Investigation 36

Games: NIM and the 15 Puzzle

Focus Questions

By the end of this investigation, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the investigation.

- How is the game of NIM related to group theory? What is a good strategy for playing NIM?
- What is the 15 Puzzle, and how can the symmetric groups tell us if a 15 Puzzle is solvable?

Games can be fun to play—and, as it turns out, to study. In fact, many games involve mathematical ideas or can be analyzed using mathematics. In this investigation, we will learn how group theory can be used to determine winning strategies in the game of NIM and to determine if a 15 Puzzle is solvable.

Preview Activity 36.1. Go to any online version of NIM and play the game a few times. Search for a winning strategy.

The Game of NIM

To play the game of NIM, one begins with a number of sets, or stacks, of objects. We can think of these stacks as piles of stones, as shown in Table 36.1. (Note that we have displayed the piles horizontally to save space.) In this example, the first pile has 6 stones, the second has 2, and the third has 3.

•	•	•	•	•	•
•	•				
•	•	•			

Table 36.1
A NIM game.

The game is played by two players alternating turns. At each turn, a player can take as many stones from a single pile as he or she wants, but the player must remove at least one stone. The object is to be the last player to remove stones.

The number of stones in each pile in a NIM game is an element in the set $\mathbb{W}$ of whole numbers. So we can think of a particular state of a NIM game with three piles as an element in the Cartesian product $\mathbb{W} \times \mathbb{W} \times \mathbb{W}$. Recall that $\mathbb{W}$ is not a group under standard addition of integers. Therefore, to relate this game to group theory, we will need to define a relevant operation under which $\mathbb{W}$ is a group. As we will see, using binary representations of whole numbers will help us to define such an operation.

To study binary representations, it will be helpful to first review how the decimal representation of a whole number works. Recall that we typically think of the digits of a whole number as representing place value. For example, in the integer 1234, the digit 4 is located in the ones place, the digit 3 is in the 10's place, 2 is in the 100's place, and 1 is in the 1000's place. In other words, the integer 1234 can also be represented by the sum

$$(1 \times 10^3) + (2 \times 10^2) + (3 \times 10^1) + (4 \times 10^0).$$

This is the *decimal representation* of the number 1234. In the decimal system, we add two whole numbers digit-by-digit, from right to left, reducing modulo 10 and carrying a 1 to the next digit whenever the sum of two digits is 10 or greater.

There is nothing particularly special about the base 10 used in the decimal representation of a number, other than the fact that it is convenient. After all, most people have 10 fingers and 10 toes, so a base 10 system is natural. However, we could just as easily replace the base 10 with any other base. In the *binary* system, we replace the base 10 with the base 2. There is an adjustment we must make, though. With the decimal system, each individual digit can be anywhere between 0 and 9 (because these integers are less than 10). With the binary system, we will only use the digits 0 and 1. For example, the binary number 10110 represents the decimal number $(1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (0 \times 2^0) = 22$. In fact, any whole number can be represented in binary format.

To represent a given whole number in the binary system, we first look for the highest power of 2 that is less than the number. Then we subtract this highest power of 2 and repeat the process with the difference. To illustrate, let's convert 219 to binary. First note that $2^7 = 128$ and $2^8 = 256$. So 2^7 is the highest power of 2 less than 219. Now $219 - 1 \times 2^7 = 91$, and so $219 = (1 \times 2^7) + 91$. Repeating the process with the integer 91, we note that the highest power of 2 in 91 is 2^6. Since $91 - 2^6 = 27$, we see that $219 = (1 \times 2^7) + (1 \times 2^6) + 27$. We then continue reducing the differences until we no longer have any powers of 2 remaining. This leaves us with

$$\begin{aligned} 219 = (1 \times 2^7) + (1 \times 2^6) + (0 \times 2^5) + (1 \times 2^4) \\ + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0). \end{aligned}$$

Therefore, the binary representation of 219 is 11011011. The standard sum of two whole numbers $a = a_n a_{n-1} a_{n-2} \cdots a_1 a_0$ and $b = b_n b_{n-1} b_{n-2} \cdots b_1 b_0$ in binary is similar to the decimal sum: we add digit-by-digit, from right to left, reducing modulo 2 and carrying a 1 to the next digit whenever the sum of two digits is 2 or greater.

Remember that $\mathbb{W}$ is not a group under standard addition of integers. If, however, we convert each whole number into its binary representation, then we can define a special operation under which $\mathbb{W}$ is a group. Let $x = x_n x_{n-1} \ldots x_1 x_0$ and $y = y_n y_{n-1} \ldots y_1 y_0$ be whole numbers in binary form. We can assume both integers have the same number of digits in their binary representations by simply appending zeros to the left end of one number if necessary. We define the "NIM sum" of x and y to be the binary number $x \oplus y = s_n s_{n-1} \ldots s_1 s_0$, where $s_i = (x_i + y_i) \pmod 2$. Note that the NIM sum of two binary numbers is the same as the normal binary sum, except that we don't allow carrying. For example, the NIM sum $101101 \oplus 00111$ is 101010. Of course, we can also add more than two numbers this way. For example, $10111 \oplus 1110 \oplus 111 = 11110$.

Activity 36.2. Let $\mathcal{B}$ be the set of binary representations of the whole numbers.

(a) Is the NIM sum operation $\oplus$ well-defined in $\mathcal{B}$? Explain.

(b) Is $\mathcal{B}$ closed under the NIM sum $\oplus$? Explain.

(c) Is there an identity element in $\mathcal{B}$ with respect to the NIM sum $\oplus$? If yes, what is the identity? If no, why not?

(d) Does $\mathcal{B}$ contain an inverse for each of its elements with respect to the NIM sum $\oplus$? If yes, what is the inverse of a given element? If no, why not?

(e) Is the NIM sum associative in $\mathcal{B}$? Prove your answer.

(f) What conclusion can we draw about $\mathcal{B}$?

Now that we have an operation under which the whole numbers in binary form are a group, we can also make a group out of $\mathcal{B}^n = \underbrace{\mathcal{B} \oplus \mathcal{B} \oplus \cdots \oplus \mathcal{B}}_{n \text{ factors}}$. The group $\mathcal{B}^n$ forms the playing field for all NIM games with n piles of stones—that is, each element in $\mathcal{B}^n$ corresponds to a particular stage in a NIM game. As such, we will call any element in $\mathcal{B}^n$ a *configuration*. If the i^{th} pile of stones contains N_i stones, then the NIM game has the configuration $(N_1, N_2, \ldots, N_n)$.

A *legal move* in a NIM game consists of removing some number of stones (at least one) from a single pile. Note that we can view a move as a configuration as well; in particular, the move that takes m stones from pile i can be thought of as the configuration $M = (0, 0, \ldots, 0, m, 0, \ldots, 0)$, with m in the i^{th} component. Since every element in $\mathcal{B}^n$ is its own inverse, the result of performing move M on configuration X is the configuration $X \oplus M$. For example, let $X = (011, 100, 001)$ be the configuration in $\mathcal{B}^3$ with 3 stones in the first pile, 4 in the second, and 1 in the third, as shown on the left in Table 36.2. Let $M = (010, 000, 000)$ be the move that takes 2 stones from the first pile. The result of applying the move M to the configuration X is the configuration $X \oplus M = (001, 100, 001)$, as shown on the right in Table 36.2.

Table 36.2
A NIM move.

In any NIM game, the last move will result in the configuration $(0, 0, \ldots, 0)$. This configuration has the special property that the NIM sum of all of the components is 0. In general, a configuration that satisfies this property is called an *even* configuration—that is, $(N_1, N_2, \ldots, N_n)$ is an *even* configuration if

$$\bigoplus_{i=1}^{n} N_i = 0.$$

Any other configuration is called an *odd* configuration. Note that every move is an odd configuration.

Activity 36.3. Let n be a positive integer, and let $\mathcal{E}^n$ be the subset of $\mathcal{B}^n$ consisting of the even configurations. Is $\mathcal{E}^n$ a subgroup of $\mathcal{B}^n$? Prove your answer.

We will now explore some strategy behind the game of NIM.

Activity 36.4.

(a) If $X \in \mathcal{E}^n$ is not the identity, how many nonzero components must X have?

(b) If $X \in \mathcal{E}^n$ is not the identity, how many nonzero components must its inverse have? Explain what this observation tells us about the possibility of winning a NIM game from a nonzero even configuration.

The result of Activity 36.4 reveals a strategy for playing NIM defensively. In particular, if we can always present our opponent with an even configuration, then he or she cannot win. The question now is how that can be done.

Activity 36.5. Let n be a positive integer, and let $X \in \mathcal{E}^n$. Determine and describe all moves M so that $(X \oplus M) \in \mathcal{E}^n$. Relate your answer to playing the game of NIM.

Activity 36.5 tells us that if we present our opponent with an even configuration, we will always be confronted with an odd configuration on our next turn. So the final question is whether we can convert an odd configuration into an even one. This is a bit more complicated, and so we will describe the process using a NIM game with three piles.

Let $X = (N_1, N_2, N_3)$ be an odd configuration in $\mathcal{B}^3$. We want to find a move $M = (M_1, M_2, M_3)$ with exactly one of M_1, M_2, M_3 nonzero so that $X \oplus M$ is even. (It is not true that the NIM sum of two odd configurations is even, and you should find a simple example to convince yourself of this.) Let $N_1 = a_m a_{m-1} \ldots a_1 a_0$, $N_2 = b_m b_{m-1} \ldots b_1 b_0$, and $N_3 = c_m c_{m-1} \ldots c_1 c_0$ (all in binary). Since $N_1 \oplus N_2 \oplus N_3 \neq 0$, there is some index i so that $a_i + b_i + c_i \equiv 1 \pmod 2$. Let k be the largest index for which this happens. At least one of a_k, b_k, c_k must be 1. Without loss of generality, assume $a_k = 1$. This means that $b_k + c_k \equiv 0 \pmod 2$. For $0 \leq i < k$, let

$$a'_i = \begin{cases} 0, & \text{if } (b_i + c_i) \equiv 0 \pmod 2 \\ 1, & \text{otherwise.} \end{cases}$$

So $a'_i + b_i + c_i \equiv 0 \pmod 2$ for $i < k$. Let M be the move that takes stones from pile 1 so that $N'_1 = a_m a_{m-1} \ldots a_{k+1} 0 a'_{k-1} a'_{k-2} \ldots a'_1 a'_0$ remain. The result of applying move M to X is the configuration $X' = (X \oplus M) = (N'_1, N_2, N_3)$. Recall that $(b_k + c_k) \equiv 0 \pmod 2$, and $(a_i + b_i + c_i) \equiv 0 \pmod 2$ if $i > k$ (by the definition of k). Since $a'_i + b_i + c_i \equiv 0 \pmod 2$ for $i < k$, we can therefore conclude that $N'_1 \oplus N_2 \oplus N_3 = 0$, and $X' \in \mathcal{E}^n$.

In terms of the NIM game, this result tells us that if we are confronted with an odd configuration, we can always change it to an even configuration.

Activity 36.6. Apply the algorithm provided above to find a move that converts the NIM configuration in Table 36.3 to an even configuration.

Based on our work up to this point, we have proved the following theorem (with $n = 3$ for part (iii), but you are asked to extend this to $\mathcal{B}^n$ for any n in Exercise (2)).

Theorem 36.7. *Let n be a positive integer.*

(i) *If $X \in \mathcal{E}^n$ is nonzero, then there is no move M so that $X \oplus M = 0$.*

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Table 36.3
A NIM configuration.

(ii) *If $X \in \mathcal{E}^n$ is nonzero, then $(X \oplus M) \notin \mathcal{E}^n$ for any move M.*

(iii) *If $X \in (\mathcal{B}^n \setminus \mathcal{E}^n)$, then there exists a move M so that $(X \oplus M) \in \mathcal{E}^n$.*

Interpreted in more natural language, Theorem 36.7 presents us with the following strategy for playing the NIM game:

(i) Since it is not possible to win from a nonzero even configuration, always present our opponent with an even configuration if possible.

(ii) If we can present our opponent with an even configuration, any move our opponent makes will present us with an odd configuration.

(iii) If we have an odd configuration at our turn, we can always turn it into an even configuration.

So our strategy is to always present our opponent with an even configuration. If the configuration is the 0 configuration, we have won. If not, then (ii) shows that our opponent cannot win because he or she will be forced to present us with an odd configuration. By part (iii), we can turn that odd configuration into an even configuration so that our opponent cannot win on the next turn. Since at least one stone is removed at each turn, our opponent will eventually have to present us with an odd configuration from which we can win.

One final note: if we are ever presented with even configuration, we cannot win the game unless our opponent makes a mistake. For this reason, it is always advantageous to be able to decide whether to move first or second after seeing the initial configuration.

We will illustrate the strategy described above with the game from Table 36.3.

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Table 36.4
Our first move (left), our opponent's move (middle), and the coup de gras (right).

As we argued before, if we move first, we should remove stones from pile 1 to leave exactly 2 stones as shown on the left of Table 36.4. Now, whatever move our opponent makes, we will be left with an odd configuration. Suppose our opponent's move leaves us with the configuration shown in the middle of Table 36.4. This configuration gives us $N_1 = 2 = 10$, $N_2 = 4 = 100$, and $N_3 = 4 = 100$. Now $10 \oplus 100 \oplus 100 = 010$, so we must change the 1 in the 2's place. We can do this by removing all of the stones from pile 1, leaving us with the NIM sum $100 \oplus 100 = 000$, which corresponds to the configuration on the right of Table 36.4. Whatever moves our opponent makes, we can now win the game by keeping both piles of the same size. This will always result in a 0 NIM sum. Thus, the game is ours.

The 15 Puzzle

Preview Activity 36.8. Go to any on-line version of the 15 Puzzle and play the game a few times. Find one that allows you to create your own 15 Puzzle. Are all 15 Puzzles solvable? If not, search for a pattern that determines which puzzles are solvable.

The classic 15 Puzzle was made famous in the 19th century by puzzleist Sam Loyd.* The game consists of a starting position, which is a 4×4 array of the integers between 1 and 15 along with a symbol # (which we interpret as a blank space), as shown in Table 36.5. We will call each entry of the array a *cell*.

2	9	7	3
10	15	12	8
1	4	#	14
6	13	5	11

Table 36.5
15 Puzzle: Configuration 1.

The game is played with one type of legal move: interchanging the blank cell with a cell either to the left or right or the cell above or below. (The children's game is usually made of sliding tiles that are numbered 1 to 15. The interchange mentioned here is done by sliding a tile to the empty cell.) In this example, we can interchange the blank with the 12, 4, 14 or 5. Interchanging the blank and the 5 leaves us with the configuration in Table 36.6. We can then interchange the blank with the 5, 13, or 11.

2	9	7	3
10	15	12	8
1	4	5	14
6	13	#	11

Table 36.6
15 Puzzle: Configuration 2.

The object of the game is to interchange the blank with other cells and transform the starting

*In 2006, Jerry Slocum and Dic Sonneveld published their book, *The 15 Puzzle* (Slocum Puzzle Foundation), in which they write, "Sam Loyd did not invent the 15 puzzle and had nothing to do with promoting or popularizing it. The puzzle craze that was created by the 15 Puzzle began in January 1880 in the US and in April in Europe. The craze ended by July 1880 and Sam Loyd's first article about the puzzle was not published until sixteen years later, January 1896. Loyd first claimed in 1891 that he invented the puzzle, and he continued until his death a 20 year campaign to falsely take credit for the puzzle. The actual inventor was Noyes Chapman, the Postmaster of Canastota, New York, and he applied for a patent in March 1880."

position to the standard position in Table 36.7. It is important to note that each interchange is reversible. So an equivalent game is to begin with the standard position and move to obtain a specified position. It is this latter version of the game that we will analyze.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	#

Table 36.7
15 Puzzle: Standard configuration.

Permutations and the 15 Puzzle

To analyze this game, we will construct a correspondence between possible configurations of the 4×4 array and elements in the symmetric group S_{16}. To do this, we will let the number 16 represent the blank cell. Recall that any permutation can be written as a product of transpositions. We will apply a transposition $(a\ b)$ to a configuration A by interchanging the labels of the cells in *positions* a and b in A. (Note that this may be different than interchanging the cells *labeled* a and b.) For example, the transposition $(12\ 16)$ applied to the standard configuration I will correspond to Table 36.8 in which the cells in positions 12 and 16 have been interchanged.

1	2	3	4
5	6	7	8
9	10	11	#
13	14	15	12

Table 36.8
The transposition $(12\ 16)$ applied to I.

To apply a permutation $\sigma \in S_{16}$ to a configuration A, we can first write σ as a product $\tau_m \tau_{m-1} \cdots \tau_2 \tau_1$ of transpositions and then apply the transpositions, in order, to A. The resulting configuration is denoted by $\sigma(A)$. It is important to note that only certain permutations in S_{16} can be applied to a given configuration. In particular, the transpositions involved must always interchange two adjacent cells. So, for example, while we could apply the transposition $(12\ 16)$ to the standard configuration I (as we did in Table 36.8), we could not apply the transposition $(12\ 13)$ to I, since cells 12 and 13 are not adjacent. For convenience, we will refer to permutations in S_{16} that can be applied to I as *valid* permutations.

Activity 36.9. Find a permutation that converts the standard configuration to the configuration shown in Table 36.9.

We can construct a one-to-one correspondence between possible configurations and valid elements in S_{16} by assigning to each valid $\sigma \in S_{16}$ the configuration $\sigma(I)$. Because of this correspon-

1	#	2	4
5	6	3	8
9	10	7	12
13	14	11	15

Table 36.9
A target configuration.

dence, we will from this point on refer to valid elements of S_{16} as configurations, and vice versa. Note that the standard configuration is represented by the identity permutation.

Solving the 15 Puzzle

To determine if a 15 Puzzle is solvable, we are interested in answering the following equivalent questions:

- From which initial configuration can we obtain the standard configuration?
- Which configurations can be obtained starting from the standard configuration?

We will answer the second of these questions and, consequently, obtain the answer to the first question at the same time. To make our work a little easier, first observe that any configuration can be reduced to one in which the blank square is at location 16.

Activity 36.10. As an example, let A be the configuration shown in Table 36.5. Find a permutation σ for which $\sigma(A)$ produces the configuration shown in Table 36.10.

2	9	7	3
10	15	12	8
1	4	14	11
6	13	5	#

Table 36.10
Moving the blank to cell 16.

With this in mind, we need only to determine the configurations that have the blank in position 16 and can be obtained from the standard position I. The next theorem, which we will prove throughout the remainder of this investigation, tells us exactly which configurations meet these conditions.

Theorem 36.11. *Let H be the subset of S_{16} corresponding to all configurations that have the blank in position 16 and can be obtained from the standard position I. Then $H = A_{15}$.*

Note that even though S_{15} is not a subset of S_{16}, we can consider S_{15} to be contained in S_{16} as the set of all permutations that fix 16.

Activity 36.12.

(a) Explain how Theorem 36.11 tells us exactly which 15 Puzzles are solvable.

(b) Let σ_0 represent the configuration shown in Table 36.5. Can we find a sequence of allowable moves to transform σ_0 to I? Why or why not?

It is probably not surprising that H, the set of all permutations with the blank in position 16 that can be obtained from I, is a subgroup of S_{16}. This is left for you to prove in Exercise (4). It is also the case that every element in H is an even permutation. (See Exercise (5).) Thus, $H \subseteq A_{15}$. To complete the proof of Theorem 36.11, we need to show that $A_{15} \subseteq H$. Two facts will be useful in our argument:

- Lemma 24.16 (see page 342) shows that for $n \geq 3$, any permutation in A_n can be written as a product of 3-cycles.
- Exercise (16) of Investigation 22 (see page 318) shows that for any $\alpha \in S_{16}$, any $x, y, z \in \{1, 2, 3, \ldots, 16\}$, and any $n \in \mathbb{Z}^+$, we have

$$\alpha^n (x\ y\ z)\alpha^{-n} = (\alpha^n(x)\ \alpha^n(y)\ \alpha^n(z)). \tag{36.1}$$

Now we can show that H contains every 3-cycle of the form $(a\ b\ c)$ for $a, b, c \in S = \{1, 2, 3, \ldots, 15\}$. Since A_{15} is generated by three cycles, we will be able to conclude that $H = A_{15}$, as desired.

Lemma 36.13. *The group H contains every 3-cycle of the form $(a\ b\ c)$ for $a, b, c \in \{1, 2, 3, \ldots, 15\}$.*

Proof. Let $(a\ b\ c)$ be a 3-cycle with $a, b, c \in S = \{1, 2, 3, \ldots, 15\}$. If $\alpha \in S_{16}$ is in H and $(a\ b\ c) \in H$, then equation (36.1) shows that

$$\alpha^n (a\ b\ c)\alpha^{-n} = (\alpha^n(a)\ \alpha^n(b)\ \alpha^n(c)) \in H \tag{36.2}$$

for any $n \in \mathbb{Z}^+$.

To complete the proof of Lemma 36.13, we will apply this idea to some specific elements in H, as indicated in the next activity, to show the following:

(1) Every 3-cycle of the form $(11\ 7\ b)$ is in H, where $b \in S$ and $b \neq 7, 11, 16$.

(2) Every 3-cycle of the form $(a\ b\ 11)$ is in H, where $a, b \in S$, $a \neq 11, 16$, and $b \neq 7, 11, 16$.

(3) Every 3-cycle is in H.

Once we have established these facts, we will have proved the lemma. ■

Activity 36.14.

(a) Let $b \in S$ with $b \neq 7, 11, 16$. Here we will show that every 3-cycle of the form $(11\ 7\ b)$ is in H.

(i) Explain why

$$\alpha_1 = (11\ 15\ 12) = (16\ 12)(12\ 11)(11\ 15)(15\ 16)$$

(as shown in Table 36.11) is in H.

1	2	3	4
5	6	7	8
9	10	12	15
13	14	11	#

Table 36.11
α_1.

1	2	3	4
5	7	8	12
9	6	11	15
13	10	14	#

Table 36.12
α_2.

(ii) Explain why

$$\begin{aligned}\alpha_2 &= (16\ 12)(12\ 8)(8\ 7)(7\ 6)(6\ 10)(10\ 14)(14\ 15)(15\ 16)\\ &= (6\ 10\ 14\ 15\ 12\ 8\ 7)\end{aligned}$$

(as shown in Table 36.12) is an element of H. Then use equation (36.2) to show that $(11\ 7\ 6)$ is an element of H.

(iii) Next, note that the element α_3 defined by

$$\begin{aligned}\alpha_3 &= (16\ 12)(12\ 8)(8\ 4)(4\ 3)(3\ 2)(2\ 1)(1\ 5)(5\ 6)\\ &\qquad (6\ 10)(10\ 9)(9\ 13)(13\ 14)(14\ 15)(15\ 16)\\ &= (1\ 5\ 6\ 10\ 4\ 13\ 14\ 15\ 12\ 8\ 4\ 3\ 2)\end{aligned}$$

(as shown in Table 36.13) is another element of H.

2	3	4	8
1	5	7	12
10	6	11	15
9	13	14	#

Table 36.13
α_3.

Use equation (36.2) and the fact that α_3 fixes 7, 11, and 16 to complete the argument that every 3-cycle of the form $(11\ 7\ b)$, where $b \in S$ and $b \neq 7, 11, 16$, is in H.

(b) Let $a, b \in S$ with $a \neq 11, 16$ and $b \neq 7, 11, 16$. Show that every 3-cycle of the form $(a\ b\ 11)$ is in H. (Hint: Why is $(11\ b\ 7) \in H$?)

(c) Finally, show that every 3-cycle is in H. How does this show that $H = A_{15}$?

The work we did in Activity 36.14 completes our proof of Lemma 36.13. To summarize, we can determine if a particular 15 Puzzle is solvable by first transforming it to a puzzle A with the blank in position 16. Then if there is an even permutation σ so that $\sigma(I) = A$, we can conclude that our original 15 Puzzle is solvable.

Activity 36.15. Find a 15 Puzzle that is not solvable and that is also not easily seen to be unsolvable. Explain how you know your puzzle is not solvable.

One final note: Our analysis of the 15 Puzzle completely classifies which games can be won but does not tell us how to win. There are strategies for winning, but trial and error is often the best bet.

Concluding Activities

Activity 36.16. Go to any online site that has a NIM game and play it using the strategies we have described in this activity. Make sure you win!

Exercises

(1) Let m be a positive integer, and let $\mathcal{B}_m$ be the set of whole numbers, in binary, that are less than 2^m.

(a) Show that $\mathcal{B}_m$ is a subgroup of $\mathcal{B}$.

(b) As a finite Abelian group, the group $\mathcal{B}_m$ is isomorphic to a direct sum of cyclic groups. To what familiar direct sum is $\mathcal{B}_m$ isomorphic?

* (2) Let n be a positive integer. Show that if $X \notin \mathcal{E}^n$, then there is a move M so that $(X + M) \in \mathcal{E}^n$.

(3) There is a story that when Sam Loyd first distributed his game, the configuration of the puzzle was the standard configuration but with the 14 and 15 pieces in reversed position. Loyd offered a prize of 1000 dollars for a correct solution to that puzzle. Did he ever pay the 1000 dollar prize? Explain.

* (4) Let H be the subset of S_{16} consisting of all permutations with the blank in position 16 that can be obtained from the standard configuration I. Prove that H is a subgroup of S_{16}.

* (5) Prove that if σ is in H, the set of all permutations with the blank in position 16 that can be obtained from I, then σ is an even permutation.

Connections

Abstract algebra has many applications—some serious, and some more fun. In this investigation, we saw how group theory could be used to analyze two games: NIM and the 15 Puzzle. The structure of the game of NIM is related to the group $\mathcal{B}^n$ and its subgroup $\mathcal{E}^n$ of the even configurations, while the symmetric groups are important in determining which 15 Puzzles can be solved.