Chapter 3

Triangles and Vectors

As was stated at the start of Chapter 1, trigonometry had its origins in the study of triangles. In fact, the word *trigonometry* comes from the Greek words for triangle measurement. We will see that we can use the trigonometric functions to help determine lengths of sides of triangles or the measure of angles in triangles. As we will see in the last two sections of this chapter, triangle trigonometry is also useful in the study of vectors.

3.1 Trigonometric Functions of Angles

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we define the cosine and sine as functions of angles?
- How are the trigonometric functions defined on angles using circles of any radius?

Beginning Activity

1. How do we define an angle whose measure is one radian? See the definition on page 27.

2. Draw an angle in standard position with a measure of $\dfrac{\pi}{4}$ radians. Draw an angle in standard position with a measure of $\dfrac{5\pi}{3}$ radians.

3. What is the formula for the arc length s on a circle of radius r that is intercepted by an angle with radian measure θ? See page 36. Why does this formula imply that radians are a dimensionless quantity and that a measurement in radians can be thought of as a real number?

Some Previous Results

In Section 1.2, we defined the cosine function and the sine function using the unit circle. In particular, we learned that we could define $\cos(t)$ and $\sin(t)$ for any real number where the real number t could be thought of as the length of an arc on the unit circle.

In Section 1.3, we learned that the radian measure of an angle is the length of the arc on the unit circle that is intercepted by the angle. That is,

> An angle (in standard position) of t radians will correspond to an arc of length t on the unit circle, and this allows us to think of $\cos(t)$ and $\sin(t)$ when t is the radian measure of an angle.

So when we think of $\cos(t)$ and $\sin(t)$ (and the other trigonometric functions), we can consider t to be:

- a real number;
- the length of an arc with initial point $(1, 0)$ on the unit circle;
- the radian measure of an angle in standard position.

Figure 3.1 shows an arc on the unit circle with the corresponding angle.

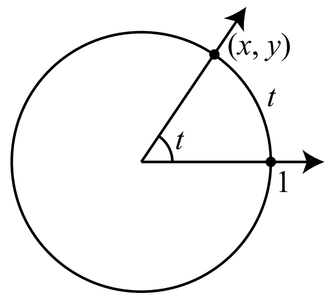


Figure 3.1: An Angle in Standard Position with the Unit Circle

Trigonometric Functions of an Angle

With the notation in Figure 3.1, we see that $\cos(t) = x$ and $\sin(t) = y$. In this context, we often call the cosine and sine *circular functions* because they are defined by points on the unit circle. Now we want to focus on the perspective of the cosine and sine as functions of angles. When using this perspective we will refer to the cosine and sine as *trigonometric functions*. Technically, we have two different types of cosines and sines: one defined as functions of arcs and the other as functions of angles. However, the connection is so close and the distinction so minor that we will often interchange the terms circular and trigonometric. One notational item is that when we think of the trigonometric functions as functions of angles, we often use Greek letters for the angles. The most common ones are θ (theta), α (alpha), β (beta), and ϕ (phi).

Although the definition of the trigonometric functions uses the unit circle, it will be quite useful to expand this idea to allow us to determine the cosine and sine of angles related to circles of any radius. The main concept we will use to do this will be similar triangles. We will use the triangles shown in Figure 3.2.

In this figure, the angle θ is in standard position, the point $P(u, v)$ is on the unit circle, and the point $Q(x, y)$ is on a circle of radius r. So we see that

$$\cos(\theta) = u \quad \text{and} \quad \sin(\theta) = v.$$

We will now use the triangles $\triangle PAO$ and $\triangle QBO$ to write $\cos(\theta)$ and $\sin(\theta)$ in terms of x, y, and r. Figure 3.3 shows these triangles by themselves without the circles.

The two triangles in Figure 3.2 are similar triangles since the corresponding angles of the two triangles are equal. (See page 426 in Appendix C.) Because of

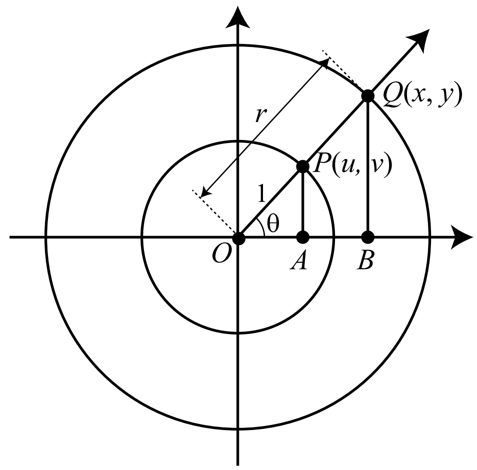


Figure 3.2: An Angle in Standard Position

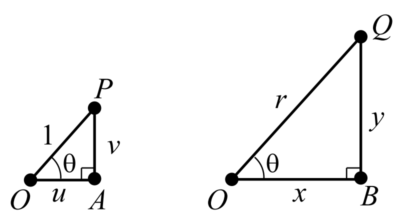


Figure 3.3: Similar Triangles from Figure 3.2

this, we can write

$$\frac{u}{1} = \frac{x}{r} \qquad \frac{v}{1} = \frac{y}{r}$$
$$u = \frac{x}{r} \qquad v = \frac{y}{r}$$
$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r}$$

In addition, note that $u^2 + v^2 = 1$ and $x^2 + y^2 = r^2$. So we have obtained the following results, which show that once we know the coordinates of one point on the terminal side of an angle θ in standard position, we can determine all six trigonometric functions of that angle.

For any point (x, y) other than the origin on the terminal side of an angle θ in standard position, the trigonometric functions of θ are defined as:

$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r} \qquad \tan(\theta) = \frac{y}{x}, x \neq 0$$

$$\sec(\theta) = \frac{r}{x}, x \neq 0 \qquad \csc(\theta) = \frac{r}{y}, y \neq 0 \qquad \cot(\theta) = \frac{x}{y}, y \neq 0$$

where $r^2 = x^2 + y^2$ and $r > 0$ and so $r = \sqrt{x^2 + y^2}$.

Notice that the other trigonometric functions can also be determined in terms of x, y, and r. For example, if $x \neq 0$, then

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \cdot \frac{r}{x} = \frac{y}{x} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\frac{x}{r}} = 1 \cdot \frac{r}{x} = \frac{r}{x}$$

For example, if the point $(3, -1)$ is on the terminal side of the angle θ, then we can use $x = 3$, $y = -1$, and $r = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$, and so

$$\cos(\theta) = \frac{3}{\sqrt{10}} \qquad \tan(\theta) = -\frac{1}{3} \qquad \sec(\theta) = \frac{\sqrt{10}}{3}$$

$$\sin(\theta) = -\frac{1}{\sqrt{10}} \qquad \cot(\theta) = -\frac{3}{1} \qquad \csc(\theta) = -\frac{\sqrt{10}}{1}$$

The next two progress checks will provide some practice with using these results.

Progress Check 3.1 (The Trigonometric Functions for an Angle)
Suppose we know that the point $P(-3, 7)$ is on the terminal side of the angle θ in standard position.

1. Draw a coordinate system, plot the point P, and draw the terminal side of the angle θ.

2. Determine the radius r of the circle centered at the origin that passes through the point $P(-3, 7)$. **Hint**: $x^2 + y^2 = r^2$.

3. Now determine the values of the six trigonometric functions of θ.

Progress Check 3.2 (The Trigonometric Functions for an Angle)
Suppose that α is an angle, that $\tan(\alpha) = \dfrac{2}{3}$, and when α is in standard position, its terminal side is in the first quadrant.

1. Draw a coordinate system and draw the terminal side of the angle α in standard position.

2. Determine a point that lies on the terminal side of α.

3. Determine the six trigonometric functions of α.

The Pythagorean Identity

Perhaps the most important identity for the circular functions is the so-called Pythagorean Identity, which states that for any real number t,

$$\cos^2(t) + \sin^2(t) = 1.$$

It should not be surprising that this identity also holds for the trigonometric functions when we consider these to be functions of angles. This will be verified in the next progress check.

Progress Check 3.3 (The Pythagorean Identity)
Let θ be an angle and assume that (x, y) is a point on the terminal side of θ in standard position. We then let $r^2 = x^2 + y^2$. So we see that

$$\cos^2(\theta) + \sin^2(\theta) = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2.$$

1. Use algebra to rewrite $\left(\dfrac{x}{r}\right)^2 + \left(\dfrac{y}{r}\right)^2$ as a single fraction with denominator r^2.

2. Now use the fact that $x^2 + y^2 = r^2$ to prove that $\left(\dfrac{x}{r}\right)^2 + \left(\dfrac{y}{r}\right)^2 = 1$.

3. Finally, conclude that

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

The next progress check shows how to use the Pythagorean Identity to help determine the trigonometric functions of an angle.

Progress Check 3.4 (Using the Pythagorean Identity)

Assume that θ is an angle in standard position and that $\sin(\theta) = \dfrac{1}{3}$ and $\dfrac{\pi}{2} < \theta < \pi$.

1. Use the Pythagorean Identity to determine $\cos^2(\theta)$ and then use the fact that $\dfrac{\pi}{2} < \theta < \pi$ to determine $\cos(\theta)$.
2. Use the identity $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ to determine the value of $\tan(\theta)$.
3. Determine the values of the other three trigonometric functions of θ.

The Inverse Trigonometric Functions

In Section 2.5, we studied the inverse trigonometric functions when we considered the trigonometric (circular) functions to be functions of a real number t. At the start of this section, however, we saw that t could also be considered to be the length of an arc on the unit circle, or the radian measure of an angle in standard position. At that time, we were using the unit circle to determine the radian measure of an angle but now we can use any point on the terminal side of the angle to determine the angle. The important thing is that these are now functions of angles and so we can use the inverse trigonometric functions to determine angles. We can use either radian measure or degree measure for the angles. The results we need are summarized below.

1. $\theta = \arcsin(x) = \sin^{-1}(x)$ means $\sin(\theta) = x$ and $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$ or $-90° \leq \theta \leq 90°$.
2. $\theta = \arccos(x) = \cos^{-1}(x)$ means $\cos(\theta) = x$ and $0 \leq \theta \leq \pi$ or $0° \leq \theta \leq 180°$.
3. $\theta = \arctan(x) = \tan^{-1}(x)$ means $\tan(\theta) = x$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ or $-90° < \theta < 90°$.

The important things to remember are that an equation involving the inverse trigonometric function can be translated to an equation involving the corresponding trigonometric function and that the angle must be in a certain range. For example, if we know that the point $(5, 3)$ is on the terminal side of an angle θ and that $0 \leq \theta < \pi$, then we know that

$$\tan(\theta) = \frac{y}{x} = \frac{3}{5}.$$

We can use the inverse tangent function to determine (and approximate) the angle θ since the inverse tangent function gives an angle (in radian measure) between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Since $\tan(\theta) > 0$, we will get an angle between 0 and $\frac{\pi}{2}$. So

$$\theta = \arctan\left(\frac{3}{5}\right) \approx 0.54042.$$

If we used degree measure, we would get

$$\theta = \arctan\left(\frac{3}{5}\right) \approx 30.96376°.$$

It is important to note that in using the inverse trigonometric functions, we must be careful with the restrictions on the angles. For example, if we had stated that $\tan(\alpha) = \frac{5}{3}$ and $\pi < \alpha < \frac{3\pi}{2}$, then the inverse tangent function would not give the correct result. We could still use

$$\theta = \arctan\left(\frac{3}{5}\right) \approx 0.54042,$$

but now we would have to use this result and the fact that the terminal side of α is in the third quadrant. So

$$\begin{aligned} \alpha &= \theta + \pi \\ \alpha &= \arctan\left(\frac{3}{5}\right) + \pi \\ \alpha &\approx 3.68201 \end{aligned}$$

We should now use a calculator to verify that $\tan(\alpha) = \frac{3}{5}$.

The relationship between the angles α and θ is shown in Figure 3.4.

Progress Check 3.5 (Finding an Angle)
Suppose that the point $(-2, 5)$ is on the terminal side of the angle θ in standard position and that $0 \leq \theta < 360°$. We then know that $\tan(\theta) = -\frac{5}{2} = -2.5$.

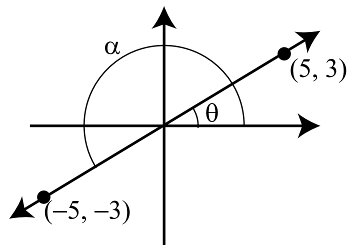


Figure 3.4: Two Angles with the Same Tangent Value

1. Draw a picture of the angle θ.

2. Use a calculator to approximate the value of $\tan^{-1}(-2.5)$ to three decimal places.

3. Notice that $\tan^{-1}(-2.5)$ is a negative angle and cannot equal θ since θ is a positive angle. Use the approximation for $\tan^{-1}(-2.5)$ to determine an approximation for θ to three decimal places.

In the following example, we will determine the exact value of an angle that is given in terms of an inverse trigonometric function.

Example 3.6 Determining an Exact Value
We will determine the exact value of $\cos\left(\arcsin\left(-\frac{2}{7}\right)\right)$. Notice that we can use a calculator to determine that

$$\cos\left(\arcsin\left(-\frac{2}{7}\right)\right) \approx 0.958315.$$

Even though this is correct to six decimal places, it is not the exact value. We can use this approximation, however, to check our work below.

We let $\theta = \arcsin\left(-\frac{2}{7}\right)$. We then know that

$$\sin(\theta) = -\frac{2}{7} \quad \text{and} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

We note that since $\sin(\theta) < 0$, we actually know that $-\frac{\pi}{2} \leq \theta \leq 0$.

So we can use the Pythagorean Identity to determine $\cos^2(\theta)$ as follows:

$$\begin{aligned}\cos^2(\theta) + \sin^2(\theta) &= 1\\ \cos^2(\theta) &= 1 - \left(-\frac{2}{7}\right)^2\\ \cos^2(\theta) &= \frac{45}{49}\end{aligned}$$

Since $-\dfrac{\pi}{2} \leq \theta \leq 0$, we see that $\cos(\theta) = \dfrac{\sqrt{45}}{7}$. That is

$$\cos\left(\arcsin\left(-\frac{2}{7}\right)\right) = \frac{\sqrt{45}}{7}.$$

We can now use a calculator to verify that $\dfrac{\sqrt{45}}{7} \approx 0.958315$.

Summary of Section 3.1

In this section, we studied the following important concepts and ideas:

The trigonometric functions can be defined using any point on the terminal side of an angle in standard position. For any point (x, y) other than the origin on the terminal side of an angle θ in standard position, the trigonometric functions of θ are defined as:

$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r} \qquad \tan(\theta) = \frac{y}{x},\ x \neq 0$$

$$\sec(\theta) = \frac{r}{x},\ x \neq 0 \qquad \csc(\theta) = \frac{r}{y},\ y \neq 0 \qquad \cot(\theta) = \frac{x}{y},\ y \neq 0$$

where $r^2 = x^2 + y^2$ and $r > 0$ and so $r = \sqrt{x^2 + y^2}$. The Pythagorean Identity is still true when we use the trigonometric functions of an angle. That is, for any angle θ,

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

In addition, we still have the inverse trigonometric functions. In particular,

- $\theta = \arcsin(x) = \sin^{-1}(x)$ means $\sin(\theta) = x$ and $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$ or $-90° \leq \theta \leq 90°$.
- $\theta = \arccos(x) = \cos^{-1}(x)$ means $\cos(\theta) = x$ and $0 \leq \theta \leq \pi$ or $0° \leq \theta \leq 180°$.

- $\theta = \arctan(x) = \tan^{-1}(x)$ means $\tan(\theta) = x$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ or $-90° < \theta < 90°$.

Exercises for Section 3.1

1. In each of the following, the coordinates of a point P on the terminal side of an angle θ are given. For each of the following:

- Plot the point P in a coordinate system and draw the terminal side of the angle.
- Determine the radius r of the circle centered at the origin that passes through the point P.
- Determine the values of the six trigonometric functions of the angle θ.

⋆ **(a)** $P(3, 3)$
⋆ **(b)** $P(5, 8)$
(c) $P(-2, -2)$
(d) $P(5, -2)$
⋆ **(e)** $P(-1, -4)$
(f) $P(2\sqrt{3}, 2)$
(g) $P(-3, 4)$
(h) $P(3, -3\sqrt{3})$
(i) $P(2, -1)$

2. For each of the following, draw the terminal side of the indicated angle on a coordinate system and determine the values of the six trigonometric functions of that angle

(a) The terminal side of the angle α is in the first quadrant and $\sin(\alpha) = \dfrac{1}{\sqrt{3}}$.

⋆ **(b)** The terminal side of the angle β is in the second quadrant and $\cos(\beta) = -\dfrac{2}{3}$.

(c) The terminal side of the angle γ is in the second quadrant and $\tan(\gamma) = -\dfrac{1}{2}$.

(d) The terminal side of the angle θ is in the second quadrant and $\sin(\theta) = \dfrac{1}{3}$.

3. For each of the following, determine an approximation for the angle θ in degrees (to three decimal places) when $0° \leq \theta < 360°$.

(a) The point $(3, 5)$ is on the terminal side θ.

(**b**) The point $(2, -4)$ is on the terminal side of θ.

⋆ **(c)** $\sin(\theta) = \dfrac{2}{3}$ and the terminal side of θ is in the second quadrant.

(d) $\sin(\theta) = -\dfrac{2}{3}$ and the terminal side of θ is in the fourth quadrant.

⋆ **(e)** $\cos(\theta) = -\dfrac{1}{4}$ and the terminal side of θ is in the second quadrant.

(f) $\cos(\theta) = -\dfrac{3}{4}$ and the terminal side of θ is in the third quadrant.

4. For each of the angles in Exercise (3), determine the radian measure of θ if $0 \le \theta < 2\pi$.

5. Determine the exact value of each of the following. Check all results with a calculator.

(**a**) $\cos\left(\arcsin\left(\dfrac{1}{5}\right)\right)$.

⋆ **(b)** $\tan\left(\cos^{-1}\left(\dfrac{2}{3}\right)\right)$.

(c) $\sin\left(\tan^{-1}(2)\right)$.

(d) $\cos\left(\arcsin\left(-\dfrac{1}{5}\right)\right)$.

(e) $\sin\left(\arccos\left(-\dfrac{3}{5}\right)\right)$.

3.2 Right Triangles

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How does the cosine relate sides and acute angles in a right triangle? Why?
- How does the sine relate sides and acute angles in a right triangle? Why?
- How does the tangent relate sides and acute angles in a right triangle? Why?
- How can we use the cosine, sine, and tangent of an angle in a right triangle to help determine unknown parts of that triangle?

Beginning Activity

Figure 3.5 shows a typical right triangle. The lengths of the three sides of the right triangle are labeled as a, b, and c. The angles opposite the sides of lengths a, b, and c are labeled α (alpha), β (beta), and γ (gamma), respectively. (Alpha, beta, and gamma are the first three letters in the Greek alphabet.) The small square with the angle γ indicates that this is the right angle in the right triangle. The triangle, of course, has three sides. We call the side opposite the right angle (the side of length c in the diagram) the **hypotenuse** of the right triangle.

When we work with triangles, the angles are usually measured in degrees and so we would say that γ is an angle of $90°$.

1. What can we conclude about a, b, and c from the Pythagorean Theorem?

When working with triangles, we usually measure angles in degrees. For the fractional part of the degree measure of an angle, we often used decimals but we also frequently use minutes and seconds.

2. What is the sum of the angles in a triangle? In this case, what is $\alpha + \beta + \gamma$?

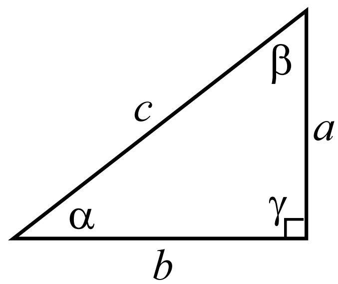


Figure 3.5: A typical right triangle

3. What is the sum of the two acute angles in a right triangle. In this case, what is $\alpha + \beta$?

4. How many minutes are in a degree? How many seconds are in a minute?

5. Determine the solution of the equation $7.3 = \dfrac{118.8}{x}$ correct to the nearest thousandth. (You should be able to show that $x \approx 16.274$.)

6. Determine the solution of the equation $\sin(32°) = \dfrac{5}{x}$ correct to the nearest ten-thousandth. (You should be able to show that $x \approx 9.4354$.)

Introduction

Suppose you want to find the height of a tall object such as a flagpole (or a tree or a building). It might be inconvenient (or even dangerous) to climb the flagpole and measure it, so what can you do? It might be easy to measure the length the shadow the flagpole casts and also the angle θ determined by the ground level to the sun (called the **angle of elevation of the object**) as in Figure 3.6. In this section, we will learn how to use the trigonometric functions to relate lengths of sides to angles in right triangles and solve this problem as well as many others.

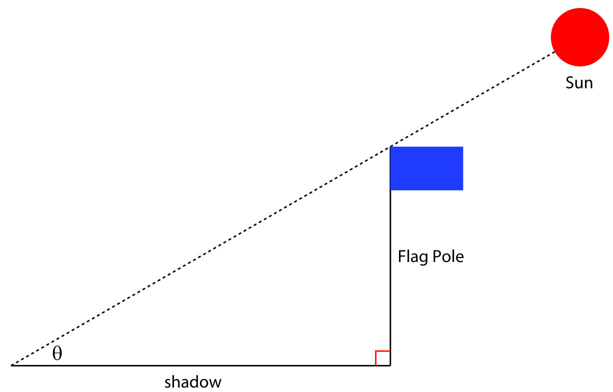


Figure 3.6: Finding the height of a flagpole (drawing not to scale)

Trigonometric Functions and Right Triangles

We have seen how we determine the values of the trigonometric functions of an angle θ by placing θ in standard position and letting (x, y) be the point of intersection of the terminal side of angle θ with a circle of radius r. Then

$$\cos(\theta) = \frac{x}{r}, \qquad \sec(\theta) = \frac{r}{x} \text{ if } x \neq 0,$$
$$\sin(\theta) = \frac{y}{r}, \qquad \csc(\theta) = \frac{r}{y} \text{ if } y \neq 0,$$
$$\tan(\theta) = \frac{y}{x} \text{ if } x \neq 0, \qquad \cot(\theta) = \frac{x}{y} \text{ if } y \neq 0.$$

In our work with right triangles, we will use only the sine, cosine, and tangent functions.

Now we will see how to relate the trigonometric functions to angles in right triangles. Suppose we have a right triangle with sides of length x and y and hypotenuse of length r. Let θ be the angle opposite the side of length y as shown in Figure 3.7. We can now place our triangle such that the angle θ is in standard position in the plane and the triangle will fit into the circle of radius r as shown at right in Figure 3.8. By the definition of our trigonometric functions we then have

$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r} \qquad \tan(\theta) = \frac{y}{x}$$

If instead of using x, y, and r, we label y as the length of the side opposite the acute angle θ, x as the length of the side adjacent to the acute angle θ, and r as the length of the hypotenuse, we get Figure 3.9.

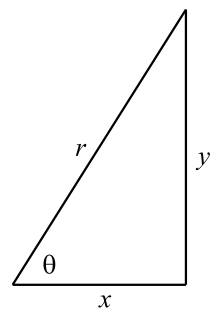


Figure 3.7: A right triangle

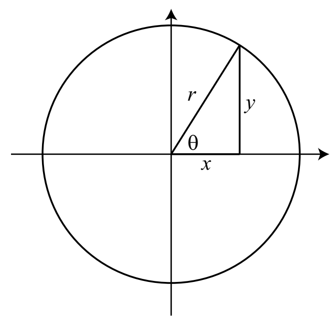


Figure 3.8: Right triangle in standard position

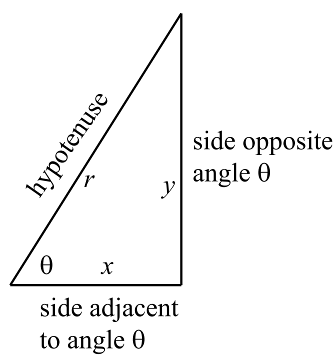


Figure 3.9: A right triangle

So we see that

$$\sin(\theta) = \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}} \qquad \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}} \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta} \qquad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

The equations on the right are convenient abbreviations of the correct equations on the left.

Progress Check 3.7 (**Labeling a Right Triangle**)
We must be careful when we use the terms opposite and adjacent because the meaning of these terms depends on the angle we are using. Use the diagrams in Figure 3.10 to determine formulas for each of the following in terms of a, b, and c.

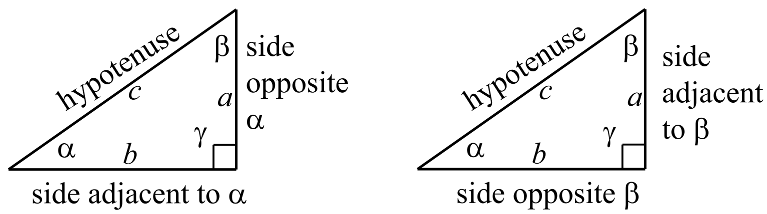


Figure 3.10: Labels for a right triangle

$\cos(\alpha) =$ __________ $\cos(\beta) =$ __________

$\sin(\alpha) =$ __________ $\sin(\beta) =$ __________

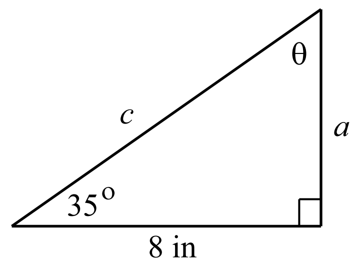
$\tan(\alpha) =$ __________ $\tan(\beta) =$ __________

We should also note that with the labeling of the right triangle shown in Figure 3.10, we can use the Pythagorean Theorem and the fact that the sum of the angles of a triangle is 180 degrees to conclude that

$$a^2 + b^2 = c^2 \quad \text{and} \quad \begin{aligned} \alpha + \beta + \gamma &= 180^\circ \\ \gamma &= 90^\circ \\ \alpha + \beta &= 90^\circ \end{aligned}$$

Example 3.8 Suppose that one of the acute angles of a right triangle has a measure of 35° and that the side adjacent to this angle is 8 inches long. Determine the other acute angle of the right triangle and the lengths of the other two sides.

Solution. The first thing we do is draw a picture of the triangle. (The picture does not have to be perfect but it should reasonably reflect the given information.) In making the diagram, we should also label the unknown parts of the triangle. One way to do this is shown in the diagram.



One thing we notice is that $35^\circ + \theta = 90^\circ$ and so $\theta = 55^\circ$. We can also use the cosine and tangent of 35° to determine the values of a and c.

$$\begin{aligned} \cos\left(35^\circ\right) &= \frac{8}{c} \\ c\cos\left(35^\circ\right) &= 8 \\ c &= \frac{8}{\cos\left(35^\circ\right)} \\ c &\approx 9.76620 \end{aligned} \qquad \begin{aligned} \tan\left(35^\circ\right) &= \frac{a}{8} \\ 8\tan\left(35^\circ\right) &= a \\ a &\approx 5.60166 \end{aligned}$$

Before saying that this example is complete, we should check our results. One way to do this is to verify that the lengths of the three sides of the right triangle satisfy the formula for the Pythagorean Theorem. Using the given value for one side and the calculated values of a and c, we see that

$$\begin{aligned} 8^2 + a^2 &\approx 95.379 \\ c^2 &\approx 95.379 \end{aligned}$$

So we see that our work checks with the Pythagorean Theorem.

Solving Right Triangles

What we did in Example 3.8 is what is called **solving a right triangle**. Please note that this phrase is misleading because you cannot really "solve" a triangle. However, since this phrase is a traditional part of the vernacular of trigonometry and so we will continue to use it. The idea is that if we are given enough information about the lengths of sides and measures of angles in a right triangle, then we can determine all of the other values. The next progress check is also an example of "solving a right triangle."

Progress Check 3.9 (**Solving a Right Triangle**)
The length of the hypotenuse of a right triangle is 17 feet and the length of one side of this right triangle is 5 feet. Determine the length of the other side and the two acute angles for this right triangle.

Hint: Draw a picture and label the third side of the right triangle with a variable and label the two acute angles as α and β.

Applications of Right Triangles

As the examples have illustrated up to this point, when working on problems involving right triangles (including application problems), we should:

- Draw a diagram for the problem.
- Identify the things you know about the situation. If appropriate, include this information in your diagram.
- Identify the quantity that needs to be determined and give this quantity a variable name. If appropriate, include this information in your diagram.
- Find an equation that relates what is known to what must be determined. This will often involve a trigonometric function or the Pythagorean Theorem.
- Solve the equation for the unknown. Then think about this solution to make sure it makes sense in the context of the problem.
- If possible, find a way to check the result.

We return to the example given in the introduction to this section on page 179. In this example, we used the term *angle of elevation*. This is a common term (as well as *angle of depression*) in problems involving triangles. We can define an **angle of elevation** of an object to be an angle whose initial side is horizontal and has a rotation so that the terminal side is above the horizontal. An **angle of depression** is then an angle whose initial side is horizontal and has a rotation so that the terminal side is below the horizontal. See Figure 3.11.

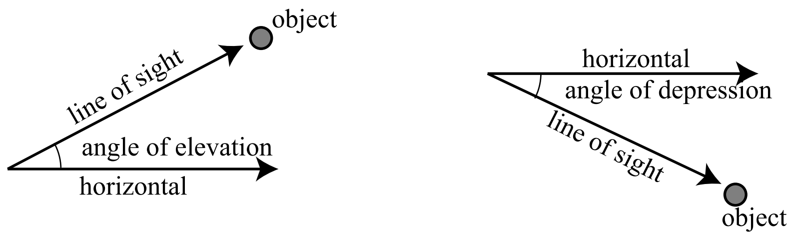


Figure 3.11: Angle of Elevation and Angle of Depression

Example 3.10 Determining the Height of a Flagpole
Suppose that we want to determine the height of a flagpole and cannot measure the height directly. Suppose that we measure the length of the shadow of the flagpole to be 44 feet, 5 inches. In addition, we measure the angle of elevation of the sun to be $33°15'$.

Solution. The first thing we do is to draw the diagram. In the diagram, we let h

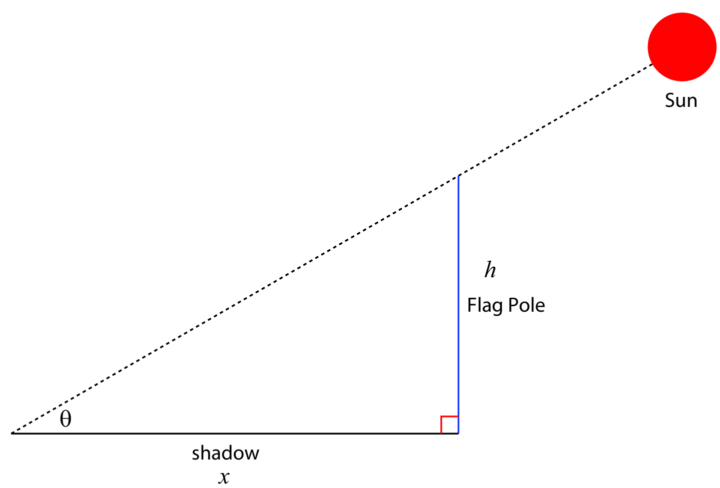


Figure 3.12: Finding the height of a flagpole (drawing not to scale)

be the height of the flagpole, x be the length of the shadow, and θ be the angle of elevation. We are given values for x and θ, and we see that

$$\tan(\theta) = \frac{h}{x}$$
$$x\tan(\theta) = h \qquad (1)$$

So we can now determine the value of h, but we must be careful to use a decimal (or fractional) value for x (equivalent to 44 feet, 5 inches) and a decimal (or fractional) value for θ (equivalent to $33°15'$). So we will use

$$x = 44 + \frac{5}{12} \quad \text{and} \quad \theta = \left(33 + \frac{15}{60}\right)^{\circ}.$$

Using this and equation (1), we see that

$$h = \left(44 + \frac{5}{12}\right)\tan\left(33 + \frac{15}{60}\right)^{\circ}$$
$$h \approx 29.1208 \text{ feet.}$$

The height of the flagpole is about 29.12 feet or 29 feet, 1.4 inches.

Progress Check 3.11 (Length of a Ramp)
A company needs to build a wheelchair accessible ramp to its entrance. The Americans with Disabilities Act Guidelines for Buildings and Facilities for ramps state the "The maximum slope of a ramp in new construction shall be 1:12."

1. The 1:12 guideline means that for every 1 foot of rise in the ramp there must be 12 feet of run. What is the angle of elevation (in degrees) of such a ramp?
2. If the company's entrance is 7.5 feet above the level ground, use trigonometry to approximate the length of the ramp that the company will need to build using the maximum slope. Explain your process.

Progress Check 3.12 (Guided Activity – Using Two Right Triangles)
This is a variation of Example 3.19. Suppose that the flagpole sits on top a hill and that we cannot directly measure the length of the shadow of the flagpole as shown in Figure 3.19.

Some quantities have been labeled in the diagram. Angles α and β are angles of elevation to the top of the flagpole from two different points on level ground. These points are d feet apart and directly in line with the flagpole. The problem

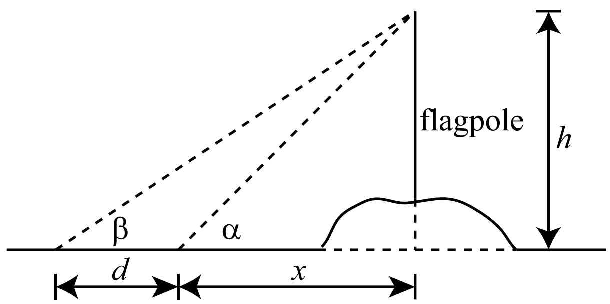


Figure 3.13: Flagpole on a hill

is to determine h, the height from level ground to the top of the flagpole. The following measurements have been recorded.

$$\alpha = 43.2^\circ \qquad d = 22.75\text{feet}$$
$$\beta = 34.7^\circ$$

Notice that a value for x was not given because it is the distance from the first point to an imaginary point directly below the flagpole and even with level ground.

Please keep in mind that it is probably easier to write formulas in terms of α, β, and γ and wait until the end to use the numerical values. For example, we see that

$$\tan(\alpha) = \frac{h}{x} \quad \text{and} \tag{1}$$
$$\tan(\beta) = \frac{h}{d + x}. \tag{2}$$

In equation (1), notice that we know the value of α. This means if we can determine a value for either x or h, we can use equation (1) to determine the value of the other. We will first determine the value of x.

1. Solve equation (1) for h and then substitute this into equation (2). Call this equation (3).

2. One of the terms in equation (3) has a denominator. Multiply both sides of equation (3) by this denominator.

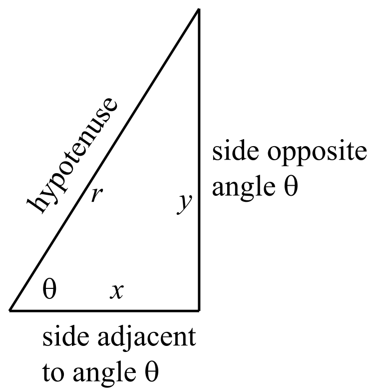
3. Now solve the resulting equation for x (in terms of α, β, and d).

4. Substitute the given values for α, β, and d to determine the value of x and then use this value and equation (1) to determine the value of h.

5. Is there a way to check to make sure the result is correct?

Summary of Section 3.2

In this section, we studied the following important concepts and ideas:
Given enough information about the lengths of sides and measures of angles in a right triangle, we can determine all of the other values using the following relationships:



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

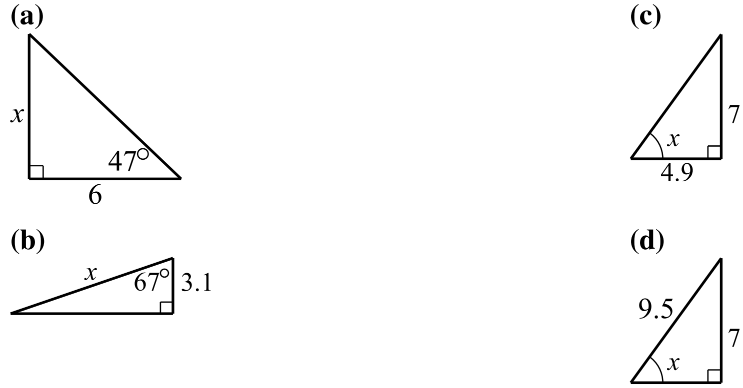
$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

$$x^2 + y^2 = r^2$$

Exercises for Section 3.2

⋆ **1.** For each of the following right triangles, determine the value of x correct to the nearest thousandth.



2. One angle in a right triangle is $55°$ and the side opposite that angle is 10 feet long. Determine the length of the other side, the length of the hypotenuse, and the measure of the other acute angle.

3. One angle in a right triangle is $37.8°$ and the length of the hypotenuse is 25 inches. Determine the length of the other two sides of the right triangle.

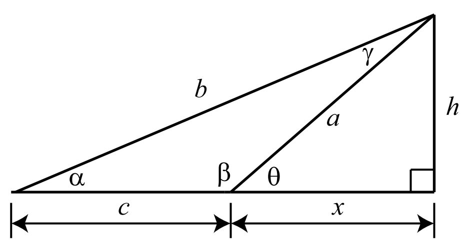
⋆ 4. One angle in a right triangle is $27°12'$ and the length of the side adjacent to this angle is 4 feet. Determine the other acute angle in the triangle, the length of the side opposite this angle, and the length of the hypotenuse.
Note: The notation means that the angle is 27 degrees, 12 seconds. Recall that 1 second is $\dfrac{1}{60}$ of a degree.

5. If we only know the measures of the three angles of a right triangle, explain why it is not possible to determine the lengths of the sides of this right triangle.

6. Suppose that we know the measure θ of one of the acute angles in a right triangle and we know the length x of the side opposite the angle θ. Explain how to determine the length of the side adjacent to the angle θ and the length of the hypotenuse.

⋆ 7. In the diagram to the right, determine the values of a, b, and h to the nearest thousandth.
The given values are:

$$\alpha = 23°$$
$$\beta = 140°$$
$$c = 8$$

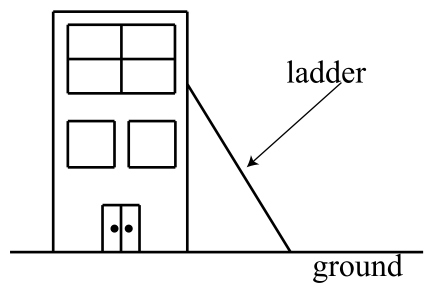


8. A tall evergreen tree has been damaged in a strong wind. The top of the tree is cracked and bent over, touching the ground as if the trunk were hinged. The tip of the tree touches the ground 20 feet 6 inches from the base of the tree (where the tree and the ground meet). The tip of the tree forms an angle of 17 degrees where it touches the ground. Determine the original height of the tree (before it broke) to the nearest tenth of a foot. Assume the base of the tree is perpendicular to the ground.

9. Suppose a person is standing on the top of a building and that she has an instrument that allows her to measure angles of depression. There are two points that are 100 feet apart and lie on a straight line that is perpendicular

to the base of the building. Now suppose that she measures the angle of depression to the closest point to be 35.5° and that she measures the angle of depression to the other point to be 29.8°. Determine the height of the building.

10. A company has a 35 foot ladder that it uses for cleaning the windows in their building. For safety reasons, the ladder must never make an angle of more than 50° with the ground.



(**a**) What is the greatest height that the ladder can reach on the building if the angle it makes with the ground is no more than 50°.

(**b**) Suppose the building is 40 feet high. Again, following the safety guidelines, what length of ladder is needed in order to have the ladder reach the top of the building?

3.3 Triangles that Are Not Right Triangles

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is the Law of Sines?
- What information do we need about a triangle to apply the Law of Sines?
- What do we mean by the ambiguous case for the Law of Sines? Why is it ambiguous?
- What is the Law of Cosines?
- What information do we need about a triangle to apply the Law of Cosines?

Introduction

In Section 3.2, we learned how to use the trigonometric functions and given information about a right triangle to determine other parts of that right triangle. Of course, there are many triangles without right angles (these triangles are called *oblique triangles*). Our next task is to develop methods to relate sides and angles of oblique triangles. In this section, we will develop two such methods, the Law of Sines and the Law of Cosines. In the next section, we will learn how to use these methods in applications.

As with right triangles, we will want some standard notation when working with general triangles. Our notation will be similar to the what we used for right triangles. In particular, we will often let the lengths of the three sides of a triangle be a, b, and c. The angles opposite the sides of length a, b, and c will be labeled α, β, and γ, respectively. See Figure 3.14.

We will sometimes label the vertices of the triangle as A, B, and C as shown in Figure 3.14.

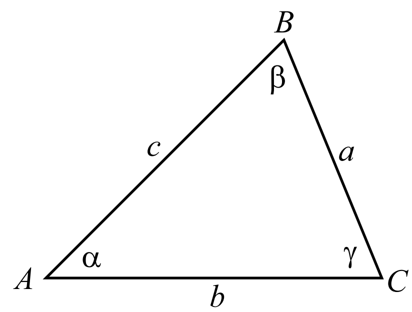


Figure 3.14: Standard Labeling for a Triangle

Beginning Activity

Before we state the Law of Sines and the Law of Cosines, we are going to use two Geogebra apps to explore the relationships about the parts of a triangle. In each of these apps, a triangle is drawn. The lengths of the sides of the triangle and the measure for each of the angles in shown. The size and shape of the triangle can be changed by dragging one (or all) of the points that form the vertices of the triangle.

1. Open the Geogebra app called *The Law of Sines* at

http://gvsu.edu/s/01B

(**a**) Experiment by moving the vertices of the triangle and observing what happens with the lengths and the angles and the computations shown in the lower left part of the screen.

(**b**) Use a particular triangle and verify the computations shown in the lower left part of the screen. Round your results to the nearest thousandth as is done in the app.

(**c**) Write an equation (or equations) that this app is illustrating. This will be part of the Law of Sines.

2. Open the Geogebra app called *The Law of Cosines* at

http://gvsu.edu/s/01C

(**a**) Experiment by moving the vertices of the triangle and observing what happens with the lengths and the angles and the computations shown in the lower left part of the screen.

(**b**) Use a particular triangle and verify the computations shown in the lower left part of the screen. Round your results to the nearest thousandth as is done in the app.

(**c**) Write an equation that this app is illustrating. This will be part of the Law of Cosines.

The Law of Sines

The first part of the beginning activity was meant to illustrate the Law of Sines. Following is a formal statement of the Law of Sines.

Law of Sines
In a triangle, if a, b, and c are the lengths of the sides opposite angles α, β, and γ, respectively, then

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

This is equivalent to

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}.$$

A proof of the Law of Sines is included at the end of this section.

Please note that the Law of Sines actually has three equations condensed into a single line. The three equations are:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} \qquad \frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c} \qquad \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

The key to using the Law of Sines is that each equation involves 4 quantities, and if we know 3 of these quantities, we can use the Law of Sines to determine the fourth. These 4 quantities are actually two different pairs, where one element of a pair is an angle and the other element of that pair is the length of the side opposite that angle. In Figure 3.15, θ and x form one such pair, and ϕ and y are another such pair. We can write the Law of Sines as follows:

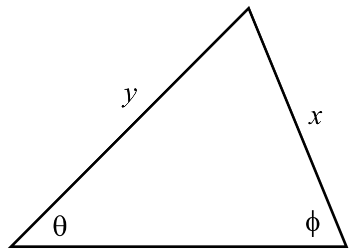
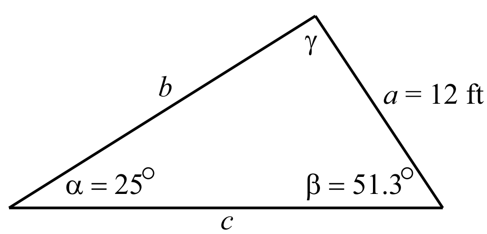


Figure 3.15: Diagram for the Law of Sines

Law of Sines
In a triangle, if x is the length of the side opposite angle θ and y is the length of the side opposite angle ϕ, then

$$\frac{x}{\sin(\theta)} = \frac{y}{\sin(\phi)} \quad \text{or} \quad \frac{\sin(\theta)}{x} = \frac{\sin(\phi)}{y}.$$

Example 3.13 (Using the Law of Sines)
Suppose that the measures of two angles of a triangle are 25° and 51.3° and that the side opposite the 25° angle is 12 feet long. We will use the Law of Sines to determine the other three parts of the triangle. (Remember that we often say that we are "solving the triangle.") The first step is to draw a reasonably accurate diagram of the triangle and label the parts. This is shown in the following diagram.



We notice that we know the values of the length of a side and its opposite angles (a and α). Since we also know the value of β, we can use the Law of Sines to determine b. This is done as follows:

$$\begin{aligned}\frac{a}{\sin(\alpha)} &= \frac{b}{\sin(\beta)}\\ b &= \frac{a\sin(\beta)}{\sin(\alpha)}\\ b &= \frac{12\sin(51.3^\circ)}{\sin(25^\circ)}\\ b &\approx 22.160\end{aligned}$$

So we see that the side opposite the 51.3° angle is about 22.160 feet in length. We still need to determine γ and c. We will use the fact that the sum of the angles of a triangle is equal to 180° to determine γ.

$$\begin{aligned}\alpha + \beta + \gamma &= 180^\circ\\ 25^\circ + 51.3^\circ + \gamma &= 180^\circ\\ \gamma &= 103.7^\circ\end{aligned}$$

Now that we know γ, we can use the Law of Sines again to determine c. To do this, we solve the following equation for c.

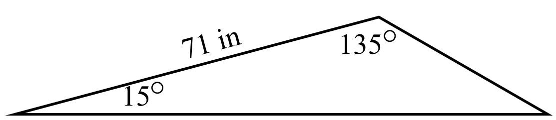
$$\frac{a}{\sin(\alpha)} = \frac{c}{\sin(\gamma)}.$$

We should verify that the result is $c \approx 27.587$ feet. To check our results, we should verify that for this triangle,

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \approx 0.035.$$

Progress Check 3.14 (Using the Law of Sines)

Suppose that the measures of two angles of a triangle are 15° and 135° and that the side that is common to these two angles is 71 inches long. Following is a reasonably accurate diagram for this triangle.



Determine the lengths of the other two sides of the triangle and the measure of the third angle. **Hint**: First introduce some appropriate notation, determine the measure of the third angle, and then use the Law of Sines.

Using the Law of Sines to Determine an Angle

As we have stated, an equation for the Law of Sines involves four quantities, two angles and the lengths of the two sides opposite these angles. In the examples we have looked at, two angles and one side has been given. We then used the Law of Sines to determine the length of the other side.

We can run into a slight complication when we want to determine an angle using the Law of Sines. This can occur when we are given the lengths of two sides and the measure of an angle opposite one of these sides. The problem is that there are two different angles between 0° and 180° that are solutions of an equation of the form

$$\sin(\theta) = \text{“a number between 0 and 1”}.$$

For example, consider the equation $\sin(\theta) = 0.7$. We can use the inverse sine function to determine one solution of this equation, which is

$$\theta_1 = \sin^{-1}(0.7) \approx 44.427°.$$

The inverse sine function gives us the solution that is between 0° and 90°, that is, the solution in the first quadrant. There is a second solution to this equation in the second quadrant, that is, between 90° and 180°. This second solution is $\theta_2 = 180° - \theta_1$. So in this case,

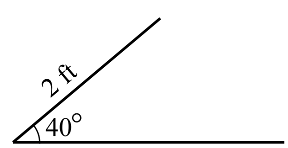
$$\theta_2 = 180° - \sin^{-1}(0.7) \approx 135.573°.$$

The next two progress checks will be guided activities through examples where we will need to use the Law of Sines to determine an angle.

Progress Check 3.15 (Using the Law of Sines for an Angle)

Suppose a triangle has a side of length 2 feet that is an adjacent side for an angle of 40°. Is it possible for the side opposite the 40° angle to have a length of 1.7 feet?

To try to answer this, we first draw a reasonably accurate diagram of the situation as shown below.



The horizontal line is not a side of the triangle (yet). For now, we are just using it as one of the sides of the 40° angle. In addition, we have not drawn the side opposite the 40° angle since just by observation, it appears there could be two possible ways to draw a side of length 1.7 feet. Now we get to the details.

1. Let θ be the angle opposite the side of length 2 feet. Use the Law of Sines to determine $\sin(\theta)$.

2. Use the inverse sine function to determine one solution (rounded to the nearest tenth of a degree) for θ. Call this solution θ_1.

3. Let $\theta_2 = 180° - \theta_1$. Explain why (or verify that) θ_2 is also a solution of the equation in part (1).

This means that there could be two triangles that satisfy the conditions of the problem.

4. Determine the third angle and the third side when the angle opposite the side of length 2 is θ_1.

5. Determine the third angle and the third side when the angle opposite the side of length 2 is θ_2.

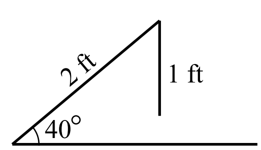
There are times when the Law of Sines will show that there are no triangles that meet certain conditions. We often see this when an equation from the Law of Sines produces an equation of the form

$$\sin(\theta) = p,$$

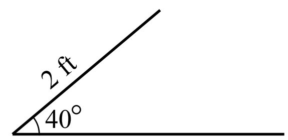
where p is real number but is not between 0 and 1. For example, changing the conditions in Progress Check 3.15 so that we want a triangle that has a side of length 2 feet that is an adjacent side for an angle of 40° and the side opposite the 40° angle is to have a length of 1 foot. As in Progress Check 3.15, we let θ be the angle opposite the side of length 2 feet and use the Law of Sines to obtain

$$\begin{aligned}\frac{\sin(\theta)}{2} &= \frac{\sin(40°)}{1}\\ \sin(\theta) &= \frac{2\sin(40°)}{1} \approx 1.2856\end{aligned}$$

There is no such angle θ and this shows that there is no triangle that meets the specified conditions. The diagram on the right illustrates the situation.



Progress Check 3.16 (Using the Law of Sines for an Angle)
Suppose a triangle has a side of length 2 feet that is an adjacent side for an angle of 40°. Is it possible for the side opposite the 40° angle to have a length of 3 feet?



The only difference between this and Progress Check 3.15 is in the length of the side opposite the 40° angle. We can use the same diagram. By observation, it appears there is likely only way to draw a side of length 3 feet. Now we get to the details.

1. Let θ be the angle opposite the side of length 2 feet. Use the Law of Sines to determine $\sin(\theta)$.

2. Use the inverse sine function to determine one solution (rounded to the nearest tenth of a degree) for θ. Call this solution θ_1.

3. Let $\theta_2 = 180° - \theta_1$. Explain why (or verify that) θ_2 is also a solution of the equation in part (1).

This means that there could be two triangles that satisfy the conditions of the problem.

4. Determine the third angle and the third side when the angle opposite the side of length 2 is θ_1.

5. Now determine the sum $40° + \theta_2$ and explain why this is not possible in a triangle.

Law of Cosines

We have seen how the Law of Sines can be used to determine information about sides and angles in oblique triangles. However, to use the Law of Since we need to know three pieces of information. We need to know an angle and the length of its opposite side, and in addition, we need to know another angle or the length of another side. If we have three different pieces of information such as the lengths of two sides and the included angle between them or the lengths of the three sides, then we need a different method to determine the other pieces of information about the triangle. This is where the Law of Cosines is useful.

We first explored the Law of Cosines in the beginning activity for this section. Following is the usual formal statement of the Law of Cosines. A proof of the Law of Cosines is included at the end of this section.

Law of Cosines
In a triangle, if a, b, and c are the lengths of the sides opposite angles α, β, and γ, respectively, then

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$
$$b^2 = a^2 + c^2 - 2ac\cos(\beta)$$
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$

As with the Law of Sines, there are three equations in the Law of Cosines. However, we can remember this with only one equation since the key to using the Law of Cosines is that this law involves 4 quantities. These 4 quantities are the lengths of the three sides and the measure of one of the angles of the triangle as shown in Figure 3.16.

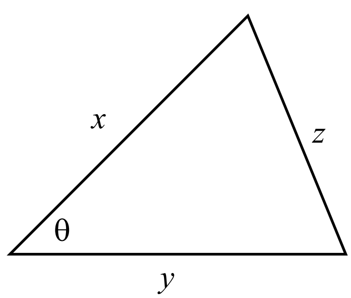


Figure 3.16: Diagram for the Law of Cosines

In this diagram, x, y, and z are the lengths of the three sides and θ is the angle between the sides x and y. Theta can also be thought of as the angle opposite side z. So we can write the Law of Cosines as follows:

Law of Cosines
In a triangle, if x, y, and z are the lengths of the sides of a triangle and θ is the angle between the sides x and y as in Figure 3.16, then

$$z^2 = x^2 + y^2 - 2xy\cos(\theta).$$

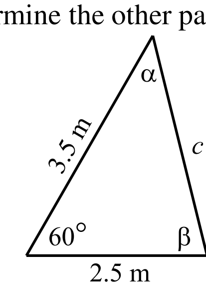
The idea is that if you know 3 of these 4 quantities, you can use the Law of Cosines to determine the fourth quantity. The Law of Cosines involves the lengths of all three sides of a triangle and one angle. It states that:

The square of the side opposite an angle is the sum of the squares of the two sides of the angle minus two times the product of the two sides of the angle and the cosine of the angle.

We will explore the use of the Law of Cosines in the next progress check.

Progress Check 3.17 (Using the Law of Cosines
Two sides of a triangle have length 2.5 meters and 3.5 meters, and the angle formed by these two sides has a measure of 60°. Determine the other parts of the triangle.

The first step is to draw a reasonably accurate diagram of the triangle and label the parts. This is shown in the diagram on the right.



1. Use the Law of Cosines to determine the length of the side opposite the 60° angle. (c).

We now know an angle (60°) and the length of its opposite side. We can use the Law of Sines to determine the other two angles. However, remember that we must be careful when using the Law of Sines to determine an angle since the equation may produce two angles.

2. Use the Law of Sines to determine $\sin(\alpha)$. Determine the two possible values for α and explain why one of them is not possible.

3. Use the fact that the sum of the angles of a triangle is 180° to determine the angle β.

4. Use the Law of Sines to check the results.

We used the Law of Sines to determine two angles in Progress Check 3.17 and saw that we had to be careful since the equation for the Law of Sines often produces two possible angles. We can avoid this situation by using the Law of Cosines to determine the angles instead. This is because an equation of the form $\cos(\theta) = p$, where p is a real number between 0 and 1 has only one solution for θ between $0°$ and $180°$. The idea is to solve an equation from the Law of Cosines for the cosine of the angle. In Progress Check 3.17, we first determined $c^2 = 9.75$ or $c \approx 3.12250$.. We then could have proceeded as follows:

$$\begin{aligned} 2.5^2 &= 3.5^2 + 3.12250^2 - 2(3.5)(3.12250)\cos(\alpha) \\ 2(3.5)(3.12250)\cos(\alpha) &= 3.5^2 + 3.12250^2 - 2.5^2 \\ \cos(\alpha) &= \frac{15.75}{21.8575} \approx 0.720577 \end{aligned}$$

We can then use the inverse cosine function and obtain $\alpha \approx 43.898°$, which is what we obtained in Progress Check 3.17.

We can now use the fact that the sum of the angles in a triangle is $180°$ to determine β but for completeness, we could also use the Law of Cosines to determine β and then use the angle sum for the triangle as a check on our work.

Progress Check 3.18 (Using the Law of Cosines)
The three sides of a triangle have lengths of 3 feet, 5 feet, and 6 feet. Use the Law of Cosines to determine each of the three angles.

Appendix – Proof of the Law of Sines

We will use what we know about right triangles to prove the Law of Sines. The key idea is to create right triangles from the diagram for a general triangle by drawing an altitude of length h from one of the vertices. We first note that if α, β, and γ are the three angles of a triangle, then

$$\alpha + \beta + \gamma = 180°.$$

This means that at most one of the three angles can be an obtuse angle (between $90°$ and $180°$), and hence, at least two of the angles must be acute (less than $90°$). Figure 3.17 shows the two possible cases for a general triangle. The triangle on the left has three acute angles and the triangle on the right has two acute angles (α and β) and one obtuse angle (γ).

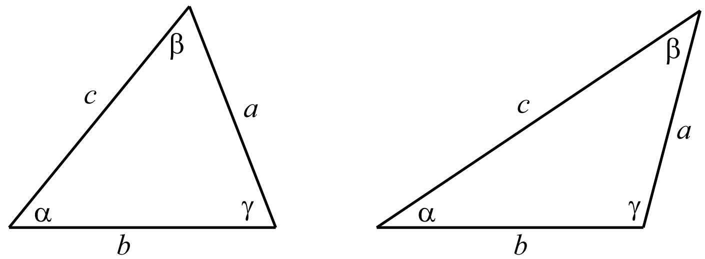


Figure 3.17: General Triangles

We will now prove the Law of Sines for the case where all three angles of the triangle are acute angles. The proof for the case where one angle of the triangle is obtuse is included in the exercises. The key idea is to create right triangles from the diagram for a general triangle by drawing altitudes in the triangle as shown in Figure 3.18 where an altitude of length h is drawn from the vertex of angle β and an altitude of length k is drawn from the vertex of angle γ.

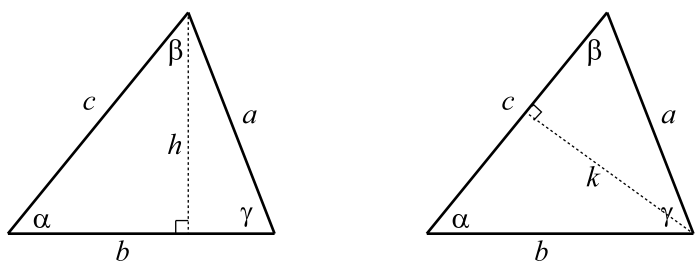


Figure 3.18: Diagram for the Proof of the Law of Sines

Using the right triangles in the diagram on the left, we see that

$$\sin(\alpha) = \frac{h}{c} \qquad \sin(\gamma) = \frac{h}{a}$$

From this, we can conclude that

$$h = c\sin(\alpha) \qquad h = a\sin(\gamma) \tag{1}$$

Using the two equations in (1), we can use the fact that both of the right sides are equal to h to conclude that

$$c\sin(\alpha) = a\sin(\gamma).$$

Now, dividing both sides of the last equation by ac, we see that

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}. \qquad (2)$$

We now use a similar argument using the triangle on the right in Figure 3.18. We see that

$$\sin(\alpha) = \frac{k}{b} \qquad \sin(\beta) = \frac{k}{a}$$

From this, we obtain

$$k = b\sin(\alpha) \qquad k = a\sin(\beta)$$

and so

$$b\sin(\alpha) = a\sin(\beta)$$
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} \qquad (3)$$

We can now use equations (2) and (3) to complete the proof of the Law of Sines, which is

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

Appendix – Proof of the Law of Cosines

As with the Law of Sines, we will use results about right triangles to prove the Law of Cosines. We will also use the distance formula. We will start with a general triangle with a, b, and c representing the lengths of the sides opposite the angles α, β, and γ, respectively. We will place the angle γ in standard position in the coordinate system as shown in Figure 3.19.

In this diagram, the angle γ is shown as an obtuse angle but the proof would be the same if γ was an acute angle. We have labeled the vertex of angle α as A with coordinates (x, y) and we have drawn a line from A perpendicular to the x-axis. So from the definitions of the trigonometric functions in Section 3.1, we see that

$$\cos(\gamma) = \frac{x}{b} \qquad \sin(\gamma) = \frac{y}{b}$$
$$x = b\cos(\gamma) \qquad y = b\sin(\gamma) \qquad (4)$$

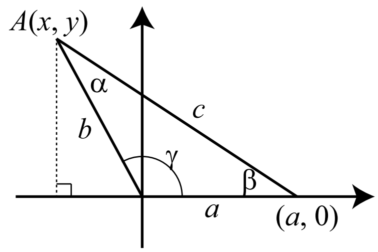


Figure 3.19: Diagram for the Law of Cosines

We now use the distance formula with the points A and the vertex of angle β, which has coordinates $(a, 0)$. This gives

$$\begin{aligned} c &= \sqrt{(x-a)^2 + (y-0)^2} \\ c^2 &= (x-a)^2 + y^2 \\ c^2 &= x^2 - 2ax + a^2 + y^2 \end{aligned}$$

We now substitute the values for x and y in equation (4) and obtain

$$\begin{aligned} c^2 &= b^2\cos^2(\gamma) - 2ab\cos(\gamma) + a^2 + b^2\sin^2(\gamma) \\ c^2 &= a^2 + b^2\cos^2(\gamma) + b^2\sin^2(\gamma) - 2ab\cos(\gamma) \\ c^2 &= a^2 + b^2\left(\cos^2(\gamma) + \sin^2(\gamma)\right) - 2ab\cos(\gamma) \end{aligned}$$

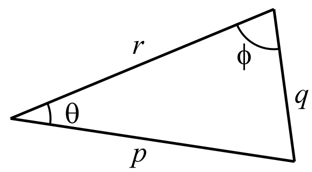
We can now use the last equation and the fact that $\cos^2(\gamma) + \sin^2(\gamma) = 1$ to conclude that

$$c^2 = a^2 + b^2 \; - 2ab\cos(\gamma).$$

This proves one of the equations in the Law of Cosines. The other two equations can be proved in the same manner by placing each of the other two angles in standard position.

Summary of Section 3.3

In this section, we studied the following important concepts and ideas:
The Law of Sines and the Law of Cosines can be used to determine the lengths of sides of a triangle and the measure of the angles of a triangle.



The **Law of Sines** states that if q is the length of the side opposite the angle θ and p is the length of the side opposite the angle ϕ, then

$$\frac{\sin(\theta)}{q} = \frac{\sin(\phi)}{p}.$$

The **Law of Cosines** states that if p, q, and r are the lengths of the sides of a triangle and θ is the angles opposite the side q, then

$$q^2 = p^2 + r^2 - 2pr\cos(\theta).$$

Each of the equations in the Law of Sines and the Law of Cosines involves four variables. So if we know the values of three of the variables, then we can use the appropriate equation to solve for the fourth variable.

Exercises for Section 3.3

For Exercises (1) through (4), use the Law of Sines.

⋆ **1.** Two angles of a triangle are 42° and 73°. The side opposite the 73° angle is 6.5 feet long. Determine the third angle of the triangle and the lengths of the other two sides.

2. A triangle has a side that is 4.5 meters long and this side is adjacent to an angle of 110°. In addition, the side opposite the 110° angle is 8 meters long. Determine the other two angles of the triangle and the length of the third side.

⋆ **3.** A triangle has a side that is 5 inches long that is adjacent to an angle of 61°. The side opposite the 61° angle is 4.5 inches long. Determine the other two angles of the triangle and the length of the third side.

4. In a given triangle, the side opposite an angle of 107° is 18 inches long. One of the sides adjacent to the 107° angle is 15.5 inches long. Determine the other two angles of the triangle and the length of the third side.

For Exercises (5) through (6), use the Law of Cosines.

⋆ **5.** The three sides of a triangle are 9 feet long, 5 feet long, and 7 feet long. Determine the three angles of the triangle.

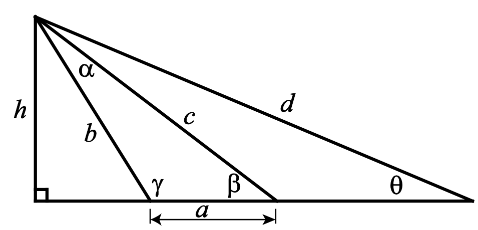
6. A triangle has two sides of lengths 8.5 meters and 6.8 meters. The angle formed by these two sides is 102°. Determine the length of the third side and the other two angles of the triangle.

For the remaining exercises, use an appropriate method to solve the problem.

7. Two angles of a triangle are 81.5° and 34°. The length of the side opposite the third angle is 8.8 feet. Determine the third angle and the lengths of the other two sides of the triangle.

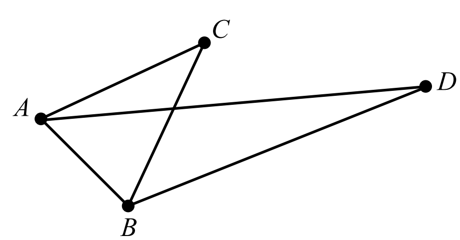
8. In the diagram to the right, determine the value of γ (to the nearest hundredth of a degree) and determine the values of h and d (to the nearest thousandth) if it is given that

$$a = 4 \qquad b = 8$$
$$c = 10 \qquad \theta = 26°$$



9. In the diagram to the right, it is given that:

- The length of AC is 2.
- The length of BC is 2.
- $\angle ACB = 40°$.
- $\angle CAD = 20°$.
- $\angle CBD = 45°$.



Determine the lengths of AB and AD to the nearest thousandth.

3.4 Applications of Triangle Trigonometry

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we use the Law of Sines and the Law of Cosines to help solve applied problems that involve triangles?
- How do we determine the area of a triangle?
- What is Heron's Law for the area of a triangle?

In Section 3.2, we used right triangles to solve some applied problems. It should then be no surprise that we can use the Law of Sines and the Law of Cosines to solve applied problems involving triangles that are not right triangles.

In most problems, we will first draw a rough diagram or picture showing the triangle or triangles involved in the problem. We then need to label the known quantities. Once that is done, we can see if there is enough information to use the Law of Sines or the Law of Cosines. Remember that each of these laws involves four quantities. If we know the value of three of those four quantities, we can use that law to determine the fourth quantity.

We begin with the example in Progress Check 3.12. The solution of this problem involved some complicated work with right triangles and some algebra. We will now solve this problem using the results from Section 3.3.

Example 3.19 (Height to the Top of a Flagpole)
Suppose that the flagpole sits on top a hill and that we cannot directly measure the length of the shadow of the flagpole as shown in Figure 3.19.

Some quantities have been labeled in the diagram. Angles α and β are angles of elevation to the top of the flagpole from two different points on level ground. These points are d feet apart and directly in line with the flagpole. The problem is to determine h, the height from level ground to the top of the flagpole. The following measurements have been recorded.

$$\alpha = 43.2^\circ \qquad d = 22.75\text{feet}$$
$$\beta = 34.7^\circ$$

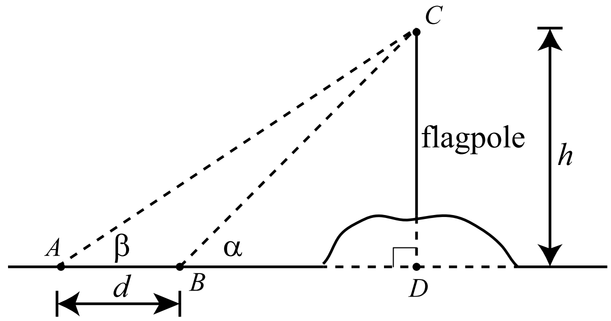


Figure 3.20: Flagpole on a hill

We notice that if we knew either length BC or BD in $\triangle BDC$, then we could use right triangle trigonometry to determine the length CD, which is equal to h. Now look at $\triangle ABC$. We are given the measure of angle β. However, we also know the measure of angle α. Because they form a straight angle, we have

$$\angle ABC + \alpha = 180°.$$

Hence, $\angle ABC = 180° - 43.2° = 136.8°$. We now know two angles in $\triangle ABC$ and hence, we can determine the third angle as follows:

$$\begin{aligned} \beta + \angle ABC + \angle ACB &= 180° \\ 34.7° + 136.8° + \angle ACB &= 180° \\ \angle ACB &= 8.5° \end{aligned}$$

We now know all angles in $\triangle ABC$ and the length of one side. We can use the Law of Sines. We have

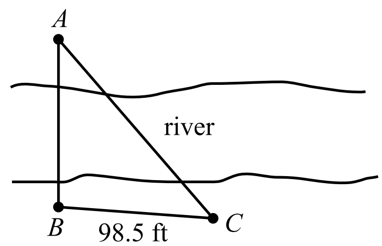
$$\begin{aligned} \frac{BC}{\sin(34.7°)} &= \frac{22.75}{\sin(8.5°)} \\ BC &= \frac{22.75\sin(34.7°)}{\sin(8.5°)} \approx 87.620 \end{aligned}$$

We can now use the right triangle $\triangle BDC$ to determine h as follows:

$$\begin{aligned} \frac{h}{BC} &= \sin\left(43.2°\right) \\ h &= BC \cdot \sin\left(43.2°\right) \approx 59.980 \end{aligned}$$

So the top of the flagpole is 59.980 feet above the ground. This is the same answer we obtained in Progress Check 3.12.

Progress Check 3.20 (An Application)
A bridge is to be built across a river. The bridge will go from point A to point B in the diagram on the right. Using a transit (an instrument to measure angles), a surveyor measures angle ABC to be 94.2° and measures angle BCA to be 48.5°. In addition, the distance from B to C is measured to be 98.5 feet. How long will the bridge from point B to point A be?



Area of a Triangle

We will now develop a few different ways to calculate the area of a triangle. Perhaps the most familiar formula for the area is the following:

> The area A of a triangle is
> $$A = \frac{1}{2}bh,$$
> where b is the length of the base of a triangle and h is the length of the altitude that is perpendicular to that base.

The triangles in Figure 3.21 illustrate the use of the variables in this formula.

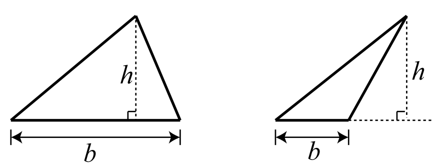
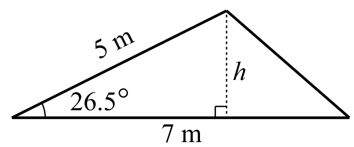


Figure 3.21: Diagrams for the Formula for the Area of a Triangle

A proof of this formula for the area of a triangle depends on the formula for the area of a parallelogram and is included in Appendix C.

Progress Check 3.21 (The Area of a Triangle)

Suppose that the length of two sides of a triangle are 5 meters and 7 meters and that the angle formed by these two sides is 26.5°. See the diagram on the right.



For this problem, we are using the side of length 7 meters as the base. The altitude of length h that is perpendicular to this side is shown.

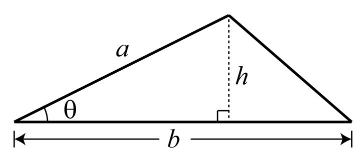
1. Use right triangle trigonometry to determine the value of h.
2. Determine the area of this triangle.

One purpose of Progress Check 3.21 was to illustrate that if we know the length of two sides of a triangle and the angle formed by these two sides, then we can determine the area of that triangle.

> **The Area of a Triangle**
> The area of a triangle equals one-half the product of two of its sides times the sine of the angle formed by these two sides.

Progress Check 3.22 (Proof of the Formula for the Area of a Triangle)

In the diagram on the right, b is the length of the base of a triangle, a is the length of another side, and θ is the angle formed by these two sides. We let A be the area of the triangle.



Follow the procedure illustrated in Progress Check 3.21 to prove that

$$A = \frac{1}{2}ab\sin(\theta).$$

Explain why this proves the formula for the area of a triangle.

There is another common formula for the area of a triangle known as Heron's Formula named after Heron of Alexandria (circa 75 CE). This formula shows that the area of a triangle can be computed if the lengths of the three sides of the triangle are known.

Heron's Formula
The area A of a triangle with sides of length a, b, and c is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \dfrac{1}{2}(a+b+c)$.

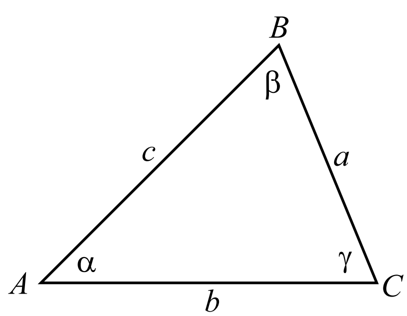
For example, suppose that the lengths of the three sides of a triangle are $a = 3$ ft, $b = 5$ ft, and $c = 6$ ft. Using Heron's Formula, we get

$$\begin{aligned} s &= \frac{1}{2}(a+b+c) & A &= \sqrt{s(s-a)(s-b)(s-c)} \\ s &= 7 & A &= \sqrt{7(7-3)(7-5)(7-6)} \\ & & A &= \sqrt{42} \end{aligned}$$

This fairly complex formula is actually derived from the previous formula for the area of a triangle and the Law of Cosines. We begin our exploration of the proof of this formula in Progress Check 3.23.

Progress Check 3.23 (Heron's Formula)

Suppose we have a triangle as shown in the diagram on the right.



1. Use the Law of Cosines that involves the angle γ and solve this formula for $\cos(\gamma)$. This gives a formula for $\cos(\gamma)$ in terms of a, b, and c.

2. Use the Pythagorean Identity $\cos^2(\gamma) + \sin^2(\gamma) = 1$ to write $\sin(\gamma)$ in terms of $\cos^2(\gamma)$. Substitute for $\cos^2(\gamma)$ using the formula in part (1). This gives a formula for $\sin(\gamma)$ in terms of a, b, and c. (Do not do any algebraic implication.)

3. We also know that a formula for the area of this triangle is $A = \dfrac{1}{2}ab\sin(\gamma)$. Substitute for $\sin(\gamma)$ using the formula in (2). (Do not do any algebraic simplification.) This gives a formula for the area A in terms of a, b, and c.

The formula obtained in Progress Check 3.23 was

$$A = \frac{1}{2}ab\sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2}$$

This is a formula for the area of a triangle in terms of the lengths of the three sides of the triangle. It does not look like Heron's Formula, but we can use some substantial algebra to rewrite this formula to obtain Heron's Formula. This algebraic work is completed in the appendix for this section.

Appendix – Proof of Heron's Formula

The formula for the area of a triangle obtained in Progress Check 3.23 was

$$A = \frac{1}{2}ab\sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2}$$

We now complete the algebra to show that this is equivalent to Heron's formula. The first step is to rewrite the part under the square root sign as a single fraction.

$$\begin{aligned}
A &= \frac{1}{2}ab\sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2} \\
&= \frac{1}{2}ab\sqrt{\frac{(2ab)^2 - \left(a^2 + b^2 - c^2\right)^2}{(2ab)^2}} \\
&= \frac{1}{2}ab\frac{\sqrt{(2ab)^2 - \left(a^2 + b^2 - c^2\right)^2}}{2ab} \\
&= \frac{\sqrt{(2ab)^2 - \left(a^2 + b^2 - c^2\right)^2}}{4}
\end{aligned}$$

Squaring both sides of the last equation, we obtain

$$A^2 = \frac{(2ab)^2 - \left(a^2 + b^2 - c^2\right)^2}{16}.$$

The numerator on the right side of the last equation is a difference of squares. We will now use the difference of squares formula, $x^2 - y^2 = (x - y)(x + y)$ to factor the numerator.

$$\begin{aligned}
A^2 &= \frac{(2ab)^2 - \left(a^2 + b^2 - c^2\right)^2}{16} \\
&= \frac{\left(2ab - \left(a^2 + b^2 - c^2\right)\right)\left(2ab + \left(a^2 + b^2 - c^2\right)\right)}{16} \\
&= \frac{\left(-a^2 + 2ab - b^2 + c^2\right)\left(a^2 + 2ab + b^2 - c^2\right)}{16}
\end{aligned}$$

We now notice that $-a^2 + 2ab - b^2 = -(a-b)^2$ and $a^2 + 2ab + b^2 = (a+b)^2$. So using these in the last equation, we have

$$\begin{aligned}
A^2 &= \frac{\left(-(a-b)^2 + c^2\right)\left((a+b)^2 - c^2\right)}{16} \\
&= \frac{\left(-\left[(a-b)^2 - c^2\right]\right)\left((a+b)^2 - c^2\right)}{16}
\end{aligned}$$

We can once again use the difference of squares formula as follows:

$$\begin{aligned}
(a-b)^2 - c^2 &= (a-b-c)(a-b+c) \\
(a+b)^2 - c^2 &= (a+b-c)(a+b+c)
\end{aligned}$$

Substituting this information into the last equation for A^2, we obtain

$$A^2 = \frac{-(a-b-c)(a-b+c)(a+b-c)(a+b+c)}{16}.$$

Since $s = \dfrac{1}{2}(a+b+c)$, $2s = a+b+c$. Now notice that

$$\begin{aligned}
-(a-b-c) &= -a+b+c \\
&= a+b+c-2a \\
&= 2s-2a
\end{aligned}
\qquad
\begin{aligned}
a-b+c &= a+b+c-2b \\
&= 2s-2b
\end{aligned}$$

$$\begin{aligned}
a+b-c &= a+b+c-2c \\
&= 2s-2c
\end{aligned}
\qquad
a+b+c = 2s$$

So

$$
\begin{aligned}
A^2 &= \frac{-(a-b-c)(a-b+c)(a+b-c)(a+b+c)}{16} \\
&= \frac{(2s-2a)(2s-2b)(2s-2c)(2s)}{16} \\
&= \frac{16s(s-a)(s-b)(s-c)}{16} \\
&= s(s-a)(s-b)(s-c) \\
A &= \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}
$$

This completes the proof of Heron's formula.

Summary of Section 3.4

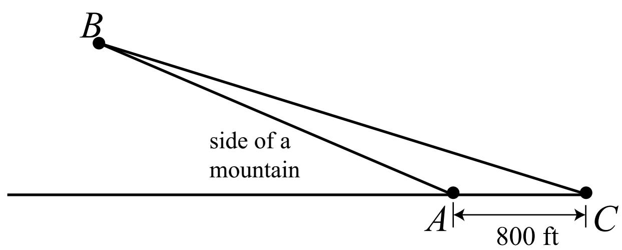
In this section, we studied the following important concepts and ideas:

- How to use right triangle trigonometry, the Law of Sines, and the Law of Cosines to solve applied problems involving triangles.
- Three ways to determine the area A of a triangle.
  - $A = \frac{1}{2}bh$, where b is the length of the base and h is the length of the altitude.
  - $A = \frac{1}{2}ab\sin(\theta)$, where a and b are the lengths of two sides of the triangle and θ is the angle formed by the sides of length a and b.
  - **Heron's Formula**. If a, b, and c are the lengths of the sides of a triangle and $s = \frac{1}{2}(a+b+c)$, then

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

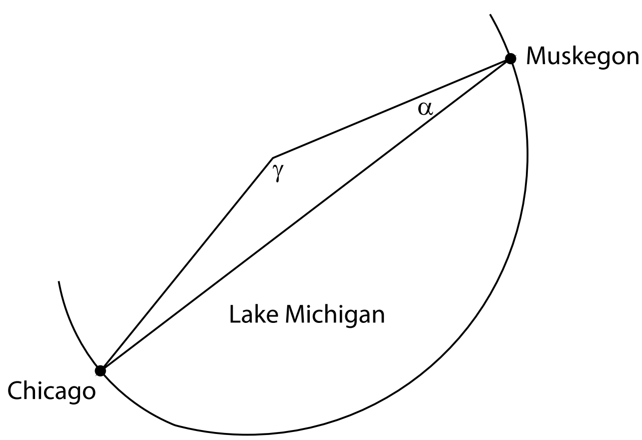
Exercises for Section 3.4

⋆ **1.** A ski lift is to be built along the side of a mountain from point A to point B in the following diagram. We wish to determine the length of this ski lift.



A surveyor determines the measurement of angle BAC to be 155.6° and then measures a distance of 800 feet from Point A to Point C. Finally, she determines the measurement of angle BCA to be 17.2°. What is the length of the ski lift (from point A to point B)?

⋆ **2.** A boat sails from Muskegon bound for Chicago, a sailing distance of 121 miles. The boat maintains a constant speed of 15 miles per hour. After encountering high cross winds the crew finds itself off course by 20° after 4 hours. A crude picture is shown in the following diagram, where $\alpha = 20°$.

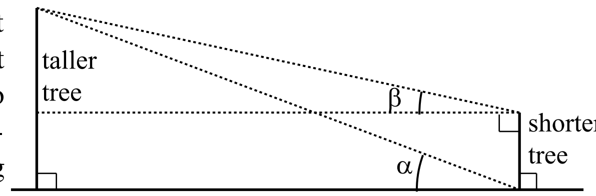


(**a**) How far is the sailboat from Chicago at this time?

(**b**) What is the degree measure of the angle γ (to the nearest tenth) in the diagram? Through what angle should the boat turn to correct its course and be heading straight to Chicago?

(**c**) Assuming the boat maintains a speed of 15 miles per hour, how much time have they added to their trip by being off course?

3. Two trees are on opposite sides of a river. It is known that the height of the shorter of the two trees is 13 meters. A person makes the following angle measurements:

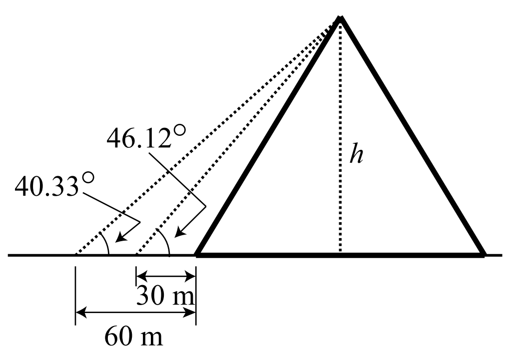


- The angle of elevation from the base of the shorter tree to the top of the taller tree is $\alpha = 20°$.
- The angle of elevation from the top of the shorter tree to the top of the taller tree is $\beta = 12°$.

Determine the distance between the bases of the two trees and the height of the taller tree.

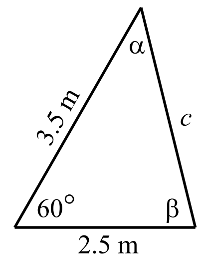
4. One of the original Seven Wonders of the World, the Great Pyramid of Giza (also known as the Pyramid of Khufu or the Pyramid of Cheops), was believed to have been built in a 10 to 20 year period concluding around 2560 B.C.E. It is also believed that the original height of the pyramid was 146.5 meters but that it is now shorter due to erosion and the loss of some topmost stones. [1]

To determine its current height, the angle of elevation from a distance of 30 meters from the base of the pyramid was measured to be 46.12°, and then the angle of elevation was measured to be 40.33° from a distance of 60 meters from the base of the pyramid as shown in the following diagram. Use this information to determine the height h of the pyramid. (138.8 meters)

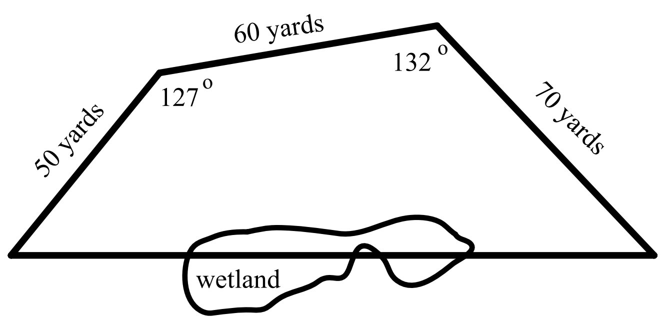


[1] https://en.wikipedia.org/wiki/Great_Pyramid_of_Giza

5. Two sides of a triangle have length 2.5 meters and 3.5 meters, and the angle formed by these two sides has a measure of 60°. Determine the area of the triangle. **Note**: This is the triangle in Progress Check 3.17 on page 200.



6. A field has the shape of a quadrilateral that is not a parallelogram. As shown in the following diagram, three sides measure 50 yards, 60 yards, and 70 yards. Due to some wetland along the fourth side, the length of the fourth side could not be measured directly. Two angles shown in the diagram measure 127° and 132°.



Determine the length of the fourth side of the quadrilateral, the measures of the other two angles in the quadrilateral, and the area of the quadrilateral. Lengths must be accurate to the nearest hundredth of a yard, angle measures must be correct to the nearest hundredth of a degree, and the area must be correct to the nearest hundredth of a square yard.

3.5 Vectors from a Geometric Point of View

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is a vector?
- How do we use the geometric form of vectors to find the sum of two vectors?
- How do we use the geometric form of vectors to find a scalar multiple of a vector?
- How do we use the geometric form of vectors to find the difference of two vectors?
- What is the angle between two vectors?
- Why is force a vector and how do we use vectors and triangles to determine forces acting on an object?

We have all had the experience of dropping something and watching it fall to the ground. What is happening, of course, is that the force of gravity is causing the object to fall to the ground. In fact, we experience the force of gravity everyday simply by being on Earth. Each person's weight is a measure of the force of gravity since pounds are a unit of force. So when a person weighs 150 pounds, it means that gravity is exerting a force of 150 pounds straight down on that person. Notice that we described this with a quantity and a direction (straight down). Such a quantity (with magnitude and direction) is called a vector.

Now suppose that person who weighs 150 pounds is standing on a hill. In mathematics, we simplify the situation and say that the person is standing on an inclined plane as shown in Figure 3.22. (By making the hill a straight line, we simplify the mathematics involved.) In the diagram in Figure 3.22, an object is on the inclined plane at the point P. The inclined plane makes an angle of θ with the horizontal. The vector **w** shows the weight of the object (force of gravity, straight down). The diagram also shows two other vectors. The vector **b** is perpendicular

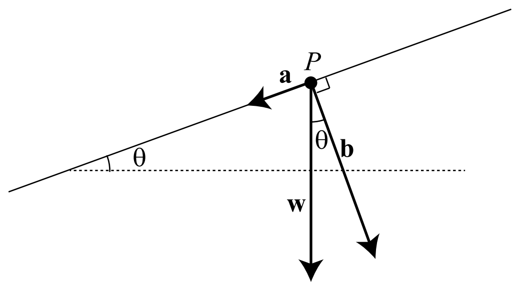


Figure 3.22: Inclined Plane

to the plane and represents the force that the object exerts on the plane. The vector **a** is perpendicular to **b** and parallel to the inclined plane. This vector represents the force of gravity along the plane. In this and the next section, we will learn more about these vectors and how to determine the magnitudes of these vectors. We will also see that with our definition of the addition of two vectors that $\mathbf{w} = \mathbf{a} + \mathbf{b}$.

Definitions

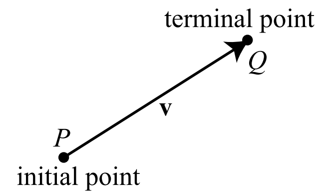
There are some quantities that require only a number to describe them. We call this number the magnitude of the quantity. One such example is temperature since we describe this with only a number such as 68 degrees Fahrenheit. Other such quantities are length, area, and mass. These types of quantities are often called **scalar quantities**. However, there are other quantities that require both a magnitude and a direction. One such example is force, and another is velocity. We would describe a velocity with something like 45 miles per hour northwest. Velocity and force are examples of a **vector quantity**. Other examples of vectors are acceleration and displacement.

Some vectors are closely associated with scalars. In mathematics and science, we make a distinction between *speed* and *velocity*. Speed is a scalar and we would say something like our speed is 65 miles per hour. However, if we used a velocity, we would say something like 65 miles per hour east. This is different than a velocity of 65 miles per hour north even though in both cases, the speed is 65 miles per hour.

Definition. A **vector** is a quantity that has both magnitude and direction. A **scalar** is a quantity that has magnitude only.

Geometric Representation of Vectors

Vectors can be represented geometrically by arrows (directed line segments). The **arrowhead** indicates the direction of the vector, and the **length** of the arrow describes the magnitude of the vector. A vector with initial point P (the tail of the arrow) and terminal point Q (the tip of the arrowhead) can be represented by



$$\overrightarrow{PQ}, \mathbf{v}, \text{ or } \overrightarrow{v}.$$

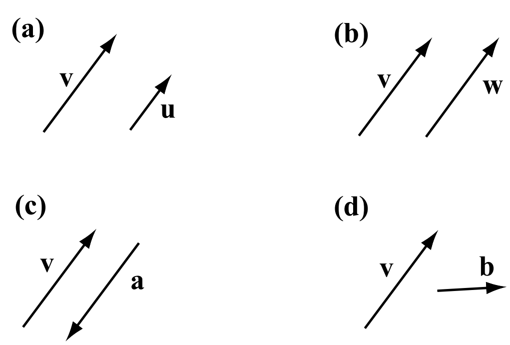
We often write $\mathbf{v} = \overrightarrow{PQ}$. In this text, we will use boldface font to designate a vector. When writing with pencil and paper, we always use an arrow above the letter (such as $\overrightarrow{v}$) to designate a vector. The **magnitude** (or **norm** or **length**) of the vector $\mathbf{v}$ is designated by $|\mathbf{v}|$. It is important to remember that $|\mathbf{v}|$ is a number that represents the magnitude or length of the vector $\mathbf{v}$.

According to our definition, a vector possesses the attributes of length (magnitude) and direction, but position is not mentioned. So we will consider two vectors to be equal if they have the same magnitude and direction. For example, if two different cars are both traveling at 45 miles per hour northwest (but in different locations), they have equal velocity vectors. We make a more formal definition.

Definition. Two vectors are **equal** if and only if they have the same magnitude and the same direction. When the vectors $\mathbf{v}$ and $\mathbf{w}$ are equal, we write $\mathbf{v} = \mathbf{w}$.

Progress Check 3.24 (Equal and Unequal Vectors)

In each of the following diagrams, a vector $\mathbf{v}$ is shown next to four other vectors. Which (if any) of these four vectors are equal to the vector $\mathbf{v}$?



Operations on Vectors

Scalar Multiple of a Vector

Doubling a scalar quantity is simply a matter of multiplying its magnitude by 2. For example, if a container has 20 ounces of water and the amount of water is doubled, it will then have 40 ounces of water. What do we mean by doubling a vector? The basic idea is to keep the same direction and multiply the magnitude by 2. So if an object has a velocity of 5 feet per second southeast and a second object has a velocity of twice that, the second object will have a velocity of 10 feet per second in the southeast direction. In this case, we say that we multiplied the vector by the scalar 2. We now make a definition that also takes into account that a scalar can be negative.

Definition. For any vector $\mathbf{v}$ and any scalar c, the vector $c\mathbf{v}$ (called a **scalar multiple** of the vector $\mathbf{v}$) is a vector whose magnitude is $|c|$ times the magnitude of the vector $\mathbf{v}$. That is,

$$|c\mathbf{v}| = |c||\mathbf{v}|.$$

Note: In this equation, $|c|$ is the absolute value of the scalar c. Care must be taken not to confuse this with the notation $|\mathbf{v}|$, which is the magnitude of the vector $\mathbf{v}$. This is one reason it is important to have a notation that clearly indicates when we are working with a vector or a scalar. Also, using this definition, we see that

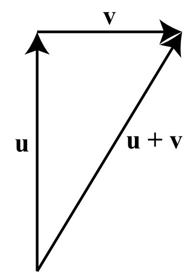
- If $c > 0$, then the direction of $c\mathbf{v}$ is the same as the direction of $\mathbf{v}$.
- If $c < 0$, then the direction of $c\mathbf{v}$ is the opposite of the direction of $\mathbf{v}$.
- If $c = 0$, then $c\mathbf{v} = 0\mathbf{v} = \mathbf{0}$.

The vector $\mathbf{0}$ is called the **zero vector** and the zero vector has no magnitude and no direction. We sometimes write $\overrightarrow{0}$ for the zero vector.

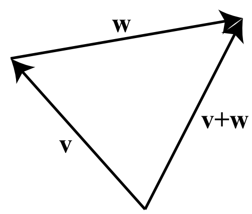
Addition of Vectors

We illustrate how to add vectors with two displacement vectors. As with velocity and speed, there is a distinction between displacement and distance. **Distance** is a scalar. So we might say that we have traveled 2 miles. **Displacement**, on the other hand, is a vector consisting of a distance and a direction. So the vectors 2 miles north and 2 miles east are different displacement vectors.

Now if we travel 3 miles north and then travel 2 miles east, we end at a point that defines a new displacement vector. See the diagram to the right. In this diagram, **u** is "3 miles north" and **v** is "2 miles east." The vector sum **u** + **v** goes from the initial point of **u** to the terminal point of **v**.



Definition. The **sum of two vectors** is defined as follows: We position the vectors so that the initial point of **w** coincides with the terminal point of **v**. The vector **v** + **w** is the vector whose initial point coincides with the initial point of **v** and whose terminal point coincides with the terminal point of **w**.



The vector **v** + **w** is called the **sum** or **resultant** of the vectors **v** and **w**.

In the definition, notice that the vectors **v**, **w**, and **v** + **w** are placed so that the result is a triangle. The lengths of the sides of that triangle are the magnitudes of these sides $|\mathbf{v}|$, $|\mathbf{w}|$, and $|\mathbf{v}+\mathbf{w}|$. If we place the two vectors **v** and **w** so that their initial points coincide, we can use a parallelogram to add the two vectors. This is shown in Figure 3.23.

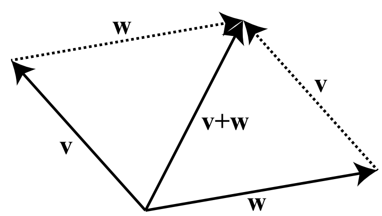
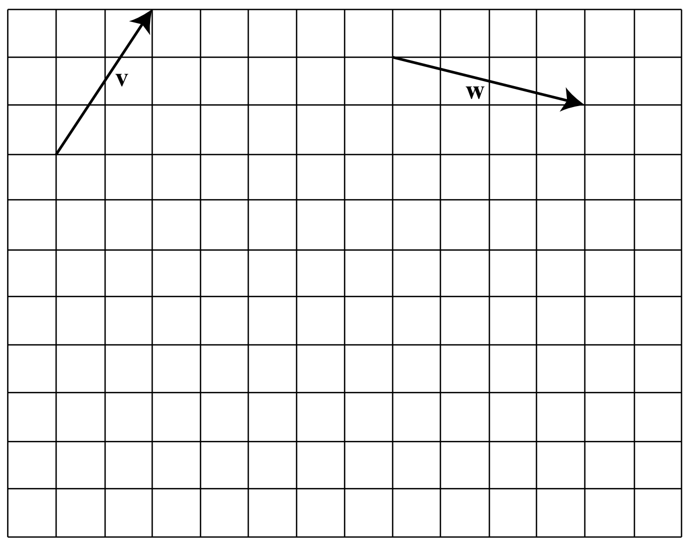


Figure 3.23: Sum of Two Vectors Using a Parallelogram

Notice that the vector **v** forms a pair of opposite sides of the parallelogram as does the vector **w**.

Progress Check 3.25 (Operations on Vectors)
The following diagram shows two vectors, **v** and **w**. Draw the following vectors:

(a) $\mathbf{v}+\mathbf{w}$ **(b)** $2\mathbf{v}$ **(c)** $2\mathbf{v}+\mathbf{w}$ **(d)** $-2\mathbf{w}$ **(e)** $-2\mathbf{w}+\mathbf{v}$



Subtraction of Vectors

Before explaining how to subtract vectors, we will first explain what is meant by the "negative of a vector." This works similarly to the negative of a real number. For example, we know that when we add -3 to 3, the result is 0. That is, $3+(-3)=0$.

We want something similar for vectors. For a vector **w**, the idea is to use the scalar multiple $(-1)\mathbf{w}$. The vector $(-1)\mathbf{w}$ has the same magnitude as **w** but has the opposite direction of **w**. We define $-\mathbf{w}$ to be $(-1)\mathbf{w}$. Figure shows that when we add $-\mathbf{w}$ to **w**, the terminal point of the sum is the same as the initial point of the sum and so the result is the zero vector. That is, $\mathbf{w}+(-\mathbf{w})=\mathbf{0}$.

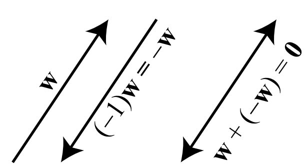
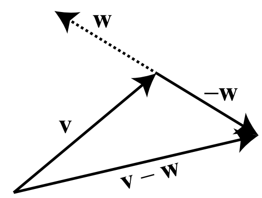


Figure 3.24: The Sum of a Vector and Its Negative

We are now in a position to define subtraction of vectors. The idea is much the same as subtraction of real numbers in that for any two real numbers a and b, $a - b = a + (-b)$.

Definition. For any two vectors **v** and **w**, the **difference between v and w** is denoted by $\mathbf{v} - \mathbf{w}$ and is defined as follows:

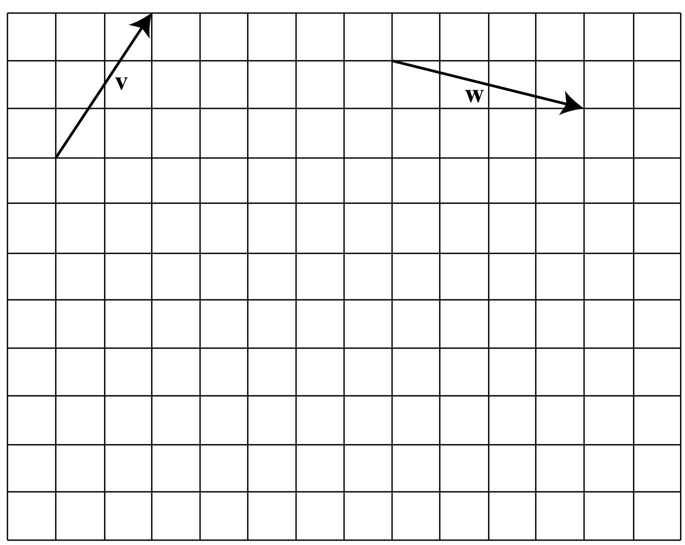
$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}).$$



We also say that we are subtracting the vector **w** from the vector **v**.

Progress Check 3.26 (Operations on Vectors)
The following diagram shows two vectors, **v** and **w**. Draw the following vectors:

(a) $-\mathbf{w}$ **(b)** $\mathbf{v} - \mathbf{w}$ **(c)** $-\mathbf{v}$ **(d)** $\mathbf{w} - \mathbf{v}$



The Angle Between Two Vectors

We have seen that we can use triangles to help us add or subtract two vectors. The lengths of the sides of the triangle are the magnitudes of certain vectors. Since we are dealing with triangles, we will also use angles determined by the vectors.

Definition. The angle θ between vectors is the angle formed by these two vectors (with $0° \leq \theta \leq 180°$) when they have the same initial point.

So in the diagram on the left in Figure 3.25, the angle θ is the angle between the vectors **v** and **w**. However, when we want to determine the sum of two angles, we often form the parallelogram determined by the two vectors as shown in the diagram on the right in Figure 3.25. (See page 427 in Appendix C for a summary of properties of a parallelogram.) We then will use the angle $180° - \theta$ and the Law of Cosines since we the two sides of the triangle are the lengths of **v** and **w** and the included angle is $180° - \theta$. We will explore this in the next progress check.

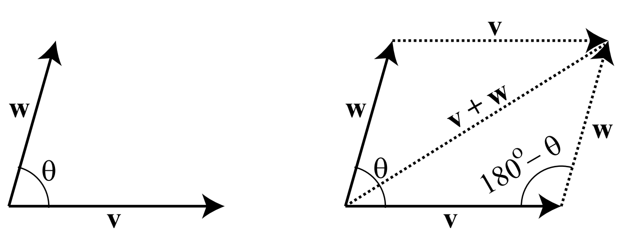


Figure 3.25: Angle Between Two Vectors

Progress Check 3.27 (The Sum of Two Vectors)
Suppose that the vectors **a** and **b** have magnitudes of 80 and 60, respectively, and that the angle θ between the two vectors is 53 degrees. In Figure 3.26, we have drawn the parallelogram determined by these two vectors and have labeled the vertices for reference.

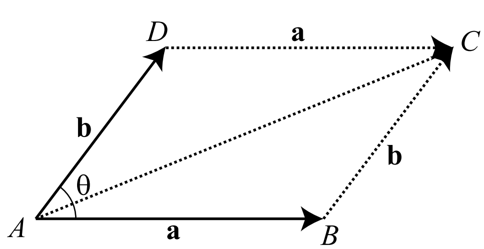


Figure 3.26: Diagram for Progress Check 3.27

Remember that a vector is determined by its magnitude and direction. We will determine $|\mathbf{a} + \mathbf{b}|$ and the measure of the angle between $\mathbf{a}$ and $\mathbf{a} + \mathbf{b}$.

1. Determine the measure of $\angle ABC$.

2. In $\triangle ABC$, the length of side AB is $|\mathbf{a}| = 80$ and the length of side BC is $|\mathbf{b}| = 60$. Use this triangle and the Law of Cosines to determine the length of the third side, which is $|\mathbf{a} + \mathbf{b}|$.

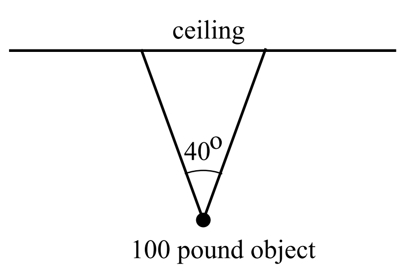
3. Determine the measure of the angle between $\mathbf{a}$ and $\mathbf{a} + \mathbf{b}$. This is $\angle CAB$ in $\triangle ABC$.

Force

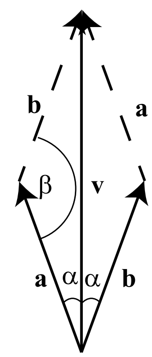
An important vector quantity is that of force. In physics, a force on an object is defined as any interaction that, when left un-opposed, will change the motion of the object. So a force will cause an object to change its velocity, that is the object will accelerate. More informally, a force is a push or a pull on an object.

One force that affects our lives is the force of gravity. The magnitude of the force of gravity on a person is that person's weight. The direction of the force of gravity is straight down. So if a person who weighs 150 pounds is standing still on the ground, then the ground is also exerting a force of 150 pounds on the person in the upward direction. The net force on the stationary person is zero. This is an example of what is known as *static equilibrium*. When an object is in static equilibrium, the sum of the forces acting on the object is equal to the zero vector.

Example 3.28 (Object Suspended from a Ceiling)
Suppose a 100 pound object is suspended from the ceiling by two wires that form a 40° angle as shown in the diagram to the right. Because the object is stationary, the two wires must exert a force on the object so that the sum of these two forces is equal to 100 pounds straight up. (The force of gravity is 100 pounds straight down.)



We will assume that the two wires exert forces of equal magnitudes and that the angle between these forces and the vertical in 20°. So our first step is to draw a picture of these forces, which is shown on the right. The vector **v** is a vector of magnitude 100 pounds. The vectors **a** and **b** are the vectors for the forces exerted by the two wires. (We have $|\mathbf{a}| = |\mathbf{b}|$.) We also know that $\alpha = 20°$ and so $2\alpha = 40°$. So the angle between the vectors **a** and **b** is 40° and so by the properties of parallelograms, $\beta = 140°$. (See page 427.)



Progress Check 3.29 (Completion of Example 3.28)
Use triangle trigonometry to determine the magnitude of the vector **a** in Example 3.28. Note that we already know the direction of this vector.

Inclined Planes

At the beginning of this section, we discussed the forces involved when an object is place on an inclined plane. Figure 3.27 is the diagram we used, but we now have added labels for some of the angles. Recall that the vector **w** shows the weight of the object (force of gravity, straight down), the vector **b** is perpendicular to the plane and represents the force that the object exerts on the plane, and the vector **a** is perpendicular to **b** and parallel to the inclined plane. This vector represents the force of gravity along the plane. Notice that we have also added a second copy of the vector **a** that begins at the tip of the vector **b**.

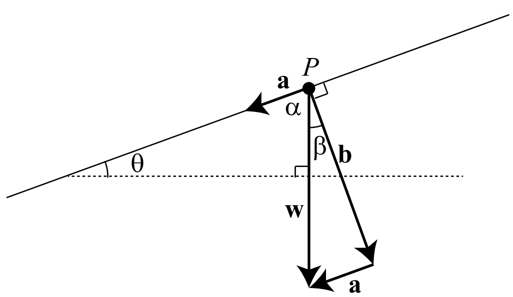


Figure 3.27: Inclined Plane

Using the angles shown, we see that $\alpha + \beta = 90°$ since they combine to form a right angle, and $\alpha + \theta = 90°$ since they are the two acute angles in a right triangle. From this, we conclude that $\beta = \theta$. This gives us the final version of the diagram of the forces on an inclined plane shown in Figure 3.28. Notice that the vectors **a**,

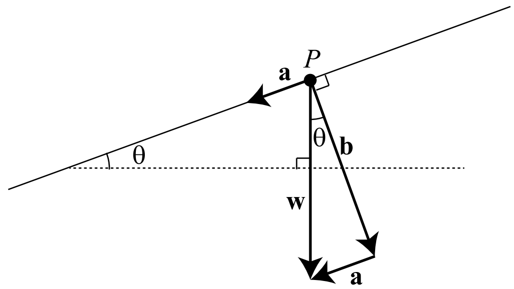


Figure 3.28: Inclined Plane

b, and **w** form a right triangle, and so we can use right triangle trigonometry for problems dealing with the forces on an inclined plane.

Progress Check 3.30 (A Problem Involving an Inclined Plane)
An object that weighs 250 pounds is placed on an inclined plane that makes an angle of 12° degrees with the horizontal. Using a diagram like the one in Figure 3.28, determine the magnitude of the force against the plane caused by the object and the magnitude of the force down the plane on the object due to gravity. **Note**: The magnitude of the force down the plane will be the force in the direction up the plane that is required to keep the object stationary.

Summary of Section 3.5

In this section, we studied the following important concepts and ideas:

Vectors and Scalars

A **vector** is a quantity that has both magnitude and direction. A **scalar** is a quantity that has magnitude only. Two vectors are **equal** if and only if they have the same magnitude and the same direction.

Scalar Multiple of a Vector

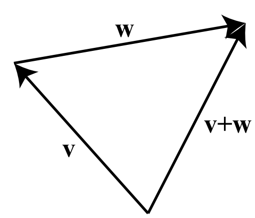
For any vector **v** and any scalar c, the vector $c\mathbf{v}$ (called a **scalar multiple** of the vector **v**) is a vector whose magnitude is $|c|$ times the magnitude of the vector **v**.

- If $c > 0$, then the direction of $c\mathbf{v}$ is the same as the direction of $\mathbf{v}$.
- If $c < 0$, then the direction of $c\mathbf{v}$ is the opposite of the direction of $\mathbf{v}$.
- If $c = 0$, then $c\mathbf{v} = 0\mathbf{v} = \mathbf{0}$.
- Using vector notation, we have $|c\mathbf{v}| = |c||\mathbf{v}|$.

The vector $\mathbf{0}$ is called the **zero vector** and the zero vector has no magnitude and no direction. We sometimes write $\overrightarrow{0}$ for the zero vector.

The Sum of Two Vectors

The **sum of two vectors** is defined as follows: We position the vectors so that the initial point of $\mathbf{w}$ coincides with the terminal point of $\mathbf{v}$. The vector $\mathbf{v} + \mathbf{w}$ is the vector whose initial point coincides with the initial point of $\mathbf{v}$ and whose terminal point coincides with the terminal point of $\mathbf{w}$.



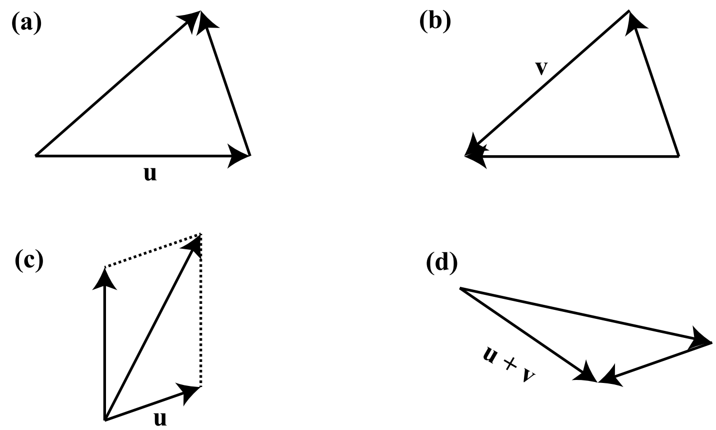
The vector $\mathbf{v} + \mathbf{w}$ is called the **sum** or **resultant** of the vectors $\mathbf{v}$ and $\mathbf{w}$.

The Angle Between Two Vectors

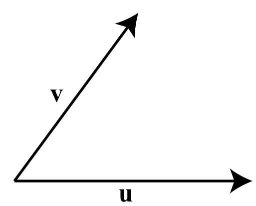
The angle θ between vectors is the angle formed by these two vectors (with $0° \leq \theta \leq 180°$) when they have the same initial point.

Exercises for Section 3.5

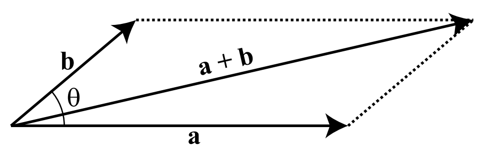
* **1.** In each of the following diagrams, one of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ is labeled. Label the other two vectors to make the diagram a valid representation of $\mathbf{u} + \mathbf{v}$.



★ 2. On the following diagram, draw the vectors $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $2\mathbf{u}+\mathbf{v}$, and $2\mathbf{u}-\mathbf{v}$.



★ 3. In the following diagram, $|\mathbf{a}| = 10$ and $|\mathbf{a}+\mathbf{b}| = 14$. In addition, the angle θ between the vectors $\mathbf{a}$ and $\mathbf{b}$ is $30°$. Determine the magnitude of the vector $\mathbf{b}$ and the angle between the vectors $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$.



4. Suppose that vectors **a** and **b** have magnitudes of 125 and 180, respectively. Also assume that the angle between these two vectors is $35°$. Determine the magnitude of the vector $\mathbf{a}+\mathbf{b}$ and the measure of the angle between the vectors $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$.

5. A car that weighs 3250 pounds is on an inclined plane that makes an angle of $4.5°$ with the horizontal. Determine the magnitude of the force of the car on the inclined plane, and determine the magnitude of the force on the car down the plane due to gravity. What is the magnitude of the smallest force necessary to keep the car from rolling down the plane?

6. An experiment determined that a force of 45 pounds is necessary to keep a 250 pound object from sliding down an inclined plane. Determine the angle the inclined plane makes with the horizontal.

7. A cable that can withstand a force of 4500 pounds is used to pull an object up an inclined plane that makes an angle of 15 degrees with the horizontal. What is the heaviest object that can be pulled up this plane with the cable? (Assume that friction can be ignored.)

3.6 Vectors from an Algebraic Point of View

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How do we find the component form of a vector?
- How do we find the magnitude and the direction of a vector written in component form?
- How do we add and subtract vectors written in component form and how do we find the scalar product of a vector written in component form?
- What is the dot product of two vectors?
- What does the dot product tell us about the angle between two vectors?
- How do we find the projection of one vector onto another?

Introduction and Terminology

We have seen that a vector is completely determined by magnitude and direction. So two vectors that have the same magnitude and direction are equal. That means that we can position our vector in the plane and identify it in different ways. For one, we can place the initial point of a vector **v** at the origin and the terminal point will wind up at some point (v_1, v_2) as illustrated in Figure 3.29.

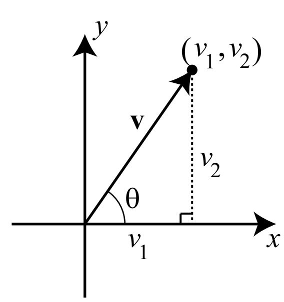


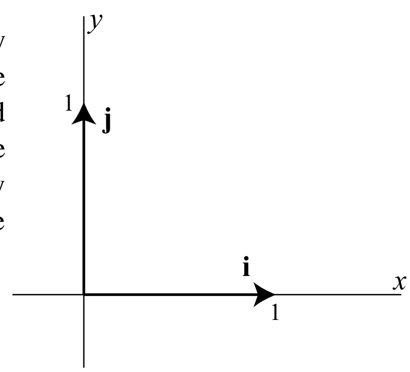
Figure 3.29: A Vector in Standard Position

A vector with its initial point at the origin is said to be in **standard position** and is represent by $\mathbf{v} = \langle v_1, v_2 \rangle$. Please note the important distinction in notation between the vector $\mathbf{v} = \langle v_1, v_2 \rangle$ and the point (v_1, v_2). The coordinates of the terminal point (v_1, v_2) are called the **components** of the vector $\mathbf{v}$. We call $\mathbf{v} = \langle v_1, v_2 \rangle$ the **component form** of the vector $\mathbf{v}$. The first coordinate v_1 is called the x-component or the **horizontal component** of the vector $\mathbf{v}$, and the second coordinate v_2 is called the y-component or the **vertical component** of the vector $\mathbf{v}$. The nonnegative angle θ between the vector and the positive x-axis (with $0 \le \theta < 360°$) is called the **direction angle** of the vector. See Figure 3.29.

Using Basis Vectors

There is another way to algebraically write a vector if the components of the vector are known. This uses the so-called **standard basis vectors** for vectors in the plane. These two vectors are denoted by **i** and **j** and are defined as follows and are shown in the diagram to the right.

$$\mathbf{i} = \langle 1, 0 \rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1 \rangle.$$



The diagram in Figure 3.30 shows how to use the vectors **i** and **j** to represent a vector $\mathbf{v} = \langle a, b \rangle$.

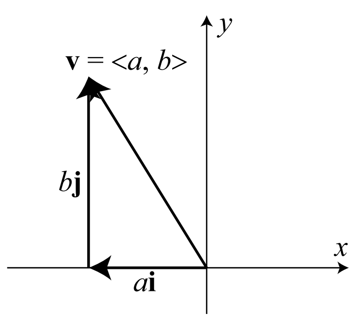


Figure 3.30: Using the Vectors **i** and **j**

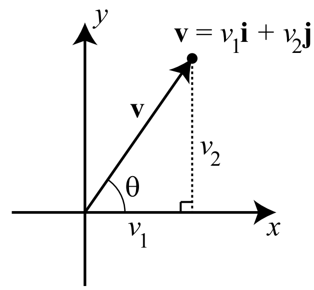
The diagram shows that if we place the tail of the vector $b\mathbf{j}$ at the tip of the vector $a\mathbf{i}$, we see that

$$\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}.$$

This is often called the **i**, **j form of a vector**, and the real number a is called the **i-component** of **v** and the real number b is called the **j-component** of **v**

Algebraic Formulas for Geometric Properties of a Vector

Vectors have certain geometric properties such as length and a direction angle. With the use of the component form of a vector, we can write algebraic formulas for these properties. We will use the diagram to the right to help explain these formulas.



- The magnitude (or length) of the vector **v** is the distance from the origin to the point (v_1, v_2) and so

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}.$$

- The direction angle of **v** is θ, where $0 \leq \theta < 360°$, and

$$\cos(\theta) = \frac{v_1}{|\mathbf{v}|} \quad \text{and} \quad \sin(\theta) = \frac{v_2}{|\mathbf{v}|}.$$

- The horizontal component and vertical component of the vector **v** with direction angle θ are

$$v_1 = |\mathbf{v}|\cos(\theta) \quad \text{and} \quad v_2 = |\mathbf{v}|\sin(\theta).$$

Progress Check 3.31 (Using the Formulas for a Vector)

1. Suppose the horizontal component of a vector **v** is 7 and the vertical component is -3. So we have $\mathbf{v} = 7\mathbf{i} + (-3)\mathbf{j} = 7\mathbf{i} - 3\mathbf{j}$. Determine the magnitude and the direction angle of **v**.

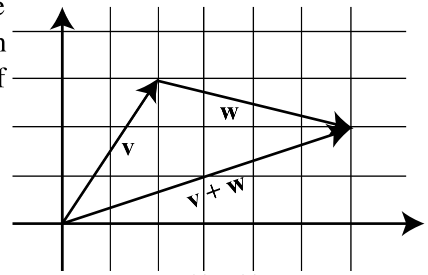
2. Suppose a vector **w** has a magnitude of 20 and a direction angle of 200°. Determine the horizontal and vertical components of **w** and write **w** in **i**, **j** form.

Operations on Vectors

In Section 3.5, we learned how to add two vectors and how to multiply a vector by a scalar. At that time, the descriptions of these operations was geometric in nature. We now know about the component form of a vector. So a good question is, "Can we use the component form of vectors to add vectors and multiply a vector by a scalar?"

To illustrate the idea, we will look at Progress Check 3.25 on page 223, where we added two vectors $\mathbf{v}$ and $\mathbf{w}$. Although we did not use the component forms of these vectors, we can now see that

$$\mathbf{v} = \langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j}, \quad \text{and}$$
$$\mathbf{w} = \langle 4, -1 \rangle = 4\mathbf{i} + (-1)\mathbf{j}.$$



v = 2**i** + 3**j**
w = 4**i** + (-1)**j**

The diagram to the right was part of the solutions for this progress check but now shows the vectors in a coordinate plane.

Notice that

$$\mathbf{v} + \mathbf{w} = 6\mathbf{i} + 2\mathbf{j}$$
$$\mathbf{v} + \mathbf{w} = (2 + 4)\mathbf{i} + (3 + (-1))\mathbf{j}$$

Figure 3.31 shows a more general diagram with

$$\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j} \text{ and } \mathbf{w} = \langle c, d \rangle = c\mathbf{i} + d\mathbf{j}$$

in standard position. This diagram shows that the terminal point of $\mathbf{v} + \mathbf{w}$ in standard position is $(a + c, b + d)$ and so

$$\mathbf{v} + \mathbf{w} = \langle a + c, b + d \rangle = (a + c)\mathbf{i} + (b + d)\mathbf{j}.$$

This means that we can add two vectors by adding their horizontal components and by adding their vertical components. The next progress check will illustrate something similar for scalar multiplication.

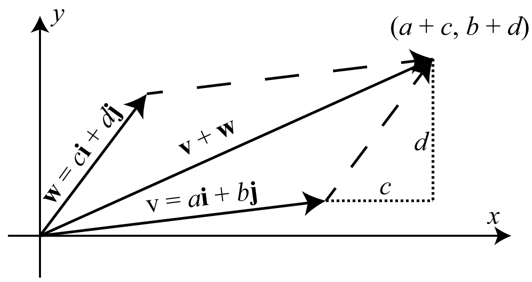


Figure 3.31: The Sum of Two Vectors

Progress Check 3.32 (Scalar Multiple of a Vector)

1. Let $\mathbf{v} = \langle 3, -2 \rangle$. Draw the vector $\mathbf{v}$ in standard position and then draw the vectors $2\mathbf{v}$ and $-2\mathbf{v}$ in standard position. What are the component forms of the vectors $2\mathbf{v}$ and $-2\mathbf{v}$?
2. In general, how do you think a scalar multiple of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ by a scalar c should be defined? Write a formal definition of a scalar multiple of a vector based on your intuition.

Based on the work we have done, we make the following formal definitions.

Definition. For vectors $\mathbf{v} = \langle v_1, v_2 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = \langle w_1, w_2 \rangle = w_1\mathbf{i} + w_2\mathbf{j}$ and scalar c, we make the following definitions:

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle \qquad \mathbf{v} + \mathbf{w} = (v_1 + w_1)\,\mathbf{i} + (v_2 + w_2)\,\mathbf{j}$$
$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2 \rangle \qquad \mathbf{v} - \mathbf{w} = (v_1 - w_1)\,\mathbf{i} + (v_2 - w_2)\,\mathbf{j}$$
$$c\mathbf{v} = \langle cv_1, cv_2 \rangle \qquad c\mathbf{v} = (cv_1)\,\mathbf{i} + (cv_2)\,\mathbf{j}$$

Progress Check 3.33 (Vector Operations
Let $\mathbf{u} = \langle 1, -2 \rangle$, $\mathbf{v} = \langle 0, 4 \rangle$, and $\mathbf{w} = \langle -5, 7 \rangle$.

1. Determine the component form of the vector $2\mathbf{u} - 3\mathbf{v}$.
2. Determine the magnitude and the direction angle for $2\mathbf{u} - 3\mathbf{v}$.
3. Determine the component form of the vector $\mathbf{u} + 2\mathbf{v} - 7\mathbf{w}$.

The Dot Product of Two Vectors

Finding optimal solutions to systems is an important problem in applied mathematics. It is often the case that we cannot find an exact solution that satisfies certain constraints, so we look instead for the “best” solution that satisfies the constraints. An example of this is fitting a least squares curve to a set of data like our calculators do when computing a sine regression curve. The dot product is useful in these situations to find “best” solutions to certain types of problems. Although we won’t see it in this course, having collections of perpendicular vectors is very important in that it allows for fast and efficient computations. The dot product of vectors allows us to measure the angle between them and thus determine if the vectors are perpendicular. The dot product has many applications, e.g., finding components of forces acting in different directions in physics and engineering. We introduce and investigate dot products in this section.

We have seen how to add vectors and multiply vectors by scalars, but we have not yet introduced a product of vectors. In general, a product of vectors should give us another vector, but there turns out to be no really useful way to define such a product of vectors. However, there is a dot “product” of vectors whose output is a scalar instead of a vector, and the dot product is a very useful product (even though it isn’t a product of vectors in a technical sense).

Recall that the magnitude (or length) of the vector $\mathbf{u} = \langle u_1, u_2\rangle$ is

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = \sqrt{u_1u_1 + u_2u_2}. \tag{1}$$

The expression under the second square root is a special case of important number we call the dot product of two vectors.

Definition. Let $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ be vectors in the plane. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2.$$

So we see that for $\mathbf{u} = \langle u_1, u_2\rangle$

$$\begin{aligned}\mathbf{u}\cdot\mathbf{u} &= u_1u_1 + u_2u_2 = u_1^2 + u_2^2\\ \mathbf{u}\cdot\mathbf{u} &= |\mathbf{u}|^2 \qquad (2)\end{aligned}$$

So the dot product is related to the length of a vector, and it turns out that the dot product of two vectors is also useful in determining the angle between two vectors.

Recall that in Progress Check 3.27 on page 225, we used the Law of Cosines to determine the sum of two vectors and then used the Law of Sines to determine the angle between the sum and one of those vectors. We have now seen how much easier it is to compute the sum of two vectors when the vectors are in component form. The dot product will allow us to determine the cosine of the angle between two vectors in component form. This is due to the following result:

The Dot Product and the Angle between Two Vectors

If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ ($0° \leq \theta \leq 180°$) as shown in Figure 3.35, then

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta) \qquad \text{or} \qquad \cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Notice that if we have written the vectors $\mathbf{u}$ and $\mathbf{v}$ in component form, then we have formulas to compute $|\mathbf{u}|$, $|\mathbf{v}|$, and $\mathbf{u}\cdot\mathbf{v}$. This result may seem surprising but it is a fairly direct consequence of the Law of Cosines. This will be shown in the appendix at the end of this section.

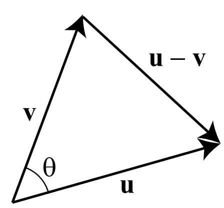


Figure 3.32: The angle between **u** and **v**

Progress Check 3.34 (Using the Dot Product)

1. Determine the angle θ between the vectors $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = -5\mathbf{i} + 2\mathbf{j}$.
2. Determine all vectors perpendicular to $\mathbf{u} = \langle 1, 3\rangle$. How many such vectors are there? **Hint**: Let $\mathbf{v} = \langle a, b\rangle$. Under what conditions will the angle between $\mathbf{u}$ and $\mathbf{v}$ be $90°$?

One purpose of Progress Check 3.34 was to use the formula

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

to determine when two vectors are perpendicular. Two vectors **u** and **v** will be perpendicular if and only if the angle θ between them is 90°. Since $\cos(90°) = 0$, we see that this formula implies that **u** and **v** will be perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. (This is because a fraction will be equal to 0 only when the numerator is equal to 0 and the denominator is not zero.) So we have

Two vectors are perpendicular if and only if their dot product is equal to 0.

Note: When two vectors are perpendicular, we also say that they are **orthogonal**.

Projections

Another useful application of the dot product is in finding the projection of one vector onto another. An example of where such a calculation is useful is the following.

> Usain Bolt from Jamaica excited the world of track and field in 2008 with his world record performances on the track. Bolt won the 100 meter race in a world record time of 9.69 seconds. He has since bettered that time with a race of 9.58 seconds with a wind assistance of 0.9 meters per second in Berlin on August 16, 2009 The wind assistance is a measure of the wind speed that is helping push the runners down the track. It is much easier to run a very fast race if the wind is blowing hard in the direction of the race. So that world records aren't dependent on the weather conditions, times are only recorded as record times if the wind aiding the runners is less than or equal to 2 meters per second. Wind speed for a race is recorded by a wind gauge that is set up close to the track. It is important to note, however, that weather is not always as cooperative as we might like. The wind does not always blow exactly in the direction of the track, so the gauge must account for the angle the wind makes with the track.

If the wind is blowing in the direction of the vector **u** and the track is in the direction of the vector **v** in Figure 3.33, then only part of the total wind vector is actually working to help the runners. This part is called the **projection of the vector u onto the vector v** and is denoted $\mathbf{proj_v u}$.

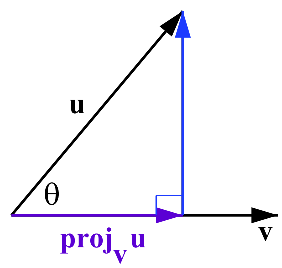


Figure 3.33: The Projection of **u** onto **v**

We can find this projection with a little trigonometry. To do so, we let θ be the angle between **u** and **v** as shown in Figure 3.33. Using right triangle trigonometry, we see that

$$\begin{aligned} |\mathbf{proj_v u}| &= |\mathbf{u}|\cos(\theta) \\ &= |\mathbf{u}|\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\ |\mathbf{v}|} \\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}. \end{aligned}$$

The quantity we just derived is called the **scalar projection (or component) of u onto v** and is denoted by $\mathbf{comp_v u}$. Thus

$$\mathbf{comp_v u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}.$$

This gives us the length of the vector projection. So to determine the vector, we use a scalar multiple of this length times a unit vector in the same direction, which is $\frac{1}{|\mathbf{v}|}\mathbf{v}$. So we obtain

$$\begin{aligned} \mathbf{proj_v u} &= |\mathbf{proj_v u}|\left(\frac{1}{|\mathbf{v}|}\mathbf{v}\right) \\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\left(\frac{1}{|\mathbf{v}|}\mathbf{v}\right) \\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} \end{aligned}$$

We can also write the projection of **u** onto **v** as

$$\mathbf{proj_v u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}.$$

The wind component that acts perpendicular to the direction of $\mathbf{v}$ in Figure 3.33 is called the **projection of u orthogonal to v** and is denoted $\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$ as shown in Figure 3.34. Since $\mathbf{u} = \mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} + \mathbf{proj}_{\mathbf{v}}\mathbf{u}$, we have that

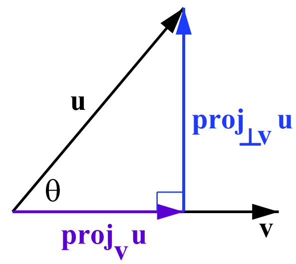


Figure 3.34: The Projection of **u** onto **v**

$$\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} = \mathbf{u} - \mathbf{proj}_{\mathbf{v}}\mathbf{u}.$$

Following is a summary of the results we have obtained.

For nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, the **projection of the vector u onto the vector v**, $\mathbf{proj}_{\mathbf{v}}\mathbf{u}$, is given by

$$\mathbf{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}.$$

See Figure 3.34. The **projection of u orthogonal to v**, denoted $\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$, is

$$\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u} = \mathbf{u} - \mathbf{proj}_{\mathbf{v}}\mathbf{u}.$$

We note that $\mathbf{u} = \mathbf{proj}_{\mathbf{v}}\mathbf{u} + \mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$.

Progress Check 3.35 (Projection of One Vector onto Another Vector
Let $\mathbf{u} = \langle 7, 5\rangle$ and $\mathbf{v} = \langle 10, -2\rangle$. Determine $\mathbf{proj}_{\mathbf{v}}\mathbf{u}$, $\mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$, and verify that $\mathbf{u} = \mathbf{proj}_{\mathbf{v}}\mathbf{u} + \mathbf{proj}_{\perp\mathbf{v}}\mathbf{u}$. Draw a picture showing all of the vectors involved in this.

Appendix – Properties of the Dot Product

The main purpose of this appendix is to provide a proof of the formula on page 238 that relates the dot product of two vectors to the angle between the two vectors. To do this, we first need to establish some properties of the dot product. The following shows the properties we will be using.

Let **a**, **b**, and **c** be vectors. Then

1. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$.
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ (Commutative Property).
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$ and $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{c}$ (Distributive Properties).

We have already established the first property on page 237. To prove the other results, we use $\mathbf{a} = \langle a_1, a_2\rangle$, $\mathbf{b} = \langle b_1, b_2\rangle$, and $\mathbf{c} = \langle c_1, c_2\rangle$. We will also use the commutative property and distributive property for real numbers.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= a_1b_1 + a_2b_2\\ &= b_1a_1 + b_2a_2\\ &= \mathbf{b}\cdot\mathbf{a}\end{aligned}$$

This establishes the commutative property for the dot product. For the distributive property, we have

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \langle a_1, a_2\rangle\cdot(\langle b_1, b_2\rangle + \langle c_1, c_2\rangle)\\ &= \langle a_1, a_2\rangle\cdot\langle b_1 + c_1, b_2 + c_2\rangle\\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2)\\ &= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2\end{aligned}$$

We now rearrange the terms on the right side of the last equation to obtain

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= (a_1b_1 + a_2b_2) + (a_1c_1 + a_2c_2)\\ &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}\end{aligned}$$

This establishes one of the distributive properties. The other is proven in a similar manner.

We are now able to provide a proof of the formula on page 238 that relates the dot product of two vectors to the angle between the two vectors. Let θ be the angle between the nonzero vectors **u** and **v** as illustrated in Figure 3.35.

We can apply the Law of Cosines to using the angle θ as follows:

$$\begin{aligned}|\mathbf{u}-\mathbf{v}|^2 &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\,\cos(\theta)\\ (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\,\cos(\theta) \qquad (3)\end{aligned}$$

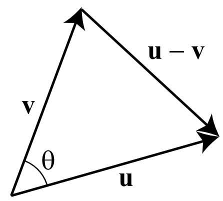


Figure 3.35: The angle between **u** and **v**

We now apply some of the properties of the dot product to the left side of equation (3).

$$\begin{aligned}(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= (\mathbf{u}-\mathbf{v})\cdot\mathbf{u}-(\mathbf{u}-\mathbf{v})\cdot\mathbf{v} \\ &= \mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} \\ &= \mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{v} \\ &= |\mathbf{u}|^2-2(\mathbf{u}\cdot\mathbf{v})+|\mathbf{v}|^2 \qquad (4)\end{aligned}$$

We can now use equations (3) and (4) to obtain

$$\begin{aligned}|\mathbf{u}|^2-2(\mathbf{u}\cdot\mathbf{v})+|\mathbf{v}|^2 &= |\mathbf{u}|^2+|\mathbf{v}|^2-2|\mathbf{u}|\,|\mathbf{v}|\ \cos(\theta) \\ -2(\mathbf{u}\cdot\mathbf{v}) &= -2\ |\mathbf{u}|\ |\mathbf{v}|\ \cos(\theta) \\ \mathbf{u}\cdot\mathbf{v} &= |\mathbf{u}|\ |\mathbf{v}|\ \cos(\theta).\end{aligned}$$

This is the formula on page 238 that relates the dot product of two vectors to the angle between the two vectors.

Summary of Section 3.6

In this section, we studied the following important concepts and ideas:
The **component form** of a vector **v** is written as $\mathbf{v} = \langle v_1, v_2\rangle$ and the **i**, **j** form of the same vector is $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ where $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$. Using this notation, we have

- The magnitude (or length) of the vector **v** is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$.
- The direction angle of **v** is θ, where $0 \le \theta < 360°$, and

$$\cos(\theta) = \frac{v_1}{|\mathbf{v}|} \quad \text{and} \quad \sin(\theta) = \frac{v_2}{|\mathbf{v}|}.$$

- The horizontal and component and vertical component of the vector **v** and direction angle θ are

$$v_1 = |\mathbf{v}|\cos(\theta) \quad \text{and} \quad v_2 = |\mathbf{v}|\sin(\theta).$$

For two vectors **v** and **w** with $\mathbf{v} = \langle v_1, v_2 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = \langle w_1, w_2 \rangle = w_1\mathbf{i} + w_2\mathbf{j}$ and a scalar c:

- $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle = (v_1 + w_1)\,\mathbf{i} + (v_2 + w_2)\,\mathbf{j}$.
- $\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2 \rangle = (v_1 - w_1)\,\mathbf{i} + (v_2 - w_2)\,\mathbf{j}$.
- $c\mathbf{v} = \langle cv_1, cv_2 \rangle = (cv_1)\,\mathbf{i} + (cv_2)\,\mathbf{j}$.
- The **dot product** of **v** and **w** is $\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2$.
- If θ is the angle between **v** and **w**, then

$$\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta) \qquad \text{or} \qquad \cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}.$$

- The **projection of the vector v onto the vector w**, $\mathbf{proj_w v}$, is given by

$$\mathbf{proj_w v} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}.$$

  The **projection of v orthogonal to w**, denoted $\mathbf{proj_{\perp w} v}$, is

$$\mathbf{proj_{\perp w} v} = \mathbf{v} - \mathbf{proj_w v}.$$

  We note that $\mathbf{v} = \mathbf{proj_w v} + \mathbf{proj_{\perp w} v}$. See Figure 3.36.

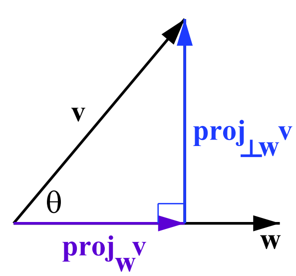


Figure 3.36: The Projection of **v** onto **w**

Exercises for Section 3.6

1. Determine the magnitude and the direction angle of each of the following vectors.

⋆ **(a)** $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$

⋆ **(b)** $\mathbf{w} = \langle -3, 6 \rangle$

(c) $\mathbf{a} = 4\mathbf{i} - 7\mathbf{j}$

(d) $\mathbf{u} = \langle -3, -5 \rangle$

2. Determine the horizontal and vertical components of each of the following vectors. Write each vector in **i**, **j** form.

⋆ **(a)** The vector **v** with magnitude 12 and direction angle 50°.

⋆ **(b)** The vector **u** with $|\mathbf{u}| = \sqrt{20}$ and direction angle 125°.

(c) The vector **w** with magnitude 5.25 and direction angle 200°.

3. Let $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + 5\mathbf{j}$, and $\mathbf{w} = 4\mathbf{i} - 2\mathbf{j}$. Determine the **i**, **j** form of each of the following:

⋆ **(a)** $5\mathbf{u} - \mathbf{v}$

(b) $2\mathbf{v} + 7\mathbf{w}$

⋆ **(c)** $\mathbf{u} + \mathbf{v} + \mathbf{w}$

(d) $3\mathbf{u} + 5\mathbf{w}$

4. Determine the value of the dot product for each of the following pairs of vectors.

⋆ **(a)** $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} - 2\mathbf{j}$.

⋆ **(b)** **a** and **b** where $|\mathbf{a}| = 6$, $|\mathbf{w}| = 3$, and the angle between **v** and **w** is 30°.

(c) **a** and **b** where $|\mathbf{a}| = 6$, $|\mathbf{w}| = 3$, and the angle between **v** and **w** is 150°.

(d) **a** and **b** where $|\mathbf{a}| = 6$, $|\mathbf{w}| = 3$, and the angle between **v** and **w** is 50°.

(e) $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j}$.

5. Determine the angle between each of the following pairs of vectors.

⋆ **(a)** $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} - 2\mathbf{j}$.

(b) $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j}$.

(c) $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = -\mathbf{i} + 4\mathbf{j}$.

6. For each pair of vectors, determine $\mathbf{proj_v w}$, $\mathbf{proj_{\perp v} w}$, and verify that $\mathbf{w} = \mathbf{proj_v w} + \mathbf{proj_{\perp v} w}$. Draw a picture showing all of the vectors involved in this.

⋆ **(a)** $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} - 2\mathbf{j}$.

(b) $\mathbf{v} = \langle -2, 3 \rangle$ and $\mathbf{w} = \langle 1, 1 \rangle$