Section 3

Metric Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a metric and what is a metric space?
- How are the Euclidean, taxicab, and max metric different and how are they similar?

Introduction

Metric spaces are particular examples of topological spaces. A metric space is a space that has a metric defined on it. A metric is a function that measures the distance between points in a metric space.

We are familiar with one special metric, the Euclidean metric d_E in $\mathbb{R}^2$ where

$$d_E((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Using this metric, the distance between two points (x_1, x_2) and (y_1, y_2) is the length of the segment connecting the points, while the unit circle (the set of points a distance 1 from the origin) looks like what we think of as a circle as illustrated in Figure 3.1.

As we will see, there are many other metrics that can be defined on $\mathbb{R}^n$, or on other sets.

Preview Activity 3.1. Consider the function d_T that assigns to each pair of points in $\mathbb{R}^2$ the real number

$$d_T((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

This function d_T is sometimes called the *taxicab metric* or *distance* because the distance between points x and y can be thought of as obtained by driving around a city block rather than going directly from point x to point y.

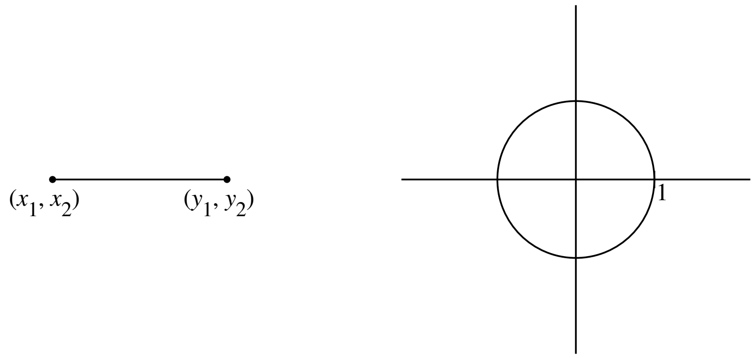

Figure 3.1: The Euclidean distance between (x_1, x_2) and (y_1, y_2) and the Euclidean unit circle in $\mathbb{R}^2$.

Any distance function should satisfy certain properties: the distance between two points should never be negative, the distance from point A to point B should be the same as the distance from point B to point A, the shortest distance between two points A and B should never be more than the distance from A to some point C plus the distance from C to B, and the distance between points should only be zero if the points are the same. In this activity, we determine if d_T has these properties. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$.

(1) Prove that $d_T(x, y) \geq 0$.

(2) Prove that $d_T(x, y) = d_T(y, x)$.

(3) Prove that $d_T(x, y) = 0$ if and only if $x = y$.

(4) Let $z = (z_1, z_2)$ in $\mathbb{R}^2$. Read the proof of Lemma 3.1 (below) and then use Lemma 3.1 to show that
$$d_T(x, y) \leq d_T(x, z) + d_T(z, y).$$
(Do you have any questions about the proof of the lemma?)

Lemma 3.1. *Let a and b be real numbers. Then*
$$|a + b| \leq |a| + |b|.$$

Proof. Let a and b be real numbers. To prove the lemma we consider cases.

Case 1: $a \geq 0$ and $b \geq 0$. In this case $a + b$ is nonnegative and so $|a| = a$, $|b| = b$, and $|a + b| = a + b$. Then
$$|a + b| = a + b = |a| + |b|.$$

Case 2: $a \leq 0$ and $b \leq 0$. In this case $a = -a'$ and $b = -b'$ where a' and b' are nonnegative. It follows from Case 1 that
$$|a + b| = | - (a' + b')| = |a' + b'| = a' + b' = |a'| + |b'| = | - a'| + | - b'| = |a| + |b|.$$

Case 3: One of a or b is positive and the other negative. Without loss of generality we assume $a > 0$ and $b < 0$. Again we consider cases. Note that $b < 0$ implies $a + b < a$.

- Suppose $b \geq -2a$. Then $a + b \geq -a$ and so $-a \leq a + b < a$. It follows that

$$|a + b| \leq a = |a| < |a| + |b|.$$

- The last case is when $b < -2a$. In this case $-b > 2a$ and so

$$|b| = -b > 2a = 2|a| > |a|.$$

Then $a + b < a = |a| < |b|$. Finally, $a > 0$ implies $a + b > b = -|b|$. So

$$-|b| < a + b < |b|$$

and

$$|a + b| \leq |b| < |a| + |b|.$$

This proves our lemma for every possible pair a, b. ■

(5) A picture to illustrate the taxicab distance d_T between (points x_1, x_2) and (y_1, y_2) is shown in Figure 3.2. Draw a picture of the unit circle (the set of points a distance 1 from the origin) using the Taxicab metric. Explain your reasoning.

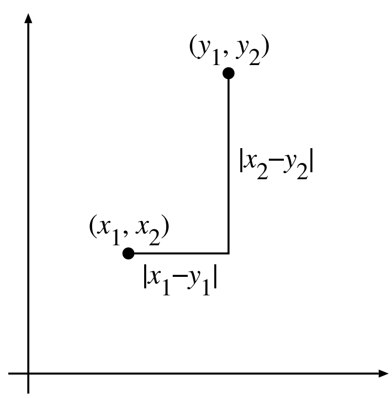

Figure 3.2: The taxicab distance between (x_1, x_2) and (y_1, y_2) in $\mathbb{R}^2$.

The taxicab metric can be extended to $\mathbb{R}^n$ for any $n \geq 1$ as follows. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{R}^n$, then the taxicab distance $d_T(x, y)$ from x to y is defined as

$$d_T(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|.$$

Metric Spaces

For most of our mathematical careers our mathematics has taken place in $\mathbb{R}^2$, where we measure the distance between points (x_1, x_2) and (y_1, y_2) with the standard Euclidean distance d_E. In our preview activity we saw that the function d_T satisfies many of the same properties as d_E. These properties allow us to use d_E or d_T as distance functions. We call any distance function a *metric*, and any space on which a metric is defined is called a *metric space*.

Definition 3.2. A **metric** on a space X is a function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ that satisfies the properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$,

(2) $d(x, y) = 0$ if and only if $x = y$ in X,

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$, and

(4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Properties 1 and 2 of a metric say that a metric is *positive definite*, while property 3 states that a metric is *symmetric*. Property 4 of the definition is usually the most difficult property to verify for a metric and is called the *triangle inequality*.

Definition 3.3. A **metric space** is a pair (X, d), where d is a metric on the space X.

When the metric is clear from the context, we just refer to X as the metric space.

Activity 3.1. For each of the following, determine if (X, d) is a metric space. If (X, d) is a metric space, explain why. If (X, d) is not a metric space, determine which properties of a metric d satisfies and which it does not. If (X, d) is a metric space, give a geometric description of the unit circle (the set of all points in X a distance 1 from the zero element) in the space.

(a) $X = \mathbb{R}$, $d(x, y) = \max\{|x|, |y|\}$.

(b) $X = \mathbb{R}$, $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$

(c) $X = \mathbb{R}^2$, $d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

(d) $X = C[0, 1]$, the set of all continuous functions on the interval $[0, 1]$,

$$d(f, g) = \int_0^1 |f(x) - g(x)|\, dx.$$

It should be noted that not all metric spaces are infinite. We discuss one metric on a finite space in the following example.

Example 3.4. Let $X = \{a, b, c\}$ and define $d : A \times A \to \mathbb{R}^+ \cup \{0\}$ with the entries in Table 3.1. By definition we have $d(x, y) \geq 0$ for all $x, y \in X$ with $d(x, y) = 0$ if and only if $x = y$. Since

	a	b	c
a	0	3	5
b	3	0	4
c	5	4	0

Table 3.1: Table of values for a function d.

the table is symmetric around the diagonal, we can see that $d(x, y) = d(y, x)$ for all $x, y \in X$. The only item to verify is the triangle inequality. If $d(x, y) = 0$, then

$$d(x, y) = 0 \leq d(x, z) + d(z, y)$$

for any $x, y \in X$. If $d(x, z) = 0$, then $x = z$ and

$$d(x, y) = d(z, y) \leq d(z, z) + d(z, y).$$

That leaves three cases to consider, when x, y, and z are distinct. Now

$$\begin{aligned} d(a, b) &= 3 \leq 5 + 4 = d(a, c) + d(c, b), \\ d(a, c) &= 5 \leq 3 + 4 = d(a, b) + d(b, c), \\ d(b, c) &= 4 \leq 3 + 5 = d(b, a) + d(a, c). \end{aligned}$$

So d is a metric on X.

Example 3.4 shows that even finite sets can be metric spaces. In fact, we can make a finite metric space by taking any finite subset S of a metric space (X, d) and use as a metric the restriction of d to S. Example 3.4 illustrates this by letting $a = (0, 0)$, $b = (3, 0)$, and $c = (3, 4)$ in $\mathbb{R}^2$. Then d is the restriction of the Euclidean metric to the set X. Another way to construct a finite metric space is to start with a finite set of points and make a graph with the points as vertices. Construct edges so that the graph is connected (that is, there is a path from any one vertex to any other) and give weights to the edges as illustrated in Figure 3.3. We then define a metric d on S by letting $d(x, y)$ be the length of a shortest path between vertices x and y in the graph. For example, $d(b, c) = d(b, e) + d(e, c) = 9$ in this example.

Just as with the Euclidean and taxicab metrics, item (c) in Activity 3.1 can be extended to $\mathbb{R}^n$ as follows. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{R}^n$, then the maximum distance $d_M(x, y)$ from x to y is defined as

$$d_M(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|, \ldots, |x_n - y_n|\} = \max_{1 \leq i \leq n}\{|x_i - y_i|\}.$$

The metric d_M is called the *max* metric. In the following section we prove that the Euclidean metric is in fact a metric. Proofs that d_T and d_M are metrics are left to Exercises (5) and (6).

The Euclidean Metric on $\mathbb{R}^n$

The metric space that is most familiar to us is the metric space $(\mathbb{R}^2, d_E)$, where

$$d_E((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

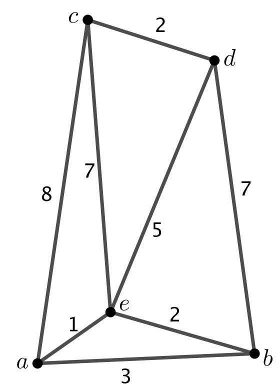

Figure 3.3: A graph to define a metric.

The metric d_E called the *standard* or *Euclidean* metric on $\mathbb{R}^2$.

We can generalize this Euclidean metric from $\mathbb{R}^2$ to any dimensional real space. Let n be a positive integer and let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$. We define $d_E : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d_E(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

In the next activity we will show that d_E satisfies the first three properties of a metric.

Activity 3.2. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$.

(a) Show that $d_E(x, y) \geq 0$.

(b) Show that $d_E(x, y) = d_E(y, x)$.

(c) Show that if $x = y$, then $d_E(x, y) = 0$.

(d) Show that if $d_E(x, y) = 0$, then $x = y$.

Proving that the triangle inequality is satisfied is often the most difficult part of proving that a function is a metric. We will work through this proof with the help of the Cauchy-Schwarz Inequality.

Lemma 3.5 (Cauchy-Schwarz Inequality)**.** *Let n be a positive integer and $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$. Then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sqrt{\sum_{i=1}^{n} x_i^2} \right) \left(\sqrt{\sum_{i=1}^{n} y_i^2} \right). \tag{3.1}$$

Activity 3.3. Before we prove the Cauchy-Schwarz Inequality, let us analyze it in two specific situations.

(a) Let $x = (1, 4)$ and $y = (3, 2)$ in $\mathbb{R}^2$. Verify the Cauchy-Schwarz Inequality in this case.

(b) Let $x = (1, 2, -3)$ and $y = (-4, 0, -1)$ in $\mathbb{R}^3$. Verify the Cauchy-Schwarz Inequality in this case.

Now we prove the Cauchy-Schwarz Inequality.

Proof of the Cauchy-Schwarz Inequality. Let n be a positive integer and $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$. To verify (3.1) it suffices to show that

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

This is difficult to do directly, but there is a nice trick one can use. Consider the expression

$$\sum (x_i - \lambda y_i)^2.$$

(All of our sums are understood to be from 1 to n, so we will omit the limits on the sums for the remainder of the proof.) Now

$$\begin{aligned} 0 &\leq \sum (x_i - \lambda y_i)^2 \\ &= \sum \left(x_i^2 - 2\lambda x_i y_i + \lambda^2 y_i^2\right) \\ &= \left(\sum y_i^2\right)\lambda^2 - 2\left(\sum x_i y_i\right)\lambda + \left(\sum x_i^2\right). \end{aligned} \tag{3.2}$$

To interpret this last expression more clearly, let $a = \sum y_i^2$, $b = -2\sum x_i y_i$ and $c = \sum x_i^2$. The inequality defined by (3.2) can then be written in the form

$$p(\lambda) = a\lambda^2 + b\lambda + c \geq 0.$$

So we have a quadratic $p(\lambda)$ that is never negative. This implies that the quadratic $p(\lambda)$ can have at most one real zero. The quadratic formula gives the roots of $p(\lambda)$ as

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac > 0$, then $p(\lambda)$ has two real roots. Therefore, in order for $p(\lambda)$ to have at most one real zero we must have

$$0 \geq b^2 - 4ac = 4\left(\sum x_i y_i\right)^2 - 4\left(\sum y_i^2\right)\left(\sum x_i^2\right)$$

or

$$\left(\sum y_i^2\right)\left(\sum x_i^2\right) \geq \left(\sum x_i y_i\right)^2.$$

This establishes the Cauchy-Schwarz Inequality. ■

One consequence of the Cauchy-Schwarz Inequality that we will need to show that d_E is a metric is the following.

Corollary 3.6. *Let n be a positive integer and $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$. Then*

$$\sqrt{\sum_{i=1}^{n} (x_i + y_i)^2} \leq \sqrt{\sum_{i=1}^{n} x_i^2} + \sqrt{\sum_{i=1}^{n} y_i^2}.$$

Activity 3.4. Before we prove the corollary, let us analyze it in two specific situations.

(a) Let $x = (1, 4)$ and $y = (3, 2)$ in $\mathbb{R}^2$. Verify Corollary 3.6 in this case.

(b) Let $x = (1, 2, -3)$ and $y = (-4, 0, -1)$ in $\mathbb{R}^3$. Verify Corollary 3.6 in this case.

Now we prove Corollary 3.6.

Proof of Corollary 3.6. Let n be a positive integer and $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$. Now

$$\begin{aligned}
\sum (x_i + y_i)^2 &= \sum \left(x_i^2 + 2x_iy_i + y_i^2\right) \\
&= \sum x_i^2 + 2\sum x_iy_i + \sum y_i^2 \\
&\leq \sum x_i^2 + 2\left(\sqrt{\sum x_i^2}\right)\left(\sqrt{\sum y_i^2}\right) + \sum y_i^2 \\
&= \left(\sqrt{\sum x_i^2} + \sqrt{\sum y_i^2}\right)^2.
\end{aligned}$$

Taking the square roots of both sides yields the desired inequality. ■

We can now complete the proof that d_E is a metric.

Activity 3.5. Let n be a positive integer and $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, and $z = (z_1, z_2, \ldots, z_n)$ be in $\mathbb{R}^n$. Use Corollary 3.6 to show that

$$d_E(x, y) \leq d_E(x, z) + d_E(z, y).$$

This concludes our proof that the Euclidean metric is in fact a metric.

We have seen several metrics in this section, some of which are given special names. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$

- The Euclidean metric d_E, where

$$d_E(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$

- The Taxicab metric d_T, where

$$d_T(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| = \sum_{i=1}^{n}\{|x_i - y_i|\}.$$

- The max metric d_M, where

$$d_M(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|, \ldots, |x_n - y_n|\} = \max_{1\leq i\leq n}\{|x_i - y_i|\}.$$

We have only shown that d_T and d_M are metrics on $\mathbb{R}^2$, but similar arguments apply in $\mathbb{R}^n$. Proofs are left to Exercises (5) and (6). In addition, the *discrete metric*

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

makes any set X into a metric space. The proof is left to Exercise (1).

Summary

Important ideas that we discussed in this section include the following.

- A metric on a space X is a function that measures distance between elements in the space. More formally, a metric on a space X is a function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ such that

  (1) $d(x, y) \geq 0$ for all $x, y \in X$,
  (2) $d(x, y) = 0$ if and only if $x = y$ in X,
  (3) $d(x, y) = d(y, x)$ for all $x, y \in X$, and
  (4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

  A metric space is any space combined with a metric defined on that space.

- The Euclidean, taxicab, and max metric are all metrics on $\mathbb{R}^n$, so they all provide ways to measure distances between points in $\mathbb{R}^n$. These metric are different in how they define the distances.

  - The Euclidean metric is the standard metric that we have used through our mathematical careers. For elements $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, the Euclidean metric d_E is defined as

    $$d_E(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

    With this metric, the unit circle in $\mathbb{R}^2$ (the set of points a distance 1 from the origin) is the standard unit circle we know from Euclidean geometry.

  - The taxicab metric d_T is defined as

    $$d_T(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|.$$

    The unit circle in $\mathbb{R}^2$ using the taxicab metric is the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$ when viewed in Euclidean geometry.

  - The max metric d_M is defined by

    $$d_M(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|, \ldots, |x_n - y_n|\} = \max_{1 \leq i \leq n} \{|x_i - y_i|\}.$$

    Under the max metric, the unit circle in $\mathbb{R}^2$ is the square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$ when viewed in Euclidean geometry.

Exercises

(1) Let X be a set. Show that the function d (the discrete metric) defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric.

(2) Let $X = \{1, 3, 5\} \subset \mathbb{Z}$ and define $d : X \times X \to \mathbb{R}$ by $d(x, y) = xy - 1 \pmod{n}$. That is, $d(x, y)$ is the remainder when $xy - 1$ is divided by n.

- For each value of n, determine if d defines a metric on X. Prove your answers.
- The unit circle in $\mathbb{R}^2$ with metric d is the set of all points in $\mathbb{R}^2$ whose distance from the origin is 1. If we let the distance be less than 1, then we have what we call an open ball. We can make this same definition in any metric space.

  Definition 3.7. Let (Y, d_Y) be a metric space. For any positive real number r, the **open ball centered at** b **of radius** r in (Y, d_Y) is the the set

  $$B(b, r) = \{y \in Y \mid d_Y(y, b) < r\}.$$

  If (X, d) is a metric space for a given value of n, determine all of the open balls in X centered at 1. If (X, d) is not a metric space, explain why.

  (a) $n = 4$

  (b) $n = 8$

(3) Let Q be the set of all rational numbers in reduced form. A rational number $\frac{r}{s}$ is in reduced form if $s > 0$ and r and s have no common factors larger than 1. Define $d : Q \times Q \to \mathbb{R}$ by

$$d\left(\frac{a}{b}, \frac{r}{s}\right) = \max\{|a - r|, |b - s|\}.$$

  (a) Prove that d is a metric.

  (b) A metric allows us to determine which elements in our metric space are "close" together. Describe the set of elements in Q that are a distance no more than 1 from $\frac{2}{3}$ using this metric d. In other words, describe the open ball centered at $\frac{2}{3}$ with radius 1 (see Definition 3.7).

  (c) If a, b, and c are elements of a metric space (X, d_X), we say that b is between a and c if $d_X(a, c) = d_X(a, b) + d_X(b, c)$. Using the Euclidean metric on $\mathbb{Q}$, there are infinitely many different rational numbers between 0 and 1 (the rational numbers between 0 and 1 that lie on the Euclidean line through 0 and 1. Describe all of the points in $(\mathbb{Q}, d)$ that are between 0 and 1.

(4) Let $(\mathbb{Q}, d)$ be the metric space from Exercise (3). If a, b, and c are elements of a metric space (X, d_X), we say that b is *between* a and c if $d_X(a, c) = d_X(a, b) + d_X(b, c)$. Using the Euclidean metric on $\mathbb{Q}$, there are infinitely many different rational numbers between 0 and 1 (the rational numbers between 0 and 1 that lie on the Euclidean line through 0 and 1. In this exercise we explore numbers that are between others in the space $(\mathbb{Q}, d)$.

  (a) Find all of the elements in $(\mathbb{Q}, d)$ that are between 0 and 1.

  (b) Which is closer to 0 in $(\mathbb{Q}, d)$: 1 or $\frac{1}{3}$?

  (c) Now find all of the elements in $(\mathbb{Q}, d)$ that are between 0 and $\frac{1}{3}$.

(5) Prove that the taxicab metric d_T is a metric on $\mathbb{R}^n$.

(6) Let A and B be nonempty finite subsets of $\mathbb{R}^n$, and let $A + B = \{a + b \mid a \in A, b \in B\}$.

(a) Prove that $\max(A + B) \leq \max A + \max B$.

(b) Prove that the max metric d_M is a metric on $\mathbb{R}^n$.

(7) If $x = (x_1, x_2, \ldots, x_n)$, we let $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots y_n)$, define $d_H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d_H(x, y) = \begin{cases} 0 & \text{if } x = y \\ |x| + |y| & \text{otherwise.} \end{cases}$$

(a) Show that d_H is a metric (called the *hub* metric).

(b)

i. Let $a = \left(\frac{1}{2}, 0\right)$. Explicitly describe which points are in the set $B(a, 1)$ in $(\mathbb{R}^2, d_H)$. (See Exercise 2 for the definition of an open ball.)

ii. Let $a = (3, 4)$. Explicitly describe which points are in the set $B(a, 1)$ in $(\mathbb{R}^2, d_H)$.

iii. Now explicitly describe all open balls in $(\mathbb{R}^2, d_H)$.

(8) Let $\mathbb{Z}$ be the set of integers and let p be a prime. For each pair of distinct integers m and n there is an integer $t = t(m, n)$ such that $|m - n| = k \times p^t$, where p does not divide k. For example, if $p = 5$, $m = 34$, and $n = 7$, then $m - n = 27 = 27 \times 5^0$. So $t(43, 7) = 27$. However, if $m = 54$ and $n = 4$, then $m - n = 50 = 2 \times 5^2$. So $t(54, 4) = 2$.

Define $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ by

$$d(m, n) = \begin{cases} 0 & \text{if } m = n \\ \frac{1}{p^t} & \text{if } m \neq n. \end{cases}$$

(a) Determine the values of $d(62, 170)$ using $p = 3$ and $d(14008, 2003)$ using $p = 7$.

(b) Prove that if a, b, and c are in $\mathbb{Z}$, then

$$t(a, c) \geq \min\{t(a, b), t(b, c)\}.$$

(c) Prove that $(\mathbb{Z}, d)$ is a metric space.

(d) Let $p = 3$. Describe the set of all elements x in $(\mathbb{Z}, d)$ such that $d(x, 0) = 1$.

(e) Continue with $p = 3$. Describe the set of all elements x in $(\mathbb{Z}, d)$ such that $d(x, 0) < \frac{1}{2}$.

(9) Let (X, d_X) and (Y, d_Y) be metric spaces. We can make the Cartesian product $X \times Y$ into a metric space by defining a metric d' on $X \times Y$ as follows. If (x_1, y_2) and (x_2, y_2) are in $X \times Y$, then

$$d'((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

You may assume without proof that d' is a metric on $X \times Y$.

(a) Let $(X, d_X) = (\mathbb{R}^2, d_M)$ and $(Y, d_Y) = (\mathbb{R}^2, d_T)$. Let $u = ((1,2),(1,-1))$ and $v = ((0,5),(2,-2))$. What is
$$d'(u,v)?$$
Recall that
$$d_M((x_1,x_2),(y_1,y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$
and
$$d_T((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

(b) Let $(X, d_X) = (\mathbb{R}, d_E)$ and $(Y, d_Y) = (\mathbb{R}, d)$, where d is the discrete metric. Let
$$A = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid -1 \le x \le 1 \text{ and } -1 \le y \le 1\}$$
in $X \times Y$. Let $a = (0,1)$ in $X \times Y$. Describe, geometrically, what the open ball $B(a,1)$ looks like in the product space $X \times Y$. Draw a picture of this open ball.

(10) Let $X = \mathbb{R}^+$, the set of positive reals, and define $d : X \times X \to \mathbb{R}$ by
$$d(x,y) = |\ln(y/x)|.$$
Is d is a metric on X? Prove your answer.

(11) Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by
$$d(x,y) = \frac{|x-y|}{|x-y|+1}.$$
Show that d is a metric on $\mathbb{R}$. (Hint: For the triangle inequality, note that $d(x,y) = f(|x-y|)$ where $f(t) = \frac{t}{t+1}$, and f is an increasing function.)

(12) Let (X,d) be a metric space and k be a constant. Define $kd : X \times X \to \mathbb{R}$ by
$$(kd)(x,y) = kd(x,y).$$
Under what, if any, conditions is kd a metric on X. Justify your answer.

(13) A real valued function f on an interval is *concave* if
$$f((1-\alpha)x + \alpha y) \ge (1-\alpha)f(x) + \alpha f(y) \tag{3.3}$$
for all $\alpha \in [0,1]$ and all x and y in the interval. Note that the expression $(1-\alpha)x + \alpha y$ is linear in α and is equal to x when $\alpha = 0$ and y when $\alpha = 1$. So $(1-\alpha)x + \alpha y$ is a parameterization of the line segment joining x to y. As Figure 3.4 indicates, (3.3) implies that the graph of a concave function f on any interval $[x,y]$ lies above the secant line joining the points $(x, f(x))$ and $(y, f(y))$.

(a) Let $f(x) = -x^2$ map $\mathbb{R}$ to $\mathbb{R}$ with the standard Euclidean metric. Show that f is concave on the interval $[-1,1]$. (Hint: Start with the fact that $\alpha(1-\alpha)(x-y)^2 \ge 0$.)

(b) Show that if f is a concave function on $[0,\infty)$ and $f(0) \ge 0$, an interval and a and b are in the interval, then
$$f(a) + f(b) \ge f(a+b).$$
(Hint: Consider (3.3) with $y = 0$. Then use the fact that $\frac{a}{a+b}$ is in $[0,1]$.)

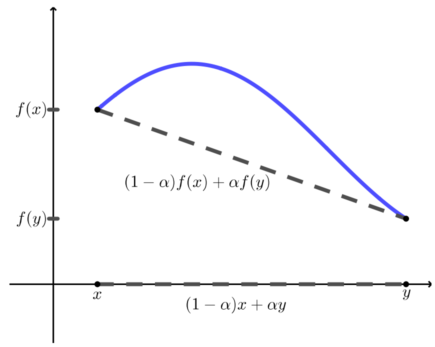

Figure 3.4: A concave function.

(c) Suppose (X, d) is a metric space and $f : [0, \infty) \to [0, \infty)$ is an increasing, concave function such that $f(x) = 0$ if and only if $x = 0$. Prove that $f \circ d$ is a metric on X.

(14) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = (x - y)^2$ is a metric on $\mathbb{R}$.

(b) Every nonempty set can be made into a metric space.

(c) It is possible to define an infinite number of metrics on every set containing at least two elements.

(d) Let (X, d_X) and (Y, d_Y) be metric spaces with $|X| \geq 2$. Then the function $d : X \times Y \to \mathbb{R}$ defined by $d((a, b), (c, d)) = d_X(a, c)d_Y(b, d)$ is a metric on $X \times Y$.

(e) Let (X, d) be a metric space. If X is infinite, then the range of d is also an infinite set.