Part IV

Eigenvalues and Eigenvectors

Section 16

The Determinant

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- How do we calculate the determinant of an $n \times n$ matrix?
- What is one important fact the determinant tells us about a matrix?

Application: Area and Volume

Consider the problem of finding the area of a parallelogram determined by two vectors $\mathbf{u}$ and $\mathbf{v}$, as illustrated at left in Figure 16.1. We could calculate this area, for example, by breaking up

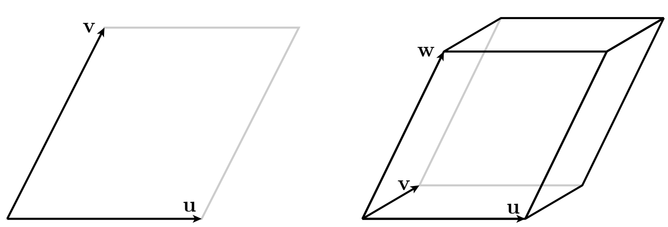

Figure 16.1: A parallelogram and a parallelepiped.

the parallelogram into two triangles and a rectangle and finding the area of each. Now consider the problem of calculating the volume of the three-dimensional analog (called a *parallelepiped*) determined by three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as illustrated at right in Figure 16.1.

It is quite a bit more difficult to break this parallelepiped into subregions whose volumes are easy to compute. However, all of these computations can be made quickly by using determinants. The details are later in this section.

Introduction

We know that a non-zero vector $\mathbf{x}$ is an eigenvector of an $n \times n$ matrix A if $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. Note that this equation can be written as $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$. Until now, we were given eigenvalues of matrices and have used the eigenvalues to find the eigenvectors. In this section we will learn an algebraic technique to find the eigenvalues ourselves. We will also be able to justify why an $n \times n$ matrix has at most n eigenvalues.

A scalar λ is an eigenvalue of A if $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ has a non-trivial solution $\mathbf{x}$, which happens if and only if $A - \lambda I_n$ is not invertible. In this section we will find a scalar whose value will tell us when a matrix is invertible and when it is not, and use this scalar to find the eigenvalues of a matrix.

Preview Activity 16.1. In this activity, we will focus on 2×2 matrices. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. To see if A is invertible, we row reduce A by replacing row 2 with $a{\cdot}$(row 2) $-c{\cdot}$(row 1):

$$\begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}.$$

So the only way A can be reduced I_2 is if $ad - bc \neq 0$. We call this quantity $ad - bc$ the *determinant* of A, and denote the determinant of A as $\det(A)$ or $|A|$. When $\det(A) \neq 0$, we know that

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We now consider how we can use the determinant to find eigenvalues and other information about the invertibility of a matrix.

(1) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find $\det(A)$ by hand. What does this mean about the matrix A? Can you confirm this with other methods?

(2) One of the eigenvalues of $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is $\lambda = 4$. Recall that we can rewrite the matrix equation $A\mathbf{x} = 4\mathbf{x}$ in the form $(A - 4I_2)\mathbf{x} = \mathbf{0}$. What must be true about $A - 4I_2$ in order for 4 to be an eigenvalue of A? How does this relate to $\det(A - 4I_2)$?

(3) Another eigenvalue of $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is $\lambda = -1$. What must be true about $A + I_2$ in order for -1 to be an eigenvalue of A? How does this relate to $\det(A + I_2)$?

(4) To find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$, we rewrite the equation $A\mathbf{x} = \lambda\mathbf{x}$ as $(A - \lambda I_2)\mathbf{x} = \mathbf{0}$. The coefficient matrix of this last system has the form $A - \lambda I_2 = \begin{bmatrix} 3 - \lambda & 2 \\ 2 & 6 - \lambda \end{bmatrix}$. The determinant of this matrix is a quadratic expression in λ. Since the eigenvalues will occur when the determinant is 0, we need to solve a quadratic equation. Find the resulting eigenvalues. (Note: One of the eigenvalues is 2.)

(5) Can you explain why a 2×2 matrix can have at most two eigenvalues?

The Determinant of a Square Matrix

Around 1900 or so determinants were deemed much more important than they are today. In fact, determinants were used even before matrices. According to Tucker[1] determinants (not matrices) developed out of the study of coefficients of systems of linear equations and were used by Leibniz 150 years before the term matrix was coined by J. J. Sylvester in 1848. Even though determinants are not as important as they once were, the determinant of a matrix is still a useful quantity. We saw in Preview Activity 16.1 that the determinant of a matrix tells us if the matrix is invertible and how it can help us find eigenvalues. In this section, we will see how to find the determinant of any size matrix and how to use this determinant to find the eigenvalues.

The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$. The matrix A is invertible if and only if $\det(A) \neq 0$. We will use a recursive approach to find the determinants of larger size matrices building from the 2×2 determinants. We present the result in the 3×3 case here – a more detailed analysis can be found at the end of this section.

To find the determinant of a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we will use the determinants of three 2×2 matrices. More specifically, the determinant of A, denoted $\det(A)$ is the quantity

$$a_{11} \det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{12} \det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) + a_{13} \det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right). \tag{16.1}$$

This sum is called a *cofactor expansion* of the determinant of A. The smaller matrices in this expansion are obtained by deleting certain rows and columns of the matrix A. In general, when finding the determinant of an $n \times n$ matrix, we find determinants of $(n-1) \times (n-1)$ matrices, which we can again reduce to smaller matrices to calculate.

We will use the specific matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 4 & 3 \\ 2 & 2 & 1 \end{bmatrix}$$

as an example in illustrating the cofactor expansion method in general.

- We first pick a row or column of A. We will pick the first row of A for this example.
- For each entry in the row (or column) we choose, in this case the first row, we will calculate the determinant of a smaller matrix obtained by removing the row and the column the entry is in. Let A_{ij} be the smaller matrix found by deleting the ith row and jth column of A. For entry a_{11}, we find the matrix A_{11} obtained by removing first row and first column:

  $$A_{11} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.$$

  For entry a_{12}, we find

  $$A_{12} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}.$$

[1] Tucker, Alan. (1993). The Growing Importance of Linear Algebra in Undergraduate Mathematics. *The College Mathematics Journal*, 1, 3-9.

Finally, for entry a_{13}, we find

$$A_{13} = \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix}.$$

- Notice that in the 3×3 determinant formula in (16.1) above, the middle term had a (-) sign. The signs of the terms in the cofactor expansion alternate within each row and each column. More specifically, the sign of a term in the ith row and jth column is $(-1)^{i+j}$. We then obtain the following pattern of the signs within each row and column:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & & & \end{bmatrix}$$

  In particular, the sign factor for a_{11} is $(-1)^{1+1} = 1$, for a_{12} is $(-1)^{1+2} = -1$, and for a_{13} is $(-1)^{1+3} = 1$.

- For each entry a_{ij} in the row (or column) of A we chose, we multiply the entry a_{ij} by the determinant of A_{ij} and the sign $(-1)^{i+j}$. In this case, we obtain the following numbers

$$a_{11}(-1)^{1+1}\det(A_{11}) = 1\det\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = 1(4-6) = -2$$

$$a_{12}(-1)^{1+2}\det(A_{12}) = -2\det\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = -2(1-6) = 10$$

$$a_{13}(-1)^{1+3}\det(A_{13}) = 0$$

  Note that in the last calculation, since $a_{13} = 0$, we did not have to evaluate the rest of the terms.

- Finally, we find the determinant by adding all these values:

$$\begin{aligned} \det(A) &= a_{11}(-1)^{1+1}\det(A_{11}) + a_{12}(-1)^{1+2}\det(A_{12}) \\ &\quad + a_{13}(-1)^{1+3}\det(A_{13}) \\ &= 8. \end{aligned}$$

Cofactors

We will now define the determinant of a general $n \times n$ matrix A in terms of a cofactor expansion as we did in the 3×3 case. To do so, we need some notation and terminology.

- We let A_{ij} be the submatrix of $A = [a_{ij}]$ found by deleting the ith row and jth column of A. The determinant of A_{ij} is called the ijth *minor* of A or the minor corresponding to the entry a_{ij}.

- Notice that in the 3×3 case, we used the opposite of the 1,2 minor in the sum. It will be the case that the terms in the cofactor expansion will alternate in sign. We can make the signs in the sum alternate by taking -1 to an appropriate power. As a result, we define the ijth *cofactor* C_{ij} of A as
$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$
- Finally, we define the determinant of A.

Definition 16.1. If $A = [a_{ij}]$ is an $n \times n$ matrix, the **determinant** of A is the scalar

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

where $C_{ij} = (-1)^{i+j}\det(A_{ij})$ is the ij-cofactor of A and A_{ij} is the matrix obtained by removing row i and column j of matrix A.

This method for computing determinants is called the *cofactor expansion* or *Laplace expansion* of A along the 1st row. The cofactor expansion reduces the computation of the determinant of an $n \times n$ matrix to n computations of determinants of $(n-1) \times (n-1)$ matrices. These smaller matrices can be reduced again using cofactor expansions, so it can be a long and grueling process for large matrices. It turns out that we can actually take this expansion along any row or column of the matrix (a proof of this fact is given in Section 21). For example, the cofactor expansion along the 2nd row is
$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + \cdots + a_{2n}C_{2n}$$
and along the 3rd column the formula is

$$\det(A) = a_{13}C_{13} + a_{23}C_{23} + \cdots + a_{n3}C_{n3}.$$

Note that when finding a cofactor expansion, choosing a row or column with many zeros makes calculations easier.

Activity 16.1.

(a) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 4 \\ 6 & 3 & 0 \end{bmatrix}$. Use the cofactor expansion along the first row to calculate the determinant of A by hand.

(b) Calculate $\det(A)$ by using a cofactor expansion along the second row where $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 0 \\ 2 & 5 & 3 \end{bmatrix}$.

(c) Calculate the determinant of $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -3 \\ 0 & 0 & 8 \end{bmatrix}$.

(d) Which determinant property can be used to calculate the determinant in part (c)? Explain how. (Determinant properties are included below for easy reference.)

(e) Consider the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. Let B be the matrix which results when c times row 1 is added to row 2 of A. Evaluate the determinant of B by hand to check that it is equal to the determinant of A, which verifies one other determinant property (in a specific case).

As with any new idea, like the determinant, we must ask what properties are satisfied. We state the following theorem without proof for the time being. For the interested reader, the proof of many of these properties is given in Section 21 and others in the exercises.

Theorem 16.2. *Given $n \times n$ matrices A, B, the following hold:*

(1) $\det(AB) = \det(A) \cdot \det(B)$*, and in particular* $\det(A^k) = (\det A)^k$ *for any positive integer* k*.*

(2) $\det(A^\mathsf{T}) = \det(A)$*.*

(3) A *is invertible if and only if* $\det(A) \neq 0$*.*

(4) If A *is invertible, then* $\det(A^{-1}) = (\det A)^{-1}$*.*

(5) For a 2×2 *matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad - bc$*.*

(6) If A *is upper/lower triangular, then* $\det(A)$ *is the product of the entries on the diagonal.*

(7) The determinant of a matrix is the product of the eigenvalues, with each eigenvalue repeated as many times as its multiplicity.

(8) Effect of row operations:

- *Adding a multiple of a row to another does NOT change the determinant of the matrix.*
- *Multiplying a row by a constant multiplies the determinant by the same constant.*
- *Row swapping multiplies the determinant by* (-1)*.*

(9) If the row echelon form U *of* A *is obtained by adding multiples of one row to another, and row swapping, then* $\det(A)$ *is equal to* $\det(U)$ *multiplied by* $(-1)^r$ *where* r *is the number of row swappings done during the row reduction.*

Note that if we were to find the determinant of a 4×4 matrix using the cofactor method, we will calculate determinants of 4 matrices of size 3×3, each of which will require 3 determinant calculations again. So, we will need a total of 12 calculations of determinants of 2×2 matrices. That is a lot of calculations. There are other, more efficient, methods for calculating determinants. For example, we can row reduce the matrix, keeping track of the effect that each row operation has on the determinant.

The Determinant of a 3×3 Matrix

Earlier we defined the determinant of a 3×3 matrix. In this section we endeavor to understand the motivation behind that definition.

We will repeat the process we went through in the 2×2 case to see how to define the determinant of a 3×3 matrix. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

To find the inverse of A we augment A by the 3×3 identity matrix

$$[A \mid I_3] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix}$$

and row reduce the matrix (using appropriate technology) to obtain

$$\begin{bmatrix} 1 & 0 & 0 & \frac{a_{33}a_{22} - a_{32}a_{23}}{d} & -\frac{a_{33}a_{12} - a_{32}a_{13}}{d} & \frac{-a_{13}a_{22} + a_{12}a_{23}}{d} \\ 0 & 1 & 0 & -\frac{a_{33}a_{21} - a_{31}a_{23}}{d} & \frac{a_{33}a_{11} - a_{31}a_{13}}{d} & -\frac{a_{23}a_{11} - a_{21}a_{13}}{d} \\ 0 & 0 & 1 & \frac{-a_{31}a_{22} + a_{32}a_{21}}{d} & -\frac{a_{32}a_{11} - a_{31}a_{12}}{d} & \frac{a_{22}a_{11} - a_{21}a_{12}}{d} \end{bmatrix},$$

where

$$\begin{aligned} d = a_{33}a_{11}a_{22} - a_{33}a_{21}a_{12} - a_{31}a_{13}a_{22} \\ - a_{32}a_{11}a_{23} + a_{32}a_{21}a_{13} + a_{31}a_{12}a_{23}. \end{aligned} \tag{16.2}$$

In this case, we can see that the inverse of the 3×3 matrix A will be defined if and only if $d \neq 0$. So, in the 3×3 case the determinant of A will be given by the value of d in Equation (16.2). What remains is for us to see how this is related to determinants of 2×2 sub-matrices of A.

To start, we collect all terms involving a_{11} in d. A little algebra shows that

$$\det(A) = a_{11}\left(a_{33}a_{22} - a_{32}a_{23}\right) - a_{33}a_{21}a_{12} - a_{31}a_{13}a_{22} + a_{32}a_{21}a_{13} + a_{31}a_{12}a_{23}.$$

Now let's collect the remaining terms involving a_{12}:

$$\det(A) = a_{11}\left(a_{33}a_{22} - a_{32}a_{23}\right) - a_{12}\left(a_{33}a_{21} - a_{31}a_{23}\right) - a_{31}a_{13}a_{22} + a_{32}a_{21}a_{13}.$$

Finally, we collect the terms involving a_{13}:

$$\det(A) = a_{11}\left(a_{33}a_{22} - a_{32}a_{23}\right) - a_{12}\left(a_{33}a_{21} - a_{31}a_{23}\right) + a_{13}\left(a_{32}a_{21} - a_{31}a_{22}\right).$$

Now we can connect the determinant of A to determinants of 2×2 sub-matrices of A.

- Notice that
$$a_{33}a_{22} - a_{32}a_{23}$$
is the determinant of the 2×2 matrix $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ obtained from A by deleting the first row and first column.

- Similarly, the expression
$$a_{33}a_{21} - a_{31}a_{23}$$
is the determinant of the 2×2 matrix $\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ obtained from A by deleting the first row and second column.

- Finally, the expression
$$a_{32}a_{21} - a_{31}a_{22}$$
is the determinant of the 2×2 matrix $\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ obtained from A by deleting the first row and third column.

Putting this all together gives us formula (16.1) for the determinant of a 3×3 matrix as we defined earlier.

Two Devices for Remembering Determinants

There are useful ways to remember how to calculate the formulas for determinants of 2×2 and 3×3 matrices. In the 2×2 case of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we saw that

$$|A| = a_{11}a_{22} - a_{21}a_{22}.$$

This makes $|A|$ the product of the diagonal elements a_{11} and a_{22} minus the product of the off-diagonal elements a_{12} and a_{21}. We can visualize this in an array by drawing arrows across the diagonal and off-diagonal, with a plus sign on the diagonal arrow indicting that we add the product of the diagonal elements and a minus sign on the off-diagonal arrow indicating that we subtract the product of the off-diagonal elements as shown in Figure 16.2.

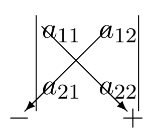

Figure 16.2: A diagram to remember the 2×2 determinant.

We can do a similar thing for the determinant of a 3×3 matrix. In this case, we extend the 3×3 array to a 3×5 array by adjoining the first two columns onto the matrix. We then add the products along the diagonals going from left to right and subtract the products along the diagonals going from right to left as indicated in Figure 16.3.

Examples

What follows are worked examples that use the concepts from this section.

Example 16.3. For each of the following

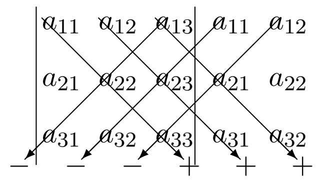

Figure 16.3: A diagram to remember the 3×3 determinant.

- Identify the sub-matrices $A_{1,j}$
- Determine the cofactors $C_{1,j}$.
- Use the cofactor expansion to calculate the determinant.

(a) $A = \begin{bmatrix} 3 & 6 & 2 \\ 0 & 4 & -1 \\ 5 & 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 0 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & -2 & 2 & -1 \\ -3 & 2 & 3 & 1 \end{bmatrix}$

Example Solution.

(a) With a 3×3 matrix, we will find the sub-matrices A_{11}, A_{12}, and A_{13}. Recall that A_{ij} is the sub-matrix of A obtained by deleting the ith row and jth column of A. Thus,

$$A_{11} = \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0 & -1 \\ 5 & 1 \end{bmatrix} \quad \text{and } A_{13} = \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}.$$

The ijth cofactor is $C_{ij} = (-1)^{i+j} \det(A_{ij})$, so

$$C_{11} = (-1)^2 \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix} = 4$$

$$C_{12} = (-1)^3 \begin{bmatrix} 0 & -1 \\ 5 & 1 \end{bmatrix} = -5$$

$$C_{13} = (-1)^4 \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix} = -20.$$

Then

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (3)(4) + (6)(-5) + (2)(-20) = -58.$$

(b) With a 4×4 matrix, we will find the sub-matrices A_{11}, A_{12}, A_{13}, and A_{14}. We see that

$$A_{11} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & -1 \\ -3 & 3 & 1 \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ -3 & 2 & 1 \end{bmatrix}$$

$$A_{14} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & -2 & 2 \\ -3 & 2 & 3 \end{bmatrix}.$$

To calculate the ijth cofactor $C_{ij} = (-1)^{i+j} \det(A_{ij})$, we need to calculate the determinants of the A_{1j}. Using the device for calculating the determinant of a 3×3 matrix we have that

$$\begin{aligned} \det(A_{11}) &= \det\left(\begin{bmatrix} 1 & 2 & 1 \\ -2 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix}\right) \\ &= (1)(2)(1) + (2)(-1)(2) + (1)(-2)(3) \\ &\quad - (1)(2)(2) - (1)(-1)(3) - (2)(-2)(1) \\ &= -5, \end{aligned}$$

$$\begin{aligned} \det(A_{12}) &= \det\left(\begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & -1 \\ -3 & 3 & 1 \end{bmatrix}\right) \\ &= (2)(2)(1) + (2)(-1)(-3) + (1)(1)(3) \\ &\quad - (1)(2)(-3) - (2)(-1)(3) - (2)(1)(1) \\ &= 23, \end{aligned}$$

$$\begin{aligned} \det(A_{13}) &= \det\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ -3 & 2 & 1 \end{bmatrix}\right) \\ &= (2)(-2)(1) + (1)(-1)(-3) + (1)(1)(2) \\ &\quad - (1)(-2)(-3) - (2)(-1)(2) - (1)(1)(1) \\ &= -2, \end{aligned}$$

and

$$\begin{aligned}\det(A_{14}) &= \det\left(\begin{bmatrix} 2 & 1 & 2 \\ 1 & -2 & 2 \\ -3 & 2 & 3 \end{bmatrix}\right) \\ &= (2)(-2)(3) + (1)(2)(-3) + (2)(1)(2) \\ &\quad - (2)(-2)(-3) - (2)(2)(2) - (1)(1)(3) \\ &= -37.\end{aligned}$$

Then

$$\begin{aligned} C_{11} &= (-1)^2 \det(A_{11}) = -5 \\ C_{12} &= (-1)^3 \det(A_{12}) = -23 \\ C_{13} &= (-1)^4 \det(A_{13}) = -2 \\ C_{14} &= (-1)^5 \det(A_{13}) = 37 \end{aligned}$$

and so

$$\begin{aligned}\det(B) &= b_{11}C_{11} + b_{12}C_{12} + b_{13}C_{13} + b_{14}C_{14} \\ &= (3)(-5) + (0)(-23) + (1)(-2) + (1)(37) \\ &= 20.\end{aligned}$$

Example 16.4. Show that for any 2×2 matrices A and B,

$$\det(AB) = \det(A)\det(B).$$

Example Solution.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Then

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}.$$

So

$$\begin{aligned}\det(AB) &= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) \\ &\quad - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21}) \\ &= (a_{11}b_{11}a_{21}b_{12} + a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} + a_{12}b_{21}a_{22}b_{22}) \\ &\quad - (a_{11}b_{12}a_{21}b_{11} + a_{11}b_{12}a_{22}b_{21} + a_{12}b_{22}a_{21}b_{11} + a_{12}b_{22}a_{22}b_{21}) \\ &= a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} - a_{11}b_{12}a_{22}b_{21} - a_{12}b_{22}a_{21}b_{11}.\end{aligned}$$

Also,

$$\begin{aligned}\det(A)\det(B) &= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) \\ &= a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}.\end{aligned}$$

We conclude that $\det(AB) = \det(A)\det(B)$ if A and B are 2×2 matrices.

Summary

- The determinant of an $n \times n$ matrix $A = [a_{ij}]$ is found by taking the cofactor expansion of A along the first row. That is

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n},$$

  where

  - A_{ij} is the sub-matrix of A found by deleting the ith row and jth column of A.
  - $C_{ij} = (-1)^{i+j} \det(A_{ij})$ is the ijth *cofactor* of A.

- The matrix A is invertible if and only if $\det(A) \neq 0$.

Exercises

(1) Use the cofactor expansion to explain why multiplying each of the entries of a 3×3 matrix A by 2 multiplies the determinant of A by 8.

(2) Use the determinant criterion to determine for which c the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & c \\ 2 & -1 & 2 \end{bmatrix}$ is invertible.

(3) Let A be a square matrix.

   (a) Explain why $\det(A^2) = [\det(A)]^2$

   (b) Expand on the argument from (a) to explain why $\det(A^k) = [\det(A)]^k$ for any positive integer k.

   (c) Suppose that A is an invertible matrix and k is a positive integer. Must A^k be an invertible matrix? Why or why not?

(4) Let A be an invertible matrix. Explain why $\det(A^{-1}) = \dfrac{1}{\det(A)}$ using determinant properties.

(5) Simplify the following determinant expression using determinant properties:

$$\det(PA^4P^{-1}A^T(A^{-1})^3)$$

(6) Find the eigenvalues of the following matrices. Find a basis for and the dimension of each eigenspace.

   (a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

(7) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** For any two $n \times n$ matrices A and B, $\det(A+B) = \det A + \det B$.

(b) **True/False** For any square matrix A, $\det(-A) = -\det(A)$.

(c) **True/False** For any square matrix A, $\det(-A) = \det(A)$.

(d) **True/False** The determinant of a square matrix with all non-zero entries is non-zero.

(e) **True/False** If the determinant of A is non-zero, then so is the determinant of A^2.

(f) **True/False** If the determinant of a matrix A is 0, then one of the rows of A is a linear combination of the other rows.

(g) **True/False** For any square matrix A, $\det(A^2) > \det(A)$.

(h) **True/False** If A and B are $n \times n$ matrices and AB is invertible, then A and B are invertible.

(i) **True/False** If A^2 is the zero matrix, then the only eigenvalue of A is 0.

(j) **True/False** If 0 is an eigenvalue of A, then 0 is an eigenvalue of AB for any B of the same size as A.

(k) **True/False** Suppose A is a 3×3 matrix. Then any three eigenvectors of A will form a basis of $\mathbb{R}^3$.

Project: Area and Volume Using Determinants

The approach we will take to connecting area (volume) to the determinant will help shed light on properties of the determinant that we will discuss from an algebraic perspective in a later section. First, we mention some basic properties of area (we focus on area for now, but these same properties are valid for volumes as well). volume). As a shorthand, we denote the area of a region R by Area(R).

- Area cannot be negative.
- If two regions R_1 and R_2 don't overlap, then the area of the union of the regions is equal to the sum of the areas of the regions. That is, if $R_1 \cap R_2 = \emptyset$, then Area$(R_1 \cup R_2) =$ Area$(R_1) +$ Area(R_2).
- Area is invariant under translation. That is, if we move a geometric region by the same amount uniformly in a given direction, the area of the original region and the area of the transformed region are the same. A translation of a region is done by just adding a fixed vector to each vector in the region. That is, a translation by a vector $\mathbf{v}$ is a function $T_{\mathbf{v}}$ such that the image $T_{\mathbf{v}}(R)$ of a region R is defined as

$$T_{\mathbf{v}}(R) = \{\mathbf{r} + \mathbf{v} : \mathbf{r} \in R\}.$$

  Since area is translation invariant, Area$(T_{\mathbf{v}}(R)) =$ Area(R).

- The area of a one-dimensional object like a line segment is 0.

Now we turn our attention to areas of parallelograms. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^2$. The parallelogram $P(\mathbf{u}, \mathbf{v})$ defined by $\mathbf{u}$ and $\mathbf{v}$ with point Q as basepoint is the set

$$P(\mathbf{u}, \mathbf{v}) = \{\overrightarrow{OQ} + r\mathbf{u} + s\mathbf{v} : 0 \leq r, s \leq 1\}.$$

An illustration of such a parallelogram is shown at left in Figure 16.4. If $\mathbf{u} = [u_1\ u_2]^{\mathsf{T}}$ and $\mathbf{v} = [v_1\ v_2]^{\mathsf{T}}$, then we will also represent $P(\mathbf{u}, \mathbf{v})$ as $P\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right)$.

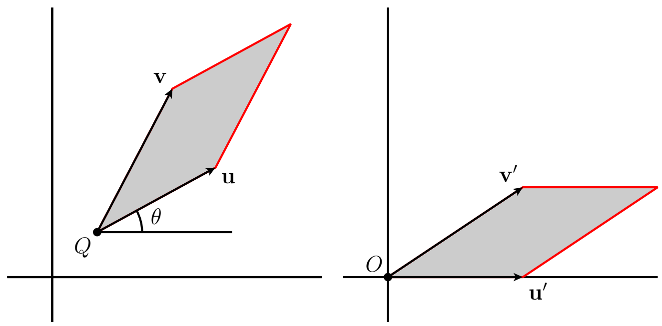

Figure 16.4: A parallelogram and a translated, rotated parallelogram.

Since area is translation and rotation invariant, we can translate our parallelogram by $-\overrightarrow{OQ}$ to place its basepoint at the origin, then rotate by an angle θ (as shown at left in Figure 16.4. This transforms the vector $\mathbf{v}$ to a vector $\mathbf{v}'$ and the vector $\mathbf{u}$ to a vector $\mathbf{u}'$ as shown at right in Figure 16.4. With this in mind we can always assume that our parallelograms have one vertex at the origin, with $\mathbf{u}$ along the x-axis, and $\mathbf{v}$ in standard position. Now we can investigate how to calculate the area of a parallelogram.

Project Activity 16.1. There are two situations to consider when we want to find the area of a parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$, both shown in Figure 16.5. The parallelogram will be determined by the lengths of these vectors.

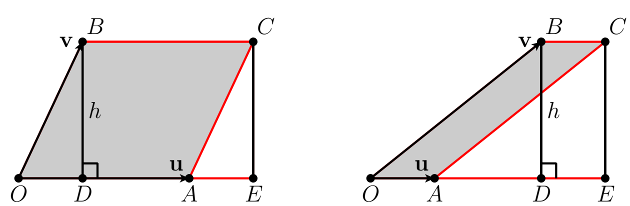

Figure 16.5: Parallelograms formed by $\mathbf{u}$ and $\mathbf{v}$

(a) In the situation depicted at left in Figure 16.5, use geometry to explain why $\text{Area}(P(\mathbf{u}, \mathbf{v})) = h|\mathbf{u}|$. (Hint: What can we say about the triangles ODB and EAC?)

(b) In the situation depicted at right in Figure 16.5, use geometry to again explain why Area$(P(\mathbf{u},\mathbf{v})) = h|\mathbf{u}|$. (Hint: What can we say about Area(AEC) and Area(ODB)?)

The result of Project Activity 16.1 is that the area of $P(\mathbf{u},\mathbf{v})$ is given by $h|\mathbf{u}|$, where h is the height of the parallelogram determined by dropping a perpendicular from the terminal point of $\mathbf{v}$ to the line determined by the vector $\mathbf{u}$.

Now we turn to the question of how the determinant is related to area of a parallelogram. Our approach will use some properties of the area of $P(\mathbf{u},\mathbf{v})$.

Project Activity 16.2. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors that determine a parallelogram in $\mathbb{R}^2$.

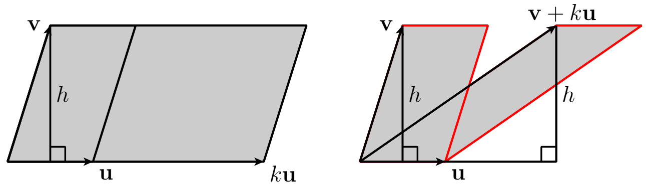

Figure 16.6: Parallelograms formed by $k\mathbf{u}$ and $\mathbf{v}$ and by $\mathbf{u}$ and $\mathbf{v}+k\mathbf{u}$.

(a) Explain why

$$\text{Area}(P(\mathbf{u},\mathbf{v})) = \text{Area}(P(\mathbf{v},\mathbf{u})) \tag{16.3}$$

(b) If k is any scalar, then $k\mathbf{u}$ either stretches or compresses $\mathbf{u}$. Use this idea, and the result of Project Activity 16.1, to explain why

$$\text{Area}(P(k\mathbf{u},\mathbf{v})) = \text{Area}(P(\mathbf{u},k\mathbf{v})) = |k|\text{Area}(P(\mathbf{u},\mathbf{v})) \tag{16.4}$$

for any real number k. A representative picture of this situation is shown at left in Figure 16.5 for a value of $k > 1$. You will also need to consider what happens when $k < 0$.

(c) Finally, use the result of Project Activity 16.1 to explain why

$$\text{Area}(P(\mathbf{u}+k\mathbf{v},\mathbf{v})) = \text{Area}(P(\mathbf{u},\mathbf{v}+k\mathbf{u})) = \text{Area}(P(\mathbf{u},\mathbf{v})) \tag{16.5}$$

for any real number k. A representative picture is shown at right in Figure 16.6.

Properties (16.4) and (16.5) will allow us to calculate the area of the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$.

Project Activity 16.3. Let $\mathbf{u} = [u_1\ u_2]^{\mathsf{T}}$ and $\mathbf{v} = [v_1\ v_2]^{\mathsf{T}}$. We will now demonstrate that

$$\text{Area}(P(\mathbf{u},\mathbf{v})) = \left|\det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right)\right|.$$

Before we begin, note that if both u_1 and v_1 are 0, then $\mathbf{u}$ and $\mathbf{v}$ are parallel. This makes $P(\mathbf{u},\mathbf{v})$ a line segment and so Area$(P(\mathbf{u},\mathbf{v})) = 0$. But if $u_1 = v_1 = 0$, it is also the case that

$$\det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) = u_1v_2 - u_2v_1 = 0$$

as well. So we can assume that at least one of u_1, v_1 is not 0. Since $P(\mathbf{u},\mathbf{v}) = P(\mathbf{v},\mathbf{u})$, we can assume without loss of generality that $u_1 \neq 0$.

(a) Explain using properties (16.4) and (16.5) as appropriate why

$$\text{Area}(P(\mathbf{u},\mathbf{v})) = \text{Area}\left(P\left(\mathbf{u}, \left[0 \; v_2 - \frac{v_1}{u_1}u_2\right]\right)\right).$$

(b) Let $\mathbf{v}_1 = \left[0 \; v_2 - \frac{v_1}{u_1}u_2\right]^{\mathsf{T}}$. Recall that our alternate representation of $P(\mathbf{u},\mathbf{v}))$ allows us to write

$$\text{Area}(P(\mathbf{u},\mathbf{v}_1)) = \text{Area}\left(P\left(\left[\begin{array}{cc} u_1 & u_2 \\ 0 & v_2 - \frac{v_1}{u_1}u_2 \end{array}\right]\right)\right).$$

This should seem very suggestive. We are essentially applying the process of Gaussian elimination to our parallelogram matrix to reduce it to a diagonal matrix. From there, we can calculate the area. The matrix form should indicate the next step – applying an operation to eliminate the entry in the first row and second column. To do this, we need to consider what happens if $v_2 - \frac{v_1}{u_1}u_2 = 0$ and if $v_2 - \frac{v_1}{u_1}u_2 \neq 0$.

i. Assume that $v_2 - \frac{v_1}{u_1}u_2 = 0$. Explain why $\text{Area}(P(\mathbf{u},\mathbf{v})) = 0$. Then explain why $\text{Area}(P(\mathbf{u},\mathbf{v})) = 0 = \det\left(\left[\begin{array}{cc} u_1 & u_2 \\ v_1 & v_2 \end{array}\right]\right)$.

ii. Now we consider the case when $v_2 - \frac{v_1}{u_1}u_2 \neq 0$. Complete the process as in part (a), using properties (16.4) and (16.5) (compare to Gaussian elimination) to continue to reduce the problem of calculating $\text{Area}(P(\mathbf{u},\mathbf{v}))$ to one of calculating $\text{Area}(P(\mathbf{e}_1,\mathbf{e}_2))$. Use this process to conclude that

$$\text{Area}(P(\mathbf{u},\mathbf{v})) = \left|\det\left(\left[\begin{array}{cc} u_1 & u_2 \\ v_1 & v_2 \end{array}\right]\right)\right|.$$

We can apply the same arguments as above using rotations, translations, shearings, and scalings to show that the properties of area given above work in any dimension. Given vectors $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_n$ in $\mathbb{R}^n$, we let

$$P(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n) = \{\overrightarrow{OQ} + x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n : 0 \leq x_i \leq 1 \text{ for each } i\}.$$

If $n = 2$, then $P(\mathbf{u}_1,\mathbf{u}_2)$ is the parallelogram determined by $\mathbf{u}_1$ and $\mathbf{u}_2$ with basepoint Q. If $n = 3$, then $P(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)$ is the parallelepiped with basepoint Q determined by $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. In higher dimensions the sets $P(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n)$ are called parallelotopes, and we use the notation $\text{Vol}(P(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n))$ for their volume. The n-dimensional volumes of these paralleotopes satisfy the following properties:

$$\begin{aligned}&\text{Vol}(P(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_{i-1},\mathbf{u}_i,\mathbf{u}_{i+1},\ldots,\mathbf{u}_{j-1},\mathbf{u}_j,\mathbf{u}_{j+1},\ldots,\mathbf{u}_n))\\ &\quad = \text{Vol}(P(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_{i-1},\mathbf{u}_j,\mathbf{u}_{i+1},\ldots,\mathbf{u}_{j-1},\mathbf{u}_i,\mathbf{u}_{j+1},\ldots,\mathbf{u}_n))\end{aligned} \tag{16.6}$$

for any i and j.

$$\text{Vol}(P(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_{i-1},k\mathbf{u}_i,\mathbf{u}_{i+1},\ldots,\mathbf{u}_n)) = |k|\text{Vol}(P(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n)) \tag{16.7}$$

for any real number k and any i.

$$\text{Vol}(P(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_{i-1},\mathbf{u}_i + k\mathbf{u}_j,\mathbf{u}_{i+1},\ldots,\mathbf{u}_n)) = \text{Vol}(P(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n)) \tag{16.8}$$

for any real number k and any distinct i and j.

Project Activity 16.4. We now show that $\text{Vol}(P(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3))$ is the absolute value of the determinant of $\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix}$. For easier notation, let $\mathbf{u} = [u_1\ u_2\ u_3]^{\mathsf{T}}$, $\mathbf{v} = [v_1\ v_2\ v_3]^{\mathsf{T}}$, and $\mathbf{w} = [w_1\ w_2\ w_3]^{\mathsf{T}}$. As we argued in the 2-dimensional case, we can assume that all terms that we need to be nonzero are nonzero, and we can do so without verification.

(a) Explain how property (16.7) shows that $\text{Vol}(P(\mathbf{u}, \mathbf{v}, \mathbf{w}))$ is equal to

$$\text{Vol}\left(P\left(\begin{bmatrix} u_1 & u_2 & u_3 \\ 0 & \frac{1}{u_1}(v_2u_1 - v_1u_2) & \frac{1}{u_1}(v_3u_1 - v_1u_3) \\ 0 & \frac{1}{u_1}(w_2u_1 - w_1u_2) & \frac{1}{u_1}(w_3u_1 - w_1u_3) \end{bmatrix}\right)\right).$$

(Hint: Think about how these properties are related to row operations.)

(b) Now let $\mathbf{v}_1 = \left[0\ \frac{1}{u_1}(v_2u_1 - v_1u_2)\ \frac{1}{u_1}(v_3u_1 - v_1u_3)\right]^{\mathsf{T}}$ and $\mathbf{w}_1 = \left[0\ \frac{1}{u_1}(w_2u_1 - w_1u_2)\ \frac{1}{u_1}(w_3u_1 - w_1u_3)\right]^{\mathsf{T}}$. Explain how property (16.7) shows that $\text{Vol}(P(\mathbf{u}, \mathbf{v}, \mathbf{w}))$ is equal to

$$\text{Vol}\left(P\left(\begin{bmatrix} u_1 & u_2 & u_3 \\ 0 & \frac{1}{u_1}(v_2u_1 - v_1u_2) & \frac{1}{u_1}(v_3u_1 - v_1u_3) \\ 0 & 0 & d \end{bmatrix}\right)\right),$$

where

$$d = \frac{1}{u_1v_2 - u_2v_1}(u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1)).$$

(c) Just as we saw in the 2-dimensional case, we can proceed to use the diagonal entries to eliminate the entries above the diagonal without changing the volume to see that

$$\text{Vol}(P(\mathbf{u}, \mathbf{v}, \mathbf{w})) = \text{Vol}\left(P\left(\begin{bmatrix} u_1 & 0 & 0 \\ 0 & \frac{1}{u_1}(v_2u_1 - v_1u_2) & 0 \\ 0 & 0 & d \end{bmatrix}\right)\right).$$

Complete the process, applying appropriate properties to explain why

$$\text{Vol}(P(\mathbf{u}, \mathbf{v}, \mathbf{w})) = x\text{Vol}(P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3))$$

for some constant x. Find the constant and, as a result, find a specific expression for $\text{Vol}(P(\mathbf{u}, \mathbf{v}, \mathbf{w}))$ involving a determinant.

Properties (16.6), (16.7), and (16.8) involve the analogs of row operations on matrices, and we will prove algebraically that the determinant exhibits the same properties. In fact, the determinant can be uniquely defined by these properties. So in a sense, the determinant is an area or volume function.

Section 17

The Characteristic Equation

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is the characteristic polynomial of a matrix?
- What is the characteristic equation of a matrix?
- How and why is the characteristic equation of a matrix useful?
- How many different eigenvalues can an $n \times n$ matrix have?
- How large can the dimension of the eigenspace corresponding to an eigenvalue be?

Application: Modeling the Second Law of Thermodynamics

Pour cream into your cup of coffee and the cream spreads out; straighten up your room and it soon becomes messy again; when gasoline is mixed with air in a car's cylinders, it explodes if a spark is introduced. In each of these cases a transition from a low energy state (your room is straightened up) to a higher energy state (a messy, disorganized room) occurs. This can be described by entropy – a measure of the energy in a system. Low energy is organized (like ice cubes) and higher energy is not (like water vapor). It is a fundamental property of energy (as described by the second law of thermodynamics) that the entropy of a system cannot decrease. In other words, in the absence of any external intervention, things never become more organized.

The Ehrenfest model[1] is a Markov process proposed to explain the statistical interpretation of the second law of thermodynamics using the diffusion of gas molecules. This process can be modeled as a problem of balls and bins, as we will do later in this section. The characteristic

[1] named after Paul and Tatiana Ehrenfest who introduced it in "Über zwei bekannte Einwände gegen das Boltzmannsche H-Theorem," *Physikalishce Zeitschrift*, vol. 8 (1907), pp. 311-314)

polynomial of the transition matrix will help us find the eigenvalues and allow us to analyze our model.

Introduction

We have seen that the eigenvalues of an $n \times n$ matrix A are the scalars λ so that $A - \lambda I_n$ has a nontrivial null space. Since a matrix has a nontrivial null space if and only if the matrix is not invertible, we can also say that λ is an eigenvalue of A if

$$\det(A - \lambda I_n) = 0. \tag{17.1}$$

This equation is called the *characteristic equation* of A. It provides us an *algebraic* way to find eigenvalues, which can then be used in finding eigenvectors corresponding to each eigenvalue. Suppose we want to find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$. Note that

$$A - \lambda I_2 = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix},$$

with determinant $(1-\lambda)(3-\lambda) - 1 = \lambda^2 - 4\lambda + 2$. Hence, the eigenvalues λ_1, λ_2 are the solutions of the characteristic equation $\lambda^2 - 4\lambda + 2 = 0$. Using quadratic formula, we find that $\lambda_1 = 2 + \sqrt{2}$ and $\lambda_2 = 2 - \sqrt{2}$ are the eigenvalues.

In this activity, our goal will be to use the characteristic equation to obtain information about eigenvalues and eigenvectors of a matrix with real entries.

Preview Activity 17.1.

(1) For each of the following parts, use the characteristic equation to determine the eigenvalues of A. Then, for each eigenvalue λ, find a basis of the corresponding eigenspace, i.e., Nul $(A - \lambda I)$. You might want to recall how to find a basis for the null space of a matrix from Section 13. Also, make sure that your eigenvalue candidate λ yields nonzero eigenvectors in Nul $(A - \lambda I)$ for otherwise λ will not be an eigenvalue.

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

(2) Use your eigenvalue and eigenvector calculations of the above problem as a guidance to answer the following questions about a matrix with real entries.

(a) At most how many eigenvalues can a 2×2 matrix have? Is it possible to have no eigenvalues? Is it possible to have only one eigenvalue? Explain.

(b) If a matrix is an upper-triangular matrix (i.e., all entries below the diagonal are 0's, as in the first two matrices of the previous problem), what can you say about its eigenvalues? Explain.

(c) How many linearly independent eigenvectors can be found for a 2×2 matrix? Is it possible to have a matrix without 2 linearly independent eigenvectors? Explain.

(3) Using the characteristic equation, determine which matrices have 0 as an eigenvalue.

The Characteristic Equation

Until now, we have been given eigenvalues or eigenvectors of a matrix and determined eigenvectors and eigenvalues from the known information. In this section we use determinants to find (or approximate) the eigenvalues of a matrix. From there we can find (or approximate) the corresponding eigenvectors. The tool we will use is a polynomial equation, the *characteristic equation*, of a square matrix whose roots are the eigenvalues of the matrix. The characteristic equation will then provide us with an *algebraic* way of finding the eigenvalues of a square matrix.

We have seen that the eigenvalues of a square matrix A are the scalars λ so that $A - \lambda I$ has a nontrivial null space. Since a matrix has a nontrivial null space if and only if the matrix is not invertible, we can also say that λ is an eigenvalue of A if

$$\det(A - \lambda I) = 0. \tag{17.2}$$

Note that if A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n. Furthermore, if A has real entries, the polynomial has real coefficients. This polynomial, and the equation (17.2) are given special names.

Definition 17.1. Let A be an $n \times n$ matrix. The **characteristic polynomial** of A is the polynomial

$$\det(A - \lambda I_n),$$

where I_n is the $n \times n$ identity matrix. The **characteristic equation** of A is the equation

$$\det(A - \lambda I_n) = 0.$$

So the characteristic equation of A gives us an algebraic way of finding the eigenvalues of A.

Activity 17.1.

(a) Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 1 \end{bmatrix}$, and use the characteristic polynomial to find all of the eigenvalues of A.

(b) Verify that 1 and 2 are the only eigenvalues of the matrix $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

As we argued in Preview Activity 17.1, a 2×2 matrix can have at most 2 eigenvalues. For an $n \times n$ matrix, the characteristic polynomial will be a degree n polynomial, and we know from algebra that a degree n polynomial can have at most n roots. Since an eigenvalue of a matrix is a root of the characteristic polynomial of that matrix, we can conclude that an $n \times n$ matrix can have at most n distinct eigenvalues. Activity 17.1 (b) shows that a 4×4 matrix may have fewer than 4 eigenvalues, however. Note that one of these eigenvalues, the eigenvalue 1, appears three times as a root of the characteristic polynomial of the matrix. The number of times an eigenvalue appears as a root of the characteristic polynomial is called the *(algebraic) multiplicity* of the eigenvalue. More formally:

Definition 17.2. The **(algebraic) multiplicity** of an eigenvalue λ of a matrix A is the largest integer m so that $(x-\lambda)^m$ divides the characteristic polynomial of A.

Thus, in Activity 17.1 (b) the eigenvalue 1 has multiplicity 3 and the eigenvalue 2 has multiplicity 1. Notice that if we count the eigenvalues of an $n \times n$ matrix with their multiplicities, the total will always be n.

If A is a matrix with real entries, then the characteristic polynomial will have real coefficients. It is possible that the characteristic polynomial can have complex roots, and that the matrix A has complex eigenvalues. The Fundamental Theorem of Algebra shows us that if a real matrix has complex eigenvalues, then those eigenvalues will appear in conjugate pairs, i.e., if $\lambda_1 = a + ib$ is an eigenvalue of A, then $\lambda_2 = a - ib$ is another eigenvalue of A. Furthermore, for an odd degree polynomial, since the complex eigenvalues will come in conjugate pairs, we will be able to find at least one real eigenvalue.

We now summarize the information we have so far about eigenvalues of an $n \times n$ real matrix:

Theorem 17.3. *Let A be an $n \times n$ matrix with real entries. Then*

(1) There are at most n eigenvalues of A. If each eigenvalue (including complex eigenvalues) is counted with its multiplicity, there are exactly n eigenvalues.

(2) If A has a complex eigenvalue λ, the complex conjugate of λ is also an eigenvalue of A.

(3) If n is odd, A has at least one real eigenvalue.

(4) If A is upper or lower-triangular, the eigenvalues are the entries on the diagonal.

Eigenspaces, A Geometric Example

Recall that for each eigenvalue λ of an $n \times n$ matrix A, the eigenspace of A corresponding to the eigenvalue λ is Nul $(A - \lambda I_n)$. These eigenspaces can tell us important information about the matrix transformation defined by A. For example, consider the matrix transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

We are interested in understanding what this matrix transformation does to vectors in $\mathbb{R}^3$. First we note that A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, with λ_1 having multiplicity 2. There is a pair $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ of linearly independent eigenvectors for A corresponding to the eigenvalue λ_1 and an eigenvector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for A corresponding to the eigenvalue λ_2. Note that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent (recall from Theorem that eigenvectors corresponding to different eigenvalues are always linearly independent). So any vector $\mathbf{b}$ in $\mathbb{R}^3$ can be written uniquely as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Let's now consider the action of the matrix transformation T on a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_2$. Note that

$$\begin{aligned} T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) \\ &= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_1\mathbf{v}_2 + c_3\lambda_2\mathbf{v}_3 \\ &= (1)(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + (2)c_3\mathbf{v}_3. \end{aligned} \tag{17.3}$$

Equation (17.3) illustrates that it is most convenient to view the action of T in the coordinate system where Span$\{\mathbf{v}_1\}$ serves as the x-axis, Span$\{\mathbf{v}_2\}$ serves as the y-axis, and Span$\{\mathbf{v}_3\}$ as the z-axis. In this case, we can visualize that when we apply the transformation T to a vector $\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ in $\mathbb{R}^3$ the result is an output vector that is unchanged in the $\mathbf{v}_1$-$\mathbf{v}_2$ plane and scaled by a factor of 2 in the $\mathbf{v}_3$ direction. For example, consider the box whose sides are determined by the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as shown in Figure 17.1. The transformation T stretches this box by a factor of 2 in the $\mathbf{v}_3$ direction and leaves everything else alone, as illustrated in Figure 17.1. So the entire Span$\{\mathbf{v}_1, \mathbf{v}_2\}$) is unchanged by T, but Span$\{\mathbf{v}_3\}$) is scaled by 2. In this situation, the eigenvalues and eigenvectors provide the most convenient perspective through which to visualize the action of the transformation T.

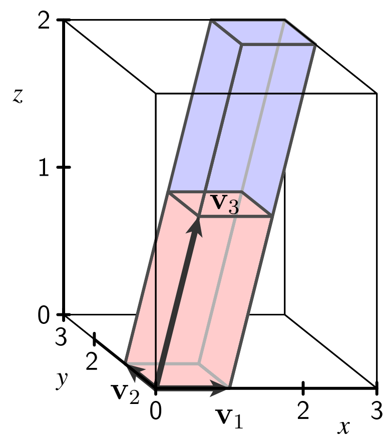

Figure 17.1: A box and a transformed box.

This geometric perspective illustrates how each eigenvalue and the corresponding eigenspace of A tells us something important about A. So it behooves us to learn a little more about eigenspaces.

Dimensions of Eigenspaces

There is a connection between the dimension of the eigenspace of a matrix corresponding to an eigenvalue and the multiplicity of that eigenvalue as a root of the characteristic polynomial. Recall that the dimension of a subspace of $\mathbb{R}^n$ is the number of vectors in a basis for the eigenspace. We investigate the connection between dimension and multiplicity in the next activity.

Activity 17.2.

(a) Find the dimension of the eigenspace for each eigenvalue of matrix $A = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 1 \end{bmatrix}$ from Activity 17.1 (a).

(b) Find the dimension of the eigenspace for each eigenvalue of matrix $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ from Activity 17.1 (b).

(c) Consider now a 3×3 matrix with 3 distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

i. Recall that a polynomial of degree can have at most three distinct roots. What does that say about the multiplicities of $\lambda_1, \lambda_2, \lambda_3$?

ii. Use the fact that eigenvectors corresponding to distinct eigenvalues are linearly independent to find the dimensions of the eigenspaces for $\lambda_1, \lambda_2, \lambda_3$.

The examples in Activity 17.2 all provide instances of the principle that the dimension of an eigenspace corresponding to an eigenvalue λ cannot exceed the multiplicity of λ. Specifically:

Theorem 17.4. *If λ is an eigenvalue of A, the dimension of the eigenspace corresponding to λ is less than or equal to the multiplicity of λ.*

The examples we have seen raise another important point. The matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ from our geometric example has two eigenvalues 1 and 2, with the eigenvalue 1 having multiplicity 2. If we let E_λ represent the eigenspace of A corresponding to the eigenvalue λ, then $\dim(E_1) = 2$ and $\dim(E_2) = 1$. If we change this matrix slightly to the matrix $B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ we see that B has two eigenvalues 1 and 2, with the eigenvalue 1 having multiplicity 2. However, in this case we have $\dim(E_1) = 1$ (like the example in from Activities 17.1 (a) and 17.2 (a)). In this case the vector $\mathbf{v}_1 = [1\ 0\ 0]^{\mathsf{T}}$ forms a basis for E_2 and the vector $\mathbf{v}_2 = [0\ 1\ 0]^{\mathsf{T}}$ forms a basis for E_1. We can visualize the action of B on the square formed by $\mathbf{v}_1$ and $\mathbf{v}_2$ in the xy-plane as a scaling by 2 in the $\mathbf{v}_1$ direction as shown in Figure 17.2, but since we do not have a third linearly independent eigenvector, the action of B in the direction of $[0\ 0\ 1]^{\mathsf{T}}$ is not so clear.

So the action of a matrix transformation can be more easily visualized if the dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue. This geometric perspective leads us to define the geometric multiplicity of an eigenvalue.

Definition 17.5. The **geometric multiplicity** of an eigenvalue of an $n \times n$ matrix A is the dimension of the corresponding eigenspace Nul $(A - \lambda I_n)$.

Examples

What follows are worked examples that use the concepts from this section.

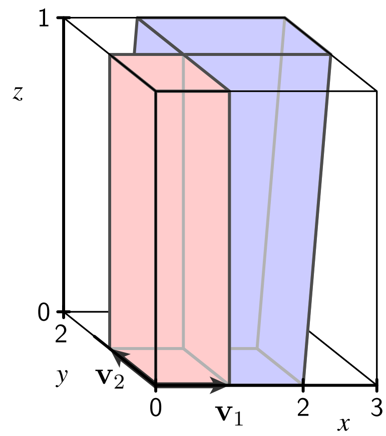

Figure 17.2: A box and a transformed box.

Example 17.6. Let $A = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find the characteristic polynomial of A.

(b) Factor the characteristic polynomial and find the eigenvalues of A.

(c) Find a basis for each eigenspace of A.

(d) Is it possible to find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A? Explain.

Example Solution.

(a) The characteristic polynomial of A is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I_3) \\ &= \det\left(\begin{bmatrix} -1-\lambda & 0 & -2 \\ 2 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix} \right) \\ &= (-1-\lambda)(1-\lambda)(1-\lambda). \end{aligned}$$

(b) The eigenvalues of A are the solutions to the characteristic equation. Since

$$p(\lambda) = (-1-\lambda)(1-\lambda)(1-\lambda) = 0$$

implies $\lambda = -1$ or $\lambda = 1$, the eigenvalues of A are 1 and -1.

(c) To find a basis for the eigenspace of A corresponding to the eigenvalue 1, we find a basis for Nul $(A - I_3)$. The reduced row echelon form of $A - I_3 = \begin{bmatrix} -2 & 0 & -2 \\ 2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ is

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $(A - I_3)\mathbf{x} = \mathbf{0}$ has general solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, $\{[0\ 1\ 0]^{\mathsf{T}}, [-1\ 0\ 1]^{\mathsf{T}}\}$ is a basis for the eigenspace of A corresponding to the eigenvalue 1.

To find a basis for the eigenspace of A corresponding to the eigenvalue -1, we find a basis for Nul $(A + I_3)$. The reduced row echelon form of $A + I_3 = \begin{bmatrix} 0 & 0 & -2 \\ 2 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $(A + I_3)\mathbf{x} = \mathbf{0}$ has general solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore, a basis for the eigenspace of A corresponding to the eigenvalue -1 is $\{[-1\ 1\ 0]^{\mathsf{T}}\}$.

(d) Let $\mathbf{v}_1 = [0\ 1\ 0]^{\mathsf{T}}, [-1\ 0\ 1]^{\mathsf{T}}$, $\mathbf{v}_2 = [-1\ 0\ 1]^{\mathsf{T}}$, and $\mathbf{v}_3 = [-1\ 1\ 0]^{\mathsf{T}}$. Since eigenvectors corresponding to different eigenvalues are linearly independent, and since neither $\mathbf{v}_1$ nor $\mathbf{v}_2$ is a scalar multiple of the other, we can conclude that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set with $3 = \dim(\mathbb{R}^3)$ vectors. Therefore, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

Example 17.7. Find a 3×3 matrix A that has an eigenvector $\mathbf{v}_1 = [1\ 0\ 1]^{\mathsf{T}}$ with corresponding eigenvalue $\lambda_1 = 2$, an eigenvector $\mathbf{v}_2 = [0\ 2\ -3]^{\mathsf{T}}$ with corresponding eigenvalue $\lambda_2 = -3$, and an eigenvector $\mathbf{v}_3 = [-4\ 0\ 5]^{\mathsf{T}}$ with corresponding eigenvalue $\lambda_3 = 5$. Explain your process.

Example Solution. We are looking for a 3×3 matrix A such that $A\mathbf{v}_1 = 2\mathbf{v}_1$, $A\mathbf{v}_2 = -3\mathbf{v}_2$ and $A\mathbf{v}_3 = 5\mathbf{v}_3$. Since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors corresponding to different eigenvalues, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent. So the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ is invertible. It follows that

$$\begin{aligned}
A[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] &= [A\mathbf{v}_1\ A\mathbf{v}_2\ A\mathbf{v}_3] \\
A\begin{bmatrix} 1 & 0 & -4 \\ 0 & 2 & 0 \\ 1 & -3 & 5 \end{bmatrix} &= [2\mathbf{v}_1\ -3\mathbf{v}_2\ 5\mathbf{v}_3] \\
A\begin{bmatrix} 1 & 0 & -4 \\ 0 & 2 & 0 \\ 1 & -3 & 5 \end{bmatrix} &= \begin{bmatrix} 2 & 0 & -20 \\ 0 & -6 & 0 \\ 2 & 9 & 25 \end{bmatrix} \\
A &= \begin{bmatrix} 2 & 0 & -20 \\ 0 & -6 & 0 \\ 2 & 9 & 25 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ 0 & 2 & 0 \\ 1 & -3 & 5 \end{bmatrix}^{-1} \\
A &= \begin{bmatrix} 2 & 0 & -20 \\ 0 & -6 & 0 \\ 2 & 9 & 25 \end{bmatrix}\begin{bmatrix} \frac{5}{9} & \frac{2}{3} & \frac{4}{9} \\ 0 & \frac{1}{2} & 0 \\ -\frac{1}{9} & \frac{1}{6} & \frac{1}{9} \end{bmatrix} \\
A &= \begin{bmatrix} \frac{10}{3} & -2 & -\frac{4}{3} \\ 0 & -3 & 0 \\ -\frac{5}{3} & 10 & \frac{11}{3} \end{bmatrix}.
\end{aligned}$$

Summary

In this section we studied the characteristic polynomial of a matrix and similar matrices.

- If A is an $n \times n$ matrix, the characteristic polynomial of A is the polynomial
$$\det(A - \lambda I_n),$$
where I_n is the $n \times n$ identity matrix.
- If A is an $n \times n$ matrix, the characteristic equation of A is the equation
$$\det(A - \lambda I_n) = 0.$$
- The characteristic equation of a square matrix provides us an algebraic method to find the eigenvalues of the matrix.
- The eigenvalues of an upper or lower-triangular matrix are the entries on the diagonal.
- There are at most n eigenvalues of an $n \times n$ matrix.
- For a real matrix A, if an eigenvalue λ of A is complex, then the complex conjugate of λ is also an eigenvalue.
- The algebraic multiplicity of an eigenvalue λ is the multiplicity of λ as a root of the characteristic equation.
- The dimension of the eigenspace corresponding to an eigenvalue λ is less than or equal to the algebraic multiplicity of λ.

Exercises

(1) There is interesting relationship[2] between a matrix and its characteristic equation that we explore in this exercise.

(a) We first illustrate with an example. Let $B = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$.

i. Show that $\lambda^2 + \lambda - 4$ is the characteristic polynomial for B.

ii. Calculate B^2. Then compute $B^2 + B - 4I_2$. What do you get?

(b) The first part of this exercise presents an example of a matrix that satisfies its own characteristic equation. Explain for a general $n \times n$ matrix, why A satisfies its characteristic equation.

(2) There is a useful relationship between the determinant and eigenvalues of a matrix A that we explore in this exercise.

(a) Let $B = \begin{bmatrix} 2 & 3 \\ 8 & 4 \end{bmatrix}$. Find the determinant of B and the eigenvalues of B, and compare $\det(B)$ to the eigenvalues of B.

(b) Let A be an $n \times n$ matrix. In this part of the exercise we argue the general case illustrated in the previous part – that $\det(A)$ is the product of the eigenvalues of A. Let $p(\lambda) = \det(A - \lambda I_n)$ be the characteristic polynomial of A.

i. Let λ_1, λ_2, ..., λ_n be the eigenvalues of A (note that these eigenvalues may not all be distinct). Recall that if r is a root of a polynomial $q(x)$, then $(x - r)$ is a factor of $q(x)$. Use this idea to explain why

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n).$$

ii. Explain why $p(0) = \lambda_1\lambda_2\cdots\lambda_n$.

iii. Why is $p(0)$ also equal to $\det(A)$. Explain how we have shown that $\det(A)$ is the product of the eigenvalues of A.

(3) Find the eigenvalues of the following matrices. For each eigenvalue, determine its algebraic and geometric multiplicity.

(a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

(4) Let A be an $n \times n$ matrix. Use the characteristic equation to explain why A and A^{T} have the same eigenvalues.

[2]This result is known as the Cayley-Hamilton Theorem and is one of the fascinating results in linear algebra.

(5) Find three 3×3 matrices whose eigenvalues are 2 and 3, and for which the dimensions of the eigenspaces for $\lambda = 2$ and $\lambda = 3$ are different.

(6) Suppose A is an $n \times n$ matrix and B is an invertible $n \times n$ matrix. Explain why the characteristic polynomial of A is the same as the characteristic polynomial of BAB^{-1}, and hence, as a result, the eigenvalues of A and BAB^{-1} are the same.

(7) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** If the determinant of a 2×2 matrix A is positive, then A has two distinct real eigenvalues.

(b) **True/False** If two 2×2 matrices have the same eigenvalues, then the have the same eigenvectors.

(c) **True/False** The characteristic polynomial of an $n \times n$ matrix has degree n.

(d) **True/False** If R is the reduced row echelon form of an $n \times n$ matrix A, then A and R have the same eigenvalues.

(e) **True/False** If R is the reduced row echelon form of an $n \times n$ matrix A, and $\mathbf{v}$ is an eigenvector of A, then $\mathbf{v}$ is an eigenvector of R.

(f) **True/False** Let A and B be $n \times n$ matrices with characteristic polynomials $p_A(\lambda)$ and $p_B(\lambda)$, respectively. If $A \neq B$, then $p_A(\lambda) \neq p_B(\lambda)$.

(g) **True/False** Every matrix has at least one eigenvalue.

(h) **True/False** Suppose A is a 3×3 matrix with three distinct eigenvalues. Then any three eigenvectors, one for each eigenvalue, will form a basis of $\mathbb{R}^3$.

(i) **True/False** If an eigenvalue λ is repeated 3 times among the eigenvalues of a matrix, then there are at most 3 linearly independent eigenvectors corresponding to λ.

Project: The Ehrenfest Model

To realistically model the diffusion of gas molecules we would need to consider a system with a large number of balls as substitutes for the gas molecules. However, the main idea can be seen in a model with a much smaller number of balls, as we will do now. Suppose we have two bins that contain a total of 4 balls between them. Label the bins as Bin 1 and Bin 2. In this case we can think of entropy as the number of different possible ways the balls can be arranged in the system. For example, there is only 1 way for all of the balls to be in Bin 1 (low entropy), but there are 4 ways that we can have one ball in Bin 1 (choose any one of the four different balls, which can be distinguished from each other) and 3 balls in Bin 2 (higher entropy). The highest entropy state has the balls equally distributed between the bins (with 6 different ways to do this).

We assume that there is a way for balls to move from one bin to the other (like having gas molecules pass through a permeable membrane). A way to think about this is that we select a ball (from ball 1 to ball 4, which are different balls) and move that ball from its current bin to the other bin. Consider a "move" to be any instance when a ball changes bins. A *state* is any configuration of balls in the bins at a given time, and the state changes when a ball is chosen at random and moved to the other bin. The possible states are to have 0 balls in Bin 1 and 4 balls in Bin 2 (State 0, entropy 1), 1 ball in Bin 1 and 3 in Bin 2 (State 1, entropy 4), 2 balls in each Bin (State 2, entropy 6), 3 balls in Bin 1 and 1 ball in Bin 2 (State 3, entropy 4), and 4 balls in Bin 1 and 0 balls in Bin 2 (State 4, entropy 1). These states are shown in Figure 17.3.

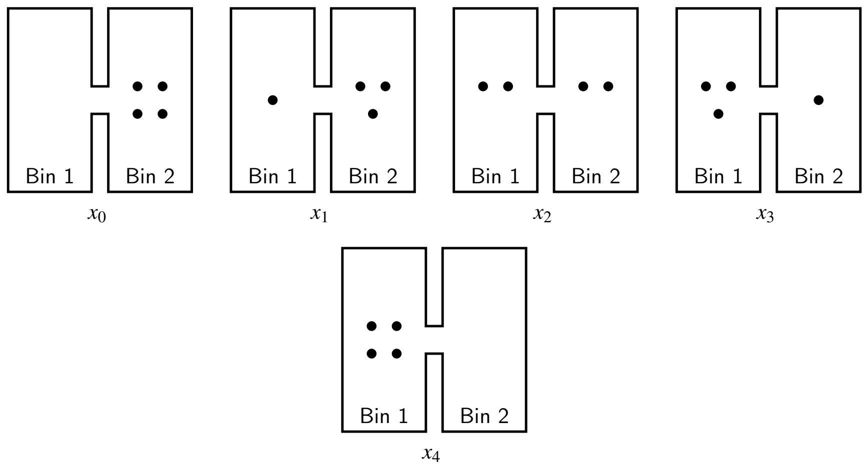

Figure 17.3: States

Project Activity 17.1. To model the system of balls in bins we need to understand how the system can transform from one state to another. It suffices to count the number of balls in Bin 1 (since the remaining balls will be in Bin 2). Even though the balls are labeled, our count only cares about how many balls are in each bin. Let $\mathbf{x}_0 = [x_0, x_1, x_2, x_3, x_4]^\mathsf{T}$, where x_i is the probability that Bin 1 contains i balls, and let $\mathbf{x}_1 = \left[x_0^1, x_1^1, x_2^1, x_3^1, x_4^1\right]^\mathsf{T}$, where x_i^1 is the probability that Bin 1 contains i balls after the first move. We will call the vectors $\mathbf{x}_0$ and $\mathbf{x}_1$ *probability distributions* of balls in bins. Note that since all four balls have to be placed in some bin, the sum of the entries in our probability distribution vectors must be 1. Recall that a move is an instance when a ball changes bins. We want to understand how $\mathbf{x}_1$ is obtained from $\mathbf{x}_0$. In other words, we want to figure out what the probability that Bin 1 contains 0, 1, 2, 3, or 4 balls after one ball changes bins if our initial probability distribution of balls in bins is $\mathbf{x}_0$.

We begin by analyzing the ways that a state can change. For example,

- Suppose there are 0 balls in Bin 1. (In our probability distribution $\mathbf{x}_0$, this happens with probability x_0.) Then there are four balls in Bin 2. The only way for a ball to change bins is if one of the four balls moves from Bin 2 to Bin 1, putting us in State 1. Regardless of which ball moves, we will always be put in State 1, so this happens with a probability of 1. In other words, if the probability that Bin 1 contains 0 balls is x_0, then there is a probability of $(1)x_0$ that Bin 1 will contain 1 ball after the move.

- Suppose we have 1 ball in Bin 1. There are four ways this can happen (since there are four balls, and the one in Bin 1 is selected at random from the four balls), so the probability of a given ball being in Bin 1 is $\frac{1}{4}$.

– If the ball in Bin 1 moves, that move puts us in State 0. In other words, if the probability that Bin 1 contains 1 ball is x_1, then there is a probability of $\frac{1}{4}x_1$ that Bin 1 will contain 0 balls after a move.
– If any of the 3 balls in Bin 2 moves (each moves with probability $\frac{3}{4}$), that move puts us in State 2. In other words, if the probability that Bin 1 contains 1 ball is x_1, then there is a probability of $\frac{3}{4}x_1$ that Bin 1 will contain 2 balls after a move.

(a) Complete this analysis to explain the probabilities if there are 2, 3, or 4 balls in Bin 1.

(b) Explain how the results of part (a) show that

$$\begin{array}{lcrcrcrcrcr} x_0^1 & = & 0x_0 & + & \frac{1}{4}x_1 & + & 0x_2 & + & 0x_3 & + & 0x_4 \\ x_1^1 & = & 1x_0 & + & 0x_1 & + & \frac{1}{2}x_2 & + & 0x_3 & + & 0x_4 \\ x_2^1 & = & 0x_0 & + & \frac{3}{4}x_1 & + & 0x_2 & + & \frac{3}{4}x_3 & + & 0x_4 \\ x_3^1 & = & 0x_0 & + & 0x_1 & + & \frac{1}{2}x_2 & + & 0x_3 & + & 1x_4 \\ x_4^1 & = & 0x_0 & + & 0x_1 & + & 0x_2 & + & \frac{1}{4}x_3 & + & 0x_4 \end{array}$$

The system we developed in Project Activity 17.1 has matrix form

$$\mathbf{x}_1 = T\mathbf{x}_0,$$

where T is the *transition matrix*

$$T = \begin{bmatrix} 0 & \frac{1}{4} & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \end{bmatrix}.$$

Subsequent moves give probability distribution vectors

$$\begin{aligned} \mathbf{x}_2 &= T\mathbf{x}_1 \\ \mathbf{x}_3 &= T\mathbf{x}_2 \\ \vdots\ &\quad \vdots \\ \mathbf{x}_k &= T\mathbf{x}_{k-1}. \end{aligned}$$

This example is an example of a Markov process (see Definition 9.4). There are several questions we can ask about this model. For example, what is the long-term behavior of this system, and how does this model relate to entropy? That is, given an initial probability distribution vector $\mathbf{x}_0$, the system will have probability distribution vectors $\mathbf{x}_1$, $\mathbf{x}_2$, ... after subsequent moves. What happens to the vectors $\mathbf{x}_k$ as k goes to infinity, and what does this tell us about entropy? To answer these questions, we will first explore the sequence $\{\mathbf{x}_k\}$ numerically, and then use the eigenvalues and eigenvectors of T to analyze the sequence $\{\mathbf{x}_k\}$.

Project Activity 17.2. Use appropriate technology to do the following.

(a) Suppose we begin with a probability distribution vector $\mathbf{x}_0 = [1\ 0\ 0\ 0\ 0]^{\mathsf{T}}$. Calculate vectors $\mathbf{x}_k$ for enough values of k so that you can identify the long term behavior of the sequence. Describe this behavior.

(b) Repeat part (a) with

i. $\mathbf{x}_0 = \left[0\ \frac{1}{2}\ \frac{1}{2}\ 0\ 0\right]^{\mathsf{T}}$

ii. $\mathbf{x}_0 = \left[0\ \frac{1}{3}\ \frac{1}{3}\ 0\ \frac{1}{3}\right]^{\mathsf{T}}$

iii. $\mathbf{x}_0 = \left[\frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\ \frac{1}{5}\right]^{\mathsf{T}}$

Describe the long term behavior of the sequence $\{\mathbf{x}_k\}$ in each case.

In what follows, we investigate the behavior of the sequence $\{\mathbf{x}_k\}$ that we uncovered in Project Activity 17.2.

Project Activity 17.3. We use the characteristic polynomial to find the eigenvalues of T.

(a) Find the characteristic polynomial of T. Factor the characteristic polynomial into a product of linear polynomials to show that the eigenvalues of T are 0, 1, -1, $\frac{1}{2}$ and $-\frac{1}{2}$.

(b) As we will see a bit later, certain eigenvectors for T will describe the end behavior of the sequence $\{\mathbf{x}_k\}$. Find eigenvectors for T corresponding to the eigenvalues 1 and -1. Explain how the eigenvector for T corresponding to the eigenvalue 1 explains the behavior of one of the sequences was saw in Project Activity 17.2. (Any eigenvector of T with eigenvalue 1 is called an *equilibrium* or *steady state* vector.)

Now we can analyze the behavior of the sequence $\{\mathbf{x}_k\}$.

Project Activity 17.4. To make the notation easier, we will let $\mathbf{v}_1$ be an eigenvector of T corresponding to the eigenvalue 0, $\mathbf{v}_2$ an eigenvector of T corresponding to the eigenvalue 1, $\mathbf{v}_3$ an eigenvector of T corresponding to the eigenvalue -1, $\mathbf{v}_4$ an eigenvector of T corresponding to the eigenvalue $\frac{1}{2}$, and $\mathbf{v}_5$ an eigenvector of T corresponding to the eigenvalue $-\frac{1}{2}$.

(a) Explain why $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is a basis of $\mathbb{R}^5$.

(b) Let $\mathbf{x}_0$ be any initial probability distribution vector. Explain why we can write $\mathbf{x}_0$ as

$$\mathbf{x}_0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 = \sum_{i=1}^{5} a_i\mathbf{v}_i$$

for some scalars a_1, a_2, a_3, a_4, and a_5.

We can now use the eigenvalues and eigenvectors of T to write the vectors $\mathbf{x}_k$ in a convenient form. Let $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$, $\lambda_4 = \frac{1}{2}$, and $\lambda_5 = -\frac{1}{2}$. Notice that

$$\begin{aligned}
\mathbf{x}_1 &= T\mathbf{x}_0 \\
&= T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5) \\
&= a_1T\mathbf{v}_1 + a_2T\mathbf{v}_2 + a_3T\mathbf{v}_3 + a_4T\mathbf{v}_4 + a_5T\mathbf{v}_5 \\
&= a_1\lambda_1\mathbf{v}_1 + a_2\lambda_2\mathbf{v}_2 + a_3\lambda_3\mathbf{v}_3 + a_4\lambda_4\mathbf{v}_4 + a_5\lambda_5\mathbf{v}_5 \\
&= \sum_{i=1}^{5} a_i\lambda_i\mathbf{v}_i.
\end{aligned}$$

Similarly

$$\mathbf{x}_2 = T\mathbf{x}_1 = T\left(\sum_{i=1}^{5} a_i\lambda_i\mathbf{v}_i\right) = \sum_{i=1}^{5} a_i\lambda_i T\mathbf{v}_i = \sum_{i=1}^{5} a_i\lambda_i^2\mathbf{v}_i.$$

We can continue in this manner to ultimately show that for each positive integer k we have

$$\mathbf{x}_k = \sum_{i=1}^{5} a_i\lambda_i^k\mathbf{v}_i \tag{17.4}$$

when $\mathbf{x}_0 = \sum_{i=1}^{5} a_i\mathbf{v}_i$.

Project Activity 17.5. Recall that we are interested in understanding the behavior of the sequence $\{\mathbf{x}_k\}$ as k goes to infinity.

(a) Equation (17.4) shows that we need to know $\lim_{k\to\infty} \lambda_i^k$ for each i in order to analyze $\lim_{k\to\infty} \mathbf{x}_k$. Calculate or describe these limits.

(b) Use the result of part (a), Equation (17.4), and Project Activity 17.3 (b) to explain why the sequence $\{\mathbf{x}_k\}$ is either eventually fixed or oscillates between two states. Compare to the results from Project Activity 17.2. How are these results related to entropy? You may use the facts that

- $\mathbf{v}_1 = [1\ 0\ -2\ 0\ 1]^{\mathsf{T}}$ is an eigenvector for T corresponding to the eigenvalue 0,
- $\mathbf{v}_2 = [1\ 4\ 6\ 4\ 1]^{\mathsf{T}}$ is an eigenvector for T corresponding to the eigenvalue 1,
- $\mathbf{v}_3 = [1\ -4\ 6\ -4\ 1]^{\mathsf{T}}$ is an eigenvector for T corresponding to the eigenvalue -1,
- $\mathbf{v}_4 = [-1\ -2\ 0\ 2\ 1]^{\mathsf{T}}$ is an eigenvector for T corresponding to the eigenvalue $\frac{1}{2}$,
- $\mathbf{v}_5 = [-1\ 2\ 0\ -2\ 1]^{\mathsf{T}}$ is an eigenvector for T corresponding to the eigenvalue $-\frac{1}{2}$.

Section 18

Diagonalization

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a diagonal matrix?
- What does it mean to diagonalize a matrix?
- What does it mean for two matrices to be similar?
- What important properties do similar matrices share?
- Under what conditions is a matrix diagonalizable?
- When a matrix A is diagonalizable, what is the structure of a matrix P that diagonalizes A?
- Why is diagonalization useful?

Application: The Fibonacci Numbers

In 1202 Leonardo of Pisa (better known as Fibonacci) published *Liber Abaci* (roughly translated as *The Book of Calculation*), in which he constructed a mathematical model of the growth of a rabbit population. The problem Fibonacci considered is that of determining the number of pairs of rabbits produced in a given time period beginning with an initial pair of rabbits. Fibonacci made the assumptions that each pair of rabbits more than one month old produces a new pair of rabbits each month, and that no rabbits die. (We ignore any issues about that might arise concerning the gender of the offspring.) If we let F_n represent the number of rabbits in month n, Fibonacci produced the model

$$F_{n+2} = F_{n+1} + F_n, \tag{18.1}$$

for $n \geq 0$ where $F_0 = 0$ and $F_1 = 1$. The resulting sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

is a very famous sequence in mathematics and is called the Fibonacci sequence. This sequence is thought to model many natural phenomena such as number of seeds in a sunflower and anything which grows in a spiral form. It is so famous in fact that it has a journal devoted entirely to it. As a note, while Fibonacci's work *Liber Abaci* introduced this sequence to the western world, it had been described earlier Sanskrit texts going back as early as the sixth century.

By definition, the Fibonacci numbers are calculated by recursion. This is a vey ineffective way to determine entries F_n for large n. Later in this section we will derive a fascinating and unexpected formula for the Fibonacci numbers using the idea of diagonalization.

Introduction

As we have seen when studying Markov processes, each state is dependent on the previous state. If $\mathbf{x}_0$ is the initial state and A is the transition matrix, then the nth state is found by $A^n\mathbf{x}_0$. In these situations, and others, it is valuable to be able to quickly and easily calculate powers of a matrix. We explore a way to do that in this section.

Preview Activity 18.1. Consider a very simplified weather forecast. Let us assume there are two possible states for the weather: rainy (R) or sunny(S). Let us also assume that the weather patterns are stable enough that we can reasonably predict the weather tomorrow based on the weather today. If is is sunny today, then there is a 70% chance that it will be sunny tomorrow, and if it is rainy today then there is a 40% chance that it will be rainy tomorrow. If $\mathbf{x}_0 = \begin{bmatrix} s \\ r \end{bmatrix}$ is a state vector that indicates a probability s that it is sunny and probability r that it is rainy on day 0, then

$$\mathbf{x}_1 = \begin{bmatrix} 0.70 & 0.40 \\ 0.30 & 0.60 \end{bmatrix} \mathbf{x}_0$$

tells us the likelihood of it being sunny or rainy on day 1. Let $A = \begin{bmatrix} 0.70 & 0.40 \\ 0.30 & 0.60 \end{bmatrix}$.

(1) Suppose it is sunny today, that is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Calculate $\mathbf{x}_1 = A\mathbf{x}_0$ and explain how this matrix-vector product tells us the probability that it will be sunny tomorrow.

(2) Calculate $\mathbf{x}_2 = A\mathbf{x}_1$ and interpret the meaning of each component of the product.

(3) Explain why $\mathbf{x}_2 = A^2\mathbf{x}_0$. Then explain in general why $\mathbf{x}_n = A^n\mathbf{x}_0$.

(4) The previous result demonstrates that to determine the long-term probability of a sunny or rainy day, we want to be able to easily calculate powers of the matrix A. Use a computer algebra system (e.g., Maple, Mathematica, Wolfram|Alpha) to calculate the entries of $\mathbf{x}_{10}$, $\mathbf{x}_{20}$, and $\mathbf{x}_{30}$. Based on this data, what do you expect the long term probability of any day being a sunny one?

Diagonalization

In Preview Activity 18.1 we saw how if we can powers of a matrix we can make predictions about the long-term behavior of some systems. In general, calculating powers of a matrix can be a very difficult thing, but there are times when the process is straightforward.

Activity 18.1. Let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(a) Show that $D^2 = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}$.

(b) Show that $D^3 = \begin{bmatrix} 2^3 & 0 \\ 0 & 3^3 \end{bmatrix}$. (Hint: $D^3 = DD^2$.)

(c) Explain in general why $D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$ for any positive integer n.

Activity 18.1 illustrates that calculating powers of square matrices whose only nonzero entries are along the diagonal is rather simple. In general, if

$$D = \begin{bmatrix} d_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & d_{22} & 0 & \cdots & 0 & 0 \\ \vdots & 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & d_{nn} \end{bmatrix},$$

then

$$D^k = \begin{bmatrix} d_{11}^k & 0 & 0 & \cdots & 0 & 0 \\ 0 & d_{22}^k & 0 & \cdots & 0 & 0 \\ \vdots & 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & d_{nn}^k \end{bmatrix}$$

for any positive integer k. Recall that a diagonal matrix is a matrix whose only nonzero elements are along the diagonal (see Definition 8.6). In this section we will see that matrices that are similar to diagonal matrices have some very nice properties, and that diagonal matrices are useful in calculations of powers of matrices.

We can utilize the method of calculating powers of diagonal matrices to also easily calculate powers of other types of matrices.

Activity 18.2. Let D be any matrix, P an invertible matrix, and let $A = P^{-1}DP$.

(a) Show that $A^2 = P^{-1}D^2P$.

(b) Show that $A^3 = P^{-1}D^3P$.

(c) Explain in general why $A^n = P^{-1}D^nP$ for positive integers n.

As Activity 18.2 illustrates, to calculate the powers of a matrix of the form $P^{-1}DP$ we only need determine the powers of the matrix D. If D is a diagonal matrix, this is especially straightforward.

Similar Matrices

Similar matrices play an important role in certain calculations. For example, Activity 18.2 showed that if we can write a square matrix A in the form $A = P^{-1}DP$ for some invertible matrix P and diagonal matrix D, then finding the powers of A is straightforward. As we will see, the relation $A = P^{-1}DP$ will imply that the matrices A and D share many properties.

Definition 18.1. The $n \times n$ matrix A is **similar** to the $n \times n$ matrix B if there is an invertible matrix P such that $A = P^{-1}BP$.

Activity 18.3. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}$. Assume that A is similar to B via the matrix $P = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$.

(a) Calculate $\det(A)$ and $\det(B)$. What do you notice?

(b) Find the characteristic polynomials of A and B. What do you notice?

(c) What can you say about the eigenvalues of A and B? Explain.

(d) Explain why $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for A with eigenvalue 2. Is $\mathbf{x}$ an eigenvector for B with eigenvalue 2? Why or why not?

Activity 18.3 suggests that similar matrices share some, but not all, properties. Note that if $A = P^{-1}BP$, then $B = Q^{-1}AQ$ with $Q = P^{-1}$. So if A is similar to B, then B is similar to A. Similarly (no pun intended), since $A = I^{-1}AI$ (where I is the identity matrix), then any square matrix is similar to itself. Also, if $A = P^{-1}BP$ and $B = M^{-1}CM$, then $A = (MP)^{-1}C(MP)$. So if A is similar to B and B is similar to C, then A is similar to C. If you have studied relations, these three properties show that similarity is an equivalence relation on the set of all $n \times n$ matrices. This is one reason why similar matrices share many important traits, as the next activity highlights.

Activity 18.4. Let A and B be similar matrices with $A = P^{-1}BP$.

(a) Use the multiplicative property of the determinant to explain why $\det(A) = \det(B)$. So similar matrices have the same determinants.

(b) Use the fact that $P^{-1}IP = I$ to show that $A - \lambda I$ is similar to $B - \lambda I$.

(c) Explain why it follows from (a) and (b) that

$$\det(A - \lambda I) = \det(B - \lambda I).$$

So similar matrices have the same characteristic polynomial, and the same eigenvalues.

We summarize some properties of similar matrices in the following theorem.

Theorem 18.2. *Let A and B be similar $n \times n$ matrices and I the $n \times n$ identity matrix. Then*

(1) $\det(A) = \det(B)$,

(2) $A - \lambda I$ is similar to $B - \lambda I$,

(3) A and B have the same characteristic polynomial,

(4) A and B have the same eigenvalues.

Similarity and Matrix Transformations

When a matrix is similar to a diagonal matrix, we can gain insight into the action of the corresponding matrix transformation. As an example, consider the matrix transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $T\mathbf{x} = A\mathbf{x}$, where

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}. \tag{18.2}$$

We are interested in understanding what this matrix transformation does to vectors in $\mathbb{R}^2$. First we note that A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If we let $P = [\mathbf{v}_1\ \mathbf{v}_2]$, then you can check that

$$P^{-1}AP = D$$

and

$$A = PDP^{-1},$$

where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Thus,

$$T(\mathbf{x}) = PDP^{-1}\mathbf{x}.$$

A simple calculation shows that

$$P^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let us apply T to the unit square whose sides are formed by the vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as shown in the first picture in Figure 18.1.

To apply T we first multiply $\mathbf{e}_1$ and $\mathbf{e}_2$ by P^{-1}. This gives us

$$P^{-1}\mathbf{e}_1 = \frac{1}{2}\mathbf{v}_1 \ \text{ and } \ P^{-1}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_2.$$

So P^{-1} transforms the standard coordinate system into a coordinate system in which $\mathbf{v}_1$ and $\mathbf{v}_2$ determine the axes, as illustrated in the second picture in Figure 18.1. Applying D to the output scales by 2 in the $\mathbf{v}_1$ direction and 4 in the $\mathbf{v}_2$ direction as depicted in the third picture in Figure 18.1. Finally, we apply P to translate back into the standard xy coordinate system as shown in the last picture in Figure 18.1.

This example illustrates that it is most convenient to view the action of T in the coordinate system where $\mathbf{v}_1$ serves as the x-direction and $\mathbf{v}_2$ as the y-direction. In this case, we can visualize that when we apply the transformation T to a vector in this system it is just scaled in both directions by the matrix D. Then the matrix P translates everything back to the standard xy coordinate system.

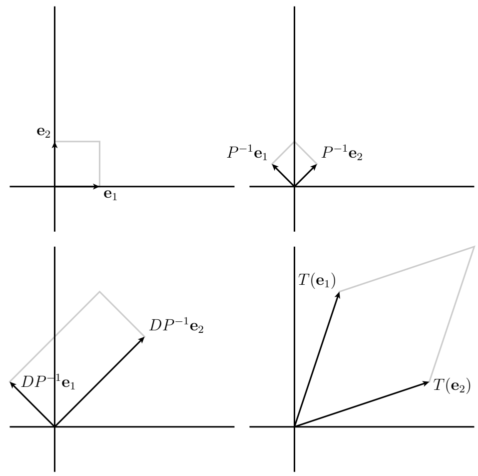

Figure 18.1: The matrix transformation.

This geometric perspective provides another example of how having a matrix similar to a diagonal matrix informs us about the situation. In what follows we determine the conditions that determine when a matrix is similar to a diagonal matrix.

Diagonalization in General

In Preview Activity 18.1 and in the matrix transformation example we found that a matrix A was similar to a diagonal matrix whose columns were eigenvectors of A. This will work for a general $n \times n$ matrix A as long as we can find an invertible matrix P whose columns are eigenvectors of A. More specifically, suppose A is an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_1$, $\ldots$, $\mathbf{v}_n$ with corresponding eigenvalues λ_1, λ_1, $\ldots$, λ_n (not necessarily distinct). Let

$$P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \cdots \ \mathbf{v}_n].$$

Then

$$\begin{aligned} AP &= [A\mathbf{v}_1 \ A\mathbf{v}_2 \ A\mathbf{v}_3 \ \cdots \ A\mathbf{v}_n] \\ &= [\lambda_1\mathbf{v}_1 \ \lambda_2\mathbf{v}_2 \ \lambda_3\mathbf{v}_3 \ \cdots \ \lambda_n\mathbf{v}_n] \\ &= [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3 \ \cdots \ \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \\ &= PD. \end{aligned}$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

Since the columns of P are linearly independent, we know P is invertible, and so

$$P^{-1}AP = D.$$

Definition 18.3. An $n \times n$ matrix A is **diagonalizable** if there is an invertible $n \times n$ matrix P so that $P^{-1}AP$ is a diagonal matrix.

In other words, a matrix A is diagonalizable if A is similar to a diagonal matrix.

IMPORTANT NOTE: The key notion to the process described above is that in order to diagonalize an $n \times n$ matrix A, we have to find n linearly independent eigenvectors for A. When A is diagonalizable, a matrix P so that $P^{-1}AP$ is diagonal is said to *diagonalize* A.

Activity 18.5. Find an invertible matrix P that diagonalizes A.

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$. (Hint: The eigenvalues of A are 8 and -1.)

It should be noted that there are square matrices that are not diagonalizable. For example, the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has 1 as its only eigenvalue and the dimension of the eigenspace of A corresponding to the eigenvalue is one. Therefore, it will be impossible to find two linearly independent eigenvectors for A.

We showed previously that eigenvectors corresponding to distinct eigenvalue are always linearly independent, so if an $n \times n$ matrix A has n distinct eigenvalues then A is diagonalizable. Activity 18.5 (b) shows that it is possible to diagonalize an $n \times n$ matrix even if the matrix does not have

n distinct eigenvalues. In general, we can diagonalize a matrix as long as the dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue. In other words, a matrix is diagonalizable if the geometric multiplicity is the same is the algebraic multiplicity for each eigenvalue.

At this point we might ask one final question. We argued that if an $n \times n$ matrix A has n linearly independent eigenvectors, then A is diagonalizable. It is reasonable to wonder if the converse is true – that is, if A is diagonalizable, must A have n linearly independent eigenvectors? The answer is yes, and you are asked to show this in Exercise 6. We summarize the result in the following theorem.

Theorem 18.4 (The Diagonalization Theorem)**.** *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. If A is diagonalizable and has linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ with $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for each i, then $n \times n$ matrix $P[\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]$ whose columns are linearly independent eigenvectors of A satisfies $P^{-1}AP = D$, where $D[d_{ij}]$ is the diagonal matrix with diagonal entries $d_{ii} = \lambda_i$ for each i.*

Examples

What follows are worked examples that use the concepts from this section.

Example 18.5. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. You should use appropriate technology to calculate determinants, perform any row reductions, or solve any polynomial equations.

(a) Determine if A is diagonalizable. If diagonalizable, find a matrix P that diagonalizes A.

(b) Determine if B is diagonalizable. If diagonalizable, find a matrix Q that diagonalizes B.

(c) Is it possible for two matrices R and S to have the same eigenvalues with the same algebraic multiplicities, but one matrix is diagonalizable and the other is not? Explain.

Example Solution.

(a) Technology shows that the characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I_3) = (4 - \lambda)(1 - \lambda)^2.$$

The eigenvalues of A are the solutions to the characteristic equation $p(\lambda) = 0$. Thus, the eigenvalues of A are 1 and 4.

To find a basis for the eigenspace of A corresponding to the eigenvalue 1, we find the general solution to the homogeneous system $(A - I_3)\mathbf{x} = \mathbf{0}$. Using technology we see that the reduced row echelon form of $A - I_3 = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then the general solution to $(A - I_3)\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} x_1 \\ \frac{1}{2}x_3 \\ x_3 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}. \end{aligned}$$

So a basis for the eigenspace of A corresponding to the eigenvalue 1 is

$$\left\{[1\ 0\ 0]^{\mathsf{T}}, [0\ 1\ 2]^{\mathsf{T}}\right\}.$$

To find a basis for the eigenspace of A corresponding to the eigenvalue 4, we find the general solution to the homogeneous system $(A - 4I_3)\mathbf{x} = \mathbf{0}$. Using technology we see that the reduced row echelon form of $A - 4I_3 = \begin{bmatrix} -3 & -2 & 1 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then the general solution to $(A - 4I_3)\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} \mathbf{x} &= [x_1\ x_2\ x_3]^{\mathsf{T}} \\ &= [x_3\ \ -x_3\ x_3]^{\mathsf{T}} \\ &= x_3\,[1\ \ -1\ 1]^{\mathsf{T}}. \end{aligned}$$

So a basis for the eigenspace of A corresponding to the eigenvalue 4 is

$$\left\{[1\ \ -1\ 0]^{\mathsf{T}}\right\}.$$

Eigenvectors corresponding to different eigenvalues are linearly independent, so the set

$$\left\{[1\ 0\ 0]^{\mathsf{T}}, [0\ 1\ 2]^{\mathsf{T}}, [1\ \ -1\ 0]^{\mathsf{T}}\right\}$$

is a basis for $\mathbb{R}^3$. Since we can find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A, we conclude that A is diagonalizable. Letting

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

gives us

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) Technology shows that the characteristic polynomial of B is

$$p(\lambda) = \det(B - \lambda I_3) = (4 - \lambda)(1 - \lambda)^2.$$

The eigenvalues of B are the solutions to the characteristic equation $p(\lambda) = 0$. Thus, the eigenvalues of B are 1 and 4.

To find a basis for the eigenspace of B corresponding to the eigenvalue 1, we find the general solution to the homogeneous system $(B - I_3)\mathbf{x} = \mathbf{0}$. Using technology we see that the reduced row echelon form of $B - I_3 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then the general solution to $(B - I_3)\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} \mathbf{x} &= [x_1\ x_2\ x_3]^{\mathsf{T}} \\ &= [x_1\ 0\ 0]^{\mathsf{T}} \\ &= x_1\,[1\ 0\ 0]^{\mathsf{T}}. \end{aligned}$$

So a basis for the eigenspace of B corresponding to the eigenvalue 1 is

$$\left\{[1\ 0\ 0]^{\mathsf{T}}\right\}.$$

To find a basis for the eigenspace of B corresponding to the eigenvalue 4, we find the general solution to the homogeneous system $(B - 4I_3)\mathbf{x} = \mathbf{0}$. Using technology we see that the reduced row echelon form of $B - 4I_3 = \begin{bmatrix} -3 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. So if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then the general solution to $(B - 4I_3)\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} \mathbf{x} &= [x_1\ x_2\ x_3]^{\mathsf{T}} \\ &= [0\ 0\ x_3]^{\mathsf{T}} \\ &= x_3\,[0\ 0\ 1]^{\mathsf{T}}. \end{aligned}$$

So a basis for the eigenspace of B corresponding to the eigenvalue 4 is

$$\left\{[0\ 0\ 1]^{\mathsf{T}}\right\}.$$

Since each eigenspace is one-dimensional, we cannot find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B. We conclude that B is not diagonalizable.

(c) Yes it is possible for two matrices R and S to have the same eigenvalues with the same multiplicities, but one matrix is diagonalizable and the other is not. An example is given by the matrices A and B in this problem.

Example 18.6.

(a) Is it possible to find diagonalizable matrices A and B such that AB is not diagonalizable? If yes, provide an example. If no, explain why.

(b) Is it possible to find diagonalizable matrices A and B such that $A+B$ is not diagonalizable? If yes, provide an example. If no, explain why.

(c) Is it possible to find a diagonalizable matrix A such that A^{T} is not diagonalizable? If yes, provide an example. If no, explain why.

(d) Is it possible to find an invertible diagonalizable matrix A such that A^{-1} is not diagonalizable? If yes, provide an example. If no, explain why.

Example Solution.

(a) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$. Since A and B are both diagonal matrices, their eigenvalues are their diagonal entries. With 2 distinct eigenvalues, both A and B are diagonalizable. In this case we have $AB = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$, whose only eigenvector is 2. The reduced row echelon form of $AB - 2I_2$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. So a basis for the eigenspace of AB is $\{[1\ 0]^{\mathsf{T}}\}$. Since there is no basis for $\mathbb{R}^2$ consisting of eigenvectors of AB, we conclude that AB is not diagonalizable.

(b) Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Since A and B are both diagonal matrices, their eigenvalues are their diagonal entries. With 2 distinct eigenvalues, both A and B are diagonalizable. In this case we have $A+B = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$, whose only eigenvector is 3. The reduced row echelon form of $(A+B) - 3I_2$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. So a basis for the eigenspace of $A+B$ is $\{[1\ 0]^{\mathsf{T}}\}$. Since there is no basis for $\mathbb{R}^2$ consisting of eigenvectors of $A+B$, we conclude that $A+B$ is not diagonalizable.

(c) It is not possible to find a diagonalizable matrix A such that A^{T} is not diagonalizable. To see why, suppose that matrix A is diagonalizable. That is, there exists a matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix. Recall that $\left(P^{-1}\right)^{\mathsf{T}} = \left(P^{\mathsf{T}}\right)^{-1}$. So

$$\begin{aligned} D &= D^{\mathsf{T}} \\ &= \left(P^{-1}AP\right)^{\mathsf{T}} \\ &= P^{\mathsf{T}}A^{\mathsf{T}}\left(P^{-1}\right)^{\mathsf{T}} \\ &= P^{\mathsf{T}}A^{\mathsf{T}}\left(P^{\mathsf{T}}\right)^{-1}. \end{aligned}$$

Letting $A = \left(P^{\mathsf{T}}\right)^{-1}$, we conclude that

$$Q^{-1}A^{\mathsf{T}}Q = D.$$

Therefore, Q diagonalizes A^{T}.

(d) It is not possible to find an invertible diagonalizable matrix A such that A^{-1} is not diagonalizable. To see why, suppose that matrix A is diagonalizable. That is, there exists a matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix. Thus, $A = PDP^{-1}$. Since A is invertible, $\det(A) \neq 0$. It follows that $\det(D) \neq 0$. So none of the diagonal entries of D can be 0. Thus, D is invertible and D^{-1} is a diagonal matrix. Then

$$D^{-1} = \left(P^{-1}AP\right)^{-1} = PA^{-1}P^{-1}$$

and so P^{-1} diagonalizes A^{-1}.

Summary

- A matrix $D = [d_{ij}]$ is a diagonal matrix if $d_{ij} = 0$ whenever $i \neq j$.
- A matrix A is diagonalizable if there is an invertible matrix P so that $P^{-1}AP$ is a diagonal matrix.
- Two matrices A and B are similar if there is an invertible matrix P so that

$$B = P^{-1}AP.$$

- Similar matrices have the same determinants, same characteristic polynomials, and same eigenvalues. Note that similar matrices do not necessarily have the same eigenvectors corresponding to the same eigenvalues.
- An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- When an $n \times n$ matrix A is diagonalizable, then $P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_n]$ is invertible and $P^{-1}AP$ is diagonal, where $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ are n linearly independent eigenvectors for A.
- One use for diagonalization is that once we have diagonalized a matrix A we can quickly and easily compute powers of A. Diagonalization can also help us understand the actions of matrix transformations.

Exercises

(1) Determine if each of the following matrices is diagonalizable or not. For diagonalizable matrices, clearly identify a matrix P which diagonalizes the matrix, and what the resulting diagonal matrix is.

(a) $A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

(2) The 3×3 matrix A has two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$. The vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ are eigenvectors for $\lambda_1 = 2$, while the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ are eigenvectors for $\lambda_2 = 3$. Find the matrix A.

(3) Find a 2×2 non-diagonal matrix A and two different pairs of P and D matrices for which $A = PDP^{-1}$.

(4) Find a 2×2 non-diagonal matrix A and two different P matrices for which $A = PDP^{-1}$ with the same D.

(5) Suppose a 4×4 matrix A has eigenvalues 2, 3 and 5 and the eigenspace for the eigenvalue 3 has dimension 2. Do we have enough information to determine if A is diagonalizable? Explain.

(6) Let A be a diagonalizable $n \times n$ matrix. Show that A has n linearly independent eigenvectors.

(7)

(a) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A and B. Conclude that it is possible for two different $n \times n$ matrices A and B to have exactly the same eigenvectors and corresponding eigenvalues.

(b) A natural question to ask is if there are any conditions under which $n \times n$ matrices that have exactly the same eigenvectors and corresponding eigenvalues must be equal. Determine the answer to this question if A and B are both diagonalizable.

(8)

(a) Show that if D and D' are $n \times n$ diagonal matrices, then $DD' = D'D$.

(b) Show that if A and B are $n \times n$ matrices and P is an invertible $n \times n$ matrix such that $P^{-1}AP = D$ and $P^{-1}BP = D'$ with D and D' diagonal matrices, then $AB = BA$.

(9) Exercise 2 in Section 17 shows that the determinant of a matrix is the product of its eigenvalues. In this exercise we show that the trace of a diagonalizable matrix is the sum of its eigenvalues.[1] First we define the trace of a matrix.

Definition 18.7. The **trace** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of the diagonal entries of A. That is,

$$\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

(a) Show that if $R = [r_{ij}]$ and $S = [s_{ij}]$ are $n \times n$ matrices, then $\text{trace}(RS) = \text{trace}(SR)$.

[1] This result is true for any matrix, but the argument is more complicated.

(b) Let A be a diagonalizable $n \times n$ matrix, and let $p(\lambda) = \det(A - \lambda I_n)$ be the characteristic polynomial of A. Let P be an invertible matrix such that $P^{-1}AP = D$, where D is the diagonal matrix whose diagonal entries are λ_1, λ_2, ..., λ_n, the eigenvalues of A (note that these eigenvalues may not all be distinct).

i. Explain why $\text{trace}(A) = \text{trace}(D)$.

ii. Show that the trace of an $n \times n$ diagonalizable matrix is the sum of the eigenvalues of the matrix.

(10) In this exercise we generalize the result of Exercise 12 in Section 8 to arbitrary diagonalizable matrices.

(a) Show that if

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

then

$$e^D = \begin{bmatrix} e^{\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix}.$$

(b) Now suppose that an $n \times n$ matrix A is diagonalizable, with $P^{-1}AP$ equal to a diagonal matrix D. Show that $e^A = Pe^DP^{-1}$.

(11) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$.

(a) Use the result of Exercise 10 to calculate e^A.

(b) Calculate e^B. (Hint: Explain why B is not diagonalizable.)

(c) Use the result of Exercise 10 to calculate e^{A+B}.

(d) The real exponential function satisfies some familiar properties. For example, $e^xe^y = e^ye^x$ and $e^{x+y} = e^xe^y$ for any real numbers x and y. Does the matrix exponential satisfy the corresponding properties. That is, if X and Y are $n \times n$ matrices, must $e^Xe^Y = e^Ye^X$ and $e^{X+Y} = e^Xe^Y$? Explain.

(12) In Exercise 11 we see that we cannot conclude that $e^{X+Y} = e^Xe^Y$ for $n \times n$ matrices X and Y. However, a more limited property is true.

(a) Follow the steps indicated to show that if A is an $n \times n$ matrix and s and t are any scalars, then $e^{As}e^{At} = e^{A(s+t)}$. (Although we will not use it, you may assume that the series for e^A converges for any square matrix A.)

i. Use the definition to show that

$$e^{As}e^{At} = \sum_{k \geq 0} \sum_{m \geq 0} \frac{s^k t^m}{k!} m! A^{k+m}.$$

ii. Relabel and reorder terms with $n = k + m$ to show that

$$e^{As}e^{At} = sum_{n \geq 0} \frac{1}{n!} A^n \sum_{m=0}^{n} \frac{n!}{(n-m)!m!} s^{n-m} t^m.$$

iii. Complete the problem using the Binomial Theorem that says

$$(s+t)^n = \sum_{m=0}^{n} \frac{n!}{(n-m)!m!} s^{n-m} t^m.$$

(b) Use the result of part (a) to show that e^A is an invertible matrix for any $n \times n$ matrix A.

(13) There is an interesting connection between the determinant of a matrix exponential and the trace of the matrix. Let A be a diagonalizable $n \times n$ matrix with real entries. Let $D = P^{-1}AP$ for some invertible matrix P, where D is the diagonal matrix with entries λ_1, λ_2, ..., λ_n the eigenvalues of A.

(a) Show that $e^A = Pe^D P^{-1}$.

(b) Use Exercise 9 to show that

$$\det\left(e^A\right) = e^{\text{trace}(A)}.$$

(14) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** If matrix A is diagonalizable, then so is A^T.

(b) **True/False** If matrix A is diagonalizable, then A is invertible.

(c) **True/False** If an $n \times n$ matrix A is diagonalizable, then A has n distinct eigenvalues.

(d) **True/False** If matrix A is invertible and diagonalizable, then so is A^{-1}.

(e) **True/False** If an $n \times n$ matrix C is diagonalizable, then there exists a basis of $\mathbb{R}^n$ consisting of the eigenvectors of C.

(f) **True/False** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

(g) **True/False** If A is an $n \times n$ diagonalizable matrix, then there is a unique diagonal matrix such that $P^{-1}AP = D$ for some invertible matrix P.

(h) **True/False** If A is an $n \times n$ matrix with eigenvalue λ, then the dimension of the eigenspace of A corresponding to the eigenvalue λ is $n - \text{rank}(A - \lambda I_n)$.

(i) **True/False** If λ is an eigenvalue of an $n \times n$ matrix A, then e^λ is an eigenvalue of e^A. (See Exercise 12 in Section 8 for information on the matrix exponential.)

Project: Binet's Formula for the Fibonacci Numbers

We return to the Fibonacci sequence F_n where $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, $F_0 = 0$, and $F_1 = 1$. Since F_{n+2} is determined by previous values F_{n+1} and F_n, the relation $F_{n+2} = F_{n+1} + F_n$ is called a *recurrence relation*. The recurrence relation $F_{n+2} = F_{n+1} + F_n$ is very time consuming to use to compute F_n for large values of n. It turns out that there is a fascinating formula that gives the nth term of the Fibonacci sequence directly, without using the relation $F_{n+2} = F_{n+1} + F_n$.

Project Activity 18.1. The recurrence relation$F_{n+2} = F_{n+1} + F_n$ gives the equations

$$F_{n+1} = F_n + F_{n-1} \tag{18.3}$$
$$F_n = F_n. \tag{18.4}$$

Let $\mathbf{x}_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ for $n \geq 0$. Explain how the equations (18.3) and (18.3) can be described with the matrix equation

$$\mathbf{x}_n = A\mathbf{x}_{n-1}, \tag{18.5}$$

where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

The matrix equation (18.5) shows us how to find the vectors $\mathbf{x}_n$ using powers of the matrix A:

$$\begin{aligned}
\mathbf{x}_1 &= A\mathbf{x}_0 \\
\mathbf{x}_2 &= A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0 \\
\mathbf{x}_3 &= A\mathbf{x}_2 = A(A^2\mathbf{x}_0) = A^3\mathbf{x}_0 \\
\vdots &\quad \vdots \\
\mathbf{x}_n &= A^n\mathbf{x}_0.
\end{aligned}$$

So if we can somehow easily find the powers of the matrix A, then we can find a convenient formula for F_n. As we have seen, we know how to do this if A is diagonalizable

Project Activity 18.2. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Show that the eigenvalues of A are $\varphi = \frac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$.

(b) Find bases for each eigenspace of A.

Now that we have the eigenvalues and know corresponding eigenvectors for A, we can return to the problem of diagonalizing A.

Project Activity 18.3.

(a) Why do we know that A is diagonalizable?

(b) Find a matrix P such that $P^{-1}AP$ is a diagonal matrix. What is the diagonal matrix?

Now we can find a formula for the nth Fibonacci number.

Project Activity 18.4. Since $P^{-1}AP = D$, where D is a diagonal matrix, we also have $A = PDP^{-1}$. Recall that when $A = PDP^{-1}$, it follows that $A^n = PD^nP^{-1}$. Use the equation $A^n = PD^nP^{-1}$ to show that

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}. \tag{18.6}$$

(Hint: We just need to calculate the second component of $A^n\mathbf{x}_0$.)

Formula (18.6) is called *Binet's formula*. It is a very surprising formula in the fact that the expression on the right hand side of (18.6) is an integer for each positive integer n. Note that with Binet's formula we can quickly compute F_n for very large values of n. For example,

$$F_{150} = 9969216677189303386214405760200.$$

The number $\varphi = \frac{1+\sqrt{5}}{2}$, called the *golden mean* or *golden ratio* is intimately related to the Fibonacci sequence. Binet's formula provides a fascinating relationship between the Fibonacci numbers and the golden ratio. The golden ratio also occurs often in other areas of mathematics. It was an important number to the ancient Greek mathematicians who felt that the most aesthetically pleasing rectangles had sides in the ratio of $\varphi : 1$.

Project Activity 18.5. You might wonder what happens if we use negative integer exponents in Binet's formula. In other words, are there negatively indexed Fibonacci numbers? For any integer n, including negative integers, let

$$F_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}}$$

There is a specific relationship between F_{-n} and F_n. Find it and verify it.

Section 19

Approximating Eigenvalues and Eigenvectors

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is the power method for?
- How does the power method work?
- How can we use the inverse power method to approximate any eigenvalue/eigenvector pair?

Application: Leslie Matrices and Population Modeling

The Leslie Matrix (also called the Leslie Model) is a powerful model for describing an age distributed growth of a population that is closed to migration. In a Leslie model, it is usually the case that only one gender (most often female) is considered. As an example, we will later consider a population of sheep that is being grown commercially. A natural question that we will address is how we can harvest the population to build a sustainable environment.

When working with populations, the matrices we use are often large. For large matrices, using the characteristic polynomial to calculate eigenvalues is too time and resource consuming to be practical, and we generally cannot find the exact values of the eigenvalues. As a result, approximation techniques are very important. In this section we will explore a method for approximating eigenvalues. The eigenvalues of a Leslie matrix are important because they describe the limiting or steady-state behavior of a population. The matrix and model were introduced by Patrick H. Leslie in "On the Use of Matrices in Certain Population Mathematics", Leslie, P.H., *Biometrika*, Volume XXXIII, November 1945, pp. 183-212.

Introduction

We have used the characteristic polynomial to find the eigenvalues of a matrix, and for each eigenvalue row reduced a corresponding matrix to find the eigenvectors This method is only practical for small matrices – for more realistic applications approximation techniques are used. We investigate one such technique in this section, the *power method*.

Preview Activity 19.1. Let $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$. Our goal is to find a scalar λ and a nonzero vector $\mathbf{v}$ so that $A\mathbf{v} = \lambda\mathbf{v}$.

(1) If we have no prior knowledge of the eigenvalues and eigenvectors of this matrix, we might just begin with a guess. Let $\mathbf{x}_0 = [1\ 0]^{\mathsf{T}}$ be such a guess for an eigenvector. Calculate $A\mathbf{x}_0$. Is $\mathbf{x}_0$ an eigenvector of A? Explain.

(2) If $\mathbf{x}_0$ is not a good approximation to an eigenvector of A, then we need to make a better guess. We have little to work with other than just random guessing, but we can use $\mathbf{x}_1 = A\mathbf{x}_0$ as another guess. We calculated $\mathbf{x}_1$ in part 1. Is $\mathbf{x}_1$ an eigenvector for A? Explain.

(3) In parts (a) and (b) you might have noticed that in some sense $\mathbf{x}_1$ is closer to being an eigenvector of A than $\mathbf{x}_0$ was. So maybe continuing this process will get us closer to an eigenvector of A. In other words, for each positive integer k we define $\mathbf{x}_k$ as $A\mathbf{x}_{k-1}$. Before we proceed, however, we should note that as we calculate the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\ldots$, the entries in the vectors get large very quickly. So it will be useful to scale the entries so that they stay at a reasonable size, which makes it easier to interpret the output. One way to do this is to divide each vector $\mathbf{x}_i$ by its largest component in absolute value so that all of the entries stay between -1 and 1.[1] So in our example we have $\mathbf{x}_0 = [1\ 0]^{\mathsf{T}}$, $\mathbf{x}_1 = [2/5\ 1]^{\mathsf{T}}$, and $\mathbf{x}_2 = [1\ 25/34]^{\mathsf{T}}$. Explain why scaling our vectors will not affect our search for an eigenvector.

(4) Use an appropriate technological tool to find the vectors $\mathbf{x}_k$ up to $k = 10$. What do you think the limiting vector $\lim_{k\to\infty} \mathbf{x}_k$ is? Is this limiting vector an eigenvector of A? If so, what is the corresponding eigenvalue?

The Power Method

While the examples we present in this text are small in order to highlight the concepts, matrices that appear in real life applications are often enormous. For example, in Google's PageRank algorithm that is used to determine relative rankings of the importance of web pages, matrices of staggering size are used (most entries in the matrices are zero, but the size of the matrices is still huge). Finding eigenvalues of such large matrices through the characteristic polynomial is impractical. In fact, finding the roots of all but the smallest degree characteristic polynomials is a very difficult problem. As a result, using the characteristic polynomial to find eigenvalues and then finding eigenvectors is not very practical in general, and it is often a better option to use a numeric approximation method. We will consider one such method in this section, the *power method*.

[1] There are several other ways to scale, but we won't consider them here.

In Preview Activity 19.1, we saw an example of a matrix $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$ so that the sequence $\{\mathbf{x}_k\}$, where $\mathbf{x}_k = A\mathbf{x}_{k-1}$, converged to a dominant eigenvector of A for an initial guess vector $\mathbf{x}_0 = [1\ 0]^{\mathsf{T}}$. The vectors $\mathbf{x}_i$ for i from 1 to 6 (with scaling) are approximately

$$\mathbf{x}_1 = \begin{bmatrix} 0.4 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0.7353 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 0.8898 \\ 1 \end{bmatrix}$$
$$\mathbf{x}_4 = \begin{bmatrix} 1 \\ 0.9575 \end{bmatrix} \quad \mathbf{x}_5 = \begin{bmatrix} 0.9838 \\ 1 \end{bmatrix} \quad \mathbf{x}_6 = \begin{bmatrix} 1 \\ 0.9939 \end{bmatrix}.$$

Numerically we can see that the sequence $\{\mathbf{x}_k\}$ approaches the vector $[1\ 1]^{\mathsf{T}}$, and Figure 19.1 illustrates this geometrically as well. This method of successive approximations $\mathbf{x}_k = A\mathbf{x}_{k-1}$ is

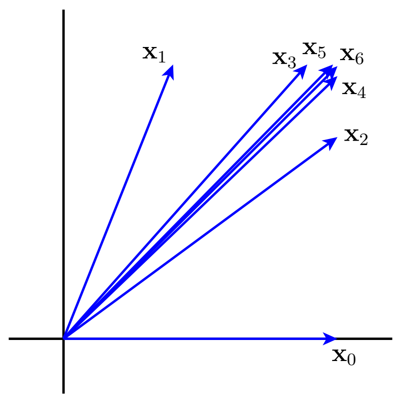

Figure 19.1: The power method.

called the *power method* (since we could write $\mathbf{x}_k$ as $A^k\mathbf{x}_0$). Our task now is to show that this method works in general. In the next activity we restrict our argument to the 2×2 case, and then discuss the general case afterwards.

Let A be an arbitrary 2×2 matrix with two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and corresponding eigenvalues λ_1 and λ_2, respectively. We will also assume $|\lambda_1| > |\lambda_2|$. An eigenvalue whose absolute value is larger than that of any other eigenvalue is called a *dominant eigenvalue*. Any eigenvector for a dominant eigenvalue is called a *dominant eigenvector*. Before we show that our method can be used to approximate a dominant eigenvector, we recall that since $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors corresponding to distinct eigenvalues, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. So there exist scalars a_1 and a_2 such that

$$\mathbf{x}_0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2.$$

We have seen that for each positive integer k we can write $\mathbf{x}_n$ as

$$\mathbf{x}_k = a_1\lambda_1^k\mathbf{v}_1 + a_2\lambda_2^k\mathbf{v}_2. \tag{19.1}$$

With this representation of $\mathbf{x}_0$ we can now see why the power method approximates a dominant eigenvector of A.

Activity 19.1. Assume as above that A is an arbitrary 2×2 matrix with two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and corresponding eigenvalues λ_1 and λ_2, respectively. (We are assuming that we don't know these eigenvectors, but we can assume that they exist.) Assume that λ_1 is the dominant eigenvalue for A, $\mathbf{x}_0$ is some initial guess to an eigenvector for A, that $\mathbf{x}_0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$, and that $\mathbf{x}_k = A\mathbf{x}_{k-1}$ for $k \geq 1$.

(a) We divide both sides of equation (19.1) by λ_1^k (since λ_1 is the dominant eigenvalue, we know that λ_1 is not 0) to obtain

$$\frac{1}{\lambda_1^k}\mathbf{x}_k = a_1\mathbf{v}_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^k \mathbf{v}_2. \tag{19.2}$$

Recall that λ_1 is the dominant eigenvalue for A. What happens to $\left(\frac{\lambda_2}{\lambda_1}\right)^k$ as $k \to \infty$? Explain what happens to the right hand side of equation (19.2) as $k \to \infty$.

(b) Explain why the previous result tells us that the vectors $\mathbf{x}_k$ are approaching a vector in the *direction* of $\mathbf{v}_1$ or $-\mathbf{v}_1$ as $k \to \infty$, assuming $a_1 \neq 0$. (Why do we need $a_1 \neq 0$? What happens if $a_1 = 0$?)

(c) What does all of this tell us about the sequence $\{\mathbf{x}_k\}$ as $k \to \infty$?

The power method is straightforward to implement, but it is not without its drawbacks. We began by assuming that we had a basis of eigenvectors of a matrix A. So we are also assuming that A is diagonalizable. We also assumed that A had a dominant eigenvalue λ_1. That is, if A is $n \times n$ we assume that A has eigenvalues λ_1, λ_2, ..., λ_n, not necessarily distinct, with

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

and with $\mathbf{v}_i$ an eigenvector of A with eigenvalue λ_i. We could then write any initial guess $\mathbf{x}_0$ in the form

$$\mathbf{x}_0 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

The initial guess is also called a *seed*.

Then

$$\mathbf{x}_k = a_1\lambda_1^k\mathbf{v}_1 + a_2\lambda_2^k\mathbf{v}_2 + \cdots + a_n\lambda_n^k\mathbf{v}_n$$

and

$$\frac{1}{\lambda_1^k}\mathbf{x}_k = a_1\mathbf{v}_1 + a_2\left(\frac{\lambda_2}{\lambda_1}\right)^k \mathbf{v}_2 + \cdots + a_n\left(\frac{\lambda_n}{\lambda_1}\right)^k \mathbf{v}_n. \tag{19.3}$$

Notice that we are not actually calculating the vectors $\mathbf{x}_k$ here – this is a theoretical argument and we don't know λ_1 and are not performing any scaling like we did in Preview Activity 19.1. We are assuming that λ_1 is the dominant eigenvalue of A, though, so for each i the terms $\left(\frac{\lambda_i}{\lambda_1}\right)^k$ converge to 0 as k goes to infinity. Thus,

$$\mathbf{x}_k \approx \lambda_1^k a_1\mathbf{v}_1$$

for large values of k, which makes the sequence $\{\mathbf{x}_k\}$ converge to a vector in the direction of a dominant eigenvector $\mathbf{v}_1$ provided $a_1 \neq 0$. So we need to be careful enough to choose a seed that has a nonzero component in the direction of $\mathbf{v}_1$. Of course, we generally don't know that our matrix is diagonalizable before we make these calculations, but for many matrices the sequence $\{\mathbf{x}_k\}$ will approach a dominant eigenvector.

Once we have an approximation to a dominant eigenvector, we can then approximate the dominant eigenvalue.

Activity 19.2. Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\mathbf{v}$.

(a) Explain why $\lambda = \frac{\lambda(\mathbf{v}\cdot\mathbf{v})}{\mathbf{v}\cdot\mathbf{v}}$.

(b) Use the result of part (a) to explain why $\lambda = \frac{(A\mathbf{v})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}$.

The result of Activity 19.2 is that, when the vectors in the sequence $\{\mathbf{x}_k\}$ approximate a dominant eigenvector of a matrix A, the quotients

$$\frac{(A\mathbf{x}_k)\cdot\mathbf{x}_k}{\mathbf{x}_k\cdot\mathbf{x}_k} = \frac{\mathbf{x}_k^{\mathsf{T}}A\mathbf{x}_k}{\mathbf{x}_k^{\mathsf{T}}\mathbf{x}_k} \tag{19.4}$$

approximate the dominant eigenvalue of A. The quotients in (19.4) are called *Rayleigh quotients*.

To summarize, the procedure for applying the power method for approximating a dominant eigenvector and dominant eigenvalue of a matrix A is as follows.

Step 1: Select an arbitrary nonzero vector $\mathbf{x}_0$ as an initial guess to a dominant eigenvector.

Step 2: Let $\mathbf{x}_1 = A\mathbf{x}_0$. Let $k = 1$.

Step 3: To avoid having the magnitudes of successive approximations become excessively large, scale this approximation $\mathbf{x}_k$. That is, find the entry α_k of $\mathbf{x}_k$ that is largest in absolute value. Then replace $\mathbf{x}_k$ by $\frac{1}{|\alpha_k|}\mathbf{x}_k$. Note that this does not change the direction of this approximation, only its magnitude.

Step 4: Calculate the Rayleigh quotient $r_k = \frac{(A\mathbf{x}_k)\cdot\mathbf{x}_k}{\mathbf{x}_k\cdot\mathbf{x}_k}$.

Step 5: Let let $\mathbf{x}_{k+1} = A\mathbf{x}_k$. Increase k by 1 and repeat Steps 3 through 5.

If the sequence $\{\mathbf{x}_k\}$ converges to a dominant eigenvector of A, then the sequence $\{r_k\}$ converges to the dominant eigenvalue of A.

The power method can be useful for approximating a dominant eigenvector as long as the successive multiplications by A are fairly simple – for example, if many entries of A are zero.[2] The rate of convergence of the sequence $\{\mathbf{x}_k\}$ depends on the ratio $\frac{\lambda_2}{\lambda_1}$. If this ratio is close to 1, then it can take many iterations before the power $\left(\frac{\lambda_2}{\lambda_1}\right)^k$ makes the $\mathbf{v}_2$ term negligible. There are other methods for approximating eigenvalues and eigenvectors, e.g., the QR factorization, that we will not discuss at this point.

[2] A matrix in which most entries are zero is called a *sparse* matrix.

The Inverse Power Method

The power method only allows us to approximate the dominant eigenvalue and a dominant eigenvector for a matrix A. It is possible to modify this method to approximate other eigenvectors and eigenvalues under certain conditions. We consider an example in the next activity to motivate the general situation.

Activity 19.3. Let $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$ be the matrix from Preview Activity 19.1. Recall that 8 is an eigenvalue for A, and a quick calculation can show that -3 is the other eigenvalue of A. Consider the matrix $B = (A - (-2)I_2)^{-1} = \frac{1}{10}\begin{bmatrix} -5 & 6 \\ 5 & -4 \end{bmatrix}$.

(a) Show that $\frac{1}{8-(-2)}$ and $\frac{1}{-3-(-2)}$ are the eigenvalues of B.

(b) Recall that $\mathbf{v}_1 = [1\ 1]^{\mathsf{T}}$ is an eigenvector of A corresponding to the eigenvalue 8 and assume that $\mathbf{v}_2 = [-6\ 5]^{\mathsf{T}}$ is an eigenvector for A corresponding to the eigenvalue -3. Calculate the products $B\mathbf{v}_1$ and $B\mathbf{v}_2$. How do the products relate to the results of part (a)?

Activity 19.3 provides evidence that we can translate the matrix A having a dominant eigenvalue to a different matrix B with the same eigenvectors as A and with a dominant eigenvalue of our choosing. To see why, let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let α be any real number distinct from the eigenvalues. Let $B = (A - \alpha I_n)^{-1}$. In our example in Activity 19.3 the numbers

$$\frac{1}{\lambda_1 - \alpha}, \frac{1}{\lambda_2 - \alpha}, \frac{1}{\lambda_3 - \alpha}, \ldots, \frac{1}{\lambda_n - \alpha}$$

were the eigenvalues of B, and that if $\mathbf{v}_i$ is an eigenvector for A corresponding to the eigenvalue λ_i, then $\mathbf{v}_i$ is an eigenvector of B corresponding to the eigenvalue $\frac{1}{\lambda_i - \alpha}$. To see why, let λ be an eigenvalue of an $n \times n$ matrix A with corresponding eigenvector $\mathbf{v}$. Let α be a scalar that is not an eigenvalue of A, and let $B = (A - \alpha I_n)^{-1}$. Now

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ A\mathbf{v} - \alpha\mathbf{v} &= \lambda\mathbf{v} - \alpha\mathbf{v} \\ (A - \alpha I_n)\mathbf{v} &= (\lambda - \alpha)\mathbf{v} \\ \frac{1}{\lambda - \alpha}\mathbf{v} &= (A - \alpha I_n)^{-1}\mathbf{v}. \end{aligned}$$

So $\frac{1}{\lambda - \alpha}$ is an eigenvalue of B with eigenvector $\mathbf{v}$.

Now suppose that A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and that we want to approximate an eigenvector and corresponding eigenvalue λ_i of A. If we can somehow find a value of α so that $|\lambda_i - \alpha| < |\lambda_j - \alpha|$ for all $j \neq i$, then $\left|\frac{1}{\lambda_i - \alpha}\right| > \left|\frac{1}{\lambda_j - \alpha}\right|$ for any $j \neq i$. Thus, the matrix $B = (A - \alpha I_n)^{-1}$ has $\frac{1}{\lambda_i - \alpha}$ as its dominant eigenvalue and we can use the power method to approximate an eigenvector and the Rayleigh quotient to approximate the eigenvalue $\frac{1}{\lambda_i - \alpha}$, and hence approximate λ_i.

Activity 19.4. Let $A = \frac{1}{8}\begin{bmatrix} 7 & 3 & 3 \\ 30 & 22 & -10 \\ 15 & -21 & 11 \end{bmatrix}$.

(a) Apply the power method to the matrix $B = (A - I_3)^{-1}$ with initial vector $\mathbf{x}_0 = [1\ 0\ 0]^{\mathsf{T}}$ to fill in Table 19.1 (to four decimal places). Use this information to estimate an eigenvalue for A and a corresponding eigenvector.

k	10	15	20
$\mathbf{x}_k$			
$\frac{\mathbf{x}_k^{\mathsf{T}} A \mathbf{x}_k}{\mathbf{x}_k^{\mathsf{T}} \mathbf{x}_k}$			

Table 19.1: Applying the power method to $(A - I_3)^{-1}$.

(b) Applying the power method to the matrix $B = (A - 0I_3)^{-1}$ with initial vector $\mathbf{x}_0 = [1\ 0\ 0]^{\mathsf{T}}$ yields the information in Table 19.2 (to four decimal places). Use this information to estimate an eigenvalue for A and a corresponding eigenvector.

k	10	15	20
$\mathbf{x}_k$	$\begin{bmatrix} 0.3344 \\ -0.6677 \\ -1.0000 \end{bmatrix}$	$\begin{bmatrix} -0.3333 \\ 0.6666 \\ 1.0000 \end{bmatrix}$	$\begin{bmatrix} 0.3333 \\ -0.6666 \\ -1.0000 \end{bmatrix}$
$\frac{\mathbf{x}_k^{\mathsf{T}} A \mathbf{x}_k}{\mathbf{x}_k^{\mathsf{T}} \mathbf{x}_k}$	-1.0014	-1.0000	-1.0000

Table 19.2: Applying the power method to $(A - 0I_3)^{-1}$.

(c) Applying the power method to the matrix $B = (A - 5I_3)^{-1}$ with initial vector $\mathbf{x}_0 = [1\ 0\ 0]^{\mathsf{T}}$ yields the information in Table 19.3 (to four decimal places). Use this information to estimate an eigenvalue for A and a corresponding eigenvector.

Examples

What follows are worked examples that use the concepts from this section.

Example 19.1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

(a) Approximate the dominant eigenvalue of A accurate to two decimal places using the power method. Use technology as appropriate.

k	10	15	20
$\mathbf{x}_k$	$\begin{bmatrix} 0.0000 \\ 1.0000 \\ -1.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0000 \\ -1.0000 \\ 1.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0000 \\ 1.0000 \\ -1.0000 \end{bmatrix}$
$\frac{\mathbf{x}_k^{\mathsf{T}} A\mathbf{x}_k}{\mathbf{x}_k^{\mathsf{T}}\mathbf{x}_k}$	-1.0000	-1.0000	-1.0000

Table 19.3: Applying the power method to $(A - 5I_3)^{-1}$.

(b) Find the characteristic polynomial $p(\lambda)$ of A. Then find the the root of $p(\lambda)$ farthest from the origin. Compare to the result of part (a). Use technology as appropriate.

Example Solution.

(a) We use technology to calculate the scaled vectors $A^k\mathbf{x}_0$ for values of k until the components don't change in the second decimal place. We start with the seed $\mathbf{x}_0 = [1\ 1\ 1]^{\mathsf{T}}$. For example, to two decimal places we have $\mathbf{x}_k = [0.28\ 0.64\ 1.00]^{\mathsf{T}}$ for $k \geq 20$. So we suspect that $[0.28\ 0.64\ 1.00]^{\mathsf{T}}$ is close to a dominant eigenvector for A.

For the dominant eigenvalue, we can calculate the Rayleigh quotients $\frac{(A\mathbf{x}_k)\cdot\mathbf{x}_k}{\mathbf{x}_k\cdot\mathbf{x}_k}$ until they do not change to two decimal places. For $k \geq 4$, our Rayleigh quotients are all (to two decimal places) equal to 16.12. So we expect that the dominant eigenvalue of A is close to 16.12. Notice that

$$A[0.28\ 0.64\ 1.00]^{\mathsf{T}} = [4.56\ 10.32\ 16.08]^{\mathsf{T}},$$

which is not far off from $16.12[0.28\ 0.64\ 1.00]^{\mathsf{T}}$.

(b) The characteristic polynomial of A is

$$p(\lambda) = -\lambda^3 + 15\lambda^2 + 18\lambda = -\lambda(\lambda^2 - 15\lambda - 18).$$

The quadratic formula gives the nonzero roots of $p(\lambda)$ as

$$\frac{15 \pm \sqrt{15^2 + 4(18)}}{2} = \frac{15 \pm 3\sqrt{33}}{2}.$$

The roots farthest from the origin is approximately 16.12, as was also calculated in part (a).

Example 19.2. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

(a) Use the power method to approximate the dominant eigenvalue and a corresponding eigenvector (using scaling) accurate to two decimal places. Use $\mathbf{x}_0 = [1\ 1\ 1]^{\mathsf{T}}$ as the seed.

(b) Determine the exact value of the dominant eigenvalue of A and compare to your result from part (a).

(c) Approximate the remaining eigenvalues of A using the inverse power method. (Hint: Try $\alpha = 0.5$ and $\alpha = 1.8$.)

Example Solution.

(a) We use technology to calculate the scaled vectors $A^k\mathbf{x}_0$ for values of k until the components don't change in the second decimal place. For example, to two decimal places we have $\mathbf{x}_k = [0.50\ 1.00\ 0.50]^{\mathsf{T}}$ for $k \geq 4$. So we suspect that $\left[\frac{1}{2}\ 1\ \frac{1}{2}\right]^{\mathsf{T}}$ is a dominant eigenvector for A.

For the dominant eigenvalue, we can calculate the Rayleigh quotients $\frac{(A\mathbf{x}_k)\cdot\mathbf{x}_k}{\mathbf{x}_k\cdot\mathbf{x}_k}$ until they do not change to two decimal places. For $k \geq 2$, our Rayleigh quotients are all (to two decimal places) equal to 4. So we expect that the dominant eigenvalue of A is 4. We could also use the fact that

$$A\left[\frac{1}{2}\ 1\ \frac{1}{2}\right]^{\mathsf{T}} = [2\ 4\ 2]^{\mathsf{T}} = 4\left[\frac{1}{2}\ 1\ \frac{1}{2}\right]^{\mathsf{T}}$$

to see that $\left[\frac{1}{2}\ 1\ \frac{1}{2}\right]^{\mathsf{T}}$ is a dominant eigenvector for A with eigenvalue 4.

(b) Technology shows that the characteristic polynomial of $A - \lambda I_3$ is

$$p(\lambda) = -\lambda^3 + 7\lambda^2 - 14\lambda + 8 = -(\lambda - 1)(\lambda - 2)(\lambda - 4).$$

We can see from the characteristic polynomial that 4 is the dominant eigenvalue of A.

(c) Applying the power method to $B = (A - 0.5I_3)^{-1}$ with seed $\mathbf{x}_0 = [1\ 1\ 1]^{\mathsf{T}}$ gives $\mathbf{x}_k \approx [0.50\ 1.00\ 0.50]^{\mathsf{T}}$ for $k \geq 5$, with Rayleigh quotients of 2 (to several decimal places). So 2 is the dominant eigenvalue of B. But $\frac{1}{\lambda-0.5}$ is also the dominant eigenvalue of B, where λ is the corresponding eigenvalue of A. . So to find λ, we note that $\frac{1}{\lambda-0.5} = 2$ implies that $\lambda = 1$ is an eigenvalue of A.

Now applying the power method to $B = (A - 1.8I_3)^{-1}$ with seed $\mathbf{x}_0 = [1\ 1\ 1]^{\mathsf{T}}$ gives $\mathbf{x}_k \approx [1.00\ \ -1.00\ 1.00]^{\mathsf{T}}$ for large enough k, with Rayleigh quotients of 5 (to several decimal places). To find the corresponding eigenvalue λ for A, we note that $\frac{1}{\lambda-1.8} = 5$, or $\lambda = 2$ is an eigenvalue of A.

Admittedly, this method is very limited. Finding good choices for α often depends on having some information about the eigenvalues of A. Choosing α close to an eigenvalue provides the best chance of obtaining that eigenvalue.

Summary

- The power method is an iterative method that can be used to approximate the dominant eigenvalue of an $n \times n$ matrix A that has n linearly independent eigenvectors and a dominant eigenvalue.

- To use the power method we start with a seed $\mathbf{x}_0$ and then calculate the sequence $\{\mathbf{x}_k\}$ of vectors, where $\mathbf{x}_k = A\mathbf{x}_{k-1}$. If $\mathbf{x}_0$ is chosen well, then the sequence $\{\mathbf{x}_k\}$ converges to a dominant eigenvector of A.
- If A is an $n \times n$ matrix with eigenvalues λ_1, λ_2, ..., λ_n, to approximate an eigenvector of A corresponding to the eigenvalue λ_i, we apply the power method to the matrix $B = (A - \alpha I_n)^{-1}$, where α is not a eigenvalue of A and $\left|\frac{1}{\lambda_i - \alpha}\right| > \left|\frac{1}{\lambda_j - \alpha}\right|$ for any $j \neq i$.

Exercises

(1) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Let $\mathbf{x}_0 = [1\ 0]^{\mathsf{T}}$.

(a) Find the eigenvalues and corresponding eigenvectors for A.

(b) Use appropriate technology to calculate $\mathbf{x}_k = A^k\mathbf{x}_0$ for k up to 10. Compare to a dominant eigenvector for A.

(c) Use the eigenvectors from part (b) to approximate the dominant eigenvalue for A. Compare to the exact value of the dominant eigenvalue of A.

(d) Assume that the other eigenvalue for A is close to 0. Apply the inverse power method and compare the results to the remaining eigenvalue and eigenvectors for A.

(2) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$. Use the power method to approximate a dominant eigenvector for A. Use $\mathbf{x}_0 = [1\ 1\ 1]^{\mathsf{T}}$ as the seed. Then approximate the dominant eigenvalue of A.

(3) Let $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$. Use the power method starting with $\mathbf{x}_0 = [1\ 1]^{\mathsf{T}}$. Explain why the method fails in this case to approximate a dominant eigenvector, and how you could adjust the seed to make the process work.

(4) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Find the eigenvalues and an eigenvector for each eigenvalue.

(b) Apply the power method with an initial starting vector $\mathbf{x}_0 = [0\ 1]^{\mathsf{T}}$. What is the resulting sequence?

(c) Use equation (19.3) to explain the sequence you found in part (b).

(5) Let $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$. Fill in the entries in Table 19.4, where $\mathbf{x}_k$ is the kth approximation to a dominant eigenvector using the power method, starting with the seed $\mathbf{x}_0 = [1\ 0]^{\mathsf{T}}$. Compare the results of this table to the eigenvalues of A and $\lim_{k\to\infty} \frac{\mathbf{x}_{k+1}\cdot\mathbf{x}_k}{\mathbf{x}_k\cdot\mathbf{x}_k}$. What do you notice?

(6) Let $A = \begin{bmatrix} 4 & -5 \\ 2 & 15 \end{bmatrix}$. The power method will approximate the dominant eigenvalue $\lambda = 14$. In this exercise we explore what happens if we apply the power method to A^{-1}.

$\mathbf{v}$	$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$
$\frac{\mathbf{v}^{\mathsf{T}}A\mathbf{v}}{\mathbf{v}^{\mathsf{T}}\mathbf{v}}$						
$\mathbf{v}$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$	$\mathbf{x}_9$	$\mathbf{x}_{10}$	$\mathbf{x}_{11}$
$\frac{\mathbf{v}^{\mathsf{T}}A\mathbf{v}}{\mathbf{v}^{\mathsf{T}}\mathbf{v}}$						

Table 19.4: Values of the Rayleigh quotient.

(a) Apply the power method to A^{-1} to approximate the dominant eigenvalue of A^{-1}. Use $[1\ 1]^{\mathsf{T}}$ as the seed. How is this eigenvalue related to an eigenvalue of A?

(b) Explain in general why applying the power method to the inverse of an invertible matrix B might give an approximation to an eigenvalue of B of smallest magnitude. When might this not work?

(7) There are other algebraic methods that do not rely on the determinant of a matrix that can be used to find eigenvalues of a matrix. We examine one such method in this exercise. Let A be any $n \times n$ matrix, and let $\mathbf{v}$ be any vector in $\mathbb{R}^n$.

(a) Explain why the vectors

$$\mathbf{v},\ A\mathbf{v},\ A^2\mathbf{v},\ \ldots,\ A^n\mathbf{v}$$

are linearly independent.

(b) Let c_0, c_1, …, c_n be scalars, not all 0, so that

$$c_0\mathbf{v} + c_1A\mathbf{v} + c_2A^2\mathbf{v} + \cdots + c_nA^n\mathbf{v} = \mathbf{0}.$$

Explain why there must be a smallest positive integer k so that there are scalars a_0, a_1, …, a_k with $a_k \neq 0$. such that

$$a_0\mathbf{v} + a_1A\mathbf{v} + a_2A^2\mathbf{v} + \cdots + a_kA^k\mathbf{v} = \mathbf{0}.$$

(c) Let

$$q(t) = a_0 + a_1t + a_2t^2 + \cdots + a_kt^k.$$

Then

$$q(A) = a_0 + a_1A + a_2A^2 + \cdots + a_kA^k$$

and

$$\begin{aligned} q(A)\mathbf{v} &= (a_0 + a_1A + a_2A^2 + \cdots + a_kA^k)\mathbf{v} \\ &= a_0\mathbf{v} + a_1A\mathbf{v} + a_2A^2\mathbf{v} + \cdots + a_kA^k\mathbf{v} \\ &= \mathbf{0}. \end{aligned}$$

Suppose the polynomial $q(t)$ has a linear factor, say $q(t) = (t - \lambda)Q(t)$ for some degree $k-1$ polynomial $Q(t)$. Explain why, if $Q(A)\mathbf{v}$ is non-zero, λ is an eigenvalue of A with eigenvector $Q(A)\mathbf{v}$.

(d) This method allows us to find certain eigenvalues and eigenvectors, the roots of the polynomial $q(t)$. Any other eigenvector must lie outside the eigenspaces we have already found, so repeating the process with a vector $\mathbf{v}$ not in any of the known eigenspaces will produce different eigenvalues and eigenvectors. Let $A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & 2 \\ 0 & 0 & 6 \end{bmatrix}$.

i. Find the polynomial $q(t)$. Use $\mathbf{v} = [1\ 1\ 1]^{\mathsf{T}}$.

ii. Find all of the roots of $q(t)$.

iii. For each root λ of $q(t)$, find the polynomial $Q(t)$ and use this polynomial to determine an eigenvector of A. Verify your work.

(8) We don't need to use the Rayleigh quotients to approximate the dominant eigenvalue of a matrix A if we instead keep track of the scaling factors. Recall that the scaling in the power method can be used to make the magnitudes of the successive approximations smaller and easier to work with. Let A be an $n \times n$ matrix and begin with a non-zero seed $\mathbf{v}_0$. We now want to keep track of the scaling factors, so let α_0 be the component of $\mathbf{v}_0$ with largest absolute value and let $\mathbf{x}_0 = \frac{1}{|\alpha_0|}\mathbf{v}_0$. For $k \geq 0$, let $\mathbf{v}_k = A\mathbf{x}_{k-1}$, let α_k be the component of $\mathbf{v}_k$ with largest absolute value and let $\mathbf{x}_k = \frac{1}{\alpha_k}\mathbf{v}_k$.

(a) Let $A = \begin{bmatrix} 0 & 1 \\ -8 & 6 \end{bmatrix}$. Use $\mathbf{x}_0 = [1\ 1]^{\mathsf{T}}$ as the seed and calculate α_k for k from 1 to 10. Compare to the dominant eigenvalue of A.

(b) Assume that for large k the vectors $\mathbf{x}_k$ approach a dominant eigenvector with dominant eigenvalue λ. Show now in general that the sequence of scaling factors α_k approaches λ.

(9) Let A be an $n \times n$ matrix and let α be a scalar that is not an eigenvalue of A. Suppose that $\mathbf{x}$ is an eigenvector of $B = (A - \alpha I_n)^{-1}$ with eigenvalue β. Find an eigenvalue of A in terms of β and α with corresponding eigenvector $\mathbf{x}$.

(10) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** The largest eigenvalue of a matrix is a dominant eigenvalue.

(b) **True/False** If an $n \times n$ matrix A has n linearly independent eigenvectors and a dominant eigenvalue, then the sequence $\{A^k\mathbf{x}_0\}$ converges to a dominant eigenvector of A for any initial vector $\mathbf{x}_0$.

(c) **True/False** If λ is an eigenvalue of an $n \times n$ matrix A and α is not an eigenvalue of A, then $\lambda - \alpha$ is an eigenvalue of $A - \alpha I_n$.

(d) **True/False** Every square matrix has a dominant eigenvalue.

Project: Managing a Sheep Herd

Sheep farming is a significant industry in New Zealand. New Zealand is reported to have the highest density of sheep in the world. Sheep can begin to reproduce after one year, and give birth only once per year. Table 19.5 gives Birth and Survival Rates for Female New Zealand Sheep (from G. Caughley, "Parameters for Seasonally Breeding Populations," *Ecology*, **48**, (1967), 834-839). Since sheep hardly ever live past 12 years, we will only consider the population through 12 years.

Age (years)	Birth Rate	Survival Rate
0-1	0.000	0.845
1-2	0.045	0.975
2-3	0.391	0.965
3-4	0.472	0.950
4-5	0.484	0.926
5-6	0.546	0.895
6-7	0.543	0.850
7-8	0.502	0.786
8-9	0.468	0.691
9-10	0.459	0.561
10-11	0.433	0.370
11-12	0.421	0.000

Table 19.5: New Zealand female sheep data by age group.

As sheep reproduce, they add to the 0-1 sheep (lamb) population. The potential to produce offspring is called *fecundity* (derived from the word *fecund* which generally refers to reproductive ability) and determines how many lamb are added to the population. Let F_k (the fecundity rate) be the rate at which females in age class k give birth to female offspring. Not all members of a given age group survive to the next age groups, so let s_k be the fraction of individuals that survives from age group k to age group $k+1$. With these ideas in mind, we can create a life cycle chart as in Figure 19.2 that illustrates how the population of sheep changes on a farm (for the sake of space, we illustrate with four age classes).

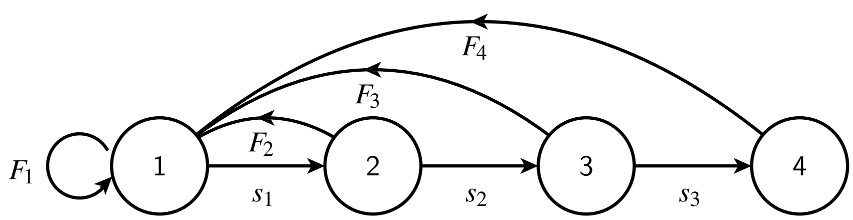

Figure 19.2: Life cycle with four age classes.

To model the sheep population, we need a few variables. Let $n_1^{(0)}$ be the number of sheep in age group 0-1, $n_2^{(0)}$ the number in age group 1-2, n_3 the number in age group 2-3 and, in general, $n_k^{(0)}$ the number of sheep in age group $(k-1)$-k at some initial time (time 0), and let

$$\mathbf{x}_0 = \left[n_1^{(0)} \; n_2^{(0)} \; n_3^{(0)} \; \cdots \; n_{12}^{(0)}\right]^{\mathsf{T}}.$$

We wish to determine the populations in the different groups after one year. Let

$$\mathbf{x}_1 = \left[n_1^{(1)}\ n_2^{(1)}\ n_3^{(1)}\ \cdots\ n_{12}^{(1)}\right]^{\mathsf{T}},$$

where $n_1^{(1)}$ denotes the number of sheep in age group 0-1, $n_2^{(1)}$ the number of sheep in age group 1-2 and, in general, $n_k^{(1)}$ the number of tilapia in age group $(k-1)$-k after one year.

Project Activity 19.1. Table 19.5 shows that, on average, each female in age group 1-2 produces 0.045 female offspring in a year. Since there are n_2 females in age group 1-2, the lamb population increases by $0.045n_2$ in a year.

(a) Continue this analysis to explain why

$$\begin{aligned} n_1^{(1)} = {} & 0.045n_2 + 0.391n_3 + 0.472n_4 + 0.484n_5 + 0.546n_6 + 0.543n_7 \\ & + 0.502n_8 + 0.468n_9 + 0.459n_{10} + 0.433n_{11} + 0.421n_{12}. \end{aligned}$$

(b) Explain why $n_2^{(1)} = 0.845n_1$.

(c) Now explain why

$$\mathbf{x}_1 = L\mathbf{x}_0, \tag{19.5}$$

where L is the matrix

$$\begin{bmatrix} 0 & 0.045 & 0.391 & 0.472 & 0.484 & 0.546 & 0.543 & 0.502 & 0.468 & 0.459 & 0.433 & 0.421 \\ 0.845 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.975 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.965 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.950 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.926 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.895 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.850 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.786 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.691 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.561 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.370 & 0 \end{bmatrix}. \tag{19.6}$$

Notice that our matrix L has the form

$$\begin{bmatrix} F_1 & F_2 & F_3 & \cdots & F_{n-1} & F_n \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix}.$$

Such a matrix is called a *Leslie matrix*.

Leslie matrices have certain useful properties, and one eigenvalue of a Leslie matrix can tell us a lot about the long-term behavior of the situation being modeled. You can take these properties as fact unless otherwise directed.

(1) A Leslie matrix L has a unique positive eigenvalue λ_1 with a corresponding eigenvector $\mathbf{v}_1$ whose entries are all positive.

(2) If λ_i $(i > 1)$ is any other eigenvalue (real or complex) of L, then $|\lambda_i| \leq \lambda_1$. If λ_1 is the largest magnitude eigenvalue of a matrix L, we call λ_1 a *dominant eigenvalue* of L.

(3) If any two successive entries in the first row of L are both positive, then $|\lambda_i| < \lambda_1$ for every $i > 1$. In this case we say that λ_1 is a *strictly dominant eigenvalue* of L. In a Leslie model, this happens when the females in two successive age classes are fertile, which is almost always the case.

(4) If λ_1 is a strictly dominant eigenvalue, then $\mathbf{x}_k$ is approximately a scalar multiple of $\mathbf{v}_1$ for large values of k, regardless of the initial state $\mathbf{x}_0$. In other words, large state vectors are close to eigenvectors for λ_1.

We can use these properties to determine the long-term behavior of the sheep herd.

Project Activity 19.2. Assume that L is defined by (19.6), and let

$$\mathbf{x}_m = \left[n_1^{(m)} \ n_2^{(m)} \ n_3^{(m)} \ \cdots \ n_{12}^{(m)} \right]^{\mathsf{T}},$$

where $n_1^{(m)}$ denotes the number of sheep in age group 0-1, $n_2^{(m)}$ the number of sheep in age group 1-2 and, in general, $n_k^{(m)}$ the number of sheep in age group $(k-1)$-k after k years.

(a) Assume that $\mathbf{x}_0 = [100 \ 100 \ 100 \ \cdots \ 100]^{\mathsf{T}}$. Use appropriate technology to calculate $\mathbf{x}_{22}$, $\mathbf{x}_{23}$, $\mathbf{x}_{24}$, and $\mathbf{x}_{25}$. Round to the nearest whole number. What do you notice about the sheep population? You may use the GeoGebra applet at `https://www.geogebra.org/m/yqss88xq`.

(b) We can use the third and fourth properties of Leslie matrices to better understand the long-term behavior of the sheep population. Since successive entries in the first row of the Leslie matrix in (19.6) are positive, our Leslie matrix has a strictly dominant eigenvalue λ_1. Given the dimensions of our Leslie matrix, finding this dominant eigenvalue through algebraic means is not feasible. Use the power method to approximate the dominant eigenvalue λ_1 of the Leslie matrix in (19.6) to five decimal places. Explain your process. Then explain how this dominant eigenvalue tells us that, unchecked, the sheep population grows at a rate that is roughly exponential. What is the growth rate of this exponential growth? You may use the GeoGebra applet at `https://www.geogebra.org/m/yqss88xq`.

Project Activity 19.2 indicates that, unchecked, the sheep population will grow without bound, roughly exponentially with ratio equal to the dominant eigenvalue of our Leslie matrix L. Of course, a sheep farmer cannot provide the physical environment or the resources to support an unlimited population of sheep. In addition, most sheep farmers cannot support themselves only by shearing sheep for the wool. Consequently, some harvesting of the sheep population each year for meat and skin is necessary. A sustainable harvesting policy allows for the regular harvesting of some sheep while maintaining the population at a stable level. It is necessary for the farmer to find an optimal harvesting rate to attain this stable population and the following activity leads us through an analysis of how such a harvesting rate can be determined.

Project Activity 19.3. The Leslie model can be modified to consider harvesting. It is possible to harvest different age groups at different rates, and to harvest only some age groups and not others.

In the case of sheep, it might make sense to only harvest from the youngest population since lamb is more desirable than mutton and the lamb population grows the fastest. Assume that this is our harvesting strategy and that we harvest our sheep from only the youngest age group at the start of each year. Let h be the fraction of sheep we harvest from the youngest age group each year after considering growth.

(a) If we begin with an initial population $\mathbf{x}_0$, then the state vector after births and expected deaths is $L\mathbf{x}_0$. Now we harvest. Explain why if we harvest a fraction h from the youngest age group after considering growth, then the state vector after 1 year will be

$$\mathbf{x}_1 = L\mathbf{x}_0 - HL\mathbf{x}_0,$$

where

$$H = \begin{bmatrix} h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Our goal is to find a harvesting rate that will lead to a steady state in which the sheep population remains the same each year. In other words, we want to find a value of h, if one exists, that satisfies

$$\mathbf{x} = L\mathbf{x} - HL\mathbf{x}. \tag{19.7}$$

Show that (19.7) is equivalent to the matrix equation

$$\mathbf{x} = (I_{12} - H)L\mathbf{x}. \tag{19.8}$$

(c) Use appropriate technology to experiment numerically with different values of h to find the value you think gives the best uniform harvest rate. Explain your reasoning. You may use the GeoGebra applet at `https://www.geogebra.org/m/yqss88xq`.

(d) Now we will use some algebra to find an equation that explicitly gives us the harvest rate in the general setting. This will take a bit of work, but none of it is too difficult. To simplify our work but yet illustrate the overall idea, let us consider the general 4×4 case with arbitrary Leslie matrix

$$L = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \end{bmatrix}.$$

Recall that we want to find a value of h that satisfies (19.8) with $H = \begin{bmatrix} h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Let $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^{\mathsf{T}}$.

i. Calculate the matrix product $(I_4 - H)L$. Explain why this product is again a Leslie matrix and why $(I_4 - H)L$ will have a dominant eigenvalue of 1.

ii. Now calculate $(I_4 - H)L\mathbf{x}$ and set it equal to $\mathbf{x}$. Write down the resulting system of 4 equations that must be satisfied. Be sure that your first equation is

$$x_1 = (1-h)F_1x_1 + (1-h)F_2x_2 + (1-h)F_3x_3 + (1-h)F_4x_4. \qquad (19.9)$$

iii. Equation (19.9) as written depends on the entries of the vector $\mathbf{x}$, but we should be able to arrive at a result that is independent of $\mathbf{x}$. To see how we do this, we assume the population of the youngest group is never 0, so we can divide both sides of (19.9) by x_1 to obtain

$$1 = (1-h)F_1 + (1-h)F_2\frac{x_2}{x_1} + (1-h)F_3\frac{x_3}{x_1} + (1-h)F_4\frac{x_4}{x_1}. \qquad (19.10)$$

Now we need to write the fractions $\frac{x_2}{x_1}$, $\frac{x_3}{x_1}$, and $\frac{x_4}{x_1}$ so that they do not involve the x_i. Use the remaining equations in your system to show that

$$\begin{aligned} \frac{x_2}{x_1} &= s_1 \\ \frac{x_3}{x_1} &= s_1s_2 \\ \frac{x_4}{x_1} &= s_1s_2s_3. \end{aligned}$$

iv. Now conclude that the harvesting value h must satisfy the equation

$$1 = (1-h)[F_1 + F_2s_1 + F_3s_1s_2 + F_4s_1s_2s_3]. \qquad (19.11)$$

The value $R = F_1 + F_2s_1 + F_3s_1s_2 + F_4s_1s_2s_3$ is called the *net reproduction rate of the population* and turns out to be the average number of daughters born to a female in her expected lifetime.

(e) Extend (19.11) to the 12 age group case of the sheep herd. Calculate the value of R for this sheep herd and then find the value of h. Compare this h to the value you obtained through experimentation earlier. Find the fraction of the lambs that should be harvested each year and explain what the stable population state vector $\mathbf{x}$ tells us about the sheep population for this harvesting policy.

Section 20

Complex Eigenvalues

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What properties do complex eigenvalues of a real matrix satisfy?
- What properties do complex eigenvectors of a real matrix satisfy?
- What is a rotation-scaling matrix?
- How do we find a rotation-scaling matrix within a matrix with complex eigenvalues?

Application: The Gershgorin Disk Theorem

We have now seen different methods for calculating/approximating eigenvalues of a matrix. The algebraic method using the characteristic polynomial can provide exact values, but only in cases where the size of the matrix is small. Methods like the power method allow us to approximate eigenvalues in many, but not all, cases. These approximation techniques can be made more efficient if we have some idea of where the eigenvalues are. The Gershgorin Disc Theorem is a useful tool that can quickly provide bounds on the location of eigenvalues using elementary calculations. For example, using the Gershsgorin Disk Theorem we can quickly tell that the real parts of the eigenvalues of the matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & -1+i & i \\ 2 & 1 & -2i \end{bmatrix}$$

lie between -4 and 5 and the imaginary parts lie between -5 and 2. Even more, we can say that the eigenvalues lie in the disks (called *Gershgorin disks*) shown in Figure 20.1. We will learn more details about the Gershgorin Disk Theorem at the end of this section.

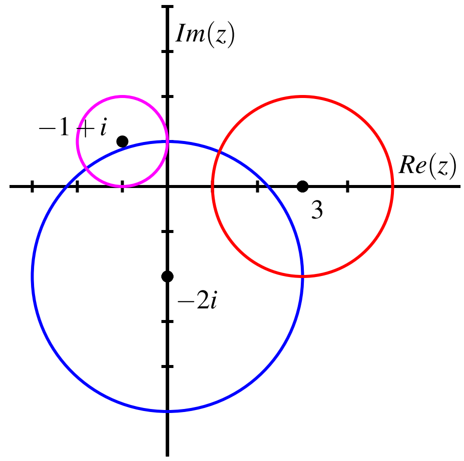

Figure 20.1: Gershgorin disks.

Introduction

So far we have worked with real matrices whose eigenvalues are all real. However, the characteristic polynomial of a matrix with real entries can have complex roots. In this section we investigate the properties of these complex roots and their corresponding eigenvectors, how these complex eigenvectors are found, and the geometric interpretation of the transformations defined by matrices with complex eigenvalues. Although we can consider matrices that have complex numbers as entries, we will restrict ourselves to matrices with real entries.

Preview Activity 20.1. Let $A = \begin{bmatrix} 2 & 4 \\ -2 & 2 \end{bmatrix}$.

(1) Find the characteristic polynomial of A.

(2) Find the eigenvalues of A. You should get two complex numbers. How are these complex numbers related?

(3) Find an eigenvector corresponding to each eigenvalue of A. You should obtain vectors with complex entries.

Complex Eigenvalues

As you noticed in Preview Activity 20.1, the complex roots of the characteristic equation of a real matrix A come in complex conjugate pairs. This should come as no surprise since we know through our use of the quadratic formula that complex roots of (real) quadratic polynomials come in complex conjugate pairs. More generally, if $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial with real coefficients and z is a root of this polynomial, meaning $p(z) = 0$, then

$$0 = \overline{p(z)} = \overline{a_0 + a_1z + a_2z^2 + \cdots + a_nz^n} = a_0 + a_1\overline{z} + a_2\overline{z}^2 + \cdots + a_n\overline{z}^n = p(\overline{z})\,.$$

Therefore, $\overline{z}$ is also a root of $p(x)$.

Activity 20.1. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(a) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a rotation transformation. What is the angle of rotation?

(b) Find the eigenvalues of A. For each eigenvalue, find an eigenvector.

In Preview Activity 20.1 and in Activity 20.1, you found that if $\mathbf{v}$ is an eigenvector of A corresponding to λ, then $\overline{\mathbf{v}}$ obtained by taking the complex conjugate of each entry in $\mathbf{v}$ is an eigenvector of A corresponding to $\overline{\lambda}$. Specifically, if $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ where both $\mathbf{u}$ and $\mathbf{w}$ are real vectors is an eigenvector of A, then so is $\overline{\mathbf{v}} = \mathbf{u} - i\mathbf{w}$. We can justify this property using matrix algebra as follows:

$$A\overline{\mathbf{v}} = \overline{A}\overline{\mathbf{v}} = \overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}\,.$$

In the first equality, we used the fact that A is a real matrix, so $\overline{A} = A$. In all the other equalities, we used the properties of the conjugation operation in complex numbers.

Rotation and Scaling Matrices

Recall that a rotation matrix is of the form

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

where the rotation is counterclockwise about the origin by an angle of θ radians. In Activity 20.1, we considered the rotation matrix with angle $\pi/2$ in counterclockwise direction. We will soon see that rotation matrices play an important role in the geometry of a matrix transformation for a matrix that has complex eigenvalues. In this activity, we will restrict ourselves to the 2×2 case, but similar arguments can be made in higher dimensions.

Activity 20.2. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) Explain why A is not a rotation matrix.

(b) Although A is not a rotation matrix, there is a rotation matrix B inside A. To find the matrix B, factor out $\sqrt{2}$ from all entries of A. In other words, write A as a product of two matrices in the form

$$A = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} B\,.$$

(c) The B matrix is a rotation matrix with an appropriate θ. Find this θ.

(d) If we think about the product of two matrices as applying one transformation after another, describe the effect of the matrix transformation defined by A geometrically.

More generally, if we have a matrix A of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then

$$A = \begin{bmatrix} \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} \end{bmatrix} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix}.$$

The first matrix in the decomposition is a scaling matrix with a scaling factor of $s = \sqrt{a^2+b^2}$. So if $s > 1$, the transformation stretches vectors, and if $s < 1$, the transformation shrinks vectors. The second matrix in the decomposition is a rotation matrix with angle θ such that $\cos(\theta) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\theta) = \frac{b}{\sqrt{a^2+b^2}}$. This angle is also the angle between the positive x-axis and the vector $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. We will refer to the matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ as *rotation-scaling matrices*.

Matrices with Complex Eigenvalues

Now we will investigate how a general 2×2 matrix with complex eigenvalues can be seen to be similar (both in a linear algebra and a colloquial meaning) to a rotation-scaling matrix.

Activity 20.3. Let $B = \begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$. The eigenvalues of B are $2 \pm 3i$. An eigenvector for the eigenvalue $2 - 3i$ is $\mathbf{v} = \begin{bmatrix} -5 \\ 1-3i \end{bmatrix}$. We will use this eigenvector to show that B is similar to a rotation-scaling matrix.

(a) Any complex vector $\mathbf{v}$ can be written as $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ where both $\mathbf{u}$ and $\mathbf{w}$ are real vectors. What are these real vectors $\mathbf{u}$ and $\mathbf{w}$ for the eigenvector $\mathbf{v}$ above?

(b) Let $P = [\mathbf{u}\ \mathbf{w}]$ be the matrix whose first column is the real part of $\mathbf{v}$ and whose second column is the imaginary part of $\mathbf{v}$ (without the i). Find $R = P^{-1}BP$.

(c) Express R as a product of a rotation and a scaling matrix. What is the factor of scaling? What is the rotation angle?

In Activity 20.3, we saw that the matrix B with complex eigenvalues $2 \pm 3i$ is similar to a rotation-scaling matrix. Specifically $R = P^{-1}BP$, where the columns of P are the real and imaginary parts of an eigenvector of B, is the rotation-scaling matrix with a factor of scaling by $\sqrt{2^2+3^2}$ and a rotation by angle $\theta = \arccos(\frac{2}{\sqrt{2^2+3^2}})$.

Does a similar decomposition result hold for a general 2×2 matrix with complex eigenvalues? We investigate this question in the next activity.

Activity 20.4. Let A be a 2×2 matrix with complex eigenvalue $\lambda = a - bi$, $b \neq 0$, and corresponding complex eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$.

(a) Explain why $A\mathbf{v} = A\mathbf{u} + iA\mathbf{w}$.

(b) Explain why $\lambda\mathbf{v} = (a\mathbf{u} + b\mathbf{w}) + i(a\mathbf{w} - b\mathbf{u})$.

(c) Use the previous two results to explain why

- $A\mathbf{u} = a\mathbf{u} + b\mathbf{w}$ and
- $A\mathbf{w} = a\mathbf{w} - b\mathbf{u}$.

(d) Let $P = [\mathbf{u}\ \mathbf{w}]$. We will now show that $AP = PR$ where $R = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

i. Without any calculation, explain why

$$AP = [A\mathbf{u}\ A\mathbf{w}].$$

ii. Recall that if M is an $m \times n$ matrix and $\mathbf{x}$ is an $n \times 1$ vector, then the matrix product $M\mathbf{x}$ is a linear combination of the columns of M with weights the corresponding entries of the vector $\mathbf{x}$. Use this idea to show that

$$PR = [a\mathbf{u} + b\mathbf{w}\ \ -b\mathbf{u} + a\mathbf{w}].$$

iii. Now explain why $AP = PR$.

iv. Assume for the moment that P is an invertible matrix. Show that $A = PRP^{-1}$.

Your work in Activity 20.4 shows that any 2×2 matrix is similar to a rotation-scaling matrix with a factor of scaling by $\sqrt{a^2+b^2}$ and a rotation by angle $\theta = \arccos(\frac{a}{\sqrt{a^2+b^2}})$ if $b \geq 0$, and $\theta = -\arccos(\frac{a}{\sqrt{a^2+b^2}})$ if $b < 0$. Geometrically, this means that every 2×2 real matrix with complex eigenvalues is just a scaled rotation (R) with respect to the basis $\mathcal{B}$ formed by $\mathbf{u}$ and $\mathbf{w}$ from the complex eigenvector $\mathbf{v}$. Multiplying by P^{-1} and P simply provides the change of basis from the standard basis to the basis $\mathcal{B}$, as we will see in detail when we learn about linear transformations.

Theorem 20.1. *Let A be a real 2×2 matrix with complex eigenvalue $a - bi$ and corresponding eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$. Then*

$$A = PRP^{-1}\ ,\ \textit{where } P = [\mathbf{u}\ \mathbf{w}]\ \textit{ and } R = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

The one fact that we have not yet addressed is why the matrix $P = [\mathbf{u}\ \mathbf{w}]$ is invertible. We do that now to complete the argument.

Let A be a real 2×2 matrix with $A\mathbf{v} = \lambda\mathbf{v}$, where $\lambda = a - bi$, $b \neq 0$ and $\mathbf{v} = \mathbf{u} + i\mathbf{w}$. To show that $\mathbf{u}$ and $\mathbf{w}$ are linearly independent, we need to show that no nontrivial linear combination of $\mathbf{u}$ and $\mathbf{w}$ can be the zero vector. Suppose

$$x_1\mathbf{u} + x_2\mathbf{w} = \mathbf{0}$$

for some scalars x_1 and x_2. We will show that $x_1 = x_2 = 0$. Assume to the contrary that one of x_1, x_2 is not zero. First, assume $x_1 \neq 0$. Then $\mathbf{u} = -\frac{x_2}{x_1}\mathbf{w}$. Let $c = -\frac{x_2}{x_1}$. Then

$$\begin{aligned} A\mathbf{u} &= A(c\mathbf{w}) \\ A\mathbf{u} &= cA\mathbf{w} \\ a\mathbf{u} + b\mathbf{w} &= c(a\mathbf{u} - b\mathbf{w}) \\ (a + cb)\mathbf{u} &= (ca - b)\mathbf{w} \\ (a + cb)(c\mathbf{u}) &= (ca - b)\mathbf{w}. \end{aligned}$$

So we must have $(a+cb)c = ca-b$. This equation simplifies to $c^2b = -b$. Since $b \neq 0$, we conclude that $c^2 = -1$ which is impossible for a real constant c. Therefore, we cannot have $x_1 \neq 0$. A similar argument (left to the reader) shows that $x_2 = 0$. Thus we can conclude that $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

Examples

What follows are worked examples that use the concepts from this section.

Example 20.2. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

(a) Without doing any computations, explain why not all of the eigenvalues of A can be complex.

(b) Find all of the eigenvalues of A.

Example Solution.

(a) Since complex eigenvalues occur in conjugate pairs, the complex eigenvalues with nonzero imaginary parts occur in pairs. Since A can have at most 3 different eigenvalues, at most two of them can have nonzero imaginary parts. So at least one eigenvalue of A is real.

(b) For this matrix A we have $A - \lambda I_3 = \begin{bmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & -1 \\ 1 & 1 & -\lambda+1 \end{bmatrix}$. Using a cofactor expansion along the first row gives us

$$\begin{aligned} \det(A - \lambda I_3) &= (-\lambda)\left((-\lambda)(1-\lambda)+1\right) - \left((-1)(1-\lambda)+1\right) \\ &= -\lambda^3 + \lambda^2 - \lambda + 1 - \lambda - 1 \\ &= \lambda^3 + \lambda^2 - 2\lambda \\ &= -\lambda(\lambda^2 - \lambda + 2). \end{aligned}$$

The roots of the characteristic polynomial are $\lambda = 0$ and

$$\lambda = \frac{1 \pm \sqrt{1-4(2)}}{2} = \frac{1}{2}(1 \pm \sqrt{7}i).$$

Example 20.3. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Find a rotation scaling matrix R that is similar to A. Identify the rotation and scaling factor.

Example Solution. The eigenvalues of A are the roots of the characteristic polynomial

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I_2) \\ &= \det\left(\begin{bmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)(3-\lambda) + 2 \\ &= \lambda^2 - 4\lambda + 5. \end{aligned}$$

The quadratic formula shows that the roots of $p(\lambda)$ are

$$\frac{4 \pm \sqrt{-4}}{2} = 2 \pm i.$$

To find an eigenvector for A with eigenvalue $2 - i$, we row reduce

$$A - (2-i)I_3 = \begin{bmatrix} -1+i & 2 \\ -1 & 1+i \end{bmatrix}$$

to

$$\begin{bmatrix} 1 & -i-1 \\ 0 & 0 \end{bmatrix}.$$

An eigenvector for A with eigenvalue $2 - i$ is then

$$[1+i \ \ 1]^{\mathsf{T}} = [1\ 1]^{\mathsf{T}} + i[1\ 0]^{\mathsf{T}}.$$

Letting $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, we have

$$R = P^{-1}AP = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

The scaling is determined by the determinant of R which is 5, and the angle θ of rotation satisfies $\sin(\theta) = \frac{1}{5}$. This makes $\theta \approx 0.2014$ radians or approximately 11.5370° counterclockwise.

Summary

- For a real matrix, complex eigenvalues appear in conjugate pairs. Specifically, if $\lambda = a + ib$ is an eigenvalue of a real matrix A, then $\overline{\lambda} = a - ib$ is also an eigenvalue of A.

- For a real matrix, if a $\mathbf{v}$ is an eigenvector corresponding to λ, then the vector $\overline{\mathbf{v}}$ obtained by taking the complex conjugate of each entry in $\mathbf{v}$ is an eigenvector corresponding to $\overline{\lambda}$.

- The rotation-scaling matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ can be written as

$$\begin{bmatrix} \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} \end{bmatrix} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix}.$$

  This decomposition geometrically means that the transformation corresponding to A can be viewed as a rotation by angle $\theta = \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$ if $b \geq 0$, or $\theta = -\arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right)$ if $b < 0$, followed by a scaling by factor $\sqrt{a^2+b^2}$.

- If A is a real 2×2 matrix with complex eigenvalue $a - bi$ and corresponding eigenvector $\mathbf{v} = \mathbf{u} + i\mathbf{w}$, then A is similar to the rotation-scaling matrix $R = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. More specifically,

$$A = PRP^{-1}\ , \text{ where } P = [\mathbf{u}\ \mathbf{w}]\,.$$

Exercises

(1) Find eigenvalues and eigenvectors of each of the following matrices.

(a) $\begin{bmatrix} 2 & 4 \\ -2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$

(2) Find a rotation-scaling matrix where the rotation angle is $\theta = 3\pi/4$ and scaling factor is less than 1.

(3) Determine which rotation-scaling matrices have determinant equal to 1. Be as specific as possible.

(4) Determine the rotation-scaling matrix inside the matrix $\begin{bmatrix} 2 & 4 \\ -2 & 2 \end{bmatrix}$.

(5) Find a real 2×2 matrix with eigenvalue $1 + 2i$.

(6) Find a real 2×2 matrix which is not a rotation-scaling matrix with eigenvalue $-1 + 2i$.

(7) We have seen how to find the characteristic polynomial of an $n \times n$ matrix. In this exercise we consider the revers question. That is, given a polynomial $p(\lambda)$ of degree n, can we find an $n \times n$ matrix whose characteristic polynomial is $p(\lambda)$?

(a) Find the characteristic polynomial of the 2×2 matrix $C = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix}$. Use this result to find a real valued matrix whose eigenvalues are $1 + i$ and $1 - i$.

(b) Repeat part (a) by showing that $-p(\lambda) = -\left(\lambda^2 + a_2\lambda^2 + a_1\lambda + a_0\right)$ is the characteristic polynomial of the 3×3 matrix $C = \begin{bmatrix} 0 & 0 & -a_0 \\ 0 & 1 & -a_1 \\ 0 & 0 & -a_2 \end{bmatrix}$.

(c) We can generalize this argument. Prove, using mathematical induction, that the polynomial

$$p(\lambda) = (-1)^n \left(\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0\right)$$

is the characteristic polynomial of the matrix

$$C = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}.$$

The matrix C is called the *companion matrix* for $p(\lambda)$.

(8) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** If $3 - 4i$ is an eigenvalue of a real matrix, then so is $3 + 4i$.

(b) **True/False** If $2 + 3i$ is an eigenvalue of a 3×3 real matrix A, then A has three distinct eigenvalues.

(c) **True/False** Every 2×2 real matrix with complex eigenvalues is a rotation-scaling matrix.

(d) **True/False** Every square matrix with real entries has real number eigenvalues.

(e) **True/False** If A is a 2×2 matrix with complex eigenvalues similar to a rotation-scaling matrix R, the eigenvalues of A and R are the same.

(f) **True/False** If A is a real matrix with complex eigenvalues, all eigenvectors of A must be non-real.

Project: Understanding the Gershgorin Disk Theorem

To understand the Gershgorin Disk Theorem, we need to recall how to visualize a complex number in the plane. Recall that a complex number z is a number of the form $z = a + bi$ where a and b are real numbers and $i^2 = -1$. The number a is the real part of z, denoted as $\Re(z)$, and b is the imaginary part of z, denoted $\Im(z)$. The set of all complex numbers is denoted $\mathbb{C}$. We define addition and multiplication on $\mathbb{C}$ as follows. For $a + bi, c + di \in \mathbb{C}$,

$$(a + bi) + (c + di) = (a + c) + (b + d)i \quad \text{and} \quad (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Note that the product is what we would expect if we "expanded" the product in the normal way and used the fact that $i^2 = -1$. The set of complex numbers forms a field – that is, $\mathbb{C}$ satisfies all of the same properties as $\mathbb{R}$ as stated in Theorem 4.2.

We can visualize the complex number $a + bi$ in the plane as the point (a, b). Here we are viewing the horizontal axis as the real axis and the vertical axis as the imaginary axis. The length (or magnitude) of the complex number $z = a + bi$, which we denote as $|z|$, is the distance from the origin to z. So by the Pythagorean Theorem we have $|a + bi| = \sqrt{a^2 + b^2}$. Note that the magnitude of $z = a + bi$ can be written as a complex product

$$|z| = \sqrt{(a + bi)(a - bi)}.$$

The complex number $a - bi$ is called the *complex conjugate* of $z = a + bi$ and is denoted as $\overline{z}$. A few important properties of real numbers and their conjugates are the following. Let $z = a + bi$ and $w = c + di$ be complex numbers. Then

- $\overline{z + w} = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i = (a - bi) + (c - di) = \overline{z} + \overline{w}$,
- $\overline{zw} = \overline{(ac - bd) + (ad + bc)i} = (ac - bd) - (ad + bc)i = (a - bi)(c - di) = \overline{z}\,\overline{w}$,
- $\overline{\overline{z}} = z$,

- $|z| = \sqrt{a^2+b^2} \geq \sqrt{a^2} = |a| = |\Re(z)|$,
- $|z| = \sqrt{a^2+b^2} \geq \sqrt{b^2} = |b| = |\Im(z)|$,
- $|\overline{z}| = |z|$,
- $|z| = 0$ if and only if $z = 0$,
- If $p(x)$ is a polynomial with real coefficients and the complex number z satisfies $p(z) = 0$, then $p(\overline{z}) = 0$ as well.

Using these facts we can show that the triangle inequality is true for complex numbers. That is,

$$|z+w| \leq |z| + |w|.$$

To see why, notice that

$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) \\
&= (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\
&= z\overline{z} + z\overline{w} + \overline{z\overline{w}} + w\overline{w} \\
&= |z|^2 + 2\Re(z\overline{w}) + |w|^2 \\
&\leq |z|^2 + 2|zw| + |w|^2 \\
&= |z|^2 + 2|z||w| + |w|^2 \\
&= (|z| + |w|)^2.
\end{aligned}$$

Since $|z+w|$, $|z|$, and $|w|$ are all non-negative, taking square roots of both sides gives us $|z+w| \leq |z| + |w|$ as desired. We can extend this triangle inequality to any number of complex numbers. That is, if z_1, z_2, ..., z_k are complex numbers, then

$$|z_1 + z_2 + \cdots + z_k| \leq |z_1| + |z_2| + \cdots + |z_k|. \tag{20.1}$$

We can prove Equation (20.1) by mathematical induction. We have already done the $k = 2$ case and so we assume that Equation (20.1) is true for any sum of k complex numbers. Now suppose that z_1, z_2, ..., z_k, z_{k+1} are complex numbers. Then

$$\begin{aligned}
|z_1 + z_2 + \cdots + z_k + z_{k+1}| &= |(z_1 + z_2 + \cdots + z_k) + z_{k+1}| \\
&\leq |z_1 + z_2 + \cdots + z_k| + |z_{k+1}| \\
&\leq (|z_1| + |z_2| + \cdots + |z_k|) + |z_{k+1}| \\
&= |z_1| + |z_2| + \cdots + |z_k| + |z_{k+1}|.
\end{aligned}$$

To prove the Gershgorin Disk Theorem, we will use the Levy-Desplanques Theorem, which gives conditions that guarantee that a matrix is invertible. We illustrate with an example in the following activity.

Project Activity 20.1. Let $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$. Since $\det(A) \neq 0$, we know that A is an invertible matrix. Let us assume for a moment that we don't know that A is invertible and try to determine if 0 is an eigenvalue of A. In other words, we want to know if there is a nonzero vector $\mathbf{v}$ so that $A\mathbf{v} = \mathbf{0}$. Assuming the existence of such a vector $\mathbf{v} = [v_1 \ v_2]^\mathsf{T}$, for $A\mathbf{v}$ to be $\mathbf{0}$ it must be the case that

$$3v_1 + 2v_2 = 0 \ \text{ and } \ -v_1 + 4v_2 = 0.$$

Since the vector $\mathbf{v}$ is not the zero vector, at least one of v_1, v_2 is not zero. Note that if one of v_1, v_2 is zero, the so is the other. So we can assume that v_1 and v_2 are nonzero.

(a) Use the fact that $3v_1 + 2v_2 = 0$ to show that $|v_2| > |v_1|$.

(b) Use the fact that $-v_1 + 4v_2 = 0$ to show that $|v_1| > |v_2|$. What conclusion can we draw about whether 0 is an eigenvalue of A? Why does this mean that A is invertible?

What makes the arguments work in Project Activity 20.1 is that $|3| > |2|$ and $|4| > |-1|$. This argument can be extended to larger matrices, as described in the following theorem.

Theorem 20.4 (Levy-Desplanques Theorem)**.** *Any square matrix $A = [a_{ij}]$ satisfying $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i is invertible.*

Proof. Let $A = [a_{ij}]$ be an $n \times n$ matrix satisfying $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i. Let us assume that A is not invertible, that is that there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \mathbf{0}$. Let $\mathbf{v} = [v_1 \ v_2 \ \cdots \ v_n]$ and t be between 1 and n so that $|v_t| \geq |v_i|$ for all i. That is, choose v_t to be the component of $\mathbf{v}$ with the largest absolute value.

Expanding the product $A\mathbf{v}$ using the row-column product along the tth row shows that

$$a_{t1}v_1 + a_{t2}v_2 + \cdots a_{tn}v_n = 0.$$

Solving for the a_{tt} term gives us

$$a_{tt}v_t = -(a_{t1}v_1 + a_{t2}v_2 + \cdots a_{t(t-1)}v_{t-1} + a_{t(t+1)}v_{t+1} + \cdots + a_{tn}v_n).$$

Then

$$\begin{aligned}
|a_{tt}||v_t| &= |-(a_{t1}v_1 + a_{t2}v_2 + \cdots a_{t(t-1)}v_{t-1} + a_{t(t+1)}v_{t+1} + \cdots + a_{tn}v_n| \\
&= |a_{t1}v_1 + a_{t2}v_2 + \cdots a_{t(t-1)}v_{t-1} + a_{t(t+1)}v_{t+1} + \cdots + a_{tn}v_n| \\
&\leq |a_{t1}||v_1| + |a_{t2}||v_2| + \cdots |a_{t(t-1)}||v_{t-1}| + |a_{t(t+1)}||v_{t+1}| + \cdots + |a_{tn}||v_n| \\
&\leq |a_{t1}||v_t| + |a_{t2}||v_t| + \cdots |a_{t(t-1)}||v_t| + |a_{t(t+1)}||v_t| + \cdots + |a_{tn}||v_t| \\
&= (|a_{t1}| + |a_{t2}| + \cdots |a_{t(t-1)}| + |a_{t(t+1)}| + \cdots + |a_{tn}|)|v_t|.
\end{aligned}$$

Since $|v_t| \neq 0$, we cancel the $|v_t|$ term to conclude that

$$|a_{tt}| \leq |a_{t1}| + |a_{t2}| + \cdots |a_{t(t-1)}| + |a_{t(t+1)}| + \cdots + |a_{tn}|.$$

But this contradicts the condition that $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i. We conclude that 0 is not an eigenvalue for A and A is invertible. ■

Any matrix $A = [a_{ij}]$ satisfying the condition of the Levy-Desplanques Theorem is given a special name.

Definition 20.5. A square matrix $A = [a_{ij}]$ is **strictly diagonally dominant** if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i.

So any strictly diagonally dominant matrix is invertible. A quick glance can show that a matrix is strictly diagonally dominant. For example, since $|3| > |1| + |-1|$, $|12| > |5| + |6|$, and $|-8| < |-2| + |4|$, the matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 12 & 6 \\ -2 & 4 & -8 \end{bmatrix}$$

is strictly diagonally dominant and therefore invertible. However, just because a matrix is not strictly diagonally dominant, it does not follow that the matrix is non-invertible. For example, the matrix $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is invertible, but not strictly diagonally dominant.

Now we can address the Gershgorin Disk Theorem.

Project Activity 20.2. Let A be an arbitrary $n \times n$ matrix and assume that λ is an eigenvalue of A.

(a) Explain why the matrix $A - \lambda I$ is singular.

(b) What does the Levy-Desplanques Theorem tell us about the matrix $A - \lambda I$?

(c) Explain how we can conclude the Gershgorin Disk Theorem.

Theorem 20.6 (Gershgorin Disk Theorem)**.** *Let $A = [a_{ij}]$ be an $n \times n$ matrix with complex entries. Then every eigenvalue of A lies in one of the Gershgorin discs*

$$\{z \in \mathbb{C} : |z - a_{ii}| \leq r_i\},$$

where $r_i = \sum_{j \neq i} |a_{ij}|$.

Based on this theorem, we define a Gershgorin disk to be $D(a_{ii}, r_i)$, where $r_i = \sum_{j \neq i} |a_{ij}|$.

(d) Use the Gershgorin Disk Theorem to give estimates on the locations of the eigenvalues of the matrix $A = \begin{bmatrix} -1 & 2 \\ -3 & 2 \end{bmatrix}$.

The Gershgorin Disk Theorem has a consequence that gives additional information about the eigenvalues if some of the Gershgorin disks do not overlap.

Theorem 20.7. *If S is a union of m Gershgorin disks of a matrix A such that S does not intersect any other Gershgorin disk, then S contains exactly m eigenvalues (counting multiplicities) of A.*

Proof. Most proofs of this theorem require some results from topology. For that reason, we will not present a completely rigorous proof but rather give the highlights. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let D_i be a collection of Gershgorin disks of A for $1 \leq i \leq m$ such that $S = \cup_{1 \leq i \leq m} D_i$ does not intersect any other Gershgorin disk of A, and let S' be the union of the Gershgorin disks of A that are different from the D_i. Note that $S \cap S' = \emptyset$. Let C be the matrix whose ith column is $a_{ii}\mathbf{e}_i$, that is C is the diagonal matrix whose diagonal entries are the corresponding diagonal entries of A. Note that the eigenvalues of C are a_{ii} and the Gershgorin disks of C are just the points a_{ii}. So our theorem is true for this matrix C. To prove the result, we build a continuum of matrices from C to A as follows: let $B = A - C$ (so that B is the matrix whose off-diagonal entries are those of A and whose diagonal entries are 0), and let $A(t) = tB + C$ for t in the interval $[0, 1]$. Note that $A(1) = A$. Since the diagonal entries of $A(t)$ are the same as those of A, the Gershgorin disks of $A(t)$ have the same centers as the corresponding Gershgorin disks of A, while the radii of the Gershgorin disks of $A(t)$ are those of A but scaled by t. So the Gershgorin disks of $A(t)$ increase from points (the a_{ii}) to the Gershgorin disks of A as t increases from 0 to 1. While the centers of the disks all remain fixed, it is important to recognize that the eigenvalues of $A(t)$ move as t changes. An illustration of this is shown in Figure 20.2 with the eigenvalues as the black points and the changing Gershgorin disks dashed in magenta, using the matrix $\left[\begin{array}{cc} i & \frac{1}{2} \\ 1 & -2+i \end{array}\right]$. We can learn about how the eigenvalues move with the characteristic polynomial.

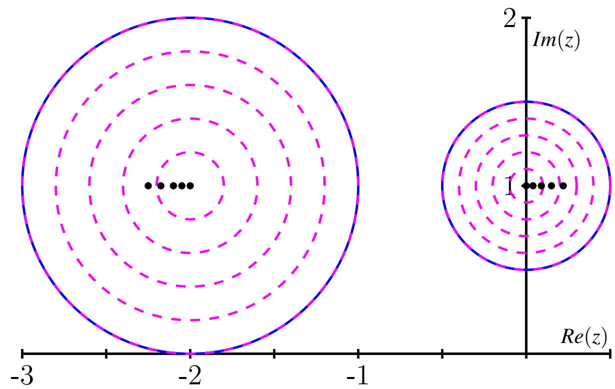

Figure 20.2: How eigenvalues move.

Let $p(t, x)$ be the characteristic polynomial of $A(t)$. Note that these characteristic polynomials are functions of both t and x. Since polynomials are continuous functions, their roots (the eigenvalues of $A(t)$) are continuous for $t \in [0, 1]$ as well. Let $\lambda(t)$ be an eigenvalue of $A(t)$. Note that $\lambda(1)$ is an eigenvalue of A, and $\lambda(0)$ is one of the a_{ii} and is therefore in S. We will argue that $\lambda(t)$ is in S for every value of t in $[0, 1]$. Let r_i be the radius of D_i and let $D(t)_i$ be the Gershgorin disk of $A(t)$ with the same center as D_i and radius $r(t)_i = tr_i$. Let $S(t) = \cup_{1 \leq i \leq m} D(s)_i$. Since $r(s)_i \leq r_i$, it follows that $D(s)_i \subseteq D_i$ and so $S(t) \cap S' = \emptyset$ as well. From topology, we know that since the disks D_i are closed, the union S of these disks is also closed. Similarly, $S(t)$ and S' are closed. Thus, $\lambda(t)$ is continuous in a closed set and so does not leave the set. Thus, $\lambda(t)$ is in S for every value of t in $[0, 1]$.

■

Section 21

Properties of Determinants

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- How do elementary row operations change the determinant?
- How can we represent elementary row operations via matrix multiplication?
- How can we use elementary row operations to calculate the determinant more efficiently?
- What is the Cramer's rule for the explicit formula for the inverse of a matrix?
- How can we interpret determinants from a geometric perspective?

Introduction

This section is different than others in that it contains mainly proofs of previously stated results and only a little new material. Consequently, there is no application attached to this section.

We have seen that an important property of the determinant is that it provides an easy criteria for the invertibility of a matrix. As a result, we obtained an algebraic method for finding the eigenvalues of a matrix, using the characteristic equation. In this section, we will investigate other properties of the determinant related to how elementary row operations change the determinant. These properties of the determinant will help us evaluate the determinant in a more efficient way compared to using the cofactor expansion method, which is computationally intensive for large n values due to it being a recursive method. Finally, we will derive a geometrical interpretation of the determinant.

Preview Activity 21.1.

(1) We first consider how the determinant changes if we multiply a row of the matrix by a constant.

(a) Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Pick a few different values for the constant k and compare the determinant of A and that of $\begin{bmatrix} 2k & 3k \\ 1 & 4 \end{bmatrix}$. What do you conjecture that the effect of multiplying a row by a constant on the determinant is?

(b) If we want to make sure our conjecture is valid for any 2×2 matrix, we need to show that for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the relationship between $\det(A)$ and the determinant of $\begin{bmatrix} a \cdot k & b \cdot k \\ c & d \end{bmatrix}$ follows our conjecture. We should also check that the relationship between $\det(A)$ and the determinant of $\begin{bmatrix} a & b \\ c \cdot k & d \cdot k \end{bmatrix}$ follows our conjecture. Verify this.

(c) Make a similar conjecture for what happens to the determinant when a row of a 3×3 matrix A is multiplied by a constant k, and explain why your conjecture is true using the cofactor expansion definition of the determinant.

(2) The second type of elementary row operation we consider is row swapping.

(a) Take a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and determine how row swapping effects the determinant.

(b) Now choose a few different 3×3 matrices and see how row swapping changes the determinant in these matrices by evaluating the determinant with a calculator or any other appropriate technology.

(c) Based on your results so far, conjecture how row swapping changes the determinant in general.

(3) The last type of elementary row operation is adding a multiple of a row to another. Determine the effect of this operation on a 2×2 matrix by evaluating the determinant of a general 2×2 matrix after a multiple of one row is added to the other row.

(4) All of the elementary row operations we discussed above can be achieved by matrix multiplication with *elementary matrices*. For each of the following elementary matrices, determine what elementary operation it corresponds to by calculating the product EA, where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a general 3×3 matrix.

(a) $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Elementary Row Operations and Their Effects on the Determinant

In Preview Activity 21.1, we conjectured how elementary row operations affect the determinant of a matrix. In the following activity, we prove how the determinant changes when a row is multiplied by a constant using the cofactor expansion definition of the determinant.

Activity 21.1. In this activity, assume that the determinant of A can be determined by a cofactor expansion along any row or column. (We will prove this result independently later in this section.) Consider an arbitrary $n \times n$ matrix $A = [a_{ij}]$.

(a)

(b) Write the expression for $\det(A)$ using the cofactor expansion along the second row.

(c) Let B be obtained by multiplying the second row of A by k. Write the expression for $\det(B)$ if the cofactor expansion along the second row is used.

(d) Use the expressions you found above, to express $\det(B)$ in terms of $\det(A)$.

(e) Explain how this method generalizes to prove the relationship between the determinant of a matrix A and that of the matrix obtained by multiplying a row by a constant k.

Your work in Activity 21.1 proves the first part of the following theorem on how elementary row operations change the determinant of a matrix.

Theorem 21.1. *Let A be a square matrix.*

(1) If B is obtained by multiplying a row of A by a constant k, then $\det(B) = k\det(A)$.

(2) If B is obtained by swapping two rows of A, then $\det(B) = -\det(A)$.

(3) If B is obtained by adding a multiple of a row of A to another, then $\det(B) = \det(A)$.

In the next section, we will use elementary matrices to prove the last two properties of Theorem 21.1.

Elementary Matrices

As we saw in Preview Activity 21.1, elementary row operations can be achieved by multiplication by *elementary matrices*.

Definition 21.2. An **elementary matrix** is a matrix obtained by performing a single elementary row operation on an identity matrix.

The following elementary matrices correspond, respectively, to an elementary row operation which swaps rows 2 and 4; an elementary row operation which multiplies the third row by 5; and an elementary row operation which adds four times the third row to the first row on any 4×4 matrix:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To obtain an elementary matrix corresponding an elementary row operation, we simply perform the elementary row operation on the identity matrix. For example, E_1 above is obtained by swapping rows 2 and 4 of the identity matrix.

With the use of elementary matrices, we can now prove the result about how the determinant is affected by elementary row operations. We first rewrite Theorem 21.1 in terms of elementary matrices:

Theorem 21.3. *Let A be an $n \times n$ matrix. If E is an $n \times n$ elementary matrix, then* $\det(EA) = \det(E)\det(A)$ *where*

$$\det(E) = \begin{cases} r & \text{if } E \text{ corresponds to multiplying a row by } r \\ -1 & \text{if } E \text{ corresponds to swapping two rows} \\ 1 & \text{if } E \text{ corresponds to adding a multiple of a row to another.} \end{cases}$$

Notes on Theorem 21.3. An elementary matrix E obtained by multiplying a row by r is a diagonal matrix with one r along the diagonal and the rest 1s, so $\det(E) = r$. Similarly, an elementary matrix E obtained by adding a multiple of a row to another is a triangular matrix with 1s along the diagonal, so $\det(E) = 1$. The fact that the the determinant of an elementary matrix obtained by swapping two rows is -1 is a bit more complicated and is verified independently later in this section. Also, the proof of 21.3 depends on the fact that the cofactor expansion of a matrix is the same along any two rows. A proof of this can also be found later in this section.

Proof of Theorem 21.3. We will prove the result by induction on n, the size of the matrix A. We verified these results in Preview Activity 21.1 for $n = 2$ using elementary row operations. The elementary matrix versions follow immediately.

Now assume the theorem is true for $k \times k$ matrices with $k \geq 2$ and consider an $n \times n$ matrix A where $n = k + 1$. If E is an $n \times n$ elementary matrix, we want to show that $\det(EA) = \det(E)\det(A)$. Let $EA = B$. (Although it is an abuse of language, we will refer to both the elementary matrix and the elementary row operation corresponding to it by E.)

When finding $\det(B) = \det(EA)$ we will use a cofactor expansion along a row which is not affected by the elementary row operation E. Since E affects at most two rows and A has $n \geq 3$ rows, it is possible to find such a row, say row i. The cofactor expansion along row i of B is

$$b_{i1}(-1)^{i+1}\det(B_{i1}) + b_{i2}(-1)^{i+2}\det(B_{i2}) + \cdots + b_{in}(-1)^{i+n}\det(B_{in}). \tag{21.1}$$

Since we chose a row of A which was not affected by the elementary row operation, it follows that $b_{ij} = a_{ij}$ for $1 \leq j \leq n$. Also, the matrix B_{ij} obtained by removing row i and column j from matrix $B = EA$ can be obtained from A_{ij} by an elementary row operation of the same type as E. Hence there is an elementary matrix E_k of the same type as E with $B_{ij} = E_k A_{ij}$. Therefore, by induction, $\det(B_{ij}) = \det(E_k)\det(A_{ij})$ and $\det(E_k)$ is equal to 1, -1 or r depending on the type of elementary row operation. If we substitute this information into equation (21.1), we obtain

$$\begin{aligned}\det(B) &= a_{i1}(-1)^{i+1}\det(E_k)\det(A_{i1}) + a_{i2}(-1)^{i+2}\det(E_k)\det(A_{i2}) \\ &\quad + \cdots + a_{in}(-1)^{i+n}\det(E_k)\det(A_{in}) \\ &= \det(E_k)\det(A)\,.\end{aligned}$$

This equation proves $\det(EA) = \det(E_k)\det(A)$ for any $n \times n$ matrix A where E_k is the corresponding elementary row operation on the $k \times k$ matrices obtained in the cofactor expansion.

The proof of the inductive step will be finished if we show that $\det(E_k) = \det(E)$. This equality follows if we let $A = I_n$ in $\det(EA) = \det(E_k)\det(A)$. Therefore, $\det(E)$ is equal to r, or 1, or -1, depending on the type of the elementary row operation E since the same is true of $\det(E_k)$ by inductive hypothesis.

Therefore, by the principle of induction, the claim is true for every $n \geq 2$. ■

As a corollary of this theorem, we can prove the multiplicativity of determinants:

Theorem 21.4. *Let A and B be $n \times n$ matrices. Then*

$$\det(AB) = \det(A)\det(B)\,.$$

Proof. If A is non-invertible, then AB is also non-invertible and both $\det(A)$ and $\det(AB)$ are 0, proving the equality in this case.

Suppose now that A is invertible. By the Invertible Matrix Theorem, we know that A is row equivalent to I_n. Expressed in terms of elementary matrices, this means that there are elementary matrices $E_1, E_2, \ldots, E_\ell$ such that

$$A = E_1E_2\cdots E_\ell I_n = E_1E_2\cdots E_\ell\,. \tag{21.2}$$

Therefore, repeatedly applying Theorem 21.3, we find that

$$\det(A) = \det(E_1)\det(E_2)\cdots\det(E_\ell)\,. \tag{21.3}$$

If we multiply equation (21.2) by B on the right, we obtain

$$AB = E_1E_2\cdots E_\ell B\,.$$

Again, by repeatedly applying Theorem 21.3 with this product of matrices, we find

$$\det(AB) = \det(E_1E_2\cdots E_\ell B) = \det(E_1)\det(E_2)\cdots\det(E_\ell)\det(B)\,.$$

From equation (21.3), the product of $\det(E_i)$'s equals $\det(A)$, so

$$\det(AB) = \det(A)\det(B)$$

which finishes the proof of the theorem. ■

We can use the multiplicative property of the determinant and the determinants of elementary matrices to calculate the determinant of a matrix in a more efficient way than using the cofactor expansion. The next activity provides an example.

Activity 21.2. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 6 \\ -1 & 2 & 1 \end{bmatrix}$.

(a) Use elementary row operations to reduce A to a row echelon form. Keep track of the elementary row operation you use.

(b) Taking into account how elementary row operations affect the determinant, use the row echelon form of A to calculate $\det(A)$.

Your work in Activity 21.2 provides an efficient method for calculating the determinant. If A is a square matrix, we use row operations given by elementary matrices E_1, E_2, ..., E_k to row reduce A to row echelon form R. That is

$$R = E_k E_{k-1} \cdots E_2 E_1 A.$$

We know $\det(E_i)$ for each i, and since R is a triangular matrix we can find its determinant. Then

$$\det(A) = \det(E_1)^{-1} \det(E_2)^{-1} \cdots \det(E_2)^{-1} \det(R).$$

In other words, if we keep track of how the row operations affect the determinant, we can calculate the determinant of a matrix A using row operations.

Activity 21.3. Theorems 21.3 and 21.4 can be used to prove the following (part c of Theorem 16.2) that A is invertible if and only if $\det(A) \neq 0$. We see how in this activity. Let A be an $n \times n$ matrix. We can row reduce A to its reduced row echelon form R by elementary matrices E_1, E_2, ..., E_k so that

$$R = E_1 E_2 \cdots E_k A.$$

(a) Suppose A is invertible. What, then, is R? What is $\det(R)$? Can the determinant of an elementary matrix ever be 0? How do we conclude that $\det(A) \neq 0$?

(b) Now suppose that $\det(A) \neq 0$. What can we conclude about $\det(R)$? What, then, must R be? How do we conclude that A is invertible?

Summary: Let A be an $n \times n$ matrix. Suppose we swap rows s times and divide rows by constants $k_1, k_2, \ldots, k_r$ while computing a row echelon form REF(A) of A. Then $\det(A) = (-1)^s k_1 k_2 \cdots k_r \det(\text{REF}(A))$.

Geometric Interpretation of the Determinant

Determinants have interesting and useful applications from a geometric perspective. To understand the geometric interpretation of the determinant of an $n \times n$ matrix A, we consider the image of the unit square under the transformation $T(\mathbf{x}) = A\mathbf{x}$ and see how its area changes based on A.

Activity 21.4.

(a) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Start with the unit square in $\mathbb{R}^2$ with corners at the origin and at $(1,1)$. In other words, the unit square we are considering consists of all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ where $0 \leq x \leq 1$ and $0 \leq y \leq 1$, visualized as points in the plane.

i. Consider the collection of image vectors $A\mathbf{v}$ obtained by multiplying $\mathbf{v}$'s by A. Sketch the rectangle formed by these image vectors.

ii. Explain how the area of this image rectangle and the unit square is related via $\det(A)$.

iii. Does the relationship you found above generalize to an arbitrary $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$? If not, modify the relationship to hold for all diagonal matrices.

(b) Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

i. Sketch the image of the unit square under the transformation $T(\mathbf{v}) = A\mathbf{v}$. To make the sketching easier, find the images of the vectors $[0\ 0]^{\mathsf{T}}, [1\ 0]^{\mathsf{T}}, [0\ 1]^{\mathsf{T}}, [1\ 1]^{\mathsf{T}}$ as points first and then connect these images to find the image of the unit square.

ii. Check that the area of the parallelogram you obtained in the above part is equal to $\det(A)$.

iii. Does the relationship between the area and $\det(A)$ still hold if $A = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$? If not, how will you modify the relationship?

It can be shown that for all 2×2 matrices a similar relationship holds.

Theorem 21.5. *For a 2×2 matrix A, the area of the image of the unit square under the transformation $T(\mathbf{x}) = A\mathbf{x}$ is equal to $|\det(A)|$. This is equivalent to saying that $|\det(A)|$ is equal to the area of the parallelogram defined by the columns of A. The area of the parallelogram is also equal to the lengths of the column vectors of A multiplied by $|\sin(\theta)|$ where θ is the angle between the two column vectors.*

There is a similar geometric interpretation of the determinant of a 3×3 matrix in terms of volume.

Theorem 21.6. *For a 3×3 matrix A, the volume of the image of the unit cube under the transformation $T(\mathbf{x}) = A\mathbf{x}$ is equal to $|\det(A)|$. This is equivalent to saying that $|\det(A)|$ is equal to the volume of the parallelepiped defined by the columns of A.*

The sign of $\det(A)$ can be interpreted in terms of the orientation of the column vectors of A. See the project in Section 16 for details.

An Explicit Formula for the Inverse and Cramer's Rule

In Section 10 we found the inverse A^{-1} using row reduction of the matrix obtained by augmenting A with I_n. However, in theoretical applications, having an explicit formula for A^{-1} can be handy.

Such an explicit formula provides us with an algebraic expression for A^{-1} in terms of the entries of A. A consequence of the formula we develop is Cramer's Rule, which can be used to provide formulas that give solutions to certain linear systems.

We begin with an interesting connection between a square matrix and the matrix of its cofactors that we explore in the next activity.

Activity 21.5. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 2 & -1 & 2 \end{bmatrix}$.

(a) Calculate the $(1,1)$, $(1,2)$, and $(1,3)$ cofactors of A.

(b) If C_{ij} represents the (i,j) cofactor of A, then the cofactor matrix C is the matrix $C = [C_{ij}]$. The *adjugate* matrix of A is the transpose of the cofactor matrix. In our example, the adjugate matrix of A is

$$\text{adj}(A) = \begin{bmatrix} 13 & -5 & -7 \\ 8 & -2 & -7 \\ -9 & 4 & 7 \end{bmatrix}.$$

Check the entries of this adjugate matrix with your calculations from part (a). Then calculate the matrix product

$$A \text{ adj}(A).$$

(c) What do you notice about the product $A \text{ adj}(A)$? How is this product related to $\det(A)$?

The result of Activity 21.5 is rather surprising, but it is valid in general. That is, if $A = [a_{ij}]$ is an invertible $n \times n$ matrix and C_{ij} is the (i,j) cofactor of A, then $A \text{ adj}(A) = \det(A) I_n$. In other words, $A\left(\frac{\text{adj}(A)}{\det(A)}\right) = I_n$ and so

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

This gives us another formulation of the inverse of a matrix. To see why $A \text{ adj}(A) = \det(A) I_n$, we use the row-column version of the matrix product to find the ijth entry of $A \text{ adj}(A)$ as indicated by the shaded row and column

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & & & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}.$$

Thus the ijth entry of $A \text{ adj}(A)$ is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}. \tag{21.4}$$

Notice that if $i = j$, then expression (21.4) is the cofactor expansion of A along the ith row. So the iith entry of $A \text{ adj}(A)$ is $\det(A)$. It remains to show that the ijth entry of $A \text{ adj}(A)$ is 0 when $i \neq j$.

When $i \neq j$, the expression (21.4) is the cofactor expansion of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j-11} & a_{j-12} & \cdots & a_{j-1n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j+11} & a_{i+12} & \cdots & a_{j+1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

along the jth row. This matrix is the one obtained by replacing the jth row of A with the ith row of A. Since this matrix has two identical rows, it is not row equivalent to the identity matrix and is therefore not invertible. Thus, when $i \neq j$ expression (21.4) is 0. This makes $A \text{ adj}(A) = \det(A)I_n$.

One consequence of the formula $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$ is Cramer's rule, which describes the solution to the equation $A\mathbf{x} = \mathbf{b}$.

Activity 21.6. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

(a) Solve the equation $A\mathbf{x} = \mathbf{b}$ using the inverse of A.

(b) Let $A_1 = \begin{bmatrix} 2 & 1 \\ 6 & 2 \end{bmatrix}$, the matrix obtained by replacing the first column of A with $\mathbf{b}$. Calculate $\frac{\det(A_1)}{\det(A)}$ and compare to your solution from part (a). What do you notice?

(c) Now let $A_2 = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}$, the matrix obtained by replacing the second column of A with $\mathbf{b}$. Calculate $\frac{\det(A_2)}{\det(A)}$ and compare to your solution from part (a). What do you notice?

The result from Activity 21.6 may seem a bit strange, but turns out to be true in general. The result is called *Cramer's Rule*.

Theorem 21.7 (Cramer's Rule)**.** *Let A be an $n \times n$ invertible matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries*

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i represents the matrix formed by replacing ith column of A with $\mathbf{b}$.

To see why Cramer's Rule works in general, let A be an $n \times n$ invertible matrix and $\mathbf{b} = [b_1 \ b_2 \ \cdots \ b_n]^{\mathsf{T}}$. The solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\text{adj}(A)\mathbf{b} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Expanding the product gives us

$$\mathbf{x} = \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}.$$

The expression

$$b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}$$

is the cofactor expansion of the matrix

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & & & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

along the jth column, giving us the formula in Cramer's Rule.

Cramer's Rule is not a computationally efficient method. To find a solution to a linear system of n equations in n unknowns using Cramer's Rule requires calculating $n+1$ determinants of $n \times n$ matrices – quite inefficient when n is 3 or greater. Our standard method of solving systems using Gaussian elimination is much more efficient. However, Cramer's Rule does provide a formula for the solution to $A\mathbf{x} = \mathbf{b}$ as long as A is invertible.

The Determinant of the Transpose

In this section we establish the fact that the determinant of a square matrix is the same as the determinant of its transpose.

The result is easily verified for 2×2 matrices, so we will proceed by induction and assume that the determinant of the transpose of any $(n-1) \times (n-1)$ matrix is the same as the determinant of its transpose. Suppose $A = [a_{ij}]$ is an $n \times n$ matrix. By definition,

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

and

$$\det(A^{\mathsf{T}}) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + \cdots + a_{n1}C_{n1}.$$

Note that the only terms in either determinant that contains a_{11} is $a_{11}C_{11}$. This term is the same in both determinants, so we proceed to examine other elements. Let us consider all terms in the cofactor expansion for $\det(A^{\mathsf{T}})$ that contain $a_{i1}a_{1j}$. The only summand that contains a_{i1} is $a_{i1}C_{i1}$. Letting A_{ij} be the sub-matrix of A obtained by deleting the ith row and jth column, we see that $a_{i1}C_{i1} = (-1)^{i+1}a_{i1}\det(A_{i1})$. Now let's examine the sub-matrix A_{i1}:

$$\begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n-1} & a_{1n} \\ a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n-1} & a_{2n} \\ \vdots & & \ddots & \vdots & \ddots & & \\ a_{i-12} & a_{i-13} & \cdots & a_{i-1j} & \cdots & a_{i-1n-1} & a_{i-1n} \\ a_{i+12} & a_{i+13} & \cdots & a_{i+1j} & \cdots & a_{i+1n-1} & a_{i+1n} \\ a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn-1} & a_{nn} \end{bmatrix}$$

When we expand along the first row to calculate $\det(A_{i1})$, the only term that will involve a_{1j} is

$$(-1)^{j-1+1}a_{1j}\det(A_{i1,1j}),$$

where $A_{ik,jm}$ denotes the sub-matrix of A obtained by deleting rows i and k and columns j and m from A. So the term that contains $a_{i1}a_{1j}$ in the cofactor expansion for $\det(A^{\mathsf{T}})$ is

$$(-1)i+1a_{i1}(-1)^{j}a_{1j}\det(A_{i1_{1j}}) = (-1)^{i+j+1}a_{i1}a_{1j}\det(A_{i1,1j}). \tag{21.5}$$

Now we examine the cofactor expansion for $\det(A)$ to find the terms that contain $a_{i1}a_{1j}$. The quantity a_{1j} only appears in the cofactor expansion as

$$a_{1j}C_{1j} = (-1)^{1+j}a_{1j}\det(A_{1j}).$$

Now let's examine the sub-matrix A_{1j}:

$$\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2j-1} & a_{2j+1} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3j-1} & a_{3j+1} & \cdots & a_{3n} \\ \vdots & & \ddots & \vdots & \ddots & & \\ a_{i1} & a_{i2} & \cdots & a_{ij-1} & a_{ij+1} & \cdots & a_{in} \\ \vdots & & & \vdots & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

Here is where we use the induction hypothesis. Since A_{1j} is an $(n-1)\times(n-1)$ matrix, its determinant can be found with a cofactor expansion down the first column. The only term in this cofactor expansion that will involve a_{i1} is

$$(-1)^{i-1+1}a_{i1}\det(A_{1i,j1}).$$

So the term that contains $a_{i1}a_{1j}$ in the cofactor expansion for $\det(A)$ is

$$(-1)^{1+j}a_{1j}(-1)^{i-1+1}a_{i1}\det(A_{1j_{i1}}) = (-1)^{i+j+1}a_{i1}a_{1j}\det(A_{1i,j1}). \tag{21.6}$$

Since the quantities in (21.5) and (21.6) are equal, we conclude that the terms in the two cofactor expansions are the same and

$$\det(A^{\mathsf{T}}) = \det(A).$$

Row Swaps and Determinants

In this section we determine the effect of row swaps to the determinant. Let E_{rs} be the elementary matrix that swaps rows r and s in the $n\times n$ matrix $A = [a_{ij}]$. Applying E_{12} to a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we see that

$$\det(A) = ad - bc = -(ad-bc) = \det\left(\begin{bmatrix} c & d \\ a & b \end{bmatrix}\right) = \det(E_{12}A).$$

So swapping rows in a 2×2 matrix multiplies the determinant by -1. Suppose that row swapping on any $(n-1)\times(n-1)$ matrix multiplies the determinant by -1 (in other words, we are proving our statement by mathematical induction). Now suppose A is an $n\times n$ matrix and let $B = [b_{ij}] = E_{rs}A$. We first consider the case that $s = r + 1$ – that we swap adjacent rows. We consider two cases, $r > 1$ and $r = 1$. First let us suppose that $r > 1$. Let C_{ij} be the (i, j) cofactor of A and C'_{ij} the (i, j) cofactor of B. We have

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

and

$$\det(B) = b_{11}C'_{11} + b_{12}C'_{12} + \cdots + b_{1n}C'_{1n}.$$

Since $r > 1$,it follows that $a_{1j} = b_{1j}$ for every j. For each j the sub-matrix B_{1j} obtained from B by deleting the ith row and jth column is the same matrix as obtained from A_{ij} by swapping rows r and s. So by our induction hypothesis, we have $C'_{1j} = -C_{1j}$ for each j. Then

$$\begin{aligned}\det(B) &= b_{11}C'_{11} + b_{12}C'_{12} + \cdots + b_{1n}C'_{1n} \\ &= a_{11}(-C_{11}) + a_{12}(-C_{12}) + \cdots + a_{1n}(-C_{1n}) \\ &= -(a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}) \\ &= -\det(A).\end{aligned}$$

Now we consider the case where $r = 1$, where B is the matrix obtained from A by swapping the first and second rows. Here we will use the fact that $\det(A) = \det(A^{\mathsf{T}})$ which allows us to calculate $\det(A)$ and $\det(B)$ with the cofactor expansions down the first column. In this case we have

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1}$$

and

$$\begin{aligned}\det(B) &= b_{11}C'_{11} + b_{21}C'_{21} + \cdots + b_{n1}C'_{n1} \\ &= a_{21}C'_{11} + a_{11}C'_{21} + a_{31}C'_{31} + \cdots + a_{n1}C'_{n1}.\end{aligned}$$

For each $i \geq 3$, the sub-matrix B_{i1} is just A_{i1} with rows 1 and 2 swapped. So we have $C'_{i1} = -C_{i1}$ by our induction hypothesis. Since we swapped rows 1 and 2, we have $B_{21} = A_{11}$ and $B_{11} = A_{21}$. Thus,

$$b_{11}C'_{11} = (-1)^{1+1}b_{11}\det(A_{21}) = a_{21}\det(A_{21}) = -a_{21}C_{21}$$

and

$$b_{21}C'_{21} = (-1)^{2+1}a_{11}\det(A_{11}) = -a_{11}\det(A_{11}) = -a_{11}C_{11}.$$

Putting this all together gives us

$$\begin{aligned}\det(B) &= b_{11}C'_{11} + b_{21}C'_{21} + \cdots + b_{n1}C'_{n1} \\ &= -a_{21}C_{21} - a_{11}C_{11} + a_{31}(-C_{31}) + \cdots + a_{n1}(-C_{n1}) \\ &= -\left(a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1}\right) \\ &= -\det(A).\end{aligned}$$

So we have shown that if B is obtained from A by interchanging two adjacent rows, then $\det(B) = -\det(A)$. Now we consider the general case. Suppose B is obtained from A by interchanging rows r and s, with $r < s$. We can perform this single row interchange through a sequence of adjacent row interchanges. First we swap rows r and $r + 1$, then rows $r + 1$ and $r + 2$, and continue until we swap rows $s-1$ and s. This places the original row r into the row s position, and the process involved $s-r$ adjacent row interchanges. Each of these interchanges multiplies the determinant by a factor of -1. At the end of this sequence of row swaps, the original row s is now row $s-1$. So it will take one fewer adjacent row interchanges to move this row to be row r. This sequence of $(s-r)+(s-r-1)=2(s-r-1)-1$ row interchanges produces the matrix B. Thus,

$$\det(B) = (-1)^{2(s-r)-1}\det(A) = -\det(A),$$

and interchanging any two rows multiplies the determinant by -1.

Cofactor Expansions

We have stated that the determinant of a matrix can be calculated by using a cofactor expansion along any row or column. We use the result that swapping rows introduces a factor of -1 in the determinant to verify that result in this section. Note that in proving that $\det(A^{\mathsf{T}}) = \det(A)$, we have already shown that the cofactor expansion along the first column is the same as the cofactor expansion along the first row. If we can prove that the cofactor expansion along any row is the same, then the fact that $\det(A^{\mathsf{T}}) = \det(A)$ will imply that the cofactor expansion along any column is the same as well.

Now we demonstrate that the cofactor expansions along the first row and the ith row are the same. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The cofactor expansion of A along the first row is

$$a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

and the cofactor expansion along the ith row is

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

Let B be the matrix obtained by swapping row i with previous rows so that row i becomes the first row and the order of the remaining rows is preserved.

$$B = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{i-11} & a_{i-12} & \cdots & a_{i-1j} & \cdots & a_{i-1n} \\ a_{i+11} & a_{i+12} & \cdots & a_{i+1j} & \cdots & a_{i+1n} \\ \vdots & \ddots & \vdots & \ddots & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

Then

$$\det(B) = (-1)^{i-1}\det(A).$$

So, letting C'_{ij} be the (i,j) cofactor of B we have

$$\det(A) = (-1)^{i-1}\det(B) = (-1)^{i-1}\left(a_{i1}C'_{11} + a_{i2}C'_{12} + \cdots + a_{in}C'_{1n}\right).$$

Notice that for each j we have $B_{1j} = A_{ij}$. So

$$\begin{aligned}
\det(A) &= (-1)^{i-1}\left(a_{i1}C'_{11} + a_{i2}C'_{12} + \cdots + a_{in}C'_{1n}\right) \\
&= (-1)^{i-1}\Big(a_{i1}(-1)^{(}1+1)\det(B_{11}) + a_{i2}(-1)^{1+2}\det(B_{12}) \\
&\qquad + \cdots + a_{in}(-1)^{1+n}\det(B_{1n})\Big) \\
&= (-1)^{i-1}\Big(a_{i1}(-1)^{(}1+1)\det(A_{i1}) + a_{i2}(-1)^{1+2}\det(A_{i2}) \\
&\qquad + \cdots + a_{in}(-1)^{1+n}\det(A_{in})\Big) \\
&= a_{i1}(-1)^{(}i+1)\det(A_{i1}) + a_{i2}(-1)^{i+2}\det(A_{i2}) \\
&\qquad + \cdots + a_{in}(-1)^{i+n}\det(A_{in}) \\
&= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.
\end{aligned}$$

The LU Factorization of a Matrix

There are many instances where we have a number of systems to solve of the form $A\mathbf{x} = \mathbf{b}$, all with the same coefficient matrix. The system may evolve over time so that we do not know the constant vectors $\mathbf{b}$ in the system all at once, but only determine them as time progresses. Each time we obtain a new vector $\mathbf{b}$, we have to apply the same row operations to reduce the coefficient matrix to solve the new system. This is time repetitive and time consuming. Instead, we can keep track of the row operations in one row reduction and save ourselves a significant amount of time. One way of doing this is the LU-factorization (or decomposition).

To illustrate, suppose we can write the matrix A as a product $A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbf{b} = [3\ 1\ 1\ 3]^{\mathsf{T}}$ and $\mathbf{x} = [x_1\ x_2\ x_3\ x_4]^{\mathsf{T}}$, and consider the linear system $A\mathbf{x} = \mathbf{b}$. If $A\mathbf{x} = \mathbf{b}$, then $LU\mathbf{x} = \mathbf{b}$. We can solve this system without applying row operations as follows. Let $U\mathbf{x} = \mathbf{z}$, where $\mathbf{z} = [z_1\ z_2\ z_3\ z_4]^{\mathsf{T}}$. We can solve $L\mathbf{z} = \mathbf{b}$ by using forward substitution.

The equation $L\mathbf{z} = \mathbf{b}$ is equivalent to the system

$$\begin{aligned}
z_1 \qquad\qquad\quad &= 3 \\
-z_1 + z_2 \qquad\quad &= 1 \\
z_2 + z_3 \quad &= 1 \\
z_4 &= 3.
\end{aligned}$$

The first equation shows that $z_1 = 3$. Substituting into the second equation gives us $z_2 = 4$. Using this information in the third equation yields $z_3 = -3$, and then the fourth equation shows that $z_4 = 0$. To return to the original system, since $U\mathbf{x} = \mathbf{z}$, we now solve this system to find the solution vector $\mathbf{x}$. In this case, since U is upper triangular, we use back substitution. The equation $U\mathbf{x} = \mathbf{z}$ is equivalent to the system

$$\begin{aligned} x_1 \qquad + \ x_3 \qquad\quad &= \ 3 \\ x_2 + 3x_3 - 2x_4 &= \ 4 \\ 3x_4 &= -3. \end{aligned}$$

Note that the third column of U is not a pivot column, so x_3 is a free variable. The last equation shows that $x_4 = -1$. Substituting into the second equation and solving for x_2 yields $x_2 = 2 - 3x_3$. The first equation then gives us $x_1 = 3 - x_3$. So the general solution

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \end{bmatrix} x_3$$

to $A\mathbf{x} = \mathbf{b}$ can be found through L and U via forward and backward substitution. If we can find a factorization of a matrix A into a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is called an *LU-factorization* or *LU-decomposition*.

We can use elementary matrices to obtain a factorization of certain matrices into products of lower triangular (the"L" in LU) and upper triangular (the "U" in LU) matrices. We illustrate with an example. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & -2 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Our goal is to find an upper triangular matrix U and a lower triangular matrix L so that $A = LU$. We begin by row reducing A to an upper triangular matrix, keeping track of the elementary matrices used to perform the row operations. We start by replacing the entries below the $(1,1)$ entry in A with zeros. The elementary matrices that perform these operations are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

and

$$E_2E_1A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We next zero out the entries below the $(2,2)$ entry as

$$E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The product $E_3E_2E_1A$ is an upper triangular matrix U. So we have

$$E_3E_2E_1A = U$$

and

$$A = E_1^{-1}E_2^{-1}E_3^{-1}U,$$

where

$$E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

is a lower triangular matrix L. So we have decomposed the matrix A into a product $A = LU$, where L is lower triangular and U is upper triangular. Since every matrix is row equivalent to a matrix in row echelon form, we can always find an upper triangular matrix U in this way. However, we may not always obtain a corresponding lower triangular matrix, as the next example illustrates.

Suppose we change the problem slightly and consider the matrix

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & -2 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Using the same elementary matrices E_1, E_2, and E_3 as earlier, we have

$$E_3E_2E_1B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

To reduce B to row-echelon form now requires a row interchange. Letting

$$E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

brings us to

$$E_4E_3E_2E_1B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

So in this case we have $U = E_4E_3E_2E_1B$, but

$$E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

is not lower triangular. The difference in this latter example is that we needed a row swap to obtain the upper triangular form.

Examples

What follows are worked examples that use the concepts from this section.

Example 21.8.

(a) If A, B are $n \times n$ matrices with $\det(A) = 3$ and $\det(B) = 2$, evaluate the following determinant values. Briefly justify.

i. $\det(A^{-1})$

ii. $\det(ABA^{\mathsf{T}})$

iii. $\det(A^3(BA)^{-1}(AB)^2)$

(b) If the determinant of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is m, find the determinant of each of the following matrices.

i. $\begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$

ii. $\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix}$

iii. $\begin{bmatrix} a & b & c \\ g-2d & h-2e & i-2f \\ a+d & b+e & c+f \end{bmatrix}$

Example Solution.

(a) Assume that $\det(A) = 3$ and $\det(B) = 2$.

i. Since $\det(A) \neq 0$, we know that A is invertible. Since $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$, it follows that $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{3}$.

ii. We know that $\det(A^\mathsf{T}) = \det(A)$, so

$$\begin{aligned}\det(ABA^\mathsf{T}) &= \det(A)\det(B)\det(A^\mathsf{T})\\ &= \det(A)\det(B)\det(A)\\ &= (3)(2)(3)\\ &= 18.\end{aligned}$$

iii. Using properties of determinants gives us

$$\begin{aligned}\det(A^3(BA)^{-1}(AB)^2) &= \det(A^3)\det((BA)^{-1})\det((AB)^2)\\ &= (\det(A))^3\left(\frac{1}{\det(AB)}\right)(\det(AB))^2\\ &= 27\left(\frac{1}{\det(A)\det(B)}\right)(\det(A)\det(B))^2\\ &= \frac{(27)(6^2)}{6}\\ &= 162.\end{aligned}$$

(b) Assume that $\det\left(\begin{bmatrix} a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\right) = m$.

i. Multiplying a row by a scalar multiples the determinant by that scalar, so

$$\det\left(\begin{bmatrix} a & b & c\\ 2d & 2e & 2f\\ g & h & i\end{bmatrix}\right) = 2m.$$

ii. Interchanging two rows multiples the determinant by -1. It takes two row swaps in the original matrix to obtain this one, so

$$\det\left(\begin{bmatrix} d & e & f\\ g & h & i\\ a & b & c\end{bmatrix}\right) = (-1)^2m = m.$$

iii. Adding a multiple of a row to another does not change the determinant of the matrix. Since there is a row swap needed to get this matrix from the original we have

$$\det\left(\begin{bmatrix} a & b & c\\ g-2d & h-2e & i-2f\\ a+d & b+e & c+f\end{bmatrix}\right) = -m.$$

Example 21.9. Let $A = \begin{bmatrix} 2 & 8 & 0\\ 2 & 2 & -3\\ 1 & 2 & 7\end{bmatrix}$.

(a) Find an LU factorization for A.

(b) Use the LU factorization with forward substitution and back substitution to solve the system $A\mathbf{x} = [18\ 3\ 12]^{\mathsf{T}}$.

Example Solution.

(a) We row reduce A to an upper triangular matrix by applying elementary matrices. First notice that if $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then

$$E_1A = \begin{bmatrix} 2 & 8 & 0 \\ 0 & -6 & -3 \\ 1 & 2 & 7 \end{bmatrix}.$$

Letting $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}$ gives us

$$E_2E_1A = \begin{bmatrix} 2 & 8 & 0 \\ 0 & -6 & -3 \\ 0 & -2 & 7 \end{bmatrix}.$$

Finally, when $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}$ we have

$$U = E_3E_2E_1A = \begin{bmatrix} 2 & 8 & 0 \\ 0 & -6 & -3 \\ 0 & 0 & 8 \end{bmatrix}.$$

This gives us $E_3E_2E_1A = U$, so we can take

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix}.$$

(b) To solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = [18\ 3\ 12]^{\mathsf{T}}$, we use the LU factorization of A and solve $LU\mathbf{x} = \mathbf{b}$. Let $\mathbf{x} = [x_1\ x_2\ x_3]^{\mathsf{T}}$ and let $\mathbf{z} = [z_1\ z_2\ z_3]^{\mathsf{T}}$ with $U\mathbf{x} = \mathbf{z}$ so that $L\mathbf{z} = L(U\mathbf{x}) = A\mathbf{x} = \mathbf{b}$. First we solve $L\mathbf{z} = [18\ 3\ 12]^{\mathsf{T}}$ to find $\mathbf{z}$ using forward substitution. The first row of L shows that $z_1 = 18$ and the second row that $z_1 + z_2 = 3$. So $z_2 = -15$. The third row of L gives us $\frac{1}{2}z_1 + \frac{1}{3}z_2 + z_3 = 12$, so $z_3 = 12 - 9 + 5 = 8$. Now to find $\mathbf{x}$ we solve $U\mathbf{x} = \mathbf{z}$ using back substitution. The third row of U tells us that $8x_3 = 8$ or that $x_3 = 1$. The second row of U shows that $-6x_2 - 3x_3 = -15$ or $x_2 = 2$. Finally, the first row of U gives us $2x_1 + 8x_2 = 18$, or $x_1 = 1$. So the solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = [1\ 2\ 1]^{\mathsf{T}}$.

Summary

- The elementary row operations have the following effects on the determinant:
  (a) If we multiply a row of a matrix by a constant k, then the determinant is multiplied by k.
  (b) If we swap two rows of a matrix, then the determinant changes sign.
  (c) If we add a multiple of a row of a matrix to another, the determinant does not change.
- Each of the elementary row operations can be achieved by multiplication by elementary matrices. To obtain the elementary matrix corresponding to an elementary row operation, we perform the operation on the identity matrix.
- Let A be an $n \times n$ invertible matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries

  $$x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}$$

  where $A_i(\mathbf{b})$ represents the matrix formed by replacing ith column of A with $\mathbf{b}$.
- Let A be an invertible $n \times n$ matrix. Then

  $$A^{-1} = \frac{1}{\det(A)} \text{adj } A$$

  where the adj A matrix, the *adjugate of* A, is defined as the matrix whose ij-th entry is C_{ji}, the ji-th cofactor of A.
- For a 2×2 matrix A, the area of the image of the unit square under the transformation $T(\mathbf{x}) = A\mathbf{x}$ is equal to $|\det(A)|$, which is also equal to the area of the parallelogram defined by the columns of A.
- For a 3×3 matrix A, the volume of the image of the unit cube under the transformation $T(\mathbf{x}) = A\mathbf{x}$ is equal to $|\det(A)|$, which is also equal to the volume of the parallelepiped defined by the columns of A.

Exercises

(1) Find a formula for $\det(rA)$ in terms of r and $\det(A)$, where A is an $n \times n$ matrix and r is a scalar. Explain why your formula is valid.

(2) Find $\det(A)$ by hand using elementary row operations where

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -1 & -2 & 3 & -1 \\ -2 & -1 & 2 & -3 \\ 1 & 8 & -3 & 8 \end{bmatrix}.$$

(3) Consider the matrix $A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}$. We will find $\det(A)$ using elementary row operations. (This matrix arises in graph theory, and its determinant gives the number of spanning trees in the complete graph with 5 vertices. This number is also equal to the number of labeled trees with 5 vertices.)

(a) Add rows R_2, R_3 and R_4 to the first row in that order.

(b) Then add the new R_1 to rows R_2, R_3 and R_4 to get a triangular matrix B.

(c) Find the determinant of B. Then use $\det(B)$ and properties of how elementary row operations affect determinants to find $\det(A)$.

(d) Generalize your work to find the determinant of the $n \times n$ matrix

$$A = \begin{bmatrix} n & -1 & -1 & \cdots & -1 & -1 \\ -1 & n & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & n \end{bmatrix}.$$

(4) For which matrices A, if any, is $\det(A) = -\det(-A)$? Justify your answer.

(5) Find the inverse A^{-1} of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ using the adjugate matrix.

(6) For an invertible $n \times n$ matrix A, what is the relationship between $\det(A)$ and $\det(\text{adj } A)$? Justify your result.

(7) Let $A = \begin{bmatrix} a & b & 1 \\ c & d & 2 \\ e & f & 3 \end{bmatrix}$, and assume that $\det(A) = 2$. Determine the determinants of each of the following.

(a) $B = \begin{bmatrix} a & b & 1 \\ 3c & 3d & 6 \\ e+a & f+b & 4 \end{bmatrix}$

(b) $C = \begin{bmatrix} 2e & 2f & 6 \\ 2c-2e & 2d-2f & -2 \\ 2a & 2b & 2 \end{bmatrix}$

(8) Find the area of the parallelogram with one vertex at the origin and adjacent vertices at $(1, 2)$ and (a, b). For which (a, b) is the area 0? When does this happen geometrically?

(9) Find the volume of the parallelepiped with one vertex at the origin and three adjacent vertices at $(3, 2, 0)$, $(1, 1, 1)$ and $(1, 3, c)$ where c is unknown. For which c, is the volume 0? When does this happen geometrically?

(10) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** If two rows are equal in A, then $\det(A) = 0$.

(b) **True/False** If A is a square matrix and R is a row echelon form of A, then $\det(A) = \det(R)$.

(c) **True/False** If a matrix A is invertible, then 0 is not an eigenvalue of A.

(d) **True/False** If A is a 2×2 matrix for which the image of the unit square under the transformation $T(\mathbf{x}) = A\mathbf{x}$ has zero area, then A is non-invertible.

(e) **True/False** Row operations do not change the determinant of a square matrix.

(f) **True/False** If A_{ij} is the matrix obtained from a square matrix $A = [a_{ij}]$ by deleting the ith row and jth column of A, then

$$\begin{aligned} a_{i1}(-1)^{i+1}\det(A_{i1}) + a_{i2}(-1)^{i+2}\det(A_{i2}) + \cdots \\ + a_{in}(-1)^{i+n}\det(A_{in}) \\ = a_{1j}(-1)^{j+1}\det(A_{1j}) + a_{2i}(-1)^{j+2}\det(A_{2i}) + \cdots \\ + a_{nj}(-1)^{j+n}\det(A_{nj}) \end{aligned}$$

for any i and j between 1 and n.

(g) **True/False** If A is an invertible matrix, then $\det\left(A^{\mathsf{T}}A\right) > 0$.