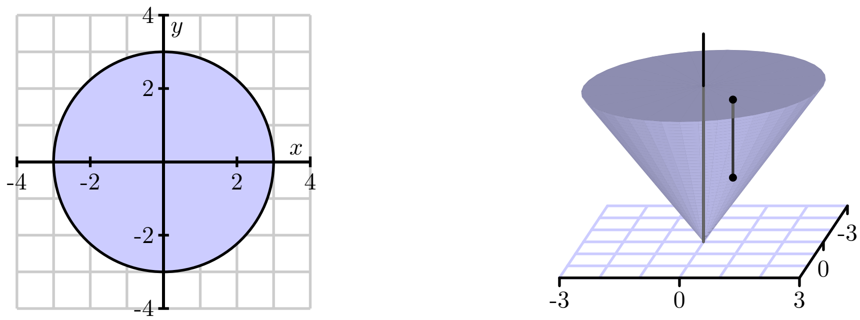


Figure 11.34: At right, the cone; at left, its projection.

i. If we think about slicing up the solid, we can consider slicing the domain of the solid's projection onto the xy-plane (just as we would slice a two-dimensional region in $\mathbb{R}^2$), and then slice in the z-direction as well. The projection of the solid is onto the xy-plane is shown at left in Figure 11.34. If we decide to first slice the domain of the solid's projection perpendicular to the x-axis, over what range of constant x-values would we have to slice?

ii. If we continue with slicing the domain, what are the limits on y on a typical slice? How do these depend on x? What, therefore, are the limits on the middle integral?

iii. Finally, now that we have thought about slicing up the two-dimensional domain that is the projection of the cone, what are the limits on z in the innermost integral? Note that over any point (x, y) in the plane, a vertical slice in the z direction will involve a range of values from the cone itself to its flat top. In particular, observe that at least one of these limits is not constant but depends on x and y.

iv. In conclusion, write an iterated integral of the form (11.5) that represents the mass of the cone S.

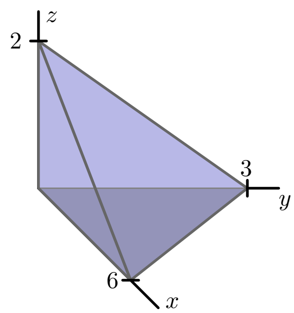
$\triangleleft$

Note well: When setting up iterated integrals, the limits on a given variable can be *only* in terms of the remaining variables. In addition, there are multiple different ways we can choose to set up such an integral. For example, two possibilities for iterated integrals that represent a triple integral $\iiint_S f(x, y, z)\, dV$ over a solid S are

- $\displaystyle \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)\, dz\, dy\, dx$
- $\displaystyle \int_r^s \int_{p_1(z)}^{p_2(z)} \int_{q_1(x,z)}^{q_2(x,z)} f(x, y, z)\, dy\, dx\, dz$

where g_1, g_2, h_1, h_2, p_1, p_2, q_1, and q_2 are functions of the indicated variables. There are four other options beyond the two stated here, since the variables x, y, and z can (theoretically) be arranged in any order. Of course, in many circumstances, an insightful choice of variable order will make it easier to set up an iterated integral, just as was the case when we worked with double integrals.

Example 11.4. Find the mass of the tetrahedron in the first octant bounded by the coordinate planes and the plane $x+2y+3z=6$ if the density at point (x,y,z) is given by $\delta(x,y,z)=x+y+z$. A picture of the solid tetrahedron is shown at left in Figure 11.35.



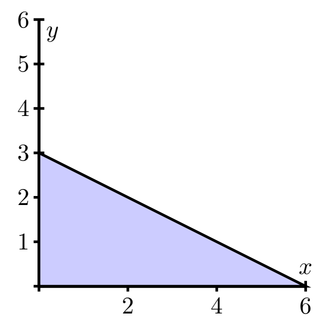


Figure 11.35: The tetrahedron and its projection.

We find the mass, M, of the tetrahedron by the triple integral

$$M=\iiint_S \delta(x,y,z)\,dV,$$

where S is the solid tetrahedron described above. In this example, we choose to integrate with respect to z first for the innermost integral. The top of the tetrahedron is given by the equation

$$x+2y+3z=6;$$

solving for z then yields

$$z=\frac{1}{3}(6-x-2y).$$

The bottom of the tetrahedron is the xy-plane, so the limits on z in the iterated integral will be $0\le z\le \frac{1}{3}(6-x-2y)$.

To find the bounds on x and y we project the tetrahedron onto the xy-plane; this corresponds to setting $z=0$ in the equation $z=\frac{1}{3}(6-x-2y)$. The resulting relation between x and y is

$$x+2y=6.$$

The right image in Figure 11.35 shows the projection of the tetrahedron onto the xy-plane.

If we choose to integrate with respect to y for the middle integral in the iterated integral, then the lower limit on y is the x-axis and the upper limit is the hypotenuse of the triangle. Note that the hypotenuse joins the points $(6,0)$ and $(0,3)$ and so has equation $y=3-\frac{1}{2}x$. Thus, the bounds on y are $0 \leq y \leq 3 - \frac{1}{2}x$. Finally, the x values run from 0 to 6, so the iterated integral that gives the mass of the tetrahedron is

$$M = \int_0^6 \int_0^{3-(1/2)x} \int_0^{(1/3)(6-x-2y)} x + y + z \, dz \, dy \, dx. \tag{11.6}$$

Evaluating the triple integral gives us

$$\begin{aligned}
M &= \int_0^6 \int_0^{3-(1/2)x} \int_0^{(1/3)(6-x-2y)} x + y + x \, dz \, dy \, dx \\
&= \int_0^6 \int_0^{3-(1/2)x} \left[xz + yz + \frac{z}{2} \right] \Big|_0^{(1/3)(6-x-2y)} \, dy \, dx \\
&= \int_0^6 \int_0^{3-(1/2)x} \frac{4}{3}x - \frac{5}{18}x^2 - \frac{}{9}xy + \frac{2}{3}y - \frac{4}{9}y^2 + 2 \, dy \, dx \\
&= \int_0^6 \left[\frac{4}{3}xy - \frac{5}{18}x^2y - \frac{7}{18}xy^2 + \frac{1}{3}y^2 - \frac{4}{27}y^3 + 2y \right] \Big|_0^{3-(1/2)x} \, dx \\
&= \int_0^6 5 + \frac{1}{2}x - \frac{7}{12}x^2 + \frac{13}{216}x^3 \, dx \\
&= \left[5x + \frac{1}{4}x^2 - \frac{7}{36}x^3 + \frac{13}{864}x^4 \right] \Big|_0^6 \\
&= \frac{33}{2}.
\end{aligned}$$

Setting up limits on iterated integrals can require considerable geometric intuition. It is important to not only create carefully labeled figures, but also to think about how we wish to slice the solid. Further, note that when we say "we will integrate first with respect to x," by "first" we are referring to the innermost integral in the iterated integral. The next activity explores several different ways we might set up the integral in the preceding example.

Activity 11.21.

There are several other ways we could have set up the integral to give the mass of the tetrahedron in Example 11.4.

(a) How many different iterated integrals could be set up that are equal to the integral in Equation (11.6)?

(b) Set up an iterated integral, integrating first with respect to z, then x, then y that is equivalent to the integral in Equation (11.6). Before you write down the integral, think about Figure 11.35, and draw an appropriate two-dimensional image of an important projection.

(c) Set up an iterated integral, integrating first with respect to y, then z, then x that is equivalent to the integral in Equation (11.6). As in (b), think carefully about the geometry first.

(d) Set up an iterated integral, integrating first with respect to x, then y, then z that is equivalent to the integral in Equation (11.6).

◁

Now that we have begun to understand how to set up iterated triple integrals, we can apply them to determine important quantities, such as those found in the next activity.

Activity 11.22.

A solid S is bounded below by the square $z = 0$, $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ and above by the surface $z = 2 - x^2 - y^2$. A picture of the solid is shown in Figure 11.36.

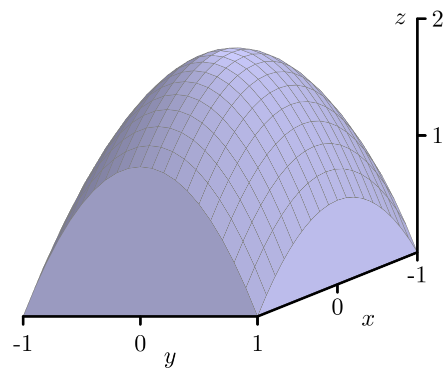


Figure 11.36: The solid bounded by the surface $z = 2 - x^2 - y^2$.

(a) Set up (but do not evaluate) an iterated integral to find the volume of the solid S.

(b) Set up (but do not evaluate) iterated integral expressions that will tell us the center of mass of S, if the density at point (x, y, z) is $\delta(x, y, z) = x^2 + 1$.

(c) Set up (but do not evaluate) an iterated integral to find the average density on S using the density function from part (b).

(d) Use technology appropriately to evaluate the iterated integrals you determined in (a), (b), and (c); does the location you determined for the center of mass make sense?

◁

Summary

- Let $f = f(x, y, z)$ be a continuous function on a box $B = [a, b] \times [c, d] \times [r, s]$. The triple integral of f over B is defined as

$$\iiint_B f(x, y, z)\, dV = \lim_{\Delta V \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \cdot \Delta V,$$

where the triple Riemann sum is defined in the usual way. The definition of the triple integral naturally extends to non-rectangular solid regions S.

- The triple integral $\iiint_S f(x, y, z)\, dV$ can tell us
  - the volume of the solid S if $f(x, y, z) = 1$,
  - the mass of the solid S if f represents the density of S at the point (x, y, z).

  Moreover,

$$f_{\text{AVG(S)}} = \frac{1}{V(S)} \iiint_S f(x, y, z)\, dV,$$

  is the average value of f over S.

Exercises

1. Consider the solid S that is bounded by the parabolic cylinder $y = x^2$ and the planes $z = 0$ and $z = 1 - y$ as shown in Figure 11.37.

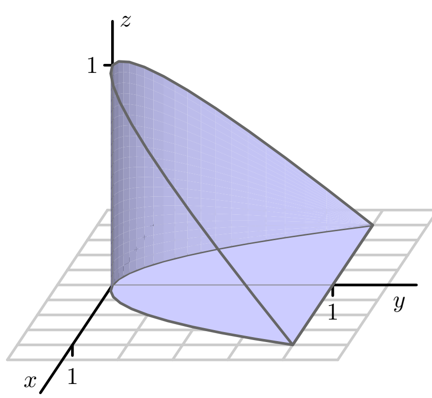


Figure 11.37: The solid bounded by $y = x^2$ and the planes $z = 0$ and $z = 1 - y$.

Assume the density of S is given by $\delta(x, y, z) = z$

(a) Set up (but do not evaluate) an iterated integral that represents the mass of S. Integrate first with respect to z, then y, then x. A picture of the projection of S onto the xy-plane is shown in Figure 11.38.

(b) Set up (but do not evaluate) an iterated integral that represents the mass of S. In this case, integrate first with respect to y, then z, then x. A picture of the projection of S onto the xz-plane is shown in Figure 11.39.

(c) Set up (but do not evaluate) an iterated integral that represents the mass of S. For this integral, integrate first with respect to x, then y, then z. A picture of the projection of S onto the yz-plane is shown in Figure 11.39.

(d) Which of these three orders of integration is the most natural to you? Why?

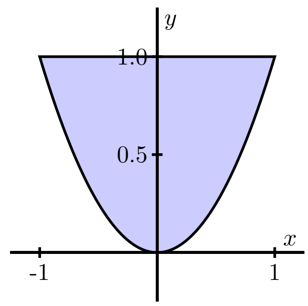


Figure 11.38: Projecting S onto the xy-plane.

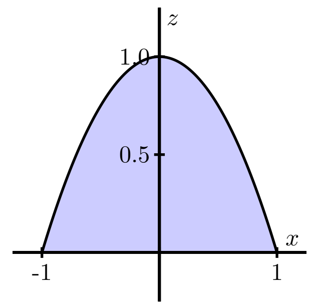


Figure 11.39: Projecting S onto the xz-plane.

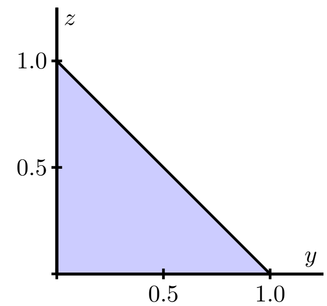


Figure 11.40: Projecting S onto the yz-plane.

2. This problem asks you to investigate the average value of some different quantities.

(a) Set up, but do not evaluate, an iterated integral expression whose value is the average sum of all real numbers x, y, and z that have the following property: y is between 0 and 2, x is greater than or equal to 0 but cannot exceed $2y$, and z is greater than or equal to 0 but cannot exceed $x + y$.

(b) Set up, but do not evaluate, an integral expression whose value represents the average value of $f(x, y, z) = x + y + z$ over the solid region in the first octant bounded by the surface $z = 4 - x - y^2$ and the coordinate planes $x = 0$, $y = 0$, $z = 0$.

(c) How are the quantities in (a) and (b) similar? How are they different?

3. Consider the solid that lies between the paraboloids $z = g(x, y) = x^2 + y^2$ and $z = f(x, y) = 8 - 3x^2 - 3y^2$.

(a) By eliminating the variable z, determine the curve of intersection between the two paraboloids, and sketch this curve in the x-y plane.

(b) Set up, but do not evaluate, an iterated integral expression whose value determine the mass of the solid, integrating first with respect to x, then y, then z. Assume the the solid's density is given by $\delta(x, y, z) = \frac{1}{x^2+y^2+z^2+1}$.

(c) Set up, but do not evaluate, iterated integral expressions whose values determine the mass of the solid using all possible remaining orders of integration. Use $\delta(x, y, z) = \frac{1}{x^2+y^2+z^2+1}$ as the density of the solid.

(d) Set up, but do not evaluate, iterated integral expressions whose values determine the center of mass of the solid. Again, assume the the solid's density is given by $\delta(x, y, z) = \frac{1}{x^2+y^2+z^2+1}$.

(e) Which coordinates of the center of mass can you determine *without* evaluating any integral expression? Why?

4. In each of the following problems, your task is to

(i) sketch, by hand, the region over which you integrate

(ii) set up iterated integral expressions which, when evaluated, will determine the value sought

(iii) use appropriate technology to evaluate each iterated integral expression you develop

Note well: in some problems you may be able to use a double rather than a triple integral, and polar coordinates may be helpful in some cases.

(a) Consider the solid created by the region enclosed by the circular paraboloid $z = 4 - x^2 - y^2$ over the region R in the x-y plane enclosed by $y = -x$ and the circle $x^2 + y^2 = 4$ in the first, second, and fourth quadrants.
Determine the solid's volume.

(b) Consider the solid region that lies beneath the circular paraboloid $z = 9 - x^2 - y^2$ over the triangular region between $y = x$, $y = 2x$, and $y = 1$. Assuming that the solid has its density at point (x, y, z) given by $\delta(x, y, z) = xyz + 1$, measured in grams per cubic cm, determine the center of mass of the solid.

(c) In a certain room in a house, the walls can be thought of as being formed by the lines $y = 0$, $y = 12 + x/4$, $x = 0$, and $x = 12$, where length is measured in feet. In addition, the ceiling of the room is vaulted and is determined by the plane $z = 16 - x/6 - y/3$. A heater is stationed in the corner of the room at $(0, 0, 0)$ and causes the temperature in the room at a particular time to be given by

$$T(x, y, z) = \frac{80}{1 + \frac{x^2}{1000} + \frac{y^2}{1000} + \frac{z^2}{1000}}$$

What is the average temperature in the room?

(d) Consider the solid enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 5$ and $z = 1$. Assuming that the solid's density is given by $\delta(x, y, z) = \sqrt{x^2 + y^2}$, find the mass and center of mass of the solid.

11.8 Triple Integrals in Cylindrical and Spherical Coordinates

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What are the cylindrical coordinates of a point, and how are they related to Cartesian coordinates?
- What is the volume element in cylindrical coordinates? How does this inform us about evaluating a triple integral as an iterated integral in cylindrical coordinates?
- What are the spherical coordinates of a point, and how are they related to Cartesian coordinates?
- What is the volume element in spherical coordinates? How does this inform us about evaluating a triple integral as an iterated integral in spherical coordinates?

Introduction

We have encountered two different coordinate systems in $\mathbb{R}^2$ – the rectangular and polar coordinates systems – and seen how in certain situations, polar coordinates form a convenient alternative. In a similar way, there there turn out to be two additional natural coordinate systems in $\mathbb{R}^3$. Given that we are already familiar with the Cartesian coordinate system for $\mathbb{R}^3$, we next investigate the cylindrical and spherical coordinate systems (each of which builds upon polar coordinates in $\mathbb{R}^2$). In what follows, we will see how to convert among the different coordinate systems, how to evaluate triple integrals using them, and some situations in which these other coordinate systems prove advantageous.

Preview Activity 11.8. In the following questions, we investigate the two new coordinate systems that are the subject of this section: cylindrical and spherical coordinates. Our goal is to consider some examples of how to convert from rectangular coordinates to each of these systems, and vice versa. Triangles and trigonometry prove to be particularly important.

The cylindrical coordinates of a point in $\mathbb{R}^3$ are given by (r, θ, z) where r and θ are the polar coordinates of the point (x, y) and z is the same z coordinate as in Cartesian coordinates. An illustration is given in Figure 11.41.

(a) Find cylindrical coordinates for the point whose Cartesian coordinates are $(-1, \sqrt{3}, 3)$. Draw a labeled picture illustrating all of the coordinates.

(b) Find the Cartesian coordinates of the point whose cylindrical coordinates are $\left(2, \frac{5\pi}{4}, 1\right)$. Draw a labeled picture illustrating all of the coordinates.

The spherical coordinates of a point in $\mathbb{R}^3$ are ρ (rho), θ, and ϕ (phi), where ρ is the distance from the point to the origin, θ has the same interpretation it does in polar coordinates, and ϕ is the angle

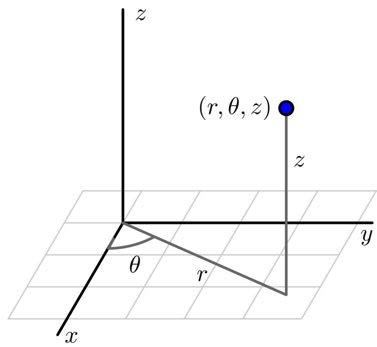


Figure 11.41: The cylindrical coordinates of a point.

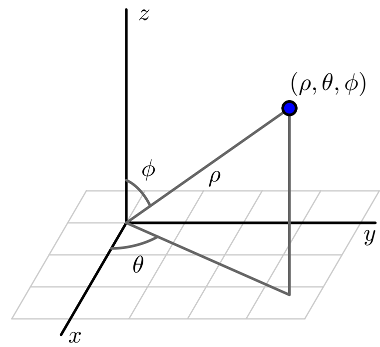


Figure 11.42: The spherical coordinates of a point.

between the positive z axis and the vector from the origin to the point, as illustrated in Figure 11.42.

For the following questions, consider the point P whose Cartesian coordinates are $(-2, 2, \sqrt{8})$.

(c) What is the distance from P to the origin? Your result is the value of ρ in the spherical coordinates of P.

(d) Determine the point that is the projection of P onto the xy-plane. Then, use this projection to find the value of θ in the polar coordinates of the projection of P that lies in the plane. Your result is also the value of θ for the spherical coordinates of the point.

(e) Based on the illustration in Figure 11.42, how is the angle ϕ determined by ρ and the z coordinate of P? Use a well-chosen right triangle to find the value of ϕ, which is the final component in the spherical coordinates of P. Draw a carefully labeled picture that clearly illustrates the values of ρ, θ, and ϕ in this example, along with the original rectangular coordinates of P.

(f) Based on your responses to (c), (d), and (e), if we are given the Cartesian coordinates (x, y, z) of a point Q, how are the values of ρ, θ, and ϕ in the spherical coordinates of Q determined by x, y, and z?

$\bowtie$

Cylindrical Coordinates

As we stated in Preview Activity 11.8, the cylindrical coordinates of a point are (r, θ, z), where r and θ are the polar coordinates of the point (x, y), and z is the same z coordinate as in Cartesian coordinates. The general situation is illustrated Figure 11.41.

Since we already know how to convert between rectangular and polar coordinates in the plane, and the z coordinate is identical in both Cartesian and cylindrical coordinates, the conversion equations between the two systems in $\mathbb{R}^3$ are essentially those we found for polar coordinates:

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = z$$

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x} \qquad z = z.$$

Just as with rectangular coordinates, where we usually write z as a function of x and y to plot the resulting surface, in cylindrical coordinates, we often express z as a function of r and θ. In the following activity, we explore several basic equations in cylindrical coordinates and the corresponding surface each generates.

Activity 11.23.

In this activity, we graph some surfaces using cylindrical coordinates. To improve your intuition and test your understanding, you should first think about what each graph should look like before you plot it using technology.[8]

(a) Plot the graph of the cylindrical equation $r = 2$, where we restrict the values of θ and z to the intervals $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 2$. What familiar shape does the resulting surface take? How does this example suggest that we call these coordinates *cylindrical coordinates*?

(b) Plot the graph of the cylindrical equation $\theta = 2$, where we restrict the other variables to the values $0 \leq r \leq 2$ and $0 \leq z \leq 2$. What familiar surface results?

(c) Plot the graph of the cylindrical equation $z = 2$, using $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. What does this surface look like?

(d) Plot the graph of the cylindrical equation $z = r$, where $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. What familiar surface results?

(e) Plot the graph of the cylindrical equation $z = \theta$ for $0 \leq \theta \leq 4\pi$. What does this surface look like?

◁

As the name and Activity 11.23 suggest, cylindrical coordinates are useful for describing surfaces that are cylindrical in nature.

Triple Integrals in Cylindrical Coordinates

To evaluate a triple integral $\iiint_S f(x, y, z)\, dV$ as an iterated integral in Cartesian coordinates, we use the fact that the volume element dV is equal to $dz\, dy\, dx$ (which corresponds to the volume of a

[8] e.g., http://www.math.uri.edu/~bkaskosz/flashmo/cylin/ – to plot $r = 2$, set r to 2, θ to s, and z to t – to plot $\theta = \pi/3$, set $\theta = \pi/3$, $r = s$, and $z = t$, for example. Thanks to Barbara Kaskosz of URI and the Flash and Math team.

small box). To evaluate a triple integral in cylindrical coordinates, we similarly must understand the volume element dV in cylindrical coordinates.

Activity 11.24.

A picture of a cylindrical box, $B = \{(r, \theta, z) : r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2, z_1 \leq z \leq z_2\}$, is shown in Figure 11.43. Let $\Delta r = r_2 - r_1$, $\Delta\theta = \theta_2 - \theta_1$, and $\Delta z = z_2 - z_1$. We want to determine the volume ΔV of B in terms of Δr, $\Delta\theta$, Δz, r, θ, and z.

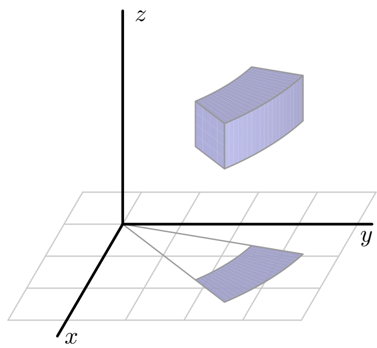


Figure 11.43: A cylindrical box.

(a) Appropriately label Δr, $\Delta\theta$, and Δz in Figure 11.43.

(b) Let ΔA be the area of the projection of the box, B, onto the xy-plane, which is shaded blue in Figure 11.43. Recall that we previously determined the area ΔA in polar coordinates in terms of r, Δr, and $\Delta\theta$. In light of the fact that we know ΔA and that z is the standard z coordinate from Cartesian coordinates, what is the volume ΔV in cylindrical coordinates?

◁

Activity 11.24 demonstrates that the volume element dV in cylindrical coordinates is given by $dV = r\,dz\,dr\,d\theta$, and hence the following rule holds in general.

Given a continuous function $f = f(x, y, z)$ over a region S in $\mathbb{R}^3$,

$$\iiint_S f(x, y, z)\,dV = \iiint_S f(r\cos(\theta), r\sin(\theta), z)\,r\,dz\,dr\,d\theta.$$

The latter expression is an **iterated integral in cylindrical coordinates**.

Of course, to complete the task of writing an iterated integral in cylindrical coordinates, we need to determine the limits on the three integrals: θ, r, and z. In the following activity, we explore how to do this in several situations where cylindrical coordinates are natural and advantageous.

Activity 11.25.

In each of the following questions, set up, but do not evaluate, the requested integral expression.

(a) Let S be the solid bounded above by the graph of $z = x^2 + y^2$ and below by $z = 0$ on the unit circle. Determine an iterated integral expression in cylindrical coordinates that gives the volume of S.

(b) Suppose the density of the cone defined by $r = 1 - z$, with $z \geq 0$, is given by $\delta(r, \theta, z) = z$. A picture of the cone is shown in Figure 11.44, and the projection of the cone onto the xy-plane in given in Figure 11.45. Set up an iterated integral in cylindrical coordinates that gives the mass of the cone.

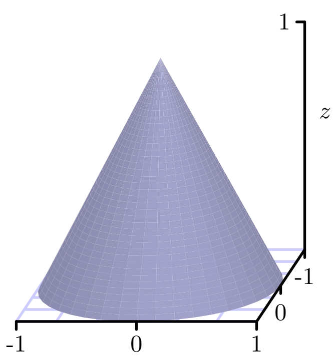


Figure 11.44: The cylindrical cone $r = 1 - z$.

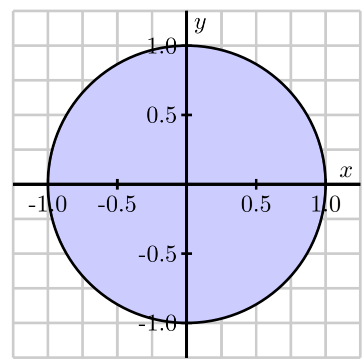


Figure 11.45: The projection into the xy-plane.

(c) Determine an iterated integral expression in cylindrical coordinates whose value is the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the cone $z = 4 - \sqrt{x^2 + y^2}$. A picture is shown in Figure 11.46.

◁

Spherical Coordinates

As we saw in Preview Activity 11.8, the spherical coordinates of a point in 3-space have the form (ρ, θ, ϕ), where ρ is the distance from the point to the origin, θ has the same meaning as in polar coordinates, and ϕ is the angle between the positive z axis and the vector from the origin to the point. The overall situation is illustrated in Figure 11.42.

The example in Preview Activity 11.8 suggests that given a point in rectangular coordinates, (x, y, z), we can find the corresponding spherical coordinates. Indeed, this holds generally by

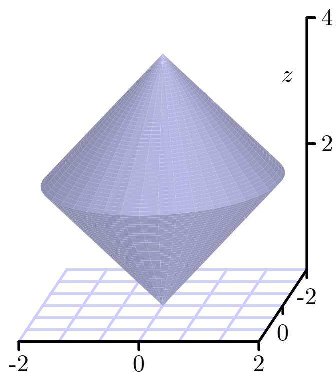


Figure 11.46: A solid bounded by the cones $z = \sqrt{x^2 + y^2}$ and $z = 4 - \sqrt{x^2 + y^2}$.

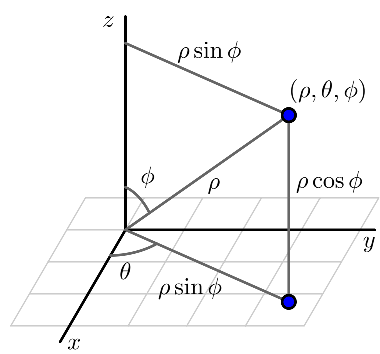


Figure 11.47: Converting from spherical to Cartesian coordinates.

using the relationships

$$\rho = \sqrt{x^2 + y^2 + z^2} \qquad \tan(\theta) = \frac{y}{x} \qquad \cos(\phi) = \frac{z}{\rho}$$

where in the latter two equations, we require $x \neq 0$ and $\rho \neq 0$.

To convert from given spherical coordinates to Cartesian coordinates, we use the equations

$$x = \rho \sin(\phi) \cos(\theta) \qquad y = \rho \sin(\phi) \sin(\theta) \qquad z = \rho \cos(\phi),$$

as illustrated in Figure 11.47.

When it comes to thinking about particular surfaces in spherical coordinates, similar to our work with cylindrical and Cartesian coordinates, we usually write ρ as a function of θ and ϕ; this is a natural analog to polar coordinates, where we often think of our distance from the origin in the plane as being a function of θ. In spherical coordinates, we likewise often view ρ as a function of θ and ϕ, thus viewing distance from the origin as a function of two key angles.

In the following activity, we explore several basic equations in spherical coordinates and the surfaces they generate.

Activity 11.26.

In this activity, we graph some surfaces using spherical coordinates. To improve your intuition and test your understanding, you should first think about what each graph should look like before you plot it using technology.[9]

(a) Plot the graph of $\rho = 1$, where θ and ϕ are restricted to the intervals $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. What is the resulting surface? How does this particular example demonstrate the reason for the name of this coordinate system?

(b) Plot the graph of $\phi = \frac{\pi}{3}$, where ρ and θ are restricted to the intervals $0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$. What familiar surface results?

(c) Plot the graph of $\theta = \frac{\pi}{6}$, for $0 \leq \rho \leq 1$ and $0 \leq \phi \leq \pi$. What familiar shape arises?

(d) Plot the graph of $\rho = \theta$, for $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. How does the resulting surface appear?

$\triangleleft$

As the name and Activity 11.26 indicate, spherical coordinates are particularly useful for describing surfaces that are spherical in nature; they are also convenient for working with certain conical surfaces.

Triple Integrals in Spherical Coordinates

As with rectangular and cylindrical coordinates, a triple integral $\iiint_S f(x,y,z)\,dV$ in spherical coordinates can be evaluated as an iterated integral once we understand the volume element dV.

Activity 11.27.

To find the volume element dV in spherical coordinates, we need to understand how to determine the volume of a spherical box of the form $\rho_1 \leq \rho \leq \rho_2$ (with $\Delta\rho = \rho_2 - \rho_1$), $\phi_1 \leq \phi \leq \phi_2$ (with $\Delta\phi = \phi_2 - \phi_1$), and $\theta_1 \leq \theta \leq \theta_2$ (with $\Delta\theta = \theta_2 - \theta_1$). An illustration of such a box is given in Figure 11.48. This spherical box is a bit more complicated than the cylindrical box we encountered earlier. In this situation, it is easier to approximate the volume ΔV than to compute it directly. Here we can approximate the volume ΔV of this spherical box with the volume of a Cartesian box whose sides have the lengths of the sides of this spherical box. In other words,

$$\Delta V \approx |PS|\,|\widehat{PR}|\,|\widehat{PQ}|,$$

where $|\widehat{PR}|$ denotes the length of the circular arc from P to R.

[9] e.g., http://www.flashandmath.com/mathlets/multicalc/paramsphere/surf_graph_sphere.html – to plot $\rho = 2$, set ρ to 2, θ to s, and ϕ to t, for example. Thanks to Barbara Kaskosz of URI and the Flash and Math team.

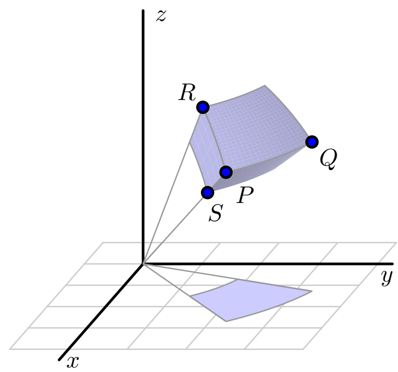


Figure 11.48: A spherical box.

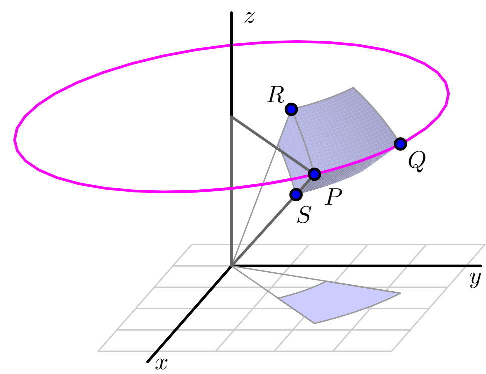


Figure 11.49: A spherical volume element.

(a) What is the length $|PS|$ in terms of ρ?

(b) What is the length of the arc $\widehat{PR}$? (Hint: The arc $\widehat{PR}$ is an arc of a circle of radius ρ_2, and arc length along a circle is the product of the angle measure (in radians) and the circle's radius.)

(c) What is the length of the arc $\widehat{PQ}$? (Hint: The arc $\widehat{PQ}$ lies on a horizontal circle as illustrated in Figure 11.49. What is the radius of this circle?)

(d) Use your work in (a), (b), and (c) to determine an approximation for ΔV in spherical coordinates.

◁

Letting $\Delta\rho$, $\Delta\phi$ and $\Delta\theta$ go to 0, it follows from the final result in Activity 11.27 that $dV = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$ in spherical coordinates, and thus allows us to state the following general rule.

Given a continuous function $f = f(x, y, z)$ over a region S in $\mathbb{R}^3$,

$$\iiint_S f(x, y, z)\, dV = \iiint_S f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

The latter expression is an **iterated integral in spherical coordinates**.

Finally, in order to actually evaluate an iterated integral in spherical coordinates, we must of course determine the limits of integration in θ, ϕ, and ρ. The process is similar to our earlier work in the other two coordinate systems.

Activity 11.28.

We can use spherical coordinates to help us more easily understand some natural geometric objects.

(a) Recall that the sphere of radius a has spherical equation $\rho = a$. Set up and evaluate an iterated integral in spherical coordinates to determine the volume of a sphere of radius a.

(b) Set up, but do not evaluate, an iterated integral expression in spherical coordinates whose value is the mass of the solid obtained by removing the cone $\phi = \frac{\pi}{4}$ from the sphere $\rho = 2$ if the density δ at the point (x, y, z) is $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. An illustration of the solid is shown in Figure 11.50.

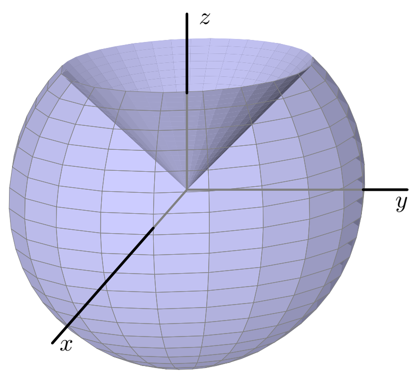


Figure 11.50: The solid cut from the sphere $\rho = 2$ by the cone $\phi = \frac{\pi}{4}$.

◁

Summary

- The cylindrical coordinates of a point P are (r, θ, z) where r is the distance from the origin to the projection of P onto the xy-plane, θ is the angle that the projection of P onto the xy-plane makes with the positive x-axis, and z is the vertical distance from P to the projection of P onto the xy-plane. When P has rectangular coordinates (x, y, z), it follows that its cylindrical coordinates are given by

$$r^2 = x^2 + y^2, \qquad \tan(\theta) = \frac{y}{x}, \qquad z = z.$$

  When P has given cylindrical coordinates (r, θ, z), its rectangular coordinates are

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = z.$$

- The volume element dV in cylindrical coordinates is $dV = r\,dz\,dr\,d\theta$. Hence, a triple integral $\iiint_S f(x, y, z)\,dA$ can be evaluated as the iterated integral

$$\iiint_S f(r\cos(\theta), r\sin(\theta), z)\,r\,dz\,dr\,d\theta.$$

- The spherical coordinates of a point P in 3-space are ρ (rho), ϕ (phi), and θ, where ρ is the distance from P to the origin, ϕ is the angle between the positive z axis and the vector from the origin to P, and θ is the angle that the projection of P onto the xy-plane makes with the positive x-axis. When P has Cartesian coordinates (x, y, z), the spherical coordinates are given by

$$\rho^2 = x^2 + y^2 + z^2, \qquad \tan(\theta) = \frac{y}{x}, \qquad \cos(\phi) = \frac{z}{\rho}.$$

Given the point P in spherical coordinates (ρ, ϕ, θ), its rectangular coordinates are

$$x = \rho \sin(\phi)\cos(\theta), \qquad y = \rho \sin(\phi)\sin(\theta), \qquad z = \rho\cos(\phi).$$

- The volume element dV in spherical coordinates is $dV = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$. Thus, a triple integral $\iiint_S f(x, y, z)\, dA$ can be evaluated as the iterated integral

$$\iiint_S f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

Exercises

1. In each of the following questions, set up an iterated integral expression whose value determines the desired result. Then, evaluate the integral first by hand, and then using appropriate technology.

(a) Find the volume of the "cap" cut from the solid sphere $x^2 + y^2 + z^2 = 4$ by the plane $z = 1$, as well as the z-coordinate of its centroid.

(b) Find the x-coordinate of the center of mass of the portion of the unit sphere that lies in the first octant (i.e., where x, y, and z are all nonnegative). Assume that the density of the solid given by $\delta(x, y, z) = \frac{1}{1+x^2+y^2+z^2}$.

(c) Find the volume of the solid bounded below by the x-y plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$.

(d) Find the z coordinate of the center of mass of the region that is bounded above by the surface $z = \sqrt{\sqrt{x^2 + y^2}}$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the x-y plane. Assume that the density of the solid is uniform and constant.

(e) Find the volume of the solid that lies outside the sphere $x^2 + y^2 + z^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 2z$.

2. For each of the following questions, (a) sketch the region of integration, (b) change the coordinate system in which the iterated integral is written to one of the remaining two, and (c) evaluate the iterated integral you deem easiest to evaluate by hand.

(a) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\, dz\, dy\, dx$

(b) $\int_0^{\pi/2} \int_0^{\pi} \int_0^1 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$

(c) $\int_0^{2\pi} \int_0^1 \int_r^1 r^2 \cos(\theta)\, dz\, dr\, d\theta$

3. Consider the solid region S bounded above by the paraboloid $z = 16 - x^2 - y^2$ and below by the paraboloid $z = 3x^2 + 3y^2$.

(a) Describe parametrically the curve in $\mathbb{R}^3$ in which these two surfaces intersect.

(b) In terms of x and y, write an equation to describe the projection of the curve onto the x-y plane.

(c) What coordinate system do you think is most natural for an iterated integral that gives the volume of the solid?

(d) Set up, but do not evaluate, an iterated integral expression whose value is average z-value of points in the solid region S.

(e) Use technology to plot the two surfaces and evaluate the integral in (c). Write at least one sentence to discuss how your computations align with your intuition about where the average z-value of the solid should fall.

11.9 Change of Variables

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a change of variables?
- What is the Jacobian, and how is it related to a change of variables?

Introduction

In single variable calculus, we encountered the idea of a change of variable in a definite integral through the method of substitution. For example, given the definite integral

$$\int_0^2 2x(x^2+1)^3\,dx,$$

we naturally consider the change of variable $u = x^2 + 1$. From this substitution, it follows that $du = 2x\,dx$, and since $x = 0$ implies $u = 1$ and $x = 2$ implies $u = 5$, we have transformed the original integral in x into a new integral in u. In particular,

$$\int_0^2 2x(x^2+1)^3\,dx = \int_1^5 u^3\,du.$$

The latter integral, of course, is far easier to evaluate.

Through our work with polar, cylindrical, and spherical coordinates, we have already implicitly seen some of the issues that arise in using a change of variables with two or three variables present. In what follows, we seek to understand the general ideas behind any change of variables in a multiple integral.

Preview Activity 11.9. Consider the double integral

$$I = \iint_D x^2 + y^2\,dA, \tag{11.7}$$

where D is the upper half of the unit disk.

(a) Write the double integral I given in Equation (11.7) as an iterated integral in rectangular coordinates.

(b) Write the double integral I given in Equation (11.7) as an iterated integral in polar coordinates.

When we write the double integral (11.7) as an iterated integral in polar coordinates we make a change of variables, namely

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta). \tag{11.8}$$

We also then have to change dA to $r\,dr\,d\theta$. This process also identifies a "polar rectangle" $[r_1, r_2] \times [\theta_1, \theta_2]$ with the original Cartesian rectangle, under the transformation[10] in Equation (11.8). The vertices of the polar rectangle are transformed into the vertices of a closed and bounded region in rectangular coordinates.

To work with a numerical example, let's now consider the polar rectangle P given by $[1, 2] \times [\frac{\pi}{6}, \frac{\pi}{4}]$, so that $r_1 = 1$, $r_2 = 2$, $\theta_1 = \frac{\pi}{6}$, and $\theta_2 = \frac{\pi}{4}$.

(c) Use the transformation determined by the equations in (11.8) to find the rectangular vertices that correspond to the polar vertices in the polar rectangle P. In other words, by substituting appropriate values of r and θ into the two equations in (11.8), find the values of the corresponding x and y coordinates for the vertices of the polar rectangle P. Label the point that corresponds to the polar vertex (r_1, θ_1) as (x_1, y_1), the point corresponding to the polar vertex (r_2, θ_1) as (x_2, y_2), the point corresponding to the polar vertex (r_1, θ_2) as (x_3, y_3), and the point corresponding to the polar vertex (r_2, θ_2) as (x_4, y_4).

(d) Draw a picture of the figure in rectangular coordinates that has the points (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) as vertices. (Note carefully that because of the trigonometric functions in the transformation, this region will not look like a Cartesian rectangle.) What is the area of this region in rectangular coordinates? How does this area compare to the area of the original polar rectangle?

⋈

Change of Variables in Polar Coordinates

The general idea behind a change of variables is suggested by Preview Activity 11.9. There, we saw that in a change of variables from rectangular coordinates to polar coordinates, a polar rectangle $[r_1, r_2] \times [\theta_1, \theta_2]$ gets mapped to a Cartesian rectangle under the transformation

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

The vertices of the polar rectangle P are transformed into the vertices of a closed and bounded region P' in rectangular coordinates. If we view the standard coordinate system as having the horizontal axis represent r and the vertical axis represent θ, then the polar rectangle P appears to

[10] A *transformation* is another name for function: here, the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ define a function T by $T(r, \theta) = (r\cos(\theta), r\sin(\theta))$ so that T is a function (transformation) from $\mathbb{R}^2$ to $\mathbb{R}^2$. We view this transformation as mapping a version of the x-y plane where the axes are viewed as representing r and θ (the r-θ plane) to the familiar x-y plane.

us at left in Figure 11.51. The image P' of the polar rectangle P under the transformation given by (11.8) is shown at right in Figure 11.51. We thus see that there is a correspondence between a simple region (a traditional, right-angled rectangle) and a more complicated region (a fraction of an annulus) under the function T given by $T(r, \theta) = (r\cos(\theta), r\sin(\theta))$.

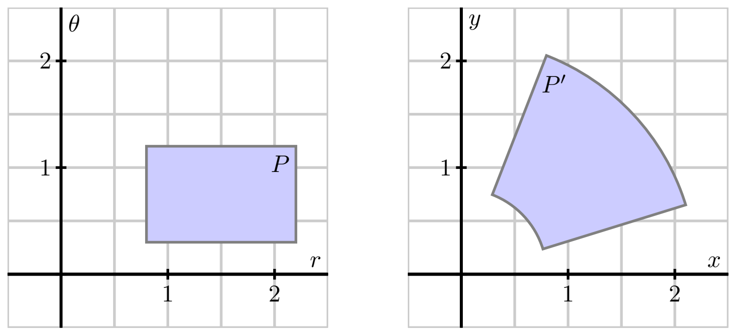


Figure 11.51: A rectangle P and its image P'.

Furthermore, as Preview Activity 11.9 suggest, it follows generally that for an original polar rectangle $P = [r_1, r_2] \times [\theta_1, \theta_2]$, the area of the transformed rectangle P' is given by $\frac{r_2+r_1}{2}\Delta r \Delta\theta$. Therefore, as Δr and $\Delta\theta$ go to 0 this area becomes the familiar area element $dA = r\,dr\,d\theta$ in polar coordinates. When we proceed to working with other transformations for different changes in coordinates, we have to understand how the transformation affects area so that we may use the correct area element in the new system of variables.

General Change of Coordinates

We first focus on double integrals. As with single integrals, we may be able to simplify a double integral of the form

$$\iint_D f(x, y)\, dA$$

by making a change of variables (that is, a substitution) of the form

$$x = x(s, t) \quad \text{and} \quad y = y(s, t)$$

where x and y are functions of new variables s and t. This transformation introduces a correspondence between a problem in the xy-plane and one in the the st-plane. The equations $x = x(s, t)$ and $y = y(s, t)$ convert s and t to x and y; we call these formulas the *change of variable* formulas. To complete the change to the new s, t variables, we need to understand the area element, dA, in this new system. The following activity helps to illustrate the idea.

Activity 11.29.

Consider the change of variables

$$x = s + 2t \quad \text{and} \quad y = 2s + \sqrt{t}.$$

Let's see what happens to the rectangle $T = [0,1] \times [1,4]$ in the st-plane under this change of variable.

(a) Draw a labeled picture of T in the st-plane.

(b) Find the image of the st-vertex $(0,1)$ in the xy-plane. Likewise, find the respective images of the other three vertices of the rectangle T: $(0,4)$, $(1,1)$, and $(1,4)$.

(c) In the xy-plane, draw a labeled picture of the image, T', of the original st-rectangle T. What appears to be the shape of the image, T'?

(d) To transform an integral with a change of variables, we need to determine the area element dA for image of the transformed rectangle. How would we find the area of the xy-figure T'? (Hint: Remember what the cross product of two vectors tells us.)

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Activity 11.29 presents the general idea of how a change of variables works. We partition a rectangular domain in the st system into subrectangles. Let $T = [a,b] \times [a+\Delta s, b+\Delta t]$ be one of these subrectangles. Then we transform this into a region T' in the standard xy Cartesian coordinate system. The region T' is called the *image* of T; the region T is the *pre-image* of T'. Although the sides of this xy region T' aren't necessarily straight (linear), we will approximate the element of area dA for this region with the area of the parallelogram whose sides are given by the vectors $\mathbf{v}$ and $\mathbf{w}$, where $\mathbf{v}$ is the vector from $(x(a,b), y(a,b))$ to $(x(a+\Delta s, b), y(a+\Delta s, b))$, and $\mathbf{w}$ is the vector from $(x(a,b), y(a,b))$ to $(x(a, b+\Delta t), y(a, b+\Delta t))$.

An example of an image T' in the xy plane that results from a transformation of a rectangle T in the st plane is shown in Figure 11.52.

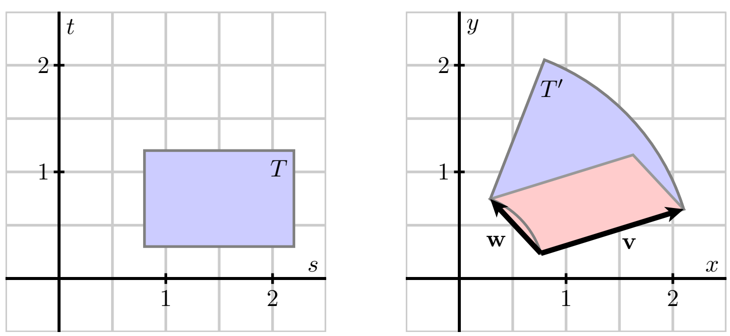

Figure 11.52: Approximating an area of an image resulting from a transformation.

The components of the vector $\mathbf{v}$ are

$$\mathbf{v} = \langle x(a+\Delta s, b) - x(a,b), y(a+\Delta s, b) - y(a,b), 0\rangle$$

and similarly those for $\mathbf{w}$ are

$$\mathbf{w} = \langle x(a, b+\Delta t) - x(a,b), y(a, b+\Delta s) - y(a,b), 0\rangle \,.$$

Slightly rewriting $\mathbf{v}$ and $\mathbf{w}$, we have

$$\mathbf{v} = \left\langle \frac{x(a+\Delta s, b) - x(a,b)}{\Delta s}, \frac{y(a+\Delta s, b) - y(a,b)}{\Delta s}, 0 \right\rangle \Delta s, \text{ and}$$
$$\mathbf{w} = \left\langle \frac{x(a, b+\Delta t) - x(a,b)}{\Delta t}, \frac{y(a, b+\Delta s) - y(a,b)}{\Delta t}, 0 \right\rangle \Delta t.$$

For small Δs and Δt, the definition of the partial derivative tells us that

$$\mathbf{v} \approx \left\langle \frac{\partial x}{\partial s}(a,b), \frac{\partial y}{\partial s}(a,b), 0 \right\rangle \Delta s \quad \text{and} \quad \mathbf{w} \approx \left\langle \frac{\partial x}{\partial t}(a,b), \frac{\partial y}{\partial t}(a,b), 0 \right\rangle \Delta t.$$

Recall that the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$ is the length of the cross product of the two vectors, $|\mathbf{v} \times \mathbf{w}|$. From this, we observe that

$$\begin{aligned}\mathbf{v} \times \mathbf{w} &\approx \left\langle \frac{\partial x}{\partial s}(a,b), \frac{\partial y}{\partial s}(a,b), 0 \right\rangle \Delta s \times \left\langle \frac{\partial x}{\partial t}(a,b), \frac{\partial y}{\partial t}(a,b), 0 \right\rangle \Delta t \\ &= \left\langle 0,\ 0,\ \frac{\partial x}{\partial s}(a,b)\frac{\partial y}{\partial t}(a,b) - \frac{\partial x}{\partial t}(a,b)\frac{\partial y}{\partial s}(a,b) \right\rangle \Delta s\, \Delta t.\end{aligned}$$

Finally, by computing the magnitude of the cross product, we see that

$$\begin{aligned}|\mathbf{v} \times \mathbf{w}| &\approx \left| \left\langle 0, 0, \frac{\partial x}{\partial s}(a,b)\frac{\partial y}{\partial t}(a,b) - \frac{\partial x}{\partial t}(a,b)\frac{\partial y}{\partial s}(a,b) \right\rangle \Delta s\, \Delta t \right| \\ &= \left| \frac{\partial x}{\partial s}(a,b)\frac{\partial y}{\partial t}(a,b) - \frac{\partial x}{\partial t}(a,b)\frac{\partial y}{\partial s}(a,b) \right| \Delta s\, \Delta t.\end{aligned}$$

Therefore, as the number of subdivisions increases without bound in each direction, Δs and Δt both go to zero, and we have

$$dA = \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right| ds\, dt. \tag{11.9}$$

Equation (11.9) hence determines the general change of variable formula in a double integral, and we can now say that

$$\iint_T f(x,y)\, dA = \iint_R f(x,y)\, dy\, dx = \iint_{T'} f(x(s,t), y(s,t)) \left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right| ds\, dt.$$

The quantity

$$\left| \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right|$$

is called the *Jacobian*, and we denote the Jacobian using the shorthand notation

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = \left|\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right|.$$ [11]

To summarize, the preceding change of variable formula that we have derived now follows.

Change of Variables in a Double Integral. Suppose a change of variables $x = x(s,t)$ and $y = y(s,t)$ transforms a closed and bounded region R in the st-plane into a closed and bounded region R' in the xy-plane. Under modest conditions (that are studied in advanced calculus), it follows that

$$\iint_{R'} f(x,y)\,dA = \iint_{R} f(x(s,t),y(s,t))\left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt.$$

Activity 11.30.

Find the Jacobian when changing from rectangular to polar coordinates. That is, for the transformation given by $x = r\cos(\theta)$, $y = r\sin(\theta)$, determine a simplified expression for the quantity

$$\left|\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right|.$$

What do you observe about your result? How is this connected to our earlier work with double integrals in polar coordinates?

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Given a particular double integral, it is natural to ask, "how can we find a useful change of variables?" There are two general factors to consider: if the integrand is particularly difficult, we might choose a change of variables that would make the integrand easier; or, given a complicated region of integration, we might choose a change of variables that transforms the region of integration into one that has a simpler form. These ideas are illustrated in the next activities.

Activity 11.31.

Consider the problem of finding the area of the region D' defined by the ellipse $x^2 + \frac{y^2}{4} = 1$. Here we will make a change of variables so that the pre-image of the domain is a circle.

(a) Let $x(s,t) = s$ and $y(s,t) = 2t$. Explain why the pre-image of the original ellipse (which lies in the xy plane) is the circle $s^2 + t^2 = 1$ in the st-plane.

[11] If you are familiar with determinants of matrices, we can can also represent $\frac{\partial(x,y)}{\partial(s,t)}$ as the determinant of a 2×2 matrix

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix}.$$

(b) Recall that the area of the ellipse D' is determined by the double integral $\iint_{D'} 1\,dA$. Explain why

$$\iint_{D'} 1\,dA = \iint_{D} 2\,ds\,dt$$

where D is the disk bounded by the circle $s^2 + t^2 = 1$. In particular, explain the source of the "2" in the st integral.

(c) Without evaluating any of the integrals present, explain why the area of the original elliptical region D' is 2π.

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Activity 11.32.

Let D' be the region in the xy-plane bounded by the lines $y = 0$, $x = 0$, and $x + y = 1$. We will evaluate the double integral

$$\iint_{D'} \sqrt{x+y}(x-y)^2\,dA \tag{11.10}$$

with a change of variables.

(a) Sketch the region D' in the xy plane.

(b) We would like to make a substitution that makes the integrand easier to antidifferentiate. Let $s = x+y$ and $t = x-y$. Explain why this should make antidifferentiation easier by making the corresponding substitutions and writing the new integrand in terms of s and t.

(c) Solve the equations $s = x + y$ and $t = x - y$ for x and y. (Doing so determines the standard form of the transformation, since we will have x as a function of s and t, and y as a function of s and t.)

(d) To actually execute this change of variables, we need to know the st-region D that corresponds to the xy-region D'.

i. What st equation corresponds to the xy equation $x + y = 1$?
ii. What st equation corresponds to the xy equation $x = 0$?
iii. What st equation corresponds to the xy equation $y = 0$?
iv. Sketch the st region D that corresponds to the xy domain D'.

(e) Make the change of variables indicated by $s = x+y$ and $t = x-y$ in the double integral (11.10) and set up an iterated integral in st variables whose value is the original given double integral. Finally, evaluate the iterated integral.

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Change of Variables in a Triple Integral

Given a function $f = f(x, y, z)$ over a region S' in $\mathbb{R}^3$, similar arguments can be used to show that a change of variables $x = x(s,t,u)$, $y = y(s,t,u)$, and $z = z(s,t,u)$ in a triple integral results in the equality

$$\iiint_{S'} f(x,y,z)\,dV = \iiint_{S} f(x(s,t,u), y(s,t,u), z(s,t,u)) \left|\frac{\partial(x,y,z)}{\partial(s,t,u)}\right| ds\,dt\,du,$$

where

$$\frac{\partial(x,y,z)}{\partial(s,t,u)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} \end{vmatrix}.$$

In expanded form,

$$\frac{\partial(x,y,z)}{\partial(s,t,u)} = \frac{\partial x}{\partial s}\left[\frac{\partial y}{\partial t}\frac{\partial z}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial z}{\partial t}\right] - \frac{\partial x}{\partial t}\left[\frac{\partial y}{\partial s}\frac{\partial z}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial z}{\partial s}\right] + \frac{\partial x}{\partial u}\left[\frac{\partial y}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial z}{\partial s}\right].$$

The expression $\left|\frac{\partial(x,y,z)}{\partial(s,t,u)}\right|$ is again called the Jacobian.

Summary

- If an integral is described in terms of one set of variables, we may write that set of variables in terms of another set of the same number of variables. If the new variables are chosen appropriately, the transformed integral may be easier to evaluate.
- The Jacobian is a scalar function that relates the area or volume element in one coordinate system to the corresponding element in a new system determined by a change of variables.

Exercises

1. Let D' be the region in the xy plane that is the parallelogram with vertices $(3,3)$, $(4,5)$, $(5,4)$, and $(6,6)$.

(a) Sketch and label the region D' in the xy plane.

(b) Consider the integral $\iint_{D'} (x+y)\,dA$. Explain why this integral would be difficult to set up as an iterated integral.

(c) Let a change of variables be given by $x = 2u + v$, $y = u + 2v$. Using substitution or elimination, solve this system of equations for u and v in terms of x and y.

(d) Use your work in (c) to find the pre-image, D, which lies in the uv plane, of the originally given region D', which lies in the xy plane. For instance, what uv point corresponds to $(3, 3)$ in the xy plane?

(e) Use the change of variables in (c) and your other work to write a new iterated integral in u and v that is equivalent to the original xy integral $\iint_{D'} (x + y)\, dA$.

(f) Finally, evaluate the uv integral, and write a sentence to explain why the change of variables made the integration easier.

2. Consider the change of variables

$$x(\rho, \theta, \phi) = \rho \sin(\phi) \cos(\theta) \quad y(\rho, \theta, \phi) = \rho \sin(\phi) \sin(\theta) \quad z(\rho, \theta, \phi) = \rho \cos(\phi),$$

which is the transformation from spherical coordinates to rectangular coordinates. Determine the Jacobian of the transformation. How is the result connected to our earlier work with iterated integrals in spherical coordinates?

3. In this problem, our goal is to find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(a) Set up an iterated integral in rectangular coordinates whose value is the volume of the ellipsoid. Do so by using symmetry and taking 8 times the volume of the ellipsoid in the first octant where x, y, and z are all nonnegative.

(b) Explain why it makes sense to use the substitution $x = as$, $y = bt$, and $z = cu$ in order to make the region of integration simpler.

(c) Compute the Jacobian of the transformation given in (b).

(d) Execute the given change of variables and set up the corresponding new iterated integral in s, t, and u.

(e) Explain why this new integral is better, but is still difficult to evaluate. What additional change of variables would make the resulting integral easier to evaluate?

(f) Convert the integral from (d) to a new integral in spherical coordinates.

(g) Finally, evaluate the iterated integral in (f) and hence determine the volume of the ellipsoid.