Chapter 2

Logical Reasoning

2.1 Statements and Logical Operators

Beginning Activity 1 (Compound Statements)
Mathematicians often develop ways to construct new mathematical objects from existing mathematical objects. It is possible to form new statements from existing statements by connecting the statements with words such as "and" and "or" or by negating the statement. A **logical operator** (or **connective**) on mathematical statements is a word or combination of words that combines one or more mathematical statements to make a new mathematical statement. A **compound statement** is a statement that contains one or more operators. Because some operators are used so frequently in logic and mathematics, we give them names and use special symbols to represent them.

- The **conjunction** of the statements P and Q is the statement "P **and** Q" and its denoted by $P \wedge Q$. The statement $P \wedge Q$ is true only when both P and Q are true.
- The **disjunction** of the statements P and Q is the statement "P **or** Q" and its denoted by $P \vee Q$. The statement $P \vee Q$ is true only when at least one of P or Q is true.
- The **negation** of the statement P is the statement "**not** P" and is denoted by $\neg P$. The negation of P is true only when P is false, and $\neg P$ is false only when P is true.
- The **implication** or **conditional** is the statement "**If** P **then** Q" and is denoted by $P \to Q$. The statement $P \to Q$ is often read as "P **implies** Q,"

and we have seen in Section 1.1 that $P \to Q$ is false only when P is true and Q is false.

Some comments about the disjunction.
It is important to understand the use of the operator "or." In mathematics, we use the **"inclusive or"** unless stated otherwise. This means that $P \vee Q$ is true when both P and Q are true and also when only one of them is true. That is, $P \vee Q$ is true when at least one of P or Q is true, or $P \vee Q$ is false only when both P and Q are false.

A different use of the word "or" is the **"exclusive or."** For the exclusive or, the resulting statement is false when both statements are true. That is, "P exclusive or Q" is true only when exactly one of P or Q is true. In everyday life, we often use the exclusive or. When someone says, "At the intersection, turn left or go straight," this person is using the exclusive or.

Some comments about the negation. Although the statement, $\neg P$, can be read as "It is not the case that P," there are often better ways to say or write this in English. For example, we would usually say (or write):

- The negation of the statement, "391 is prime" is "391 is not prime."
- The negation of the statement, "$12 < 9$" is "$12 \geq 9$."

1. For the statements

$$P\colon \text{15 is odd} \qquad Q\colon \text{15 is prime}$$

write each of the following statements as English sentences and determine whether they are true or false. Notice that P is true and Q is false.

(**a**) $P \wedge Q$. (**b**) $P \vee Q$. (**c**) $P \wedge \neg Q$. (**d**) $\neg P \vee \neg Q$.

2. For the statements

$$P\colon \text{15 is odd} \qquad R\colon 15 < 17$$

write each of the following statements in symbolic form using the operators $\wedge$, $\vee$, and $\neg$.

(**a**) $15 \geq 17$.

(**b**) 15 is odd or $15 \geq 17$.

(**c**) 15 is even or $15 < 17$.

(**d**) 15 is odd and $15 \geq 17$.

Beginning Activity 2 (Truth Values of Statements)
We will use the following two statements for all of this activity:

- P is the statement "It is raining."
- Q is the statement "Daisy is playing golf."

In each of the following four parts, a truth value will be assigned to statements P and Q. For example, in Question (1), we will assume that each statement is true. In Question (2), we will assume that P is true and Q is false. In each part, determine the truth value of each of the following statements:

(**a**) $(P \wedge Q)$ It is raining and Daisy is playing golf.

(**b**) $(P \vee Q)$ It is raining or Daisy is playing golf.

(**c**) $(P \to Q)$ If it is raining, then Daisy is playing golf.

(**d**) $(\neg P)$ It is not raining.

Which of the four statements [(a) through (d)] are true and which are false in each of the following four situations?

1. When P is true (it is raining) and Q is true (Daisy is playing golf).
2. When P is true (it is raining) and Q is false (Daisy is not playing golf).
3. When P is false (it is not raining) and Q is true (Daisy is playing golf).
4. When P is false (it is not raining) and Q is false (Daisy is not playing golf).

In the beginning activities for this section, we learned about compound statements and their truth values. This information can be summarized with the following truth tables:

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Rather than memorizing the truth tables, for many people it is easier to remember the rules summarized in Table 2.1.

Operator	**Symbolic Form**	**Summary of Truth Values**
Conjunction	$P \wedge Q$	True only when both P and Q are true
Disjunction	$P \vee Q$	False only when both P and Q are false
Negation	$\neg P$	Opposite truth value of P
Conditional	$P \to Q$	False only when P is true and Q is false

Table 2.1: Truth Values for Common Connectives

Other Forms of Conditional Statements

Conditional statements are extremely important in mathematics because almost all mathematical theorems are (or can be) stated as a conditional statement in the following form:

If "certain conditions are met," then "something happens."

It is imperative that all students studying mathematics thoroughly understand the meaning of a conditional statement and the truth table for a conditional statement.

We also need to be aware that in the English language, there are other ways for expressing the conditional statement $P \to Q$ other than "If P, then Q." Following are some common ways to express the conditional statement $P \to Q$ in the English language:

- If P, then Q.
- P implies Q.
- P only if Q.
- Q if P.
- Whenever P is true, Q is true.
- Q is true whenever P is true.
- Q is necessary for P. (This means that if P is true, then Q is necessarily true.)
- P is sufficient for Q. (This means that if you want Q to be true, it is sufficient to show that P is true.)

In all of these cases, P is the **hypothesis** of the conditional statement and Q is the **conclusion** of the conditional statement.

Progress Check 2.1 (The "Only If" Statement)
Recall that a quadrilateral is a four-sided polygon. Let S represent the following true conditional statement:

If a quadrilateral is a square, then it is a rectangle.

Write this conditional statement in English using

1. the word "whenever"

2. the phrase "only if"

3. the phrase "is necessary for"

4. the phrase "is sufficient for"

Constructing Truth Tables

Truth tables for compound statements can be constructed by using the truth tables for the basic connectives. To illustrate this, we will construct a truth table for $(P \wedge \neg Q) \to R$. The first step is to determine the number of rows needed.

- For a truth table with two different simple statements, four rows are needed since there are four different combinations of truth values for the two statements. We should be consistent with how we set up the rows. The way we will do it in this text is to label the rows for the first statement with (T, T, F, F) and the rows for the second statement with (T, F, T, F). All truth tables in the text have this scheme.

- For a truth table with three different simple statements, eight rows are needed since there are eight different combinations of truth values for the three statements. Our standard scheme for this type of truth table is shown in Table 2.2.

The next step is to determine the columns to be used. One way to do this is to work backward from the form of the given statement. For $(P \wedge \neg Q) \to R$, the last step is to deal with the conditional operator ($\to$). To do this, we need to know the truth values of $(P \wedge \neg Q)$ and R. To determine the truth values for $(P \wedge \neg Q)$, we need to apply the rules for the conjunction operator ($\wedge$) and we need to know the truth values for P and $\neg Q$.

Table 2.2 is a completed truth table for $(P \wedge \neg Q) \to R$ with the step numbers indicated at the bottom of each column. The step numbers correspond to the order in which the columns were completed.

P	Q	R	$\neg Q$	$P \wedge \neg Q$	$(P \wedge \neg Q) \to R$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T
1	1	1	2	3	4

Table 2.2: Truth Table for $(P \wedge \neg Q) \to R$

- When completing the column for $P \wedge \neg Q$, remember that the only time the conjunction is true is when both P and $\neg Q$ are true.
- When completing the column for $(P \wedge \neg Q) \to R$, remember that the only time the conditional statement is false is when the hypothesis $(P \wedge \neg Q)$ is true and the conclusion, R, is false.

The last column entered is the truth table for the statement $(P \wedge \neg Q) \to R$ using the setup in the first three columns.

Progress Check 2.2 (Constructing Truth Tables)
Construct a truth table for each of the following statements:

1. $P \wedge \neg Q$

2. $\neg (P \wedge Q)$

3. $\neg P \wedge \neg Q$

4. $\neg P \vee \neg Q$

Do any of these statements have the same truth table?

The Biconditional Statement

Some mathematical results are stated in the form "P if and only if Q" or "P is necessary and sufficient for Q." An example would be, "A triangle is equilateral if and only if its three interior angles are congruent." The symbolic form for the biconditional statement "P if and only if Q" is $P \leftrightarrow Q$. In order to determine a truth table for a biconditional statement, it is instructive to look carefully at the form of the phrase "P if and only if Q." The word "and" suggests that this statement is a conjunction. Actually it is a conjunction of the statements "P if Q" and "P only if Q." The symbolic form of this conjunction is $[(Q \to P) \wedge (P \to Q)]$.

Progress Check 2.3 (The Truth Table for the Biconditional Statement)
Complete a truth table for $[(Q \to P) \wedge (P \to Q)]$. Use the following columns: P, Q, $Q \to P$, $P \to Q$, and $[(Q \to P) \wedge (P \to Q)]$. The last column of this table will be the truth table for $P \leftrightarrow Q$.

Other Forms of the Biconditional Statement

As with the conditional statement, there are some common ways to express the biconditional statement, $P \leftrightarrow Q$, in the English language. For example,

- P if and only if Q.
- P implies Q and Q implies P.
- P is necessary and sufficient for Q.

Tautologies and Contradictions

Definition. A **tautology** is a compound statement S that is true for all possible combinations of truth values of the component statements that are part of S. A **contradiction** is a compound statement that is false for all possible combinations of truth values of the component statements that are part of S.

That is, a tautology is necessarily true in all circumstances, and a contradiction is necessarily false in all circumstances.

Progress Check 2.4 (Tautologies and Contradictions) For statements P and Q:

1. Use a truth table to show that $(P \vee \neg P)$ is a tautology.
2. Use a truth table to show that $(P \wedge \neg P)$ is a contradiction.
3. Use a truth table to determine if $P \to (P \vee Q)$ is a tautology, a contradiction, or neither.

Exercises for Section 2.1

⋆ **1.** Suppose that Daisy says, "If it does not rain, then I will play golf." Later in the day you come to know that it did rain but Daisy still played golf. Was Daisy's statement true or false? Support your conclusion.

⋆ **2.** Suppose that P and Q are statements for which $P \to Q$ is true and for which $\neg Q$ is true. What conclusion (if any) can be made about the truth value of each of the following statements?

(**a**) P (**b**) $P \wedge Q$ (**c**) $P \vee Q$

3. Suppose that P and Q are statements for which $P \to Q$ is false. What conclusion (if any) can be made about the truth value of each of the following statements?

(**a**) $\neg P \to Q$ (**b**) $Q \to P$ (**c**) $P \vee Q$

4. Suppose that P and Q are statements for which Q is false and $\neg P \to Q$ is true (and it is not known if R is true or false). What conclusion (if any) can be made about the truth value of each of the following statements?

(**a**) $\neg Q \to P$

(**b**) P

⋆ (**c**) $P \wedge R$

(**d**) $R \to \neg P$

⋆ **5.** Construct a truth table for each of the following statements:

(**a**) $P \to Q$

(**b**) $Q \to P$

(**c**) $\neg P \to \neg Q$

(**d**) $\neg Q \to \neg P$

Do any of these statements have the same truth table?

6. Construct a truth table for each of the following statements:

(**a**) $P \vee \neg Q$

(**b**) $\neg (P \vee Q)$

(**c**) $\neg P \vee \neg Q$

(**d**) $\neg P \wedge \neg Q$

Do any of these statements have the same truth table?

⋆ **7.** Construct truth tables for $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$. What do you observe?

8. Suppose each of the following statements is true.

- Laura is in the seventh grade.
- Laura got an A on the mathematics test or Sarah got an A on the mathematics test.
- If Sarah got an A on the mathematics test, then Laura is not in the seventh grade.

If possible, determine the truth value of each of the following statements. Carefully explain your reasoning.

(**a**) Laura got an A on the mathematics test.

(**b**) Sarah got an A on the mathematics test.

(**c**) Either Laura or Sarah did not get an A on the mathematics test.

9. Let P stand for "the integer x is even," and let Q stand for "x^2 is even." Express the conditional statement $P \to Q$ in English using

(**a**) The "if then" form of the conditional statement

(**b**) The word "implies"

⋆ **(c)** The "only if" form of the conditional statement

⋆ **(d)** The phrase "is necessary for"

(e) The phrase "is sufficient for"

10. Repeat Exercise (9) for the conditional statement $Q \to P$.

⋆ **11.** For statements P and Q, use truth tables to determine if each of the following statements is a tautology, a contradiction, or neither.

(**a**) $\neg Q \vee (P \to Q)$.

(**b**) $Q \wedge (P \wedge \neg Q)$.

(**c**) $(Q \wedge P) \wedge (P \to \neg Q)$.

(**d**) $\neg Q \to (P \wedge \neg P)$.

12. For statements P, Q, and R:

(**a**) Show that $[(P \to Q) \wedge P] \to Q$ is a tautology. **Note**: In symbolic logic, this is an important logical argument form called **modus ponens**.

(**b**) Show that $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$ is a tautology. **Note**: In symbolic logic, this is an important logical argument form called **syllogism**.

Explorations and Activities

13. Working with Conditional Statements. Complete the following table:

English Form	**Hypothesis**	**Conclusion**	**Symbolic Form**
If P, then Q.	P	Q	$P \to Q$
Q only if P.	Q	P	$Q \to P$
P is necessary for Q.			
P is sufficient for Q.			
Q is necessary for P.			
P implies Q.			
P only if Q.			
P if Q.			
If Q then P.			
If $\neg Q$, then $\neg P$.			
If P, then $Q \wedge R$.			
If $P \vee Q$, then R.			

14. Working with Truth Values of Statements. Suppose that P and Q are true statements, that U and V are false statements, and that W is a statement and it is not known if W is true or false.

Which of the following statements are true, which are false, and for which statements is it not possible to determine if it is true or false? Justify your conclusions.

(a) $(P \vee Q) \vee (U \wedge W)$

(b) $P \wedge (Q \to W)$

(c) $P \wedge (W \to Q)$

(d) $W \to (P \wedge U)$

(e) $W \to (P \wedge \neg U)$

(f) $(\neg P \vee \neg U) \wedge (Q \vee \neg V)$

(g) $(P \wedge \neg V) \wedge (U \vee W)$

(h) $(P \vee \neg Q) \to (U \wedge W)$

(i) $(P \vee W) \to (U \wedge W)$

(j) $(U \wedge \neg V) \to (P \wedge W)$

2.2 Logically Equivalent Statements

Beginning Activity 1 (Logically Equivalent Statements)
In Exercises (5) and (6) from Section 2.1, we observed situations where two different statements have the same truth tables. Basically, this means these statements are equivalent, and we make the following definition:

> **Definition**. Two expressions are **logically equivalent** provided that they have the same truth value for all possible combinations of truth values for all variables appearing in the two expressions. In this case, we write $X \equiv Y$ and say that X and Y are logically equivalent.

1. Complete truth tables for $\neg (P \wedge Q)$ and $\neg P \vee \neg Q$.

2. Are the expressions $\neg (P \wedge Q)$ and $\neg P \vee \neg Q$ logically equivalent?

3. Suppose that the statement "I will play golf and I will mow the lawn" is false. Then its negation is true. Write the negation of this statement in the form of a disjunction. Does this make sense?

Sometimes we actually use logical reasoning in our everyday living! Perhaps you can imagine a parent making the following two statements.

Statement 1	If you do not clean your room, then you cannot watch TV.
Statement 2	You clean your room or you cannot watch TV.

4. Let P be "you do not clean your room," and let Q be "you cannot watch TV." Use these to translate Statement 1 and Statement 2 into symbolic forms.

5. Construct a truth table for each of the expressions you determined in Part (4). Are the expressions logically equivalent?

6. Assume that Statement 1 and Statement 2 are false. In this case, what is the truth value of P and what is the truth value of Q? Now, write a true statement in symbolic form that is a conjunction and involves P and Q.

7. Write a truth table for the (conjunction) statement in Part (6) and compare it to a truth table for $\neg(P \to Q)$. What do you observe?

Beginning Activity 2 (**Converse and Contrapositive**)
We now define two important conditional statements that are associated with a given conditional statement.

Definition. If P and Q are statements, then

- The **converse** of the conditional statement $P \to Q$ is the conditional statement $Q \to P$.
- The **contrapositive** of the conditional statement $P \to Q$ is the conditional statement $\neg Q \to \neg P$.

1. For the following, the variable x represents a real number. Label each of the following statements as true or false.

(**a**) If $x = 3$, then $x^2 = 9$.
(**b**) If $x^2 = 9$, then $x = 3$.
(**c**) If $x^2 \neq 9$, then $x \neq 3$.
(**d**) If $x \neq 3$, then $x^2 \neq 9$.

2. Which statement in the list of conditional statements in Part (1) is the converse of Statement (1a)? Which is the contrapositive of Statement (1a)?

3. Complete appropriate truth tables to show that

- $P \to Q$ is logically equivalent to its contrapositive $\neg Q \to \neg P$.

- $P \to Q$ is not logically equivalent to its converse $Q \to P$.

In Beginning Activity 1, we introduced the concept of logically equivalent expressions and the notation $X \equiv Y$ to indicate that statements X and Y are logically equivalent. The following theorem gives two important logical equivalencies. They are sometimes referred to as **De Morgan's Laws**.

Theorem 2.5 (De Morgan's Laws)
For statements P and Q,

- *The statement $\neg (P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$. This can be written as $\neg (P \wedge Q) \equiv \neg P \vee \neg Q$.*
- *The statement $\neg (P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$. This can be written as $\neg (P \vee Q) \equiv \neg P \wedge \neg Q$.*

The first equivalency in Theorem 2.5 was established in Beginning Activity 1. Table 2.3 establishes the second equivalency.

P	Q	$P \vee Q$	$\neg (P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Table 2.3: Truth Table for One of De Morgan's Laws

It is possible to develop and state several different logical equivalencies at this time. However, we will restrict ourselves to what are considered to be some of the most important ones. Since many mathematical statements are written in the form of conditional statements, logical equivalencies related to conditional statements are quite important.

Logical Equivalencies Related to Conditional Statements

The first two logical equivalencies in the following theorem were established in Beginning Activity 1, and the third logical equivalency was established in Beginning Activity 2.

Theorem 2.6. *For statements P and Q,*

1. *The conditional statement $P \to Q$ is logically equivalent to $\neg P \vee Q$.*
2. *The statement $\neg (P \to Q)$ is logically equivalent to $P \wedge \neg Q$.*
3. *The conditional statement $P \to Q$ is logically equivalent to its contrapositive $\neg Q \to \neg P$.*

The Negation of a Conditional Statement

The logical equivalency $\neg (P \to Q) \equiv P \wedge \neg Q$ is interesting because it shows us that **the negation of a conditional statement is not another conditional statement.** The negation of a conditional statement can be written in the form of a conjunction. So what does it mean to say that the conditional statement

If you do not clean your room, then you cannot watch TV,

is false? To answer this, we can use the logical equivalency $\neg (P \to Q) \equiv P \wedge \neg Q$. The idea is that if $P \to Q$ is false, then its negation must be true. So the negation of this can be written as

You do not clean your room and you can watch TV.

For another example, consider the following conditional statement:

$$\text{If } -5 < -3, \text{ then } (-5)^2 < (-3)^2.$$

This conditional statement is false since its hypothesis is true and its conclusion is false. Consequently, its negation must be true. Its negation is not a conditional statement. The negation can be written in the form of a conjunction by using the logical equivalency $\neg (P \to Q) \equiv P \wedge \neg Q$. So, the negation can be written as follows:

$$-5 < -3 \text{ and } \neg \left((-5)^2 < (-3)^2 \right).$$

However, the second part of this conjunction can be written in a simpler manner by noting that "not less than" means the same thing as "greater than or equal to." So we use this to write the negation of the original conditional statement as follows:

$$-5 < -3 \text{ and } (-5)^2 \geq (-3)^2.$$

This conjunction is true since each of the individual statements in the conjunction is true.

Another Method of Establishing Logical Equivalencies

We have seen that it is often possible to use a truth table to establish a logical equivalency. However, it is also possible to prove a logical equivalency using a sequence of previously established logical equivalencies. For example,

- $P \to Q$ is logically equivalent to $\neg P \vee Q$. So
- $\neg (P \to Q)$ is logically equivalent to $\neg (\neg P \vee Q)$.
- Hence, by one of De Morgan's Laws (Theorem 2.5), $\neg (P \to Q)$ is logically equivalent to $\neg (\neg P) \wedge \neg Q$.
- This means that $\neg (P \to Q)$ is logically equivalent to $P \wedge \neg Q$.

The last step used the fact that $\neg (\neg P)$ is logically equivalent to P.

When proving theorems in mathematics, it is often important to be able to decide if two expressions are logically equivalent. Sometimes when we are attempting to prove a theorem, we may be unsuccessful in developing a proof for the original statement of the theorem. However, in some cases, it is possible to prove an equivalent statement. Knowing that the statements are equivalent tells us that if we prove one, then we have also proven the other. In fact, once we know the truth value of a statement, then we know the truth value of any other logically equivalent statement. This is illustrated in Progress Check 2.7.

Progress Check 2.7 (Working with a Logical Equivalency)
In Section 2.1, we constructed a truth table for $(P \wedge \neg Q) \to R$. See page 38.

1. Although it is possible to use truth tables to show that $P \to (Q \vee R)$ is logically equivalent to $(P \wedge \neg Q) \to R$, we instead use previously proven logical equivalencies to prove this logical equivalency. In this case, it may be easier to start working with $(P \wedge \neg Q) \to R$. Start with

   $$(P \wedge \neg Q) \to R \equiv \neg (P \wedge \neg Q) \vee R,$$

   which is justified by the logical equivalency established in Part (5) of Beginning Activity 1. Continue by using one of De Morgan's Laws on $\neg (P \wedge \neg Q)$.

2. Let a and b be integers. Suppose we are trying to prove the following:

- If 3 is a factor of $a \cdot b$, then 3 is a factor of a or 3 is a factor of b.

Explain why we will have proven this statement if we prove the following:

- If 3 is a factor of $a \cdot b$ and 3 is not a factor of a, then 3 is a factor of b.

As we will see, it is often difficult to construct a direct proof for a conditional statement of the form $P \to (Q \vee R)$. The logical equivalency in Progress Check 2.7 gives us another way to attempt to prove a statement of the form $P \to (Q \vee R)$. The advantage of the equivalent form, $(P \wedge \neg Q) \to R$, is that we have an additional assumption, $\neg Q$, in the hypothesis. This gives us more information with which to work.

Theorem 2.8 states some of the most frequently used logical equivalencies used when writing mathematical proofs.

Theorem 2.8 (**Important Logical Equivalencies**)
For statements P, Q, and R,

De Morgan's Laws	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
Conditional Statements	$P \to Q \equiv \neg Q \to \neg P$ *(contrapositive)*
	$P \to Q \equiv \neg P \vee Q$
	$\neg(P \to Q) \equiv P \wedge \neg Q$
Biconditional Statement	$(P \leftrightarrow Q) \equiv (P \to Q) \wedge (Q \to P)$
Double Negation	$\neg(\neg P) \equiv P$
Distributive Laws	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
Conditionals with Disjunctions	$P \to (Q \vee R) \equiv (P \wedge \neg Q) \to R$
	$(P \vee Q) \to R \equiv (P \to R) \wedge (Q \to R)$

We have already established many of these equivalencies. Others will be established in the exercises.

Exercises for Section 2.2

⋆ **1.** Write the converse and contrapositive of each of the following conditional statements.

(a) If $a = 5$, then $a^2 = 25$.

(b) If it is not raining, then Laura is playing golf.

(c) If $a \neq b$, then $a^4 \neq b^4$.

(d) If a is an odd integer, then $3a$ is an odd integer.

⋆ **2.** Write each of the conditional statements in Exercise (1) as a logically equivalent disjunction, and write the negation of each of the conditional statements in Exercise (1) as a conjunction.

3. Write a useful negation of each of the following statements. Do not leave a negation as a prefix of a statement. For example, we would write the negation of "I will play golf and I will mow the lawn" as "I will not play golf or I will not mow the lawn."

⋆ **(a)** We will win the first game and we will win the second game.

⋆ **(b)** They will lose the first game or they will lose the second game.

⋆ **(c)** If you mow the lawn, then I will pay you $20.

⋆ **(d)** If we do not win the first game, then we will not play a second game.

⋆ **(e)** I will wash the car or I will mow the lawn.

(f) If you graduate from college, then you will get a job or you will go to graduate school.

(g) If I play tennis, then I will wash the car or I will do the dishes.

(h) If you clean your room or do the dishes, then you can go to see a movie.

(i) It is warm outside and if it does not rain, then I will play golf.

4. Use truth tables to establish each of the following logical equivalencies dealing with biconditional statements:

(a) $(P \leftrightarrow Q) \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

(b) $(P \leftrightarrow Q) \equiv (Q \leftrightarrow P)$

(c) $(P \leftrightarrow Q) \equiv (\neg P \leftrightarrow \neg Q)$

5. Use truth tables to prove each of the **distributive laws** from Theorem 2.8.

(a) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

(b) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

6. Use truth tables to prove the following logical equivalency from Theorem 2.8:

$$[(P \vee Q) \to R] \equiv (P \to R) \wedge (Q \to R).$$

⋆ **7.** Use previously proven logical equivalencies to prove each of the following logical equivalencies about **conditionals with conjunctions:**

(**a**) $[(P \wedge Q) \to R] \equiv (P \to R) \vee (Q \to R)$

(**b**) $[P \to (Q \wedge R)] \equiv (P \to Q) \wedge (P \to R)$

8. If P and Q are statements, is the statement $(P \vee Q) \wedge \neg (P \wedge Q)$ logically equivalent to the statement $(P \wedge \neg Q) \vee (Q \wedge \neg P)$? Justify your conclusion.

9. Use previously proven logical equivalencies to prove each of the following logical equivalencies:

(**a**) $[\neg P \to (Q \wedge \neg Q)] \equiv P$

(**b**) $(P \leftrightarrow Q) \equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$

(**c**) $\neg (P \leftrightarrow Q) \equiv (P \wedge \neg Q) \vee (Q \wedge \neg P)$

(**d**) $(P \to Q) \to R \equiv (P \wedge \neg Q) \vee R$

(**e**) $(P \to Q) \to R \equiv (\neg P \to R) \wedge (Q \to R)$

(**f**) $[(P \wedge Q) \to (R \vee S)] \equiv [(\neg R \wedge \neg S) \to (\neg P \vee \neg Q)]$

(**g**) $[(P \wedge Q) \to (R \vee S)] \equiv [(P \wedge Q \wedge \neg R) \to S]$

(**h**) $[(P \wedge Q) \to (R \vee S)] \equiv (\neg P \vee \neg Q \vee R \vee S)$

(**i**) $\neg [(P \wedge Q) \to (R \vee S)] \equiv (P \wedge Q \wedge \neg R \wedge \neg S)$

⋆ **10.** Let a be a real number and let f be a real-valued function defined on an interval containing $x = a$. Consider the following conditional statement:

If f is differentiable at $x = a$, then f is continuous at $x = a$.

Which of the following statements have the same meaning as this conditional statement and which ones are negations of this conditional statement?

Note: This is not asking which statements are true and which are false. It is asking which statements are logically equivalent to the given statement. It might be helpful to let P represent the hypothesis of the given statement, Q represent the conclusion, and then determine a symbolic representation for each statement. Instead of using truth tables, try to use already established logical equivalencies to justify your conclusions.

(**a**) If f is continuous at $x = a$, then f is differentiable at $x = a$.

(**b**) If f is not differentiable at $x = a$, then f is not continuous at $x = a$.

(**c**) If f is not continuous at $x = a$, then f is not differentiable at $x = a$.

(**d**) f is not differentiable at $x = a$ or f is continuous at $x = a$.

(**e**) f is not continuous at $x = a$ or f is differentiable at $x = a$.

(**f**) f is differentiable at $x = a$ and f is not continuous at $x = a$.

11. Let a, b, and c be integers. Consider the following conditional statement:

If a divides bc, then a divides b or a divides c.

Which of the following statements have the same meaning as this conditional statement and which ones are negations of this conditional statement?

The note for Exercise (10) also applies to this exercise.

(**a**) If a divides b or a divides c, then a divides bc.

(**b**) If a does not divide b or a does not divide c, then a does not divide bc.

(**c**) a divides bc, a does not divide b, and a does not divide c.

⋆ (**d**) If a does not divide b and a does not divide c, then a does not divide bc.

(**e**) a does not divide bc or a divides b or a divides c.

(**f**) If a divides bc and a does not divide c, then a divides b.

(**g**) If a divides bc or a does not divide b, then a divides c.

12. Let x be a real number. Consider the following conditional statement:

$$\text{If } x^3 - x = 2x^2 + 6, \text{ then } x = -2 \text{ or } x = 3.$$

Which of the following statements have the same meaning as this conditional statement and which ones are negations of this conditional statement? Explain each conclusion. (See the note in the instructions for Exercise (10).)

(**a**) If $x \neq -2$ and $x \neq 3$, then $x^3 - x \neq 2x^2 + 6$.

(**b**) If $x = -2$ or $x = 3$, then $x^3 - x = 2x^2 + 6$.

(**c**) If $x \neq -2$ or $x \neq 3$, then $x^3 - x \neq 2x^2 + 6$.

(**d**) If $x^3 - x = 2x^2 + 6$ and $x \neq -2$, then $x = 3$.

(**e**) If $x^3 - x = 2x^2 + 6$ or $x \neq -2$, then $x = 3$.

(**f**) $x^3 - x = 2x^2 + 6$, $x \neq -2$, and $x \neq 3$.

(**g**) $x^3 - x \neq 2x^2 + 6$ or $x = -2$ or $x = 3$.

Explorations and Activities

13. Working with a Logical Equivalency. Suppose we are trying to prove the following for integers x and y:

If $x \cdot y$ is even, then x is even or y is even.

We notice that we can write this statement in the following symbolic form:

$$P \to (Q \vee R),$$

where P is "$x \cdot y$ is even," Q is "x is even," and R is "y is even."

(**a**) Write the symbolic form of the contrapositive of $P \to (Q \vee R)$. Then use one of De Morgan's Laws (Theorem 2.5) to rewrite the hypothesis of this conditional statement.

(**b**) Use the result from Part (13a) to explain why the given statement is logically equivalent to the following statement:

If x is odd and y is odd, then $x \cdot y$ is odd.

The two statements in this activity are logically equivalent. We now have the choice of proving either of these statements. If we prove one, we prove the other, or if we show one is false, the other is also false. The second statement is Theorem 1.8, which was proven in Section 1.2.

2.3 Open Sentences and Sets

Beginning Activity 1 (**Sets and Set Notation**)
The theory of sets is fundamental to mathematics in the sense that many areas of mathematics use set theory and its language and notation. This language and notation must be understood if we are to communicate effectively in mathematics. At this point, we will give a very brief introduction to some of the terminology used in set theory.

A **set** is a well-defined collection of objects that can be thought of as a single entity itself. For example, we can think of the set of integers that are greater than 4. Even though we cannot write down all the integers that are in this set, it is still a perfectly well-defined set. This means that if we are given a specific integer, we can tell whether or not it is in the set of integers greater than 4.

The most basic way of specifying the elements of a set is to list the elements of that set. This works well when the set contains only a small number of objects. The usual practice is to list these elements between braces. For example, if the set C consists of the integer solutions of the equation $x^2 = 9$, we would write

$$C = \{-3, 3\}.$$

For larger sets, it is sometimes inconvenient to list all of the elements of the set. In this case, we often list several of them and then write a series of three dots $(\ldots)$ to indicate that the pattern continues. For example,

$$D = \{1, 3, 5, 7, \ldots, 49\}$$

is the set of all odd natural numbers from 1 to 49, inclusive.

For some sets, it is not possible to list all of the elements of a set; we then list several of the elements in the set and again use a series of three dots $(\ldots)$ to indicate that the pattern continues. For example, if F is the set of all even natural numbers, we could write

$$F = \{2, 4, 6, \ldots\}.$$

We can also use the three dots before listing specific elements to indicate the pattern prior to those elements. For example, if E is the set of all even integers, we could write

$$E = \{\ldots - 6, -4, -2, 0, 2, 4, 6, \ldots\}.$$

Listing the elements of a set inside braces is called the **roster method** of specifying the elements of the set. We will learn other ways of specifying the elements of a set later in this section.

1. Use the roster method to specify the elements of each of the following sets:

(**a**) The set of real numbers that are solutions of the equation $x^2 - 5x = 0$.

(**b**) The set of natural numbers that are less than or equal to 10.

(**c**) The set of integers that are greater than -2.

2. Each of the following sets is defined using the roster method. For each set, determine four elements of the set other than the ones listed using the roster method.

$A = \{1, 4, 7, 10, \ldots\}$ $C = \{\ldots, -8, -6, -4, -2, 0\}$
$B = \{2, 4, 8, 16, \ldots\}$ $D = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$

Beginning Activity 2 (Variables)

Not all mathematical sentences are statements. For example, an equation such as

$$x^2 - 5 = 0$$

is not a statement. In this sentence, the symbol x is a **variable**. It represents a number that may be chosen from some specified set of numbers. The sentence (equation) becomes true or false when a specific number is substituted for x.

1. (**a**) Does the equation $x^2 - 25 = 0$ become a true statement if -5 is substituted for x?
   (**b**) Does the equation $x^2 - 25 = 0$ become a true statement if $\sqrt{5}$ is substituted for x?

Definition. A **variable** is a symbol representing an unspecified object that can be chosen from a given set U. The set U is called the **universal set for the variable.** It is the set of specified objects from which objects may be chosen to substitute for the variable. A **constant** is a specific member of the universal set.

Some sets that we will use frequently are the usual number systems. Recall that we use the symbol $\mathbb{R}$ to stand for the set of all **real numbers**, the symbol $\mathbb{Q}$ to stand for the set of all **rational numbers**, the symbol $\mathbb{Z}$ to stand for the set of all **integers**, and the symbol $\mathbb{N}$ to stand for the set of all **natural numbers**.

2. What real numbers will make the sentence "$y^2 - 2y - 15 = 0$" a true statement when substituted for y?

3. What natural numbers will make the sentence "$y^2 - 2y - 15 = 0$" a true statement when substituted for y?

4. What real numbers will make the sentence "$\sqrt{x}$ is a real number" a true statement when substituted for x?

5. What real numbers will make the sentence "$\sin^2 x + \cos^2 x = 1$" a true statement when substituted for x?

6. What natural numbers will make the sentence "$\sqrt{n}$ is a natural number" a true statement when substituted for n?

7. What real numbers will make the sentence

$$\int_0^y t^2 dt > 9$$

a true statement when substituted for y?

Some Set Notation

In Beginning Activity 1, we indicated that a set is a well-defined collection of objects that can be thought of as an entity itself.

- If A is a set and y is one of the objects in the set A, we write $y \in A$ and read this as "y is an element of A" or "y is a member of A." For example, if B is the set of all integers greater than 4, then we could write $5 \in B$ and $10 \in B$.
- If an object z is not an element in the set A, we write $z \notin A$ and read this as "z is not an element of A." For example, if B is the set of all integers greater than 4, then we could write $-2 \notin B$ and $4 \notin B$.

When working with a mathematical object, such as set, we need to define when two of these objects are equal. We are also often interested in whether or not one set is contained in another set.

Definition. Two sets, A and B, are **equal** when they have precisely the same elements. In this case, we write $A = B$. When the sets A and B are not equal, we write $A \neq B$.

The set A is a **subset** of a set B provided that each element of A is an element of B. In this case, we write $A \subseteq B$ and also say that A is **contained** in B. When A is not a subset of B, we write $A \not\subseteq B$.

Using these definitions, we see that for any set A, $A = A$ and since it is true that for each $x \in U$, if $x \in A$, then $x \in A$, we also see that $A \subseteq A$. That is, any set is equal to itself and any set is a subset of itself. For some specific examples, we see that:

- $\{1, 3, 5\} = \{3, 5, 1\}$
- $\{5, 10\} = \{5, 10, 5\}$
- $\{4, 8, 12\} = \{4, 4, 8, 12, 12\}$

- $\{5, 10\} \neq \{5, 10, 15\}$ but $\{5, 10\} \subseteq \{5, 10, 15\}$ and $\{5, 10, 15\} \not\subseteq \{5, 10\}$.

In each of the first three examples, the two sets have exactly the same elements even though the elements may be repeated or written in a different order.

Progress Check 2.9 (Set Notation)

1. Let $A = \{-4, -2, 0, 2, 4, 6, 8, \ldots\}$. Use correct set notation to indicate which of the following integers are in the set A and which are not in the set A. For example, we could write $6 \in A$ and $5 \notin A$.

$$10 \qquad 22 \qquad 13 \qquad -3 \qquad 0 \qquad -12$$

2. Use correct set notation (using $=$ or $\subseteq$) to indicate which of the following sets are equal and which are subsets of one of the other sets.

$$\begin{array}{ll} A = \{3, 6, 9\} & B = \{6, 9, 3, 6\} \\ C = \{3, 6, 9, \ldots\} & D = \{3, 6, 7, 9\} \\ E = \{9, 12, 15, \ldots\} & F = \{9, 7, 6, 2\} \end{array}$$

Variables and Open Sentences

As we have seen in the beginning activities, not all mathematical sentences are statements. This is often true if the sentence contains a variable. The following terminology is useful in working with sentences and statements.

Definition. An **open sentence** is a sentence $P(x_1, x_2, \ldots, x_n)$ involving variables $x_1, x_2, \ldots, x_n$ with the property that when specific values from the universal set are assigned to $x_1, x_2, \ldots, x_n$, then the resulting sentence is either true or false. That is, the resulting sentence is a statement. An open sentence is also called a **predicate** or a **propositional function**.

Notation: One reason an open sentence is sometimes called a propositional function is the fact that we use function notation $P(x_1, x_2, \ldots, x_n)$ for an open sentence in n variables. When there is only one variable, such as x, we write $P(x)$, which is read "P of x." In this notation, x represents an arbitrary element of the universal set, and $P(x)$ represents a sentence. When we substitute a specific element of the universal set for x, the resulting sentence becomes a statement. This is illustrated in the next example.

Example 2.10 (Open Sentences)
If the universal set is $\mathbb{R}$, then the sentence "$x^2 - 3x - 10 = 0$" is an open sentence involving the one variable x.

- If we substitute $x = 2$, we obtain the false statement "$2^2 - 3 \cdot 2 - 10 = 0$."
- If we substitute $x = 5$, we obtain the true statement "$5^2 - 3 \cdot 5 - 10 = 0$."

In this example, we can let $P(x)$ be the predicate "$x^2 - 3x - 10 = 0$ " and then say that $P(2)$ is false and $P(5)$ is true.

Using similar notation, we can let $Q(x, y)$ be the predicate "$x + 2y = 7$." This predicate involves two variables. Then,

- $Q(1, 1)$ is false since "$1 + 2 \cdot 1 = 7$" is false; and
- $Q(3, 2)$ is true since "$3 + 2 \cdot 2 = 7$" is true.

Progress Check 2.11 (Working with Open Sentences)

1. Assume the universal set for all variables is $\mathbb{Z}$ and let $P(x)$ be the predicate "$x^2 \leq 4$."
   (**a**) Find two values of x for which $P(x)$ is false.
   (**b**) Find two values of x for which $P(x)$ is true.
   (**c**) Use the roster method to specify the set of all x for which $P(x)$ is true.
2. Assume the universal set for all variables is $\mathbb{Z}$, and let $R(x, y, z)$ be the predicate "$x^2 + y^2 = z^2$."
   (**a**) Find two different examples for which $R(x, y, z)$ is false.
   (**b**) Find two different examples for which $R(x, y, z)$ is true.

Without using the term, Example 2.10 and Progress Check 2.11 (and Beginning Activity 2) dealt with a concept called the truth set of a predicate.

Definition. The **truth set of an open sentence with one variable** is the collection of objects in the universal set that can be substituted for the variable to make the predicate a true statement.

One part of elementary mathematics consists of learning how to solve equations. In more formal terms, the process of solving an equation is a way to determine the truth set for the equation, which is an open sentence. In this case, we often call the truth set the **solution set**. Following are three examples of truth sets.

- If the universal set is $\mathbb{R}$, then the truth set of the equation $3x - 8 = 10$ is the set $\{6\}$.
- If the universal set is $\mathbb{R}$, then the truth set of the equation "$x^2 - 3x - 10 = 0$" is $\{-2, 5\}$.
- If the universal set is $\mathbb{N}$, then the truth set of the open sentence "$\sqrt{n} \in \mathbb{N}$" is $\{1, 4, 9, 16, \ldots\}$.

Set Builder Notation

Sometimes it is not possible to list all the elements of a set. For example, if the universal set is $\mathbb{R}$, we cannot list all the elements of the truth set of "$x^2 < 4$." In this case, it is sometimes convenient to use the so-called **set builder notation** in which the set is defined by stating a rule that all elements of the set must satisfy. If $P(x)$ is a predicate in the variable x, then the notation

$$\{x \in U \mid P(x)\}$$

stands for the set of all elements x in the universal set U for which $P(x)$ is true. If it is clear what set is being used for the universal set, this notation is sometimes shortened to $\{x \mid P(x)\}$. This is usually read as "the set of all x such that $P(x)$." The vertical bar stands for the phrase "such that." Some writers will use a colon (:) instead of the vertical bar.

For a non-mathematical example, P could be the property that a college student is a mathematics major. Then $\{x \mid P(x)\}$ denotes the set of all college students who are mathematics majors. This could be written as

$$\{x \mid x \text{ is a college student who is a mathematics major}\}.$$

Example 2.12 (Truth Sets)
Assume the universal set is $\mathbb{R}$ and $P(x)$ is "$x^2 < 4$." We can describe the truth set of $P(x)$ as the set of all real numbers whose square is less than 4. We can also use set builder notation to write the truth set of $P(x)$ as

$$\left\{x \in \mathbb{R} \mid x^2 < 4\right\}.$$

However, if we solve the inequality $x^2 < 4$, we obtain $-2 < x < 2$. So we could also write the truth set as

$$\{x \in \mathbb{R} \mid -2 < x < 2\}.$$

We could read this as the set of all real numbers that are greater than -2 and less than 2. We can also write

$$\left\{x \in \mathbb{R} \mid x^2 < 4\right\} = \{x \in \mathbb{R} \mid -2 < x < 2\}.$$

Progress Check 2.13 (Working with Truth Sets)
Let $P(x)$ be the predicate "$x^2 \le 9$."

1. If the universal set is $\mathbb{R}$, describe the truth set of $P(x)$ using English and write the truth set of $P(x)$ using set builder notation.
2. If the universal set is $\mathbb{Z}$, then what is the truth set of $P(x)$? Describe this set using English and then use the roster method to specify all the elements of this truth set.
3. Are the truth sets in Parts (1) and (2) equal? Explain.

So far, our standard form for set builder notation has been $\{x \in U \mid P(x)\}$. It is sometimes possible to modify this form and put the predicate first. For example, the set

$$A = \{3n + 1 \mid n \in \mathbb{N}\}$$

describes the set of all natural numbers of the form $3n+1$ for some natural number. By substituting 1, 2, 3, 4, and so on, for n, we can use the roster method to write

$$A = \{3n + 1 \mid n \in \mathbb{N}\} = \{4, 7, 10, 13, \ldots\}.$$

We can sometimes "reverse this process" by starting with a set specified by the roster method and then writing the same set using set builder notation.

Example 2.14 (Set Builder Notation)
Let $B = \{\ldots, -11, -7, -3, 1, 5, 9, 13, \ldots\}$. The key to writing this set using set builder notation is to recognize the pattern involved. We see that once we have an integer in B, we can obtain another integer in B by adding 4. This suggests that the predicate we will use will involve multiplying by 4.

Since it is usually easier to work with positive numbers, we notice that $1 \in B$ and $5 \in B$. Notice that

$$1 = 4 \cdot 0 + 1 \qquad \text{and} \qquad 5 = 4 \cdot 1 + 1.$$

This suggests that we might try $\{4n + 1 \mid n \in \mathbb{Z}\}$. In fact, by trying other integers for n, we can see that

$$B = \{\ldots, -11, -7, -3, 1, 5, 9, 13, \ldots\} = \{4n + 1 \mid n \in \mathbb{Z}\}.$$

Progress Check 2.15 (Set Builder Notation)
Each of the following sets is defined using the roster method.

$$A = \{1, 5, 9, 13, \ldots\} \qquad C = \left\{\sqrt{2}, \left(\sqrt{2}\right)^3, \left(\sqrt{2}\right)^5, \ldots\right\}$$

$$B = \{\ldots, -8, -6, -4, -2, 0\} \qquad D = \{1, 3, 9, 27, \ldots\}$$

1. Determine four elements of each set other than the ones listed using the roster method.
2. Use set builder notation to describe each set.

The Empty Set

When a set contains no elements, we say that the set is the **empty set**. For example, the set of all rational numbers that are solutions of the equation $x^2 = -2$ is the empty set since this equation has no solutions that are rational numbers.

In mathematics, the empty set is usually designated by the symbol $\emptyset$. We usually read the symbol $\emptyset$ as "the empty set" or "the null set." (The symbol $\emptyset$ is actually the next to last letter in the Danish-Norwegian alphabet.)

When the Truth Set Is the Universal Set

The truth set of a predicate can be the universal set. For example, if the universal set is the set of real numbers $\mathbb{R}$, then the truth set of the predicate "$x + 0 = x$" is $\mathbb{R}$.

Notice that the sentence "$x + 0 = x$" has not been quantified and a particular element of the universal set has not been substituted for the variable x. Even though the truth set for this sentence is the universal set, we will adopt the convention that unless the quantifier is stated explicitly, we will consider the sentence to be a predicate or open sentence. So, with this convention, if the universal set is $\mathbb{R}$, then

- $x + 0 = x$ is a predicate;
- For each real number x, $(x + 0 = x)$ is a statement.

Exercises for Section 2.3

⋆ **1.** Use the roster method to specify the elements in each of the following sets and then write a sentence in English describing the set.

(a) $\left\{x \in \mathbb{R} \mid 2x^2 + 3x - 2 = 0\right\}$

(b) $\left\{x \in \mathbb{Z} \mid 2x^2 + 3x - 2 = 0\right\}$

(c) $\left\{x \in \mathbb{Z} \mid x^2 < 25\right\}$

(d) $\left\{x \in \mathbb{N} \mid x^2 < 25\right\}$

(e) $\{y \in \mathbb{Q} \mid |y - 2| = 2.5\}$

(f) $\{y \in \mathbb{Z} \mid |y - 2| \leq 2.5\}$

⋆ **2.** Each of the following sets is defined using the roster method.

$A = \{1, 4, 9, 16, 25, \ldots\}$

$B = \left\{\ldots, -\pi^4, -\pi^3, -\pi^2, -\pi, -1\right\}$

$C = \{3, 9, 15, 21, 27, \ldots\}$

$D = \{0, 4, 8, \ldots, 96, 100\}$

(a) Determine four elements of each set other than the ones listed using the roster method.

(b) Use set builder notation to describe each set.

⋆ **3.** Let $A = \left\{x \in \mathbb{R} \mid x\,(x + 2)^2 \left(x - \frac{3}{2}\right) = 0\right\}$. Which of the following sets are equal to the set A and which are subsets of A?

(a) $\{-2, 0, 3\}$

(b) $\left\{\frac{3}{2}, -2, 0\right\}$

(c) $\left\{-2, -2, 0, \frac{3}{2}\right\}$

(d) $\left\{-2, \frac{3}{2}\right\}$

4. Use the roster method to specify the truth set for each of the following open sentences. The universal set for each open sentence is the set of integers $\mathbb{Z}$.

⋆ **(a)** $n + 7 = 4$.

⋆ **(b)** $n^2 = 64$.

(c) $\sqrt{n} \in \mathbb{N}$ and n is less than 50.

(d) n is an odd integer that is greater than 2 and less than 14.

(e) n is an even integer that is greater than 10.

5. Use set builder notation to specify the following sets:

⋆ **(a)** The set of all integers greater than or equal to 5.

(b) The set of all even integers.

⋆ **(c)** The set of all positive rational numbers.

(d) The set of all real numbers greater than 1 and less than 7.

⋆ **(e)** The set of all real numbers whose square is greater than 10.

6. For each of the following sets, use English to describe the set and when appropriate, use the roster method to specify all of the elements of the set.

(a) $\{x \in \mathbb{R} \mid -3 \le x \le 5\}$

(b) $\{x \in \mathbb{Z} \mid -3 \le x \le 5\}$

(c) $\left\{x \in \mathbb{R} \mid x^2 = 16\right\}$

(d) $\left\{x \in \mathbb{R} \mid x^2 + 16 = 0\right\}$

(e) $\{x \in \mathbb{Z} \mid x \text{ is odd }\}$

(f) $\{x \in \mathbb{R} \mid 3x - 4 \ge 17\}$

Explorations and Activities

7. **Closure Explorations**. In Section 1.1, we studied some of the closure properties of the standard number systems. (See page 11.) We can extend this idea to other sets of numbers. So we say that:

- A set A of numbers is **closed under addition** provided that whenever x and y are in the set A, $x + y$ is in the set A.

- A set A of numbers is **closed under multiplication** provided that whenever x and y are are in the set A, $x \cdot y$ is in the set A.
- A set A of numbers is **closed under subtraction** provided that whenever x and y are are in the set A, $x - y$ is in the set A.

For each of the following sets, make a conjecture about whether or not it is closed under addition and whether or not it is closed under multiplication. In some cases, you may be able to find a counterexample that will prove the set is not closed under one of these operations.

(**a**) The set of all odd natural numbers

(**b**) The set of all even integers

(**c**) $A = \{1, 4, 7, 10, 13, \ldots\}$

(**d**) $B = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$

(**e**) $C = \{3n + 1 \mid n \in \mathbb{Z}\}$

(**f**) $D = \left\{ \frac{1}{2^n} \middle| n \in \mathbb{N} \right\}$

2.4 Quantifiers and Negations

Beginning Activity 1 (**An Introduction to Quantifiers**)
We have seen that one way to create a statement from an open sentence is to substitute a specific element from the universal set for each variable in the open sentence. Another way is to make some claim about the truth set of the open sentence. This is often done by using a quantifier. For example, if the universal set is $\mathbb{R}$, then the following sentence is a statement.

For each real number x, $x^2 > 0$.

The phrase "For each real number x" is said to **quantify the variable** that follows it in the sense that the sentence is claiming that something is true for all real numbers. So this sentence is a statement (which happens to be false).

> **Definition**. The phrase "for every" (or its equivalents) is called a **universal quantifier**. The phrase "there exists" (or its equivalents) is called an **existential quantifier**. The symbol $\forall$ is used to denote a universal quantifier, and the symbol $\exists$ is used to denote an existential quantifier.

Using this notation, the statement "For each real number x, $x^2 > 0$" could be written in symbolic form as: $(\forall x \in \mathbb{R}) \left(x^2 > 0\right)$. The following is an example of a statement involving an existential quantifier.

There exists an integer x such that $3x - 2 = 0$.

This could be written in symbolic form as

$$(\exists x \in \mathbb{Z})\,(3x - 2 = 0)\,.$$

This statement is false because there are no integers that are solutions of the linear equation $3x - 2 = 0$. Table 2.4 summarizes the facts about the two types of quantifiers.

A statement involving	**Often has the form**	**The statement is true provided that**
A universal quantifier: $(\forall x, P(x))$	"For every x, $P(x)$," where $P(x)$ is a predicate.	Every value of x in the universal set makes $P(x)$ true.
An existential quantifier: $(\exists x, P(x))$	"There exists an x such that $P(x)$," where $P(x)$ is a predicate.	There is at least one value of x in the universal set that makes $P(x)$ true.

Table 2.4: Properties of Quantifiers

In effect, the table indicates that the universally quantified statement is true provided that the truth set of the predicate equals the universal set, and the existentially quantified statement is true provided that the truth set of the predicate contains at least one element.

Each of the following sentences is a statement or an open sentence. Assume that the universal set for each variable in these sentences is the set of all real numbers. If a sentence is an open sentence (predicate), determine its truth set. If a sentence is a statement, determine whether it is true or false.

1. $(\forall a \in \mathbb{R})\,(a + 0 = a)$.
2. $3x - 5 = 9$.
3. $\sqrt{x} \in \mathbb{R}$.
4. $\sin(2x) = 2(\sin x)(\cos x)$.
5. $(\forall x \in \mathbb{R})\,(\sin(2x) = 2(\sin x)(\cos x))$.

6. $(\exists x \in \mathbb{R})\left(x^2 + 1 = 0\right)$.
7. $(\forall x \in \mathbb{R})\left(x^3 \geq x^2\right)$.
8. $x^2 + 1 = 0$.
9. If $x^2 \geq 1$, then $x \geq 1$.
10. $(\forall x \in \mathbb{R})\left(\text{If } x^2 \geq 1, \text{ then } x \geq 1\right)$.

Beginning Activity 2 (**Attempting to Negate Quantified Statements**)

1. Consider the following statement written in symbolic form:
   $(\forall x \in \mathbb{Z})$ (x is a multiple of 2).
   (**a**) Write this statement as an English sentence.
   (**b**) Is the statement true or false? Why?
   (**c**) How would you write the negation of this statement as an English sentence?
   (**d**) If possible, write your negation of this statement from part (2) symbolically (using a quantifier).
2. Consider the following statement written in symbolic form:
   $(\exists x \in \mathbb{Z})\left(x^3 > 0\right)$.
   (**a**) Write this statement as an English sentence.
   (**b**) Is the statement true or false? Why?
   (**c**) How would you write the negation of this statement as an English sentence?
   (**d**) If possible, write your negation of this statement from part (2) symbolically (using a quantifier).

We introduced the concepts of open sentences and quantifiers in Section 2.3. Review the definitions given on pages 54, 58, and 63.

Forms of Quantified Statements in English

There are many ways to write statements involving quantifiers in English. In some cases, the quantifiers are not apparent, and this often happens with conditional statements. The following examples illustrate these points. Each example contains a quantified statement written in symbolic form followed by several ways to write the statement in English.

1. $(\forall x \in \mathbb{R})\left(x^2 > 0\right)$.

- For each real number x, $x^2 > 0$.
- The square of every real number is greater than 0.
- The square of a real number is greater than 0.
- If $x \in \mathbb{R}$, then $x^2 > 0$.

In the second to the last example, the quantifier is not stated explicitly. Care must be taken when reading this because it really does say the same thing as the previous examples. The last example illustrates the fact that conditional statements often contain a "hidden" universal quantifier.

If the universal set is $\mathbb{R}$, then the truth set of the open sentence $x^2 > 0$ is the set of all nonzero real numbers. That is, the truth set is

$$\{x \in \mathbb{R} \mid x \neq 0\}.$$

So the preceding statements are false. For the conditional statement, the example using $x = 0$ produces a true hypothesis and a false conclusion. This is a **counterexample** that shows that the statement with a universal quantifier is false.

2. $(\exists x \in \mathbb{R})\left(x^2 = 5\right)$.

- There exists a real number x such that $x^2 = 5$.
- $x^2 = 5$ for some real number x.
- There is a real number whose square equals 5.

The second example is usually not used since it is not considered good writing practice to start a sentence with a mathematical symbol.

If the universal set is $\mathbb{R}$, then the truth set of the predicate "$x^2 = 5$" is $\left\{-\sqrt{5},\ \sqrt{5}\right\}$. So these are all true statements.

Negations of Quantified Statements

In Beginning Activity 1, we wrote negations of some quantified statements. This is a very important mathematical activity. As we will see in future sections, it is sometimes just as important to be able to describe when some object does not satisfy a certain property as it is to describe when the object satisfies the property. Our next task is to learn how to write negations of quantified statements in a useful English form.

We first look at the negation of a statement involving a universal quantifier. The general form for such a statement can be written as $(\forall x \in U)\,(P(x))$, where $P(x)$ is an open sentence and U is the universal set for the variable x. When we write

$$\neg\,(\forall x \in U)\,[P\,(x)]\,,$$

we are asserting that the statement $(\forall x \in U)\,[P(x)]$ is false. This is equivalent to saying that the truth set of the open sentence $P(x)$ is not the universal set. That is, there exists an element x in the universal set U such that $P(x)$ is false. This in turn means that there exists an element x in U such that $\neg P(x)$ is true, which is equivalent to saying that $(\exists x \in U)\,[\neg P(x)]$ is true. This explains why the following result is true:

$$\neg\,(\forall x \in U)\,[P(x)] \equiv (\exists x \in U)\,[\neg P(x)]\,.$$

Similarly, when we write

$$\neg\,(\exists x \in U)\,[P(x)]\,,$$

we are asserting that the statement $(\exists x \in U)\,[P(x)]$ is false. This is equivalent to saying that the truth set of the open sentence $P(x)$ is the empty set. That is, there is no element x in the universal set U such that $P(x)$ is true. This in turn means that for each element x in U, $\neg P(x)$ is true, and this is equivalent to saying that $(\forall x \in U)\,[\neg P(x)]$ is true. This explains why the following result is true:

$$\neg\,(\exists x \in U)\,[P(x)] \equiv (\forall x \in U)\,[\neg P(x)]\,.$$

We summarize these results in the following theorem.

Theorem 2.16. *For any open sentence $P(x)$,*

$$\neg\,(\forall x \in U)\,[P(x)] \equiv (\exists x \in U)\,[\neg P(x)]\,, \textit{ and}$$
$$\neg\,(\exists x \in U)\,[P(x)] \equiv (\forall x \in U)\,[\neg P(x)]\,.$$

Example 2.17 (Negations of Quantified Statements)
Consider the following statement: $(\forall x \in \mathbb{R})\left(x^3 \geq x^2\right)$.

We can write this statement as an English sentence in several ways. Following are two different ways to do so.

- For each real number x, $x^3 \geq x^2$.
- If x is a real number, then x^3 is greater than or equal to x^2.

The second statement shows that in a conditional statement, there is often a hidden universal quantifier. This statement is false since there are real numbers x for which x^3 is not greater than or equal to x^2. For example, we could use $x = -1$ or $x = \frac{1}{2}$.

This means that the negation must be true. We can form the negation as follows:

$$\neg (\forall x \in \mathbb{R})\left(x^3 \geq x^2\right) \equiv (\exists x \in \mathbb{R}) \neg \left(x^3 \geq x^2\right).$$

In most cases, we want to write this negation in a way that does not use the negation symbol. In this case, we can now write the open sentence $\neg\left(x^3 \geq x^2\right)$ as $\left(x^3 < x^2\right)$. (That is, the negation of "is greater than or equal to" is "is less than.") So we obtain the following:

$$\neg (\forall x \in \mathbb{R})\left(x^3 \geq x^2\right) \equiv (\exists x \in \mathbb{R})\left(x^3 < x^2\right).$$

The statement $(\exists x \in \mathbb{R})\left(x^3 < x^2\right)$ could be written in English as follows:

- There exists a real number x such that $x^3 < x^2$.
- There exists an x such that x is a real number and $x^3 < x^2$.

Progress Check 2.18 (Negating Quantified Statements)
For each of the following statements

- Write the statement in the form of an English sentence that does not use the symbols for quantifiers.
- Write the negation of the statement in a symbolic form that does not use the negation symbol.
- Write the negation of the statement in the form of an English sentence that does not use the symbols for quantifiers.

1. $(\forall a \in \mathbb{R})\,(a + 0 = a)$.
2. $(\forall x \in \mathbb{R})\,[\sin(2x) = 2(\sin x)(\cos x)]$.
3. $(\forall x \in \mathbb{R})\,\left(\tan^2 x + 1 = \sec^2 x\right)$.
4. $(\exists x \in \mathbb{Q})\,\left(x^2 - 3x - 7 = 0\right)$.
5. $(\exists x \in \mathbb{R})\,\left(x^2 + 1 = 0\right)$.

Counterexamples and Negations of Conditional Statements

The real number $x = -1$ in the previous example was used to show that the statement $(\forall x \in \mathbb{R})\,\left(x^3 \geq x^2\right)$ is false. This is called a **counterexample** to the statement. In general, a **counterexample** to a statement of the form $(\forall x)\,[P(x)]$ is an object a in the universal set U for which $P(a)$ is false. It is an example that proves that $(\forall x)\,[P(x)]$ is a false statement, and hence its negation, $(\exists x)\,[\neg P(x)]$, is a true statement.

In the preceding example, we also wrote the universally quantified statement as a conditional statement. The number $x = -1$ is a counterexample for the statement

If x is a real number, then x^3 is greater than or equal to x^2.

So the number -1 is an example that makes the hypothesis of the conditional statement true and the conclusion false. Remember that a conditional statement often contains a "hidden" universal quantifier. Also, recall that in Section 2.2 we saw that the negation of the conditional statement "If P then Q" is the statement "P and not Q." Symbolically, this can be written as follows:

$$\neg\,(P \to Q) \equiv P \wedge \neg Q.$$

So when we specifically include the universal quantifier, the symbolic form of the negation of a conditional statement is

$$\begin{aligned}\neg\,(\forall x \in U)\,[P(x) \to Q(x)] &\equiv (\exists x \in U)\,\neg\,[P(x) \to Q(x)]\\ &\equiv (\exists x \in U)\,[P(x) \wedge \neg Q(x)]\,.\end{aligned}$$

That is,

$$\neg\,(\forall x \in U)\,[P(x) \to Q(x)] \equiv (\exists x \in U)\,[P(x) \wedge \neg Q(x)]\,.$$

Progress Check 2.19 (Using Counterexamples)
Use counterexamples to explain why each of the following statements is false.

1. For each integer n, $\left(n^2 + n + 1\right)$ is a prime number.
2. For each real number x, if x is positive, then $2x^2 > x$.

Quantifiers in Definitions

Definitions of terms in mathematics often involve quantifiers. These definitions are often given in a form that does not use the symbols for quantifiers. Not only is it important to know a definition, it is also important to be able to write a negation of the definition. This will be illustrated with the definition of what it means to say that a natural number is a perfect square.

Definition. A natural number n is a **perfect square** provided that there exists a natural number k such that $n = k^2$.

This definition can be written in symbolic form using appropriate quantifiers as follows:

A natural number n is a **perfect square** provided $(\exists k \in \mathbb{N})\left(n = k^2\right)$.

We frequently use the following steps to gain a better understanding of a definition.

1. Examples of natural numbers that are perfect squares are 1, 4, 9, and 81 since $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, and $81 = 9^2$.
2. Examples of natural numbers that are not perfect squares are 2, 5, 10, and 50.
3. This definition gives two "conditions." One is that the natural number n is a perfect square and the other is that there exists a natural number k such that $n = k^2$. The definition states that these mean the same thing. So when we say that a natural number n is not a perfect square, we need to negate the condition that there exists a natural number k such that $n = k^2$. We can use the symbolic form to do this.

$$\neg\,(\exists k \in \mathbb{N})\left(n = k^2\right) \equiv (\forall k \in \mathbb{N})\left(n \neq k^2\right)$$

Notice that instead of writing $\neg\left(n = k^2\right)$, we used the equivalent form of $\left(n \neq k^2\right)$. This will be easier to translate into an English sentence. So we can write,

> A natural number n is not a perfect square provided that for every natural number k, $n \neq k^2$.

The preceding method illustrates a good method for trying to understand a new definition. Most textbooks will simply define a concept and leave it to the reader to do the preceding steps. Frequently, it is not sufficient just to read a definition and expect to understand the new term. We must provide examples that satisfy the definition, as well as examples that do not satisfy the definition, and we must be able to write a coherent negation of the definition.

Progress Check 2.20 (Multiples of Three)

Definition. An integer n is a **multiple of 3** provided that there exists an integer k such that $n = 3k$.

1. Write this definition in symbolic form using quantifiers by completing the following:

   An integer n is a multiple of 3 provided that ....

2. Give several examples of integers (including negative integers) that are multiples of 3.

3. Give several examples of integers (including negative integers) that are not multiples of 3.

4. Use the symbolic form of the definition of a multiple of 3 to complete the following sentence: "An integer n is not a multiple of 3 provided that ...."

5. Without using the symbols for quantifiers, complete the following sentence: "An integer n is not a multiple of 3 provided that ...."

Statements with More than One Quantifier

When a predicate contains more than one variable, each variable must be quantified to create a statement. For example, assume the universal set is the set of integers, $\mathbb{Z}$, and let $P(x, y)$ be the predicate, "$x + y = 0$." We can create a statement from this predicate in several ways.

1. $(\forall x \in \mathbb{Z})\,(\forall y \in \mathbb{Z})\,(x + y = 0)$.

   We could read this as, "For all integers x and y, $x + y = 0$." This is a false statement since it is possible to find two integers whose sum is not zero $(2 + 3 \neq 0)$.

2. $(\forall x \in \mathbb{Z})\,(\exists y \in \mathbb{Z})\,(x + y = 0)$.

   We could read this as, "For every integer x, there exists an integer y such that $x + y = 0$." This is a true statement.

3. $(\exists x \in \mathbb{Z})\,(\forall y \in \mathbb{Z})\,(x + y = 0)$.

   We could read this as, "There exists an integer x such that for each integer y, $x + y = 0$." This is a false statement since there is no integer whose sum with each integer is zero.

4. $(\exists x \in \mathbb{Z})\,(\exists y \in \mathbb{Z})\,(x + y = 0)$.

   We could read this as, "There exist integers x and y such that $x + y = 0$." This is a true statement. For example, $2 + (-2) = 0$.

When we negate a statement with more than one quantifier, we consider each quantifier in turn and apply the appropriate part of Theorem 2.16. As an example, we will negate Statement (3) from the preceding list. The statement is

$$(\exists x \in \mathbb{Z})\,(\forall y \in \mathbb{Z})\,(x + y = 0)\,.$$

We first treat this as a statement in the following form: $(\exists x \in \mathbb{Z})\,(P(x))$ where $P(x)$ is the predicate $(\forall y \in \mathbb{Z})\,(x + y = 0)$. Using Theorem 2.16, we have

$$\neg\,(\exists x \in \mathbb{Z})\,(P(x)) \equiv (\forall x \in \mathbb{Z})\,(\neg P(x))\,.$$

Using Theorem 2.16 again, we obtain the following:

$$\begin{aligned} \neg P(x) &\equiv \neg\,(\forall y \in \mathbb{Z})\,(x + y = 0) \\ &\equiv (\exists y \in \mathbb{Z})\,\neg\,(x + y = 0) \\ &\equiv (\exists y \in \mathbb{Z})\,(x + y \neq 0)\,. \end{aligned}$$

Combining these two results, we obtain

$$\neg (\exists x \in \mathbb{Z}) (\forall y \in \mathbb{Z}) (x + y = 0) \equiv (\forall x \in \mathbb{Z}) (\exists y \in \mathbb{Z}) (x + y \neq 0).$$

The results are summarized in the following table.

	Symbolic Form	**English Form**
Statement	$(\exists x \in \mathbb{Z}) (\forall y \in \mathbb{Z}) (x + y = 0)$	There exists an integer x such that for each integer y, $x + y = 0$.
Negation	$(\forall x \in \mathbb{Z}) (\exists y \in \mathbb{Z}) (x + y \neq 0)$	For each integer x, there exists an integer y such that $x + y \neq 0$.

Since the given statement is false, its negation is true.

We can construct a similar table for each of the four statements. The next table shows Statement (2), which is true, and its negation, which is false.

	Symbolic Form	**English Form**
Statement	$(\forall x \in \mathbb{Z}) (\exists y \in \mathbb{Z}) (x + y = 0)$	For every integer x, there exists an integer y such that $x + y = 0$.
Negation	$(\exists x \in \mathbb{Z}) (\forall y \in \mathbb{Z}) (x + y \neq 0)$	There exists an integer x such that for every integer y, $x + y \neq 0$.

Progress Check 2.21 (Negating a Statement with Two Quantifiers)
Write the negation of the statement

$$(\forall x \in \mathbb{Z}) (\forall y \in \mathbb{Z}) (x + y = 0)$$

in symbolic form and as a sentence written in English.

Writing Guideline

Try to use English and minimize the use of cumbersome notation. Do not use the special symbols for quantifiers $\forall$ (for all), $\exists$ (there exists), $\ni$ (such that), or $\therefore$.

(therefore) in formal mathematical writing. It is often easier to write and usually easier to read, if the English words are used instead of the symbols. For example, why make the reader interpret

$$(\forall x \in \mathbb{R})\,(\exists y \in \mathbb{R})\,(x + y = 0)$$

when it is possible to write

For each real number x, there exists a real number y such that $x + y = 0$,

or, more succinctly (if appropriate),

Every real number has an additive inverse.

Exercises for Section 2.4

$\star$ **1.** For each of the following, write the statement as an English sentence and then explain why the statement is false.

(**a**) $(\exists x \in \mathbb{Q})\left(x^2 - 3x - 7 = 0\right)$.
(**b**) $(\exists x \in \mathbb{R})\left(x^2 + 1 = 0\right)$.
(**c**) $(\exists m \in \mathbb{N})\left(m^2 < 1\right)$.

2. For each of the following, use a counterexample to show that the statement is false. Then write the negation of the statement in English, without using symbols for quantifiers.

$\star$ **(a)** $(\forall m \in \mathbb{Z})\left(m^2 \text{ is even}\right)$.
$\star$ **(b)** $(\forall x \in \mathbb{R})\left(x^2 > 0\right)$.
(c) For each real number x, $\sqrt{x} \in \mathbb{R}$.
(d) $(\forall m \in \mathbb{Z})\left(\dfrac{m}{3} \in \mathbb{Z}\right)$.
(e) $(\forall a \in \mathbb{Z})\left(\sqrt{a^2} = a\right)$.
$\star$ **(f)** $(\forall x \in \mathbb{R})\left(\tan^2 x + 1 = \sec^2 x\right)$.

3. For each of the following statements

- Write the statement as an English sentence that does not use the symbols for quantifiers.

- Write the negation of the statement in symbolic form in which the negation symbol is not used.
- Write a useful negation of the statement in an English sentence that does not use the symbols for quantifiers.

⋆ **(a)** $(\exists x \in \mathbb{Q})\left(x > \sqrt{2}\right)$.

(b) $(\forall x \in \mathbb{Q})\left(x^2 - 2 \neq 0\right)$.

⋆ **(c)** $(\forall x \in \mathbb{Z})$ (x is even or x is odd).

(d) $(\exists x \in \mathbb{Q})\left(\sqrt{2} < x < \sqrt{3}\right)$. **Note**: The sentence "$\sqrt{2} < x < \sqrt{3}$" is actually a conjuction. It means $\sqrt{2} < x$ and $x < \sqrt{3}$.

⋆ **(e)** $(\forall x \in \mathbb{Z})$ (If x^2 is odd, then x is odd).

(f) $(\forall n \in \mathbb{N})$ [If n is a perfect square, then $(2^n - 1)$ is not a prime number].

(g) $(\forall n \in \mathbb{N})$ ($n^2 - n + 41$ is a prime number).

⋆ **(h)** $(\exists x \in \mathbb{R})\,(\cos(2x) = 2(\cos x))$.

4. Write each of the following statements as an English sentence that does not use the symbols for quantifiers.

⋆ **(a)** $(\exists m \in \mathbb{Z})\,(\exists n \in \mathbb{Z})\,(m > n)$

(b) $(\exists m \in \mathbb{Z})\,(\forall n \in \mathbb{Z})\,(m > n)$

(c) $(\forall m \in \mathbb{Z})\,(\exists n \in \mathbb{Z})\,(m > n)$

(d) $(\forall m \in \mathbb{Z})\,(\forall n \in \mathbb{Z})\,(m > n)$

⋆ **(e)** $(\exists n \in \mathbb{Z})\,(\forall m \in \mathbb{Z})\,\left(m^2 > n\right)$

(f) $(\forall n \in \mathbb{Z})\,(\exists m \in \mathbb{Z})\,\left(m^2 > n\right)$

⋆ **5.** Write the negation of each statement in Exercise (4) in symbolic form and as an English sentence that does not use the symbols for quantifiers.

⋆ **6.** Assume that the universal set is $\mathbb{Z}$. Consider the following sentence:

$$(\exists t \in \mathbb{Z})\,(t \cdot x = 20)\,.$$

(a) Explain why this sentence is an open sentence and not a statement.

(b) If 5 is substituted for x, is the resulting sentence a statement? If it is a statement, is the statement true or false?

(c) If 8 is substituted for x, is the resulting sentence a statement? If it is a statement, is the statement true or false?

(d) If -2 is substituted for x, is the resulting sentence a statement? If it is a statement, is the statement true or false?

(**e**) What is the truth set of the open sentence $(\exists t \in \mathbb{Z})\,(t \cdot x = 20)$?

7. Assume that the universal set is $\mathbb{R}$. Consider the following sentence:

$$(\exists t \in \mathbb{R})\,(t \cdot x = 20)\,.$$

(**a**) Explain why this sentence is an open sentence and not a statement.

(**b**) If 5 is substituted for x, is the resulting sentence a statement? If it is a statement, is the statement true or false?

(**c**) If π is substituted for x, is the resulting sentence a statement? If it is a statement, is the statement true or false?

(**d**) If 0 is substituted for x, is the resulting sentence a statement? If it is a statement, is the statement true or false?

(**e**) What is the truth set of the open sentence $(\exists t \in \mathbb{R})\,(t \cdot x = 20)$?

8. Let $\mathbb{Z}^*$ be the set of all nonzero integers.

(**a**) Use a counterexample to explain why the following statement is false:

For each $x \in \mathbb{Z}^*$, there exists a $y \in \mathbb{Z}^*$ such that $xy = 1$.

(**b**) Write the statement in part (a) in symbolic form using appropriate symbols for quantifiers.

(**c**) Write the negation of the statement in part (b) in symbolic form using appropriate symbols for quantifiers.

(**d**) Write the negation from part (c) in English without using the symbols for quantifiers.

9. An integer m is said to have the *divides property* provided that for all integers a and b, if m divides ab, then m divides a or m divides b.

(**a**) Using the symbols for quantifiers, write what it means to say that the integer m has the divides property.

(**b**) Using the symbols for quantifiers, write what it means to say that the integer m does not have the divides property.

(**c**) Write an English sentence stating what it means to say that the integer m does not have the divides property.

10. In calculus, we define a function f with domain $\mathbb{R}$ to be **strictly increasing** provided that for all real numbers x and y, $f(x) < f(y)$ whenever $x < y$. Complete each of the following sentences using the appropriate symbols for quantifiers:

* ⋆ **(a)** A function f with domain $\mathbb{R}$ is strictly increasing provided that ....
* **(b)** A function f with domain $\mathbb{R}$ is not strictly increasing provided that ....

Complete the following sentence in English without using symbols for quantifiers:

* **(c)** A function f with domain $\mathbb{R}$ is not strictly increasing provided that ....

11. In calculus, we define a function f to be **continuous** at a real number a provided that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

Note: The symbol ε is the lowercase Greek letter epsilon, and the symbol δ is the lowercase Greek letter delta.

Complete each of the following sentences using the appropriate symbols for quantifiers:

* **(a)** A function f is continuous at the real number a provided that ....
* **(b)** A function f is not continuous at the real number a provided that ....

Complete the following sentence in English without using symbols for quantifiers:

* **(c)** A function f is not continuous at the real number a provided that ....

12. The following exercises contain definitions or results from more advanced mathematics courses. Even though we may not understand all of the terms involved, it is still possible to recognize the structure of the given statements and write a meaningful negation of that statement.

* **(a)** In abstract algebra, an operation $*$ on a set A is called a **commutative operation** provided that for all $x, y \in A$, $x * y = y * x$. Carefully explain what it means to say that an operation $*$ on a set A is not a commutative operation.
* **(b)** In abstract algebra, a **ring** consists of a nonempty set R and two operations called addition and multiplication. A nonzero element a in a ring R is called a **zero divisor** provided that there exists a nonzero element b in R such that $ab = 0$ or $ba = 0$. Carefully explain what it means to say that a nonzero element a in a ring R is not a zero divisor.

(**c**) A set M of real numbers is called a **neighborhood** of a real number a provided that there exists a positive real number ε such that the open interval $(a - \varepsilon, a + \varepsilon)$ is contained in M. Carefully explain what it means to say that a set M is not a neighborhood of a real number a.

(**d**) In advanced calculus, a sequence of real numbers $(x_1, x_2, \ldots, x_k, \ldots)$ is called a **Cauchy sequence** provided that for each positive real number ε, there exists a natural number N such that for all $m, n \in \mathbb{N}$, if $m > N$ and $n > N$, then $|x_n - x_m| < \varepsilon$. Carefully explain what it means to say that the sequence of real numbers $(x_1, x_2, \ldots, x_k, \ldots)$ is not a Cauchy sequence.

Explorations and Activities

13. Prime Numbers. The following definition of a prime number is very important in many areas of mathematics. We will use this definition at various places in the text. It is introduced now as an example of how to work with a definition in mathematics.

> **Definition**. A natural number p is a **prime number** provided that it is greater than 1 and the only natural numbers that are factors of p are 1 and p. A natural number other than 1 that is not a prime number is a **composite number**. The number 1 is neither prime nor composite.

Using the definition of a prime number, we see that 2, 3, 5, and 7 are prime numbers. Also, 4 is a composite number since $4 = 2 \cdot 2$; 10 is a composite number since $10 = 2 \cdot 5$; and 60 is a composite number since $60 = 4 \cdot 15$.

(**a**) Give examples of four natural numbers other than 2, 3, 5, and 7 that are prime numbers.

(**b**) Explain why a natural number p that is greater than 1 is a prime number provided that

For all $d \in \mathbb{N}$, if d is a factor of p, then $d = 1$ or $d = p$.

(**c**) Give examples of four natural numbers that are composite numbers and explain why they are composite numbers.

(**d**) Write a useful description of what it means to say that a natural number is a composite number (other than saying that it is not prime).

14. Upper Bounds for Subsets of $\mathbb{R}$. Let A be a subset of the real numbers. A number b is called an **upper bound** for the set A provided that for each element x in A, $x \leq b$.

(**a**) Write this definition in symbolic form by completing the following:

Let A be a subset of the real numbers. A number b is called an upper bound for the set A provided that ....

(**b**) Give examples of three different upper bounds for the set $A = \{x \in \mathbb{R} \mid 1 \leq x \leq 3\}$.

(**c**) Does the set $B = \{x \in \mathbb{R} \mid x > 0\}$ have an upper bound? Explain.

(**d**) Give examples of three different real numbers that are not upper bounds for the set $A = \{x \in \mathbb{R} \mid 1 \leq x \leq 3\}$.

(**e**) Complete the following in symbolic form: "Let A be a subset of $\mathbb{R}$. A number b is not an upper bound for the set A provided that ...."

(**f**) Without using the symbols for quantifiers, complete the following sentence: "Let A be a subset of $\mathbb{R}$. A number b is not an upper bound for the set A provided that ...."

(**g**) Are your examples in Part (14d) consistent with your work in Part (14f)? Explain.

15. Least Upper Bound for a Subset of $\mathbb{R}$. In Exercise 14, we introduced the definition of an upper bound for a subset of the real numbers. Assume that we know this definition and that we know what it means to say that a number is not an upper bound for a subset of the real numbers.

Let A be a subset of $\mathbb{R}$. A real number α is the **least upper bound** for A provided that α is an upper bound for A, and if β is an upper bound for A, then $\alpha \leq \beta$.

Note: The symbol α is the lowercase Greek letter alpha, and the symbol β is the lowercase Greek letter beta.

If we define $P(x)$ to be "x is an upper bound for A," then we can write the definition for least upper bound as follows:

A real number α is the **least upper bound** for A provided that $P(\alpha) \wedge [(\forall \beta \in \mathbb{R})(P(\beta) \rightarrow (\alpha \leq \beta))]$.

(**a**) Why is a universal quantifier used for the real number β?

(**b**) Complete the following sentence in symbolic form: "A real number α is not the least upper bound for A provided that ...."

(**c**) Complete the following sentence as an English sentence: "A real number α is not the least upper bound for A provided that ...."

2.5 Chapter 2 Summary

Important Definitions

- Logically equivalent statements, page 43
- Converse of a conditional statement, page 44
- Contrapositive of a conditional statement, page 44
- Variable, page 54
- Universal set for a variable, page 54
- Constant, page 54
- Equal sets, page 55
- Predicate, page 56
- Open sentence, page 56
- Truth set of a predicate, page 58
- Universal quantifier, page 63
- Existential quantifier, page 63
- Empty set, page 60
- Counterexample, pages 66 and 69
- Perfect square, page 70
- Prime number, page 78
- Composite number, page 78

Important Theorems and Results

Theorem 2.8. Important Logical Equivalencies. *For statements P, Q, and R,*

De Morgan's Laws	$\neg (P \wedge Q) \equiv \neg P \vee \neg Q$ $\neg (P \vee Q) \equiv \neg P \wedge \neg Q$
Conditional Statements	$P \to Q \equiv \neg Q \to \neg P$ (contrapositive) $P \to Q \equiv \neg P \vee Q$ $\neg (P \to Q) \equiv P \wedge \neg Q$
Biconditional Statement	$(P \leftrightarrow Q) \equiv (P \to Q) \wedge (Q \to P)$
Double Negation	$\neg (\neg P) \equiv P$
Distributive Laws	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
Conditionals with Disjunctions	$P \to (Q \vee R) \equiv (P \wedge \neg Q) \to R$ $(P \vee Q) \to R \equiv (P \to R) \wedge (Q \to R)$

Theorem 2.16. Negations of Quantified Statements. *For any predicate $P(x)$,*

$$\neg (\forall x)\,[P(x)] \equiv (\exists x)\,[\neg P(x)]\text{, and}$$
$$\neg (\exists x)\,[P(x)] \equiv (\forall x)\,[\neg P(x)]\,.$$

Important Set Theory Notation

Notation	Description	Page
$y \in A$	y is an element of the set A.	55
$z \notin A$	z is not an element of the set A.	55
$\{\quad\}$	The roster method	53
$\{x \in U \mid P(x)\}$	Set builder notation	58
