Chapter 4

Trigonometric Identities and Equations

Trigonometric identities describe equalities between related trigonometric expressions while trigonometric equations ask us to determine the specific values of the variables that make two expressions equal. Identities are tools that can be used to simplify complicated trigonometric expressions or solve trigonometric equations. In this chapter we will prove trigonometric identities and derive the double and half angle identities and sum and difference identities. We also develop methods for solving trigonometric equations, and learn how to use trigonometric identities to solve trigonometric equations.

4.1 Trigonometric Identities

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is an identity?
- How do we verify an identity?

Consider the trigonometric equation $\sin(2x) = \cos(x)$. Based on our current knowledge, an equation like this can be difficult to solve exactly because the periods of the functions involved are different. What will allow us to solve this equation relatively easily is a trigonometric identity, and we will explicitly solve this equation in a subsequent section. This section is an introduction to trigonometric identities.

As we discussed in Section 2.6, a mathematical **equation** like $x^2 = 1$ is a relation between two expressions that may be true for some values of the variable. To solve an equation means to find all of the values for the variables that make the two expressions equal to each other. An **identity**, is an equation that is true for all allowable values of the variable. For example, from previous algebra courses, we have seen that

$$x^2 - 1 = (x+1)(x-1),$$

for all real numbers x. This is an algebraic identity since it is true for all real number values of x. An example of a trigonometric identity is

$$\cos^2(x) + \sin^2(x) = 1$$

since this is true for all real number values of x.

So while we solve equations to determine when the equality is valid, there is no reason to solve an identity since the equality in an identity is always valid. Every identity is an equation, but not every equation is an identity. To know that an equation is an identity it is necessary to provide a convincing argument that the two expressions in the equation are always equal to each other. Such a convincing argument is called a *proof* and we use proofs to verify trigonometric identities.

Definition. An **identity** is an equation that is true for all allowable values of the variables involved.

Beginning Activity

1. Use a graphing utility to draw the graph of $y = \cos\left(x - \dfrac{\pi}{2}\right)$ and $y = \sin\left(x + \dfrac{\pi}{2}\right)$ over the interval $[-2\pi, 2\pi]$ on the same set of axes. Are the two expressions $\cos\left(x - \dfrac{\pi}{2}\right)$ and $\sin\left(x + \dfrac{\pi}{2}\right)$ the same – that is, do they have the same value for every input x? If so, explain how the graphs indicate that the expressions are the same. If not, find at least one value of x at which $\cos\left(x - \dfrac{\pi}{2}\right)$ and $\sin\left(x + \dfrac{\pi}{2}\right)$ have different values.

2. Use a graphing utility to draw the graph of $y = \cos\left(x - \dfrac{\pi}{2}\right)$ and $y = \sin(x)$ over the interval $[-2\pi, 2\pi]$ on the same set of axes. Are the two expressions $\cos\left(x - \dfrac{\pi}{2}\right)$ and $\sin(x)$ the same – that is, do they have the same value for every input x? If so, explain how the graphs indicate that the expressions are the same. If not, find at least one value of x at which $\cos\left(x - \dfrac{\pi}{2}\right)$ and $\sin(x)$ have different values.

Some Known Trigonometric Identities

We have already established some important trigonometric identities. We can use the following identities to help establish new identities.

The Pythagorean Identity

This identity is fundamental to the development of trigonometry. See page 18 in Section 1.2.

$$\text{For all real numbers } t,\ \cos^2(t) + \sin^2(t) = 1.$$

Identities from Definitions

The definitions of the tangent, cotangent, secant, and cosecant functions were introduced in Section 1.6. The following are valid for all values of t for which the right side of each equation is defined.

$$\tan(t) = \frac{\sin(t)}{\cos(t)} \qquad \cot(t) = \frac{\cos(t)}{\sin(t)}$$

$$\sec(t) = \frac{1}{\cos(t)} \qquad \csc(t) = \frac{1}{\sin(t)}$$

Negative Identities

The negative were introduced in Chapter 2 when the symmetry of the graphs were discussed. (See page 82 and Exercise (2) on page 139.)

$$\cos(-t) = \cos(t) \qquad \sin(-t) = -\sin(t) \qquad \tan(-t) = -\tan(t).$$

The negative identities for cosine and sine are valid for all real numbers t, and the negative identity for tangent is valid for all real numbers t for which $\tan(t)$ is defined.

Verifying Identities

Given two expressions, say $\tan^2(x) + 1$ and $\sec^2(x)$, we would like to know if they are equal (that is, have the same values for every allowable input) or not. We can draw the graphs of $y = \tan^2(x) + 1$ and $y = \sec^2(x)$ and see if the graphs look the same or different. Even if the graphs look the same, as they do with $y = \tan^2(x)+1$ and $y = \sec^2(x)$, that is only an indication that the two expressions are equal for *every* allowable input. In order to verify that the expressions are in fact always equal, we need to provide a convincing argument that works for all possible input. To do so we use facts that we know (existing identities) to show that two trigonometric expressions are always equal. As an example, we will verify that the equation

$$\tan^2(x) + 1 = \sec^2(x) \tag{1}$$

is an identity.

A proper format for this kind of argument is to choose one side of the equation and apply existing identities that we already know to transform the chosen side into the remaining side. It usually makes life easier to begin with the more complicated looking side (if there is one). In our example of equation (1) we might begin with the expression $\tan^2(x) + 1$.

Example 4.1 (Verifying a Trigonometric Identity)
To verify that equation (1) is an identity, we work with the expression $\tan^2(x) + 1$. It can often be a good idea to write all of the trigonometric functions in terms of the cosine and sine to start. In this case, we know that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, so we could begin by making this substitution to obtain the identity

$$\tan^2(x) + 1 = \left(\frac{\sin(x)}{\cos(x)}\right)^2 + 1. \tag{2}$$

Note that this is an identity and so is valid for all allowable values of the variable. Next we can apply the square to both the numerator and denominator of the right hand side of our identity (2).

$$\left(\frac{\sin(x)}{\cos(x)}\right)^2 + 1 = \frac{\sin^2(x)}{\cos^2(x)} + 1. \tag{3}$$

Next we can perform some algebra to combine the two fractions on the right hand side of the identity (3) and obtain the new identity

$$\frac{\sin^2(x)}{\cos^2(x)} + 1 = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)}. \tag{4}$$

Now we can recognize the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$, which makes the right side of identity (4)

$$\frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}. \tag{5}$$

Recall that our goal is to verify identity (1), so we need to transform the expression into $\sec^2(x)$. Recall that $\sec(x) = \dfrac{1}{\cos(x)}$, and so the right side of identity (5) leads to the new identity

$$\frac{1}{\cos^2(x)} = \sec^2(x),$$

which verifies the identity.

An argument like the one we just gave that shows that an equation is an identity is called a *proof*. We usually leave out most of the explanatory steps (the steps should be evident from the equations) and write a proof in one long string of identities as

$$\begin{aligned}
\tan^2(x) + 1 &= \left(\frac{\sin(x)}{\cos(x)}\right)^2 + 1 \\
&= \frac{\sin^2(x)}{\cos^2(x)} + 1 \\
&= \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x).
\end{aligned}$$

To prove an identity is to show that the expressions on each side of the equation are the same for every allowable input. We illustrated this process with the equation $\tan^2(x) + 1 = \sec^2(x)$. To show that an equation isn't an identity it is enough to demonstrate that the two sides of the equation have different values at one input.

Example 4.2 (Showing that an Equation is not an Identity)

Consider the equation with the equation $\cos\left(x - \frac{\pi}{2}\right) = \sin\left(x + \frac{\pi}{2}\right)$ that we encountered in our Beginning Activity. Although you can check that $\cos\left(x - \frac{\pi}{2}\right)$ and $\sin\left(x + \frac{\pi}{2}\right)$ are equal at some values, $\frac{\pi}{4}$ for example, they are not equal at all values – $\cos\left(0 - \frac{\pi}{2}\right) = 0$ but $\sin\left(0 + \frac{\pi}{2}\right) = 1$. Since an identity must provide an equality for *all* allowable values of the variable, if the two expressions differ at one input, then the equation is not an identity. So the equation $\cos\left(x - \frac{\pi}{2}\right) = \sin\left(x + \frac{\pi}{2}\right)$ is not an identity.

Example 4.2 illustrates an important point. to show that an equation is not an identity, it is enough to find one input at which the two sides of the equation are not equal. We summarize our work with identities as follows.

- To prove that an equation is an identity, we need to apply known identities to show that one side of the equation can be transformed into the other.
- To prove that an equation is not an identity, we need to find one input at which the two sides of the equation have different values.

Important Note: When proving an identity it might be tempting to start working with the equation itself and manipulate both sides until you arrive at something you know to be true. DO NOT DO THIS! By working with both sides of the equation, we are making the assumption that the equation is an identity – but this assumes the very thing we need to show. So the proper format for a proof of a trigonometric identity is to choose one side of the equation and apply existing identities that we already know to transform the chosen side into the remaining side. It usually makes life easier to begin with the more complicated looking side (if there is one).

Example 4.3 (Verifying an Identity)
Consider the equation

$$2\cos^2(x) - 1 = \cos^2(x) - \sin^2(x).$$

Graphs of both sides appear to indicate that this equation is an identity. To prove the identity we start with the left hand side:

$$\begin{aligned} 2\cos^2(x) - 1 &= \cos^2(x) + \cos^2(x) - 1 \\ &= \cos^2(x) + (1 - \sin^2(x)) - 1 \\ &= \cos^2(x) - \sin^2(x). \end{aligned}$$

Notice that in our proof we rewrote the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$ as $\cos^2(x) = 1 - \sin^2(x)$. Any proper rearrangement of an identity is also an identity, so we can manipulate known identities to use in our proofs as well.

To reiterate, the proper format for a proof of a trigonometric identity is to choose one side of the equation and apply existing identities that we already know to transform the chosen side into the remaining side. There are no hard and fast methods for proving identities – it is a bit of an art. You must practice to become good at it.

Progress Check 4.4 (Verifying Identities)
For each of the following use a graphing utility to graph both sides of the equation. If the graphs indicate that the equation is not an identity, find one value of x at which the two sides of the equation have different values. If the graphs indicate that the equation is an identity, verify the identity.

1. $\dfrac{\sec^2(x) - 1}{\sec^2(x)} = \sin^2(x)$
2. $\cos(x)\sin(x) = 2\sin(x)$

Summary of Section 4.1

In this section, we studied the following important concepts and ideas:
An **identity** is an equation that is true for all allowable values of the variables involved.

- To prove that an equation is an identity, we need to apply known identities to show that one side of the equation can be transformed into the other.
- To prove that an equation is not an identity, we need to find one input at which the two sides of the equation have different values.

Exercises for Section 4.1

1. Use a graphing utility to graph each side of the given equation. If the equation appears to be an identity, prove the identity. If the equation appears to not be an identity, demonstrate one input at which the two sides of the equation have different values. Remember that when proving an identity, work to transform one side of the equation into the other using known identities. Some general guidelines are

   I. Begin with the more complicated side.

II. It is often helpful to use the definitions to rewrite all trigonometric functions in terms of the cosine and sine.

III. When appropriate, factor or combine terms. For example, $\sin^2(x) + \sin(x)$ can be written as $\sin(x)(\sin(x) + 1)$ and $\frac{1}{\sin(x)} + \frac{1}{\cos(x)}$ can be written as the single fraction $\frac{\cos(x)+\sin(x)}{\sin(x)\cos(x)}$ with a common denominator.

IV. As you transform one side of the equation, keep the other side of the equation in mind and use identities that involve terms that are on the other side. For example, to verify that $\tan^2(x) + 1 = \frac{1}{\cos^2(x)}$, start with $\tan^2(x) + 1$ and make use identities that relate $\tan(x)$ to $\cos(x)$ as closely as possible.

⋆ **(a)** $\cos(x)\tan(x) = \sin(x)$

⋆ **(b)** $\dfrac{\cot(s)}{\csc(s)} = \cos(s)$

(c) $\dfrac{\tan(s)}{\sec(s)} = \sin(s)$

(d) $\cot^2(x) + 1 = \csc^2(x)$

⋆ **(e)** $\sec^2(x) + \csc^2(x) = 1$

(f) $\cot(t) + 1 = \csc(t)(\cos(t) + \sin(t))$

(g) $\tan^2(\theta)(1 + \cot^2(\theta)) = \dfrac{1}{1 - \sin^2(\theta)}$

(h) $\dfrac{1 - \sin^2(\beta)}{\cos(\beta)} = \sin(\beta)$

(i) $\dfrac{1 - \sin^2(\beta)}{\cos(\beta)} = \cos(\beta)$

(j) $\sin^2(x) + \tan^2(x) + \cos^2(x) = \sec^2(x)$.

2. A student claims that $\cos(\theta) + \sin(\theta) = 1$ is an identity because $\cos(0) + \sin(0) = 1 + 0 = 0$. How would you respond to this student?

3. If a trigonometric equation has one solution, then the periodicity of the trigonometric functions implies that the equation will have infinitely many solutions. Suppose we have a trigonometric equation for which both sides of the equation are equal at infinitely many different inputs. Must the equation be an identity? Explain your reasoning.

4.2 Trigonometric Equations

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is a trigonometric equation?
- What does it mean to solve a trigonometric equation?
- How is a trigonometric equation different from a trigonometric identity?

We have already learned how to solve certain types of trigonometric equations. In Section 2.6 we learned how to use inverse trigonometric functions to solve trigonometric equations.

Beginning Activity

Refer back to the method from Section 2.6 to find all solutions to the equation $\sin(x) = 0.4$.

Trigonometric Equations

When a light ray from a point P reflects off a surface at a point R to illuminate a point Q as shown at left in Figure 4.1, the light makes two angles α and β with a perpendicular to the surface. The angle α is called the *angle of incidence* and the angle β is called the *angle of reflection*. The Law of Reflection states that when light is reflected off a surface, the angle of incidence equals the angle of reflection. What happens if the light travels through one medium (say air) from a point P, deflects into another medium (say water) to travel to a point Q? Think about what happens if you look at an object in a glass of water. See Figure 4.1 at right. Again the light makes two angles α and β with a perpendicular to the surface. The angle α is called the *angle of incidence* and the angle β is called the *angle of refraction*. If light travels from air into water, the Law of Refraction says that

$$\frac{\sin(\alpha)}{\sin(\beta)} = \frac{c_a}{c_w} \tag{6}$$

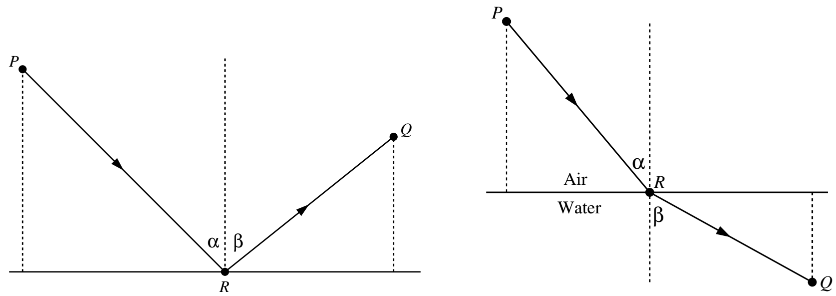


Figure 4.1: Reflection and refraction.

where c_a is the speed of light in air and c_w is the speed of light in water. The ratio $\dfrac{c_a}{c_w}$ of the speed of light in air to the speed of light in water can be calculated by experiment. In practice, the speed of light in each medium is compared to the speed of light in a vacuum. The ratio of the speed of light in a vacuum to the speed of light in water is around 1.33. This is called the index of refraction for water. The index of refraction for air is very close to 1, so the ratio $\frac{c_a}{c_w}$ is close to 1.33. We can usually measure the angle of incidence, so the Law of Refraction can tells us what the angle of refraction is by solving equation (6).

Trigonometric equations arise in a variety of situations, like in the Law of Refraction, and in a variety of disciplines including physics, chemistry, and engineering. As we develop trigonometric identities in this chapter, we will also use them to solve trigonometric equations.

Recall that Equation (6) is a *conditional* equation because it is not true for all allowable values of the variable. To *solve a conditional equation* means to find all of the values for the variables that make the two expressions on either side of the equation equal to each other.

Equations of Linear Type

Section 2.6 showed us how to solve trigonometric equations that are reducible to linear equations. We review that idea in our first example.

Example 4.5 (Solving an Equation of Linear Type)
Consider the equation

$$2\sin(x) = 1.$$

We want to find all values of x that satisfy this equation. Notice that this equation looks a lot like the linear equation $2y = 1$, with $\sin(x)$ in place of y. So this trigonometric equation is of linear type and we say that it is linear in $\sin(x)$. We know how to solve $2y = 1$, we simply divide both sides of the equation by 2 to obtain $y = \frac{1}{2}$. We can apply the same algebraic operation to $2\sin(x) = 1$ to obtain the equation

$$\sin(x) = \frac{1}{2}.$$

Now we could proceed in a couple of ways. From previous work we know that $\sin(x) = \frac{1}{2}$ when $x = \frac{\pi}{6}$. Alternatively, we could apply the inverse sine to both sides of our equation to see that one solution is $x = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Recall, however, this is not the only solution. The first task is to find all of the solutions in one complete period of the sine function. We can use the interval with $0 \leq x \leq 2\pi$ but we often use the interval $-\pi \leq x \leq \pi$. In this case, it makes no difference since the sine function is positive in the second quadrant. Using $\frac{\pi}{6}$ as a reference angle, we see that $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ is another solution of this equation. (Use a calculator to check this.)

We now use the fact that the sine function is period with a period of 2π to write formulas that can be used to generate all solutions of the equation $2\sin(x) = 1$. So the angles in the first quadrant are $\frac{\pi}{6} + k(2\pi)$ and the angles in the second quadrant are $\frac{5\pi}{6} + k(2\pi)$, where k is an integer. So for the solutions of the equation $2\sin(x) = 1$, we write

$$x = \frac{\pi}{6} + k(2\pi) \quad \text{or} \quad x = \frac{5\pi}{6} + k(2\pi),$$

where k is an integer.

We can always check our solutions by graphing both sides of the equation to see where the two expressions intersect. Figure 4.2 shows that graphs of $y = 2\sin(x)$ and $y = 1$ on the interval $[-2\pi, 3\pi]$. We can see that the points of intersection of these two curves occur at exactly the solutions we found for this equation.

Progress Check 4.6 (Solving an Equation of Linear Type)
Find the exact values of all solutions to the equation $4\cos(x) = 2\sqrt{2}$. Do this by first finding all solutions in one complete period of the cosine function and

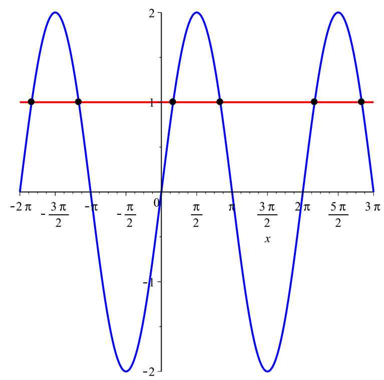


Figure 4.2: The graphs of $y = 2\sin(x)$ and $y = 1$

then using the periodic property to write formulas that can be used to generate all solutions of the equation. Draw appropriate graphs to illustrate your solutions.

Solving an Equation Using an Inverse Function

When we solved the equation $2\sin(x) = 1$, we used the fact that we know that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. When we cannot use one of the common arcs, we use the more general method of using an inverse trigonometric function. This is what we did in Section 2.6. See "A Strategy for Solving a Trigonometric Function" on page 158. We will illustrate this strategy with the equation $\cos(x) = 0.7$. We start by applying the inverse cosine function to both sides of this equation to obtain

$$\begin{aligned} \cos(x) &= 0.7 \\ \cos^{-1}(\cos(x) &= \cos^{-1}(0.7) \\ x &= \cos^{-1}(0.7) \end{aligned}$$

This gives the one solution for the equation that is in interval $[0, \pi]$. Before we use the periodic property, we need to determine the other solutions for the equation in one complete period of the cosine function. We can use the interval $[0, 2\pi]$ but it is easier to use the interval $[-\pi, \pi]$. One reason for this is the following so-called "negative arc identity" stated on page 82.

$$\cos(-x) = \cos(x) \quad \text{for every real number } x.$$

Hence, since one solution for the equation is $x = \cos^{-1}(0.7)$, another solution is $x = -\cos^{-1}(0.7)$. This means that the two solutions of the equation $x = \cos(x)$ on the interval $[-\pi, \pi]$ are

$$x = \cos^{-1}(0.7) \quad \text{and} \quad x = -\cos^{-1}(0.7).$$

Since the period iof the cosine function is 2π, we can say that any solution of the equation $\cos(x) = 0.7$ will be of the form

$$x = \cos^{-1}(0.7) + k(2\pi) \quad \text{or} \quad x = -\cos^{-1}(0.7) + k(2\pi),$$

where k is some integer.

Note: The beginning activity for this section had the equation $\sin(x) = 0.4$. The solutions for this equation are

$$x = \arcsin(0.4) + k(2\pi) \ \text{ or } \ x = (\pi - \arcsin(0.4)) + k(2\pi),$$

where k is an integer. We can write the solutions in approximate form as

$$x = 0.41152 + k(2\pi) \ \text{ or } \ x = 2.73008 + k(2\pi),$$

where k is an integer.

Progress Check 4.7 (Solving Equations of Linear Type)

1. Determine formulas that can be used to generate all solutions to the equation $5\sin(x) = 2$. Draw appropriate graphs to illustrate your solutions in one period of the sine function.
2. Approximate, to two decimal places, the angle of refraction of light passing from air to water if the angle of incidence is $40°$.

Solving Trigonometric Equations Using Identities

We can use known trigonometric identities to help us solve certain types of trigonometric equations.

Example 4.8 (Using Identities to Solve Equations)
Consider the trigonometric equation

$$\cos^2(x) - \sin^2(x) = 1. \tag{7}$$

This equation is complicated by the fact that there are two different trigonometric functions involved. In this case we use the Pythagorean Identity

$$\sin^2(x) + \cos^2(x) = 1$$

by solving for $\cos^2(x)$ to obtain

$$\cos^2(x) = 1 - \sin^2(x).$$

We can now substitute into equation (7) to get

$$(1 - \sin^2(x)) - \sin^2(x) = 1.$$

Note that everything is in terms of just the sine function and we can proceed to solve the equation from here:

$$\begin{aligned} (1 - \sin^2(x)) - \sin^2(x) &= 1 \\ 1 - 2\sin^2(x) &= 1 \\ -2\sin^2(x) &= 0 \\ \sin^2(x) &= 0 \\ \sin(x) &= 0. \end{aligned}$$

We know that $\sin(x) = 0$ when $x = \pi k$ for any integer k, so the solutions to the equation

$$\cos^2(x) - \sin^2(x) = 1$$

are

$$x = \pi k \text{ for any integer } k.$$

This is illustrated by Figure 4.3.

Progress Check 4.9 (Using Identities to Solve Equations)
Find the exact values of all solutions to the equation $\sin^2(x) = 3\cos^2(x)$. Draw appropriate graphs to illustrate your solutions.

Other Methods for Solving Trigonometric Equations

Just like we did with linear equations, we can view some trigonometric equations as quadratic in nature and use tools from algebra to solve them.

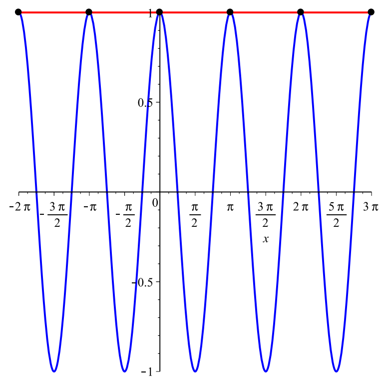


Figure 4.3: The graphs of $y = \cos^2(x) - \sin^2(x)$ and $y = 1$

Example 4.10 (Solving Trigonometric Equations of Quadratic Type)
Consider the trigonometric equation

$$\cos^2(x) - 2\cos(x) + 1 = 0.$$

This equation looks like a familiar quadratic equation $y^2 - 2y + 1 = 0$. We can solve this quadratic equation by factoring to obtain $(y - 1)^2 = 0$. So we can apply the same technique to the trigonometric equation $\cos^2(x) - 2\cos(x) + 1 = 0$. Factoring the left hand side yields

$$(\cos(x) - 1)^2 = 0.$$

The only way $(\cos(x) - 1)^2$ can be 0 is if $\cos(x) - 1$ is 0. This reduces our quadratic trigonometric equation to a linear trigonometric equation. To summarize the process so far we have

$$\begin{aligned} \cos^2(x) - 2\cos(x) + 1 &= 0 \\ (\cos(x) - 1)^2 &= 0 \\ \cos(x) - 1 &= 0 \\ \cos(x) &= 1. \end{aligned}$$

We know that $\cos(x) = 1$ when $x = 2\pi k$ for integer values of k. Therefore, the solutions to our original equation are

$$x = 2\pi k$$

where k is any integer. As a check, the graph of $y = \cos^2(x) - 2\cos(x) + 1$ is shown in Figure 4.4. The figure appears to show that the graph of $y = \cos^2(x) - 2\cos(x) + 1$ intersects the x-axis at exactly the points we found, so our solution is validated by graphical means.

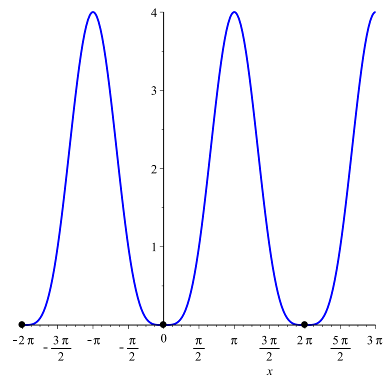


Figure 4.4: the graph of $y = \cos^2(x) - 2\cos(x) + 1$

Progress Check 4.11 (Solving Trigonometric Equations of Quadratic Type)
Find the exact values of all solutions to the equation $\sin^2(x) - 4\sin(x) = -3$. Draw appropriate graphs to illustrate your solutions.

Summary of Section 4.2

In this section, we studied the following important concepts and ideas:
A **trigonometric equation** is a conditional equation that involves trigonometric functions. If it is possible to write the equation in the form

$$\text{“some trigonometric function of } x\text{''} = \text{a number}, \qquad (1)$$

we can use the following strategy to solve the equation:

- Find all solutions of the equation within one period of the function. This is often done by using properties of the trigonometric function. Quite often, there will be two solutions within a single period.
- Use the period of the function to express formulas for all solutions by adding integer multiples of the period to each solution found in the first step. For example, if the function has a period of 2π and x_1 and x_2 are the only two

solutions in a complete period, then we would write the solutions for the equation as

$$x = x_1 + k(2\pi), \quad x = x_2 + k(2\pi), \text{ where } k \text{ is an integer.}$$

We can sometimes use trigonometric identities to help rewrite a given equation in the form of equation (1).

Exercises for Section 4.2

1. For each of the following equations, determine formulas that can be used to generate all solutions of the given equation. Use a graphing utility to graph each side of the given equation to check your solutions.

- ⋆ **(a)** $2\sin(x) - 1 = 0$
- ⋆ **(b)** $2\cos(x) + 1 = 0$
- **(c)** $2\sin(x) + \sqrt{2} = 0$
- ⋆ **(d)** $4\cos(x) - 3 = 0$
- **(e)** $3\sin^2(x) - 2\sin(x) = 0$
- **(f)** $\sin(x)\cos^2(x) = 2\sin(x)$
- **(g)** $\cos^2(x) + 4\sin(x) = 4$
- **(h)** $5\cos(x) + 4 = 2\sin^2(x)$
- **(i)** $3\tan^2(x) - 1 = 0$
- **(j)** $\tan^2(x) - \tan(x) = 6$

⋆ **2.** A student is asked to approximate all solutions in degrees (to two decimal places) to the equation $\sin(\theta) + \dfrac{1}{3} = 1$ on the interval $0° \leq \theta \leq 360°$. The student provides the answer $\theta = \sin^{-1}\left(\dfrac{2}{3}\right) \approx 41.81°$. Did the student provide the correct answer to the stated problem? Explain.

3. X-ray crystallography is an important tool in chemistry. One application of X-ray crystallography is to discover the atomic structure macromolecules. For example, the double helical structure of DNA was found using X-ray crystallography.

The basic idea behind X-ray crystallography is this: two X-ray beams with the same wavelength λ and phase are directed at an angle θ toward a crystal composed of atoms arranged in a lattice in planes separated by a distance d as illustrated in Figure 4.5.[1] The beams reflect off different atoms (labeled as P and Q in Figure 4.5) within the crystal. One X-ray beam (the lower one as

[1] The symbol λ is the Greek lowercase letter "lambda".

illustrated in Figure 4.5) must travel a longer distance than the other. When reflected, the X-rays combine but, because of the phase shift of the lower beam, the combination might have a small amplitude or a large amplitude. Bragg's Law states that the sum of the reflected rays will have maximum amplitude when the extra length the longer beam has to travel is equal to an integer multiple of the wavelength λ of the radiation. In other words,

$$n\lambda = 2d\sin(\theta),$$

for some positive integer n. Assume that $\lambda = 1.54$ angstroms and $d = 2.06$ angstroms. Approximate to two decimal places the smallest value of θ (in degrees) for which $n = 1$.

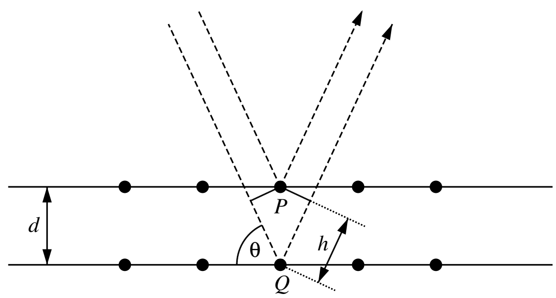


Figure 4.5: X-rays reflected from crystal atoms.

4.3 Sum and Difference Identities

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What are the Cosine Difference and Sum Identities?
- What are the Sine Difference and Sum Identities?
- What are the Tangent Difference and Sum Identities?
- What are the Cofunction Identities?
- Why are the difference and sum identities useful?

The next identities we will investigate are the sum and difference identities for the cosine and sine. These identities will help us find exact values for the trigonometric functions at many more angles and also provide a means to derive even more identities.

Beginning Activity

1. Is $\cos(A - B) = \cos(A) - \cos(B)$ an identity? Explain.

2. Is $\sin(A - B) = \sin(A) - \sin(B)$ an identity? Explain.

3. Use a graphing utility to draw the graph of $y = \sin\left(\dfrac{\pi}{2} - x\right)$ and $y = \cos(x)$ over the interval $[-2\pi, 2\pi]$ on the same set of axes. Do you think $\sin\left(\dfrac{\pi}{2} - x\right) = \cos(x)$ is an identity? Why or why not?

The Cosine Difference Identity

To this point we know the exact values of the trigonometric functions at only a few angles. Trigonometric identities can help us extend this list of angles at which we know exact values of the trigonometric functions. Consider, for example, the problem of finding the exact value of $\cos\left(\dfrac{\pi}{12}\right)$. The definitions and identities we have so far do not help us with this problem. However, we could notice that

$\dfrac{\pi}{12} = \dfrac{\pi}{3} - \dfrac{\pi}{4}$ and if we knew how the cosine behaved with respect to the difference of two angles, then we could find $\cos\left(\dfrac{\pi}{12}\right)$. In our Beginning Activity, however, we saw that the equation $\cos(A - B) = \cos(A) - \cos(B)$ is not an identity, so we need to understand how to relate $\cos(A - B)$ to cosines and sines of A and B.

We state the Cosine Difference Identity below. This identity is not obvious, and a verification of the identity is given later in this section. For now we focus on using the identity.

Cosine Difference Identity
For any real numbers A and B we have

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B).$$

Example 4.12 (Using the Cosine Difference Identity)
Let us return to our problem of finding $\cos\left(\dfrac{\pi}{12}\right)$. Since we know $\dfrac{\pi}{12} = \dfrac{\pi}{3} - \dfrac{\pi}{4}$, we can use the Cosine Difference Identity with $A = \dfrac{\pi}{3}$ and $B = \dfrac{\pi}{4}$ to obtain

$$\begin{aligned}
\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{2} + \sqrt{6}}{4}.
\end{aligned}$$

So we see that $\cos\left(\dfrac{\pi}{12}\right) = \dfrac{\sqrt{2} + \sqrt{6}}{4}$.

Progress Check 4.13 (Using the Cosine Difference Identity)

1. Determine the *exact* value of $\cos\left(\dfrac{7\pi}{12}\right)$ using the Cosine Difference Identity.

2. Given that $\dfrac{5\pi}{12} = \dfrac{\pi}{6} + \dfrac{\pi}{4} = \dfrac{\pi}{6} - \left(-\dfrac{\pi}{4}\right)$, determine the *exact* value of $\cos\left(\dfrac{5\pi}{12}\right)$ using the Cosine Difference Identity.

The Cosine Sum Identity

Since there is a Cosine Difference Identity, we might expect there to be a Cosine Sum Identity. We can use the Cosine Difference Identity along with the negative identities to find an identity for $\cos(A + B)$. The basic idea was contained in our last Progress Check, where we wrote $A + B$ as $A - (-B)$. To see how this works in general, notice that

$$\begin{aligned}\cos(A + B) &= \cos(A - (-B)) \\ &= \cos(A)\cos(-B) + \sin(A)\sin(-B) \\ &= \cos(A)\cos(B) - \sin(A)\sin(B).\end{aligned}$$

This is the Cosine Sum Identity.

Cosine Sum Identity
For any real numbers A and B we have

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B).$$

Progress Check 4.14 (Using the Cosine Sum and Difference Identities)

1. Find a simpler formula for $\cos(\pi + x)$ in terms of $\cos(x)$. Illustrate with a graph.
2. Use the Cosine Difference Identity to prove that $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$ is an identity.

Cofunction Identities

In Progress Check 4.14 we used the Cosine Difference Identity to see that $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$ is an identity. Since this is an identity, we can replace x with $\frac{\pi}{2} - x$ to see that

$$\sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) = \cos(x),$$

so $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$. The two identities

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x) \quad \text{and} \quad \sin\left(\frac{\pi}{2} - x\right) = \cos(x)$$

are called *cofunction identities*. These two cofunction identities show that the sine and cosine of the acute angles in a right triangle are related in a particular way.

Since the sum of the measures of the angles in a right triangle is π radians or $180°$, the measures of the two acute angles in a right triangle sum to $\dfrac{\pi}{2}$ radians or $90°$. Such angles are said to be complementary. Thus, the sine of an acute angle in a right triangle is the same as the cosine of its complementary angle. For this reason we call the sine and cosine *cofunctions*. The naming of the six trigonometric functions reflects the fact that they come in three sets of cofunction pairs: the sine and cosine, the tangent and cotangent, and the secant and cosecant. The cofunction identities are the same for any cofunction pair.

Cofunction Identities

For any real number x for which the expressions are defined,

- $\cos\left(\dfrac{\pi}{2} - x\right) = \sin(x)$
- $\sin\left(\dfrac{\pi}{2} - x\right) = \cos(x)$
- $\tan\left(\dfrac{\pi}{2} - x\right) = \cot(x)$
- $\cot\left(\dfrac{\pi}{2} - x\right) = \tan(x)$
- $\sec\left(\dfrac{\pi}{2} - x\right) = \csc(x)$
- $\csc\left(\dfrac{\pi}{2} - x\right) = \sec(x)$

For any angle x in degrees for which the functions are defined,

- $\cos(90° - x) = \sin(x)$
- $\sin(90° - x) = \cos(x)$
- $\tan(90° - x) = \cot(x)$
- $\cot(90° - x) = \tan(x)$
- $\sec(90° - x) = \csc(x)$
- $\csc(90° - x) = \sec(x)$

Progress Check 4.15 (Using the Cofunction Identities)
Use the cosine and sine cofuntion identities to prove the cofunction identity

$$\tan\left(\frac{\pi}{2} - x\right) = \cot(x).$$

The Sine Difference and Sum Identities

We can now use the Cosine Difference Identity and the Cofunction Identities to derive a Sine Difference Identity:

$$
\begin{aligned}
\sin(A-B) &= \cos\left(\frac{\pi}{2}-(A-B)\right)\\
&= \cos\left(\left(\frac{\pi}{2}-A\right)+B)\right)\\
&= \cos\left(\frac{\pi}{2}-A\right)\cos(B)-\sin\left(\frac{\pi}{2}-A\right)\sin(B)\\
&= \sin(A)\cos(B)-\cos(A)\sin(B).
\end{aligned}
$$

We can derive a Sine Sum Identity from the Sine Difference Identity:

$$
\begin{aligned}
\sin(A+B) &= \sin(A-(-B))\\
&= \sin(A)\cos(-B)-\cos(A)\sin(-B)\\
&= \sin(A)\cos(B)+\cos(A)\sin(B).
\end{aligned}
$$

Sine Difference and Sum Identities
For any real numbers A and B we have

$$\sin(A-B) = \sin(A)\cos(B)-\cos(A)\sin(B)$$

and

$$\sin(A+B) = \sin(A)\cos(B)+\cos(A)\sin(B).$$

Progress Check 4.16 (Using the Sine Sum and Difference Identities)
Use the Sine Sum or Difference Identities to find the *exact* values of the following.

1. $\sin\left(\dfrac{\pi}{12}\right)$

2. $\sin\left(\dfrac{5\pi}{12}\right)$

Using Sum and Difference Identities to Solve Equations

As we have done before, we can use our new identities to solve other types of trigonometric equations.

Example 4.17 (Using the Cosine Sum Identity to Solve an Equation)
Consider the equation

$$\cos(\theta)\cos\left(\frac{\pi}{5}\right) - \sin(\theta)\sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{3}}{2}.$$

On the surface this equation looks quite complicated, but we can apply an identity to simplify it to the point where it is straightforward to solve. Notice that left side of this equation has the form $\cos(A)\cos(B) - \sin(A)\sin(B)$ with $A = \theta$ and $B = \frac{\pi}{5}$. We can use the Cosine Sum Identity $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ to combine the terms on the left into a single term, and we can solve the equation from there:

$$\begin{aligned}\cos(\theta)\cos\left(\frac{\pi}{5}\right) - \sin(\theta)\sin\left(\frac{\pi}{5}\right) &= \frac{\sqrt{3}}{2} \\ \cos\left(\theta + \frac{\pi}{5}\right) &= \frac{\sqrt{3}}{2}.\end{aligned}$$

Now $\cos(x) = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6} + 2k\pi$ or $x = -\frac{\pi}{6} + 2k\pi$ for integers k. Thus, $\cos\left(\theta + \frac{\pi}{5}\right) = \frac{\sqrt{3}}{2}$ when $\theta + \frac{\pi}{5} = \frac{\pi}{6} + 2k\pi$ or $\theta + \frac{\pi}{5} = -\frac{\pi}{6} + 2k\pi$. Solving for θ gives us the solutions

$$\theta = -\frac{\pi}{30} + 2k\pi \quad \text{or} \quad \theta = -\frac{11\pi}{30} + 2k\pi$$

where k is any integer. These solutions are illustrated in Figure 4.6.

Note: Up to now, we have been using the phrase "Determine formulas that can be used to generate all the solutions of a given equation." This is not standard terminology but was used to remind us of what we have to do to solve a trigonometric equation. We will now simply say, "Determine all solutions for the given equation." When we see this, we should realize that we have to determine formulas that can be used to generate all the solutions of a given equation.

Progress Check 4.18 (Using an Identity to Help Solve an Equation)
Determine all solutions of the equation

$$\sin(x)\cos(1) + \cos(x)\sin(1) = 0.2.$$

Hint: Use a sum or difference identity and use the inverse sine function.

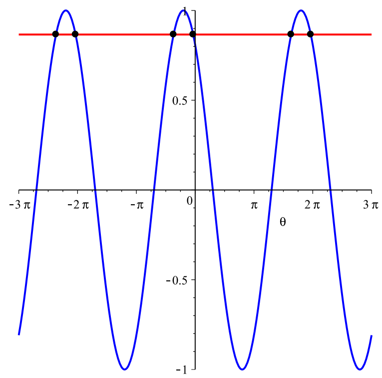


Figure 4.6: Graphs of $y = \cos(\theta)\cos\left(\frac{\pi}{5}\right) - \sin(\theta)\sin\left(\frac{\pi}{5}\right)$ and $y = \frac{\sqrt{3}}{2}$.

Appendix – Proof of the Cosine Difference Identity

To understand how to calculate the cosine of the difference of two angles, let A and B be arbitrary angles in radians. Figure 4.7 shows these angles with $A > B$, but the argument works in general. If we plot the points where the terminal sides of the angles A, B, and $A - B$ intersect the unit circle, we obtain the picture in Figure 4.7.

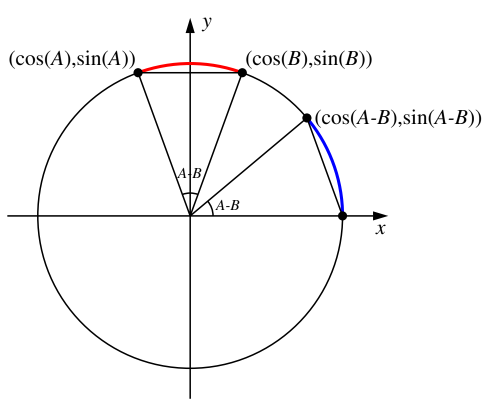


Figure 4.7: The cosine difference formula

The arc on the unit circle from the point $(\cos(B), \sin(B))$ to the point $(\cos(A), \sin(A))$ has length $A - B$, and the arc from the point (1,0) to the point $(\cos(A-B), \sin(A-B))$ also has length $A-B$. So the chord from $(\cos(B), \sin(B))$

to $(\cos(A), \sin(A))$ has the same length as the chord from (1,0) to $(\cos(A-B), \sin(A-B))$. To find the cosine difference formula, we calculate these two chord lengths using the distance formula.

The length of the chord from $(\cos(B), \sin(B))$ to $(\cos(A), \sin(A))$ is

$$\sqrt{(\cos(A) - \cos(B))^2 + (\sin(A) - \sin(B))^2}$$

and the length of the chord from (1,0) to $(\cos(A - B), \sin(A - B))$ is

$$\sqrt{(\cos(A - B) - 1)^2 + (\sin(A - B) - 0)^2}.$$

Since these two chord lengths are the same we obtain the equation

$$\begin{aligned}&\sqrt{(\cos(A - B) - 1)^2 + (\sin(A - B) - 0)^2}\\ &\qquad = \sqrt{(\cos(A) - \cos(B))^2 + (\sin(A) - \sin(B))^2}. \qquad (2)\end{aligned}$$

The cosine difference identity is found by simplifying Equation (2) by first squaring both sides:

$$\begin{aligned}&(\cos(A - B) - 1)^2 + (\sin(A - B) - 0)^2\\ &\qquad = (\cos(A) - \cos(B))^2 + (\sin(A) - \sin(B))^2.\end{aligned}$$

Then we expand both sides

$$\begin{aligned}&[\cos^2(A - B) - 2\cos(A - B) + 1] + \sin^2(A - B)\\ &= [\cos^2(A) - 2\cos(A)\cos(B) + \cos^2(B)] + [\sin^2(A) - 2\sin(A)\sin(B) + \sin^2(B)].\end{aligned}$$

We can combine some like terms:

$$\begin{aligned}&[\cos^2(A - B) + \sin^2(A - B)] - 2\cos(A - B) + 1\\ &= [\cos^2(A) + \sin^2(A)] + [\cos^2(B) + \sin^2(B)] - 2\cos(A)\cos(B) - 2\sin(A)\sin(B).\end{aligned}$$

Finally, using the Pythagorean identities yields

$$\begin{aligned}1 - 2\cos(A - B) + 1 &= 1 + 1 - 2\cos(A)\cos(B) - 2\sin(A)\sin(B)\\ -2\cos(A - B) &= -2\cos(A)\cos(B) - 2\sin(A)\sin(B)\\ \cos(A - B) &= \cos(A)\cos(B) + \sin(A)\sin(B).\end{aligned}$$

Summary of Section 4.3

In this section, we studied the following important concepts and ideas:

- **Sum and Difference Identities**

$$\begin{aligned}
\cos(A - B) &= \cos(A)\cos(B) + \sin(A)\sin(B) \\
\cos(A + B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \\
\sin(A - B) &= \sin(A)\cos(B) - \cos(A)\sin(B) \\
\sin(A + B) &= \sin(A)\cos(B) + \cos(A)\sin(B)
\end{aligned}$$

- **Cofunction Identities**
  See page 266 for a list of the cofunction identities.

Exercises for Section 4.3

1. Use an appropriate sum or difference identity to find the exact value of each of the following.

⋆ **(a)** $\cos(-10°)\cos(35°) + \sin(-10°)\sin(35°)$

⋆ **(b)** $\cos\left(\frac{7\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right) - \sin\left(\frac{7\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$

(c) $\sin\left(\frac{7\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{7\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)$

(d) $\sin(80°)\cos(55°) + \cos(80°)\sin(55°)$

2. Angles A and B are in standard position and $\sin(A) = \frac{1}{2}$, $\cos(A) > 0$, $\cos(B) = \frac{3}{4}$, and $\sin(B) < 0$. Draw a picture of the angles A and B in the plane and then find each of the following.

⋆ **(a)** $\cos(A + B)$

(b) $\cos(A - B)$

(c) $\sin(A + B)$

(d) $\sin(A - B)$

(e) $\tan(A + B)$

(f) $\tan(A - B)$

3. Identify angles A and B at which we know the values of the cosine and sine so that a sum or difference identity can be used to calculate the exact value of the given quantity. (For example, $15° = 45° - 30°$.)

- ⋆ **(a)** $\cos(15°)$
- **(b)** $\sin(75°)$
- **(c)** $\tan(105°)$
- ⋆ **(d)** $\sec(345°)$

4. Verify the sum and difference identities for the tangent:

$$\tan(A-B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

and

$$\tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

.

5. Verify the cofunction identities

- ⋆ **(a)** $\cot\left(\frac{\pi}{2} - x\right) = \tan(x)$
- **(b)** $\sec\left(\frac{\pi}{2} - x\right) = \csc(x)$
- **(c)** $\csc\left(\frac{\pi}{2} - x\right) = \sec(x)$

6. Draw graphs to determine if a given equation is an identity. Verify those equations that are identities and provide examples to show that the others are not identities.

- (**a**) $\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = 2\sin(x)\cos\left(\frac{\pi}{4}\right)$
- (**b**) $\sin(210° + x) - \cos(210° + x) = 0$

7. Determine if the following equations are identities.

- (**a**) $\dfrac{\sin(r+s)}{\cos(r)\cos(s)} = \tan(r) + \tan(s)$
- (**b**) $\dfrac{\sin(r-s)}{\cos(r)\cos(s)} = \tan(r) - \tan(s)$

8. Use an appropriate identity to solve the given equation.

- (**a**) $\sin(\theta)\cos(35°) + \cos(\theta)\sin(35°) = \dfrac{1}{2}$
- (**b**) $\cos(2x)\cos(x) + \sin(2x)\sin(x) = -1$

9. (**a**) Use a graphing device to draw the graph of $f(x) = \sin(x) + \cos(x)$ using $-\pi \leq x \leq 2\pi$ and $-2 \leq y \leq 2$. Does the graph of this function appear to be a sinusoid? If so, approximate the amplitude and phase shift of the sinusoid. What is the period of this sinusoid.

(**b**) Use one of the sum identities to rewrite the expression $\sin\left(x + \dfrac{\pi}{4}\right)$. Then use the values of $\sin\left(\dfrac{\pi}{4}\right)$ and $\cos\left(\dfrac{\pi}{4}\right)$ to further rewrite the expression.

(**c**) Use the result from part (b) to show that the function $f(x) = \sin(x) + \cos(x)$ is indeed a sinusoidal function. What is its amplitude, phase shift, and period?

10. (**a**) Use a graphing device to draw the graph of $g(x) = \sin(x) + \sqrt{3}\cos(x)$ using $-\pi \leq x \leq 2\pi$ and $-2.5 \leq y \leq 2.5$. Does the graph of this function appear to be a sinusoid? If so, approximate the amplitude and phase shift of the sinusoid. What is the period of this sinusoid.

(**b**) Use one of the sum identities to rewrite the expression $\sin\left(x + \dfrac{\pi}{3}\right)$. Then use the values of $\sin\left(\dfrac{\pi}{3}\right)$ and $\cos\left(\dfrac{\pi}{3}\right)$ to further rewrite the expression.

(**c**) Use the result from part (b) to show that the function $g(x) = \sin(x) + \sqrt{3}\cos(x)$ is indeed a sinusoidal function. What is its amplitude, phase shift, and period?

11. When two voltages are applied to a circuit, the resulting voltage in the circuit will be the sum of the individual voltages. Suppose two voltages $V_1(t) = 30\sin(120\pi t)$ and $V_2(t) = 40\cos(120\pi t)$ are applied to a circuit. The graph of the sum $V(t) = V_1(t) + V_2(t)$ is shown in Figure 4.8.

(**a**) Use the graph to estimate the values of C so that

$$y = 50\sin(120\pi(t - C))$$

fits the graph of V.

(**b**) Use the Sine Difference Identity to rewrite $50\sin(120\pi(t - C))$ as an expression of the form $50\sin(A)\cos(B) - 50\cos(A)\sin(B)$, where A and B involve t and/or C. From this, determine a value of C that will make

$$30\sin(120\pi t) + 40\cos(120\pi t) = 50\sin(120\pi(t - C)).$$

Compare this value of C to the one you estimated in part (a).

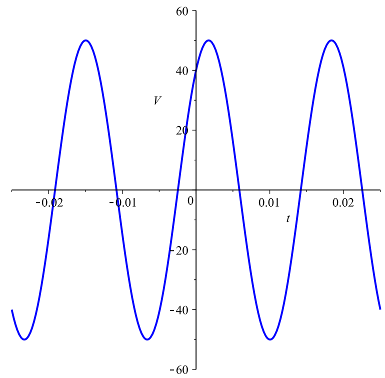


Figure 4.8: Graph of $V(t) = V_1(t) + V_2(t)$.

4.4 Double and Half Angle Identities

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What are the Double Angle Identities for the sine, cosine, and tangent?
- What are the Half Angle Identities for the sine, cosine, and tangent?
- What are the Product-to-Sum Identities for the sine and cosine?
- What are the Sum-to-Product Identities for the sine and cosine?
- Why are these identities useful?

The sum and difference identities can be used to derive the double and half angle identities as well as other identities, and we will see how in this section. Again, these identities allow us to determine exact values for the trigonometric functions at more points and also provide tools for solving trigonometric equations (as we will see later).

Beginning Activity

1. Use $B = A$ in the Cosine Sum Identity

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

to write $\cos(2A)$ in terms of $\cos(A)$ and $\sin(A)$.

2. Is the equation

$$\frac{\cos(2x)}{2} = \cos(x)$$

an identity? Verify your answer.

The Double Angle Identities

Suppose a marksman is shooting a gun with muzzle velocity $v_0 = 1200$ feet per second at a target 1000 feet away. If we neglect all forces acting on the bullet except the force due to gravity, the horizontal distance the bullet will travel depends on the angle θ at which the gun is fired. If we let r be this horizontal distance (called the range), then

$$r = \frac{v_0^2}{g}\sin(2\theta),$$

where g is the gravitational force acting to pull the bullet downward. In this context, $g = 32$ feet per second squared, giving us

$$r = 45000\sin(2\theta).$$

The marksman would want to know the minimum angle at which he should fire in order to hit the target 1000 feet away. In other words, the marksman wants to determine the angle θ so that $r = 1000$. This leads to solving the equation

$$45000\sin(2\theta) = 1000. \tag{3}$$

Equations like the range equation in which multiples of angles arise frequently, and in this section we will determine formulas for $\cos(2A)$ and $\sin(2A)$ in terms of $\cos(A)$ and $\sin(A)$. These formulas are called *double angle identities*. In our Beginning Activity we found that

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

can be derived directly from the Cosine Sum Identity. A similar identity for the sine can be found using the Sine Sum Identity:

$$\begin{aligned}
\sin(2A) &= \sin(A + A)\\
&= \sin(A)\cos(A) + \cos(A)\sin(A)\\
&= 2\cos(A)\sin(A).
\end{aligned}$$

Progress Check 4.19 (Using the Double Angle Identities)
If $\cos(\theta) = \dfrac{5}{13}$ and $\dfrac{3\pi}{2} \le \theta \le 2\pi$, find $\cos(2\theta)$ and $\sin(2\theta)$.

There is also a double angle identity for the tangent. We leave the verification of that identity for the exercises. To summarize:

Double Angle Identities

$$\begin{aligned}
\cos(2A) &= \cos^2(A) - \sin^2(A) \\
\sin(2A) &= 2\cos(A)\sin(A) \\
\tan(2A) &= \frac{2\tan(A)}{1-\tan^2(A)},
\end{aligned}$$

The first two identities are valid for all numbers A and the third is valid as long as $A \neq \dfrac{\pi}{4} + k\left(\dfrac{\pi}{2}\right)$, where k is an integer.

Progress Check 4.20 (Alternate Double Angle Identities)
Prove the alternate versions of the double angle identity for the cosine.

1. $\cos(2A) = 1 - 2\sin^2(A)$

2. $\cos(2A) = 2\cos^2(A) - 1$.

Solving Equations with Double Angles

Solving equations, like $45000\sin(2\theta) = 1000$, that involve multiples of angles, requires the same kind of techniques as solving other equations, but the multiple angle can add another wrinkle.

Example 4.21 (Solving an Equation with a Multiple Angle)
Consider the equation

$$2\cos(2\theta) - 1 = 0.$$

This is an equation that is linear in $\cos(2\theta)$, so we can apply the same ideas as we did earlier to this equation. We solve for $\cos(2\theta)$ to see that

$$\cos(2\theta) = \frac{1}{2}.$$

We know the angles at which the cosine has the value $\dfrac{1}{2}$, namely $\dfrac{\pi}{3} + 2\pi k$ and $-\dfrac{\pi}{3} + 2\pi k$ for integers k. In our case, this make

$$2\theta = \frac{\pi}{3} + 2\pi k \quad \text{or} \quad 2\theta = -\frac{\pi}{3} + 2\pi k$$

for integers k. Now we divide by 2 to find our solutions

$$\theta = \frac{\pi}{6} + \pi k \quad \text{or} \quad \theta = -\frac{\pi}{6} + \pi k$$

for integers k. These solutions are illustrated in Figure 4.9.

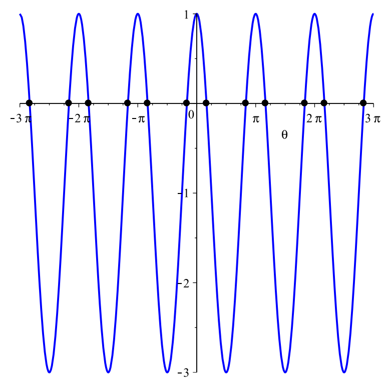


Figure 4.9: Graphs of $y = 2\cos(2\theta) - 1$.

Progress Check 4.22 (Solving Equations with Double Angles)
Approximate the smallest positive solution in degrees, to two decimal places, to the range equation

$$45000\sin(2\theta) = 1000.$$

We can also use the Double Angle Identities to solve equations with multiple angles.

Example 4.23 (Solving an Equation with a Double Angle Identity)
Consider the equation

$$\sin(2\theta) = \sin(\theta).$$

The fact that the two trigonometric functions have different periods makes this equation a little more difficult. We can use the Double Angle Identity for the sine to rewrite the equation as

$$2\sin(\theta)\cos(\theta) = \sin(\theta).$$

At this point we may be tempted to cancel the factor of $\sin(\theta)$ from both sides, but we should resist that temptation because $\sin(\theta)$ can be 0 and we can't divide by 0. Instead, let's put everything one one side and factor:

$$\begin{aligned} 2\sin(\theta)\cos(\theta) &= \sin(\theta) \\ 2\sin(\theta)\cos(\theta) - \sin(\theta) &= 0 \\ \sin(\theta)(2\cos(\theta) - 1) &= 0. \end{aligned}$$

Now we have a product that is equal to 0, so at least one of the factors must be 0. This yields the two equations

$$\sin(\theta) = 0 \ \text{ or } \ \cos(\theta) - 1 = 0.$$

We solve each equation in turn. We know that $\sin(\theta) = 0$ when $\theta = \pi k$ for integers k. Also, $\cos(\theta) - 1 = 0$ implies $\cos(\theta) = 1$, and this happens when $\theta = 2\pi k$ for integers k. Notice that these solutions are a subset of the collection πk of solutions of $\sin(\theta) = 0$. Thus, the solutions to $\sin(2\theta) = \sin(\theta)$ are $\theta = \pi k$ for integers k, as illustrated in Figure 4.10.

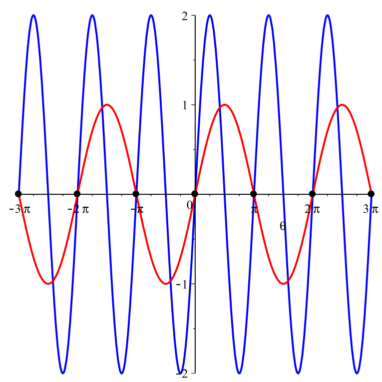


Figure 4.10: Graphs of $y = \sin(2\theta)$ and $y = \sin(\theta)$.

Progress Check 4.24 (Solving an Equation with a Double Angle Identity)
The goal is to solve the equation $\cos(2\theta) = \sin(\theta)$.

1. Use a double angle identity to help rewrite the equation in the form

$$2\sin^2(\theta) + \sin(\theta) - 1 = 0.$$

2. Solve the quadratic type equation in (1) by factoring the left side of the equation.

Half Angle Identities

Now we investigate the half angle identities, identities for $\cos\left(\dfrac{A}{2}\right)$ and $\sin\left(\dfrac{A}{2}\right)$. Here we use the double angle identities from Progress Check 4.20:

$$\begin{aligned}
\cos(A) &= \cos\left(2\left(\frac{A}{2}\right)\right)\\
\cos(A) &= 2\cos^2\left(\frac{A}{2}\right) - 1\\
\cos(A) + 1 &= 2\cos^2\left(\frac{A}{2}\right)\\
\cos^2\left(\frac{A}{2}\right) &= \frac{\cos(A)+1}{2}\\
\cos\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1+\cos(A)}{2}}.
\end{aligned}$$

The sign of $\cos\left(\dfrac{A}{2}\right)$ depends on the quadrant in which $\dfrac{A}{2}$ lies.

Example 4.25 (Using the Cosine Half Angle Identity)

We can use the Cosine Half Angle Identity to determine the exact value of $\cos\left(\dfrac{7\pi}{12}\right)$. If we let $A = \dfrac{7\pi}{6}$, then we have $\dfrac{7\pi}{12} = \dfrac{A}{2}$. The Cosine Half Angle Identity shows us that

$$\begin{aligned}
\cos\left(\frac{7\pi}{12}\right) &= \cos\left(\frac{\frac{7\pi}{6}}{2}\right)\\
&= \pm\sqrt{\frac{1+\cos\left(\frac{7\pi}{6}\right)}{2}}\\
&= \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\
&= \pm\sqrt{\frac{2-\sqrt{3}}{4}}.
\end{aligned}$$

Since the terminal side of the angle $\dfrac{7\pi}{12}$ lies in the second quadrant, we know that

$\cos\left(\frac{7\pi}{12}\right)$ is negative. Therefore,

$$\cos\left(\frac{7\pi}{12}\right) = -\sqrt{\frac{2-\sqrt{3}}{4}}.$$

We can find a similar half angle formula for the sine using the same approach:

$$\begin{aligned}
\cos(A) &= \cos\left(2\left(\frac{A}{2}\right)\right) \\
\cos(A) &= 1 - 2\sin^2\left(\frac{A}{2}\right) \\
\cos(A) - 1 &= -2\sin^2\left(\frac{A}{2}\right) \\
\sin^2\left(\frac{A}{2}\right) &= \frac{1-\cos(A)}{2} \\
\sin\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1-\cos(A)}{2}}.
\end{aligned}$$

Again, the sign of $\sin\left(\frac{A}{2}\right)$ depends on the quadrant in which $\frac{A}{2}$ lies.

To summarize,

Half Angle Identities
For any number A we have

- $\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}}$
- $\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$

where the sign depends on the quadrant in which $\frac{A}{2}$ lies.

Progress Check 4.26 (Using the Half Angle Identities)
Use a Half Angle Identity to find the exact value of $\cos\left(\frac{\pi}{8}\right)$.

Summary of Section 4.4

In this section, we studied the following important concepts and ideas:

- **Double Angle Identities**

$$\cos(2A) = \cos^2(A) - \sin^2(A) \qquad \sin(2A) = 2\cos(A)\sin(A)$$

$$\cos(2A) = 2\cos^2(A) - 1 \qquad \tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

$$\cos(2A) = 1 - 2\sin^2(A)$$

- **Half Angle Identities**

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos(A)}{2}} \qquad \sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$$

where the sign depends on the quadrant in which $\dfrac{A}{2}$ lies.

Exercises for Section 4.4

★ **1.** Given that $\cos(\theta) = \dfrac{2}{3}$ and $\sin(\theta) < 0$, determine the exact values of $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

2. Find all solutions to the given equation. Use a graphing utility to graph each side of the given equation to check your solutions.

★ **(a)** $\cos(x)\sin(x) = \dfrac{1}{2}$

(b) $\cos(2x) + 3 = 5\cos(x)$

3. Determine which of the following equations is an identity. Verify your responses.

★ **(a)** $\cot(t)\sin(2t) = 1 + \cos(2t)$

(b) $\sin(2x) = \dfrac{2 - \csc^2(x)}{\csc^2(x)}$

(c) $\cos(2x) = \dfrac{2 - \sec^2(x)}{\sec^2(x)}$

4. Find a simpler formula for $\cos(\pi + x)$ in terms of $\cos(x)$. Illustrate with a graph.

5. A classmate shares his solution to the problem of solving $\sin(2x) = 2\cos(x)$ over the interval $[0, 2\pi)$. He has written

$$\begin{aligned}\sin(2x) &= 2\cos(x)\\ \frac{\sin(2x)}{2} &= \cos(x)\\ \sin(x) &= \cos(x)\\ \tan(x) &= 1,\end{aligned}$$

so $x = \dfrac{\pi}{4}$ or $x = \dfrac{5\pi}{4}$.

(**a**) Draw graphs of $\sin(2x)$ and $2\cos(x)$ and explain why this classmates solution is incorrect.

(**b**) Find the error in this classmate's argument.

(**c**) Determine the solutions to $\sin(2x) = 2\cos(x)$ over the interval $[0, 2\pi)$.

6. Determine the exact value of each of the following:

⋆ **(a)** $\sin(22.5°)$ **(d)** $\sin(15°)$ **(g)** $\sin(195°)$

(b) $\cos(22.5°)$ **(e)** $\cos(15°)$ ⋆ **(h)** $\cos(195°)$

⋆ **(c)** $\tan(22.5°)$ **(f)** $\tan(15°)$ **(i)** $\tan(195°)$

7. Determine the exact value of each of the following:

⋆ **(a)** $\sin\left(\dfrac{3\pi}{8}\right)$ **(d)** $\sin\left(\dfrac{5\pi}{8}\right)$ **(g)** $\sin\left(\dfrac{11\pi}{12}\right)$

(b) $\cos\left(\dfrac{3\pi}{8}\right)$ **(e)** $\cos\left(\dfrac{5\pi}{8}\right)$ ⋆ **(h)** $\cos\left(\dfrac{11\pi}{12}\right)$

⋆ **(c)** $\tan\left(\dfrac{3\pi}{8}\right)$ **(f)** $\tan\left(\dfrac{5\pi}{8}\right)$ **(i)** $\tan\left(\dfrac{11\pi}{12}\right)$

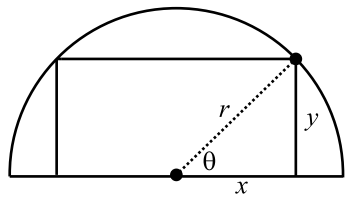
8. If $\cos(x) = \dfrac{2}{3}$ and $\sin(x) < 0$ and $0 \le x \le 2\pi$, determine the exact value of each of the following:

⋆ **(a)** $\cos\left(\dfrac{x}{2}\right)$ **(b)** $\sin\left(\dfrac{x}{2}\right)$ **(c)** $\tan\left(\dfrac{x}{2}\right)$

9. If $\sin(x) = \dfrac{2}{5}$ and $\cos(x) < 0$ and $0 \le x \le 2\pi$, determine the exact value of each of the following:

(**a**) $\cos\left(\dfrac{x}{2}\right)$ (**b**) $\sin\left(\dfrac{x}{2}\right)$ (**c**) $\tan\left(\dfrac{x}{2}\right)$

10. A rectangle is inscribed in a semicircle of radius r as shown in the diagram to the right.



We can write the area A of this rectangle as $A = (2x)y$.

(**a**) Write the area of this inscribed rectangle as a function of the angle θ shown in the diagram and then show that $A = r^2\sin(2\theta)$.

(**b**) Use the formula from part (a) to determine the angle θ that produces the largest value of A and determine the dimensions of this inscribed rectangle with the largest possible area.

11. Derive the Triple Angle Identity

$$\sin(3A) = -4\sin^3(A) + 3\sin(A)$$

for the sine with the following steps.

(**a**) Write $3A$ as $2A + A$ and apply the Sine Sum Identity to write $\sin(3A)$ in terms of $\sin(2A)$ and $\sin(A)$.

(**b**) Use Double Angle Identitys to write $\sin(2A)$ in terms of $\sin(A)$ and $\cos(A)$ and to write $\cos(2A)$ in terms of $\sin(A)$.

(**c**) Use a Pythagorean Identity to write $\cos^2(A)$ in terms of $\sin^2(A)$ and simplify.

12. Derive the Quadruple Angle Identity

$$\sin(4x) = 4\cos(x)\left[\sin(x) - 2\sin^3(x)\right]$$

as follows.

(**a**) Write $\sin(4x) = \sin(2(2x)$ and use the Double Angle Identity for sine to rewrite this formula.

(**b**) Now use the Double Angle Identity for sine and one of the Double Angle Identities for cosine to rewrite the expression from part (a).

(**c**) Algebraically rewrite the expression from part (b) to obtain the desired formula for $\sin(4x)$.

4.5 Sum and Product Identities

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What are the Product-to-Sum Identities for the sine and cosine?
- What are the Sum-to-Product Identities for the sine and cosine?
- Why are these identities useful?

In general, trigonometric equations are very difficult to solve exactly. We have been using identities to solve trigonometric equations, but there are still many more for which we cannot find exact solutions. Consider, for example, the equation

$$\sin(3x) + \sin(x) = 0.$$

The graph of $y = \sin(3x) + \sin(x)$ is shown in Figure 4.11. We can see that there are many solutions, but the identities we have so far do not help us with this equation. What would make this equation easier to solve is if we could rewrite the sum on the left as a product – then we could use the fact that a product is zero if and only if one of its factors is 0. We will later introduce the Sum-to-Product Identities that will help us solve this equation.

Beginning Activity

1. Let $A = 30°$ and $B = 45°$. Calculate

$$\cos(A)\cos(B) \text{ and } \left(\frac{1}{2}\right)[\cos(A+B) + \cos(A-B)]$$

What do you notice?

Product-to-Sum Identities

The Cosine Sum and Difference Identities

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B) \quad (4)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B) \quad (5)$$

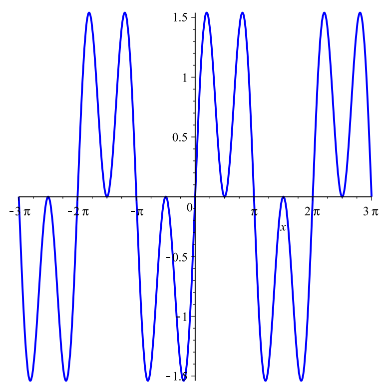


Figure 4.11: Graph of $y = \sin(3x) + \sin(x)$.

will allow us to develop identities that will express product of cosines or sines in terms of sums of cosines and sines. To see how these identities arise, we add the left and right sides of (4) and (5) to obtain

$$\cos(A - B) + \cos(A + B) = 2\cos(A)\cos(B).$$

So

$$\cos(A)\cos(B) = \left(\frac{1}{2}\right)[\cos(A + B) + \cos(A - B)].$$

Similarly, subtracting the left and right sides of (5) from (4) gives us

$$\cos(A - B) - \cos(A + B) = 2\sin(A)\sin(B).$$

So

$$\sin(A)\sin(B) = \left(\frac{1}{2}\right)[\cos(A - B) - \cos(A + B)].$$

We can similarly obtain a formula for $\cos(A)\sin(B)$. In this case we use the sine sum and difference formulas

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B) \tag{6}$$
$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B). \tag{7}$$

Adding the left and right hand sides of (6) and (7) yields

$$\sin(A - B) + \sin(A + B) = 2\sin(A)\cos(B).$$

So

$$\sin(A)\cos(B) = \left(\frac{1}{2}\right)[\sin(A+B) + \sin(A-B)].$$

Product-to-Sum Identities
For any numbers A and B we have

$$\begin{aligned}
\cos(A)\cos(B) &= \left(\frac{1}{2}\right)[\cos(A+B) + \cos(A-B)] \\
\sin(A)\sin(B) &= \left(\frac{1}{2}\right)[\cos(A-B) - \cos(A+B)] \\
\sin(A)\cos(B) &= \left(\frac{1}{2}\right)[\sin(A+B) + \sin(A-B)].
\end{aligned}$$

Progress Check 4.27 (Using the Product-to-Sum Identities)
Find the exact value of $\sin(52.5^\circ)\sin(7.5^\circ)$.

Sum-to-Product Identities

As our final identities, we derive the reverse of the Product-to-Sum identities. These identities are called the Sum-to-Product identities. For example, to verify the identity

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),$$

we first note that $A = \frac{A+B}{2} + \frac{A-B}{2}$ and $B = \frac{A+B}{2} - \frac{A-B}{2}$. So

$$\begin{aligned}
\cos(A) &= \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right) \\
&= \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
\cos(B) &= \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right) \\
&= \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right). \qquad (9)
\end{aligned}$$

Adding the left and right sides of (8) and (9) results in

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

Also, if we subtract the left and right hands sides of (9) from (8) we obtain

$$\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

Similarly,

$$\begin{aligned} \sin(A) &= \sin\left(\frac{A+B}{2} + \frac{A-B}{2}\right) \\ &= \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \end{aligned} \tag{10}$$

and

$$\begin{aligned} \sin(B) &= \sin\left(\frac{A+B}{2} - \frac{A-B}{2}\right) \\ &= \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right). \end{aligned} \tag{11}$$

Adding the left and right sides of (10) and (11) results in

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

Again, if we subtract the left and right hands sides of (11) from (10) we obtain

$$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

Sum-to-Product Identities

For any numbers A and B we have

$$\begin{aligned} \cos(A) + \cos(B) &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\ \cos(A) - \cos(B) &= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \\ \sin(A) + \sin(B) &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \\ \sin(A) - \sin(B) &= 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right). \end{aligned}$$

Progress Check 4.28 (Using the Sum-to-Product Identities)
Find the exact value of $\cos(112.5°) + \cos(67.5°)$.

We can use these Sum-to-Product and Product-to-Sum Identities to solve even more types of trigonometric equations.

Example 4.29 (Solving Equations Using the Sum-to-Product Identity)
Let us return to the problem stated at the beginning of this section to solve the equation

$$\sin(3x) + \sin(x) = 0.$$

Using the Sum-to-Product

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

with $A = x$ and $B = 3x$ we can rewrite the equation as follows:

$$\begin{aligned}
\sin(3x) + \sin(x) &= 0 \\
2\sin\left(\frac{4x}{2}\right)\cos\left(\frac{x}{2}\right) &= 0 \\
2\sin(2x)\cos\left(\frac{x}{2}\right) &= 0.
\end{aligned}$$

The advantage of this form is that we now have a product of functions equal to 0, and the only way a product can equal 0 is if one of the factors is 0. This reduces our original problem to two equations we can solve:

$$\sin(2x) = 0 \;\text{ or }\; \cos\left(\frac{x}{2}\right) = 0.$$

We know that $\sin(2x) = 0$ when $2x = k\pi$ or $x = k\dfrac{\pi}{2}$, where k is any integer, and $\cos\left(\dfrac{x}{2}\right) = 0$ when $\dfrac{x}{2} = \dfrac{\pi}{2} + k\pi$ or $x = \pi + k2\pi$, where k is any integer. These solutions can be seen where the graph of $y = \sin(3x) + \sin(x)$ intersects the x-axis as illustrated in Figure 4.12.

Summary of Trigonometric Identities

Trigonometric identities are useful in that they allow us to determine exact values for the trigonometric functions at more points than before and also provide tools for deriving new identities and for solving trigonometric equations. Here we provide a summary of our trigonometric identities.

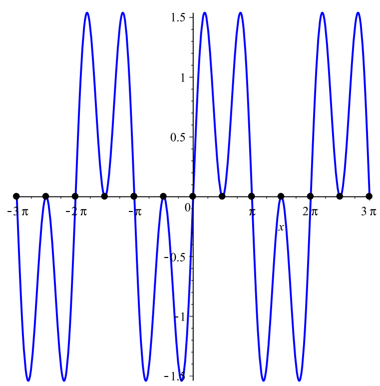


Figure 4.12: Graph of $y = \sin(3x) + \sin(x)$.

Cofunction Identities

$$\cos\left(\frac{\pi}{2} - A\right) = \sin(A)$$
$$\sin\left(\frac{\pi}{2} - A\right) = \cos(A)$$
$$\tan\left(\frac{\pi}{2} - A\right) = \cot(A).$$

Double Angle Identities

$$\sin(2A) = 2\cos(A)\sin(A)$$
$$\cos(2A) = \cos^2(A) - \sin^2(A)$$
$$\cos(2A) = 1 - 2\sin^2(A)$$
$$\cos(2A) = 2\cos^2(A) - 1$$
$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}.$$

Half Angle Identities

$$\cos^2\left(\frac{A}{2}\right) = \frac{1 + \cos(A)}{2}$$
$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos(A)}{2}}$$
$$\sin^2\left(\frac{A}{2}\right) = \frac{1 - \cos(A)}{2}$$
$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos(A)}{2}}$$
$$\tan\left(\frac{A}{2}\right) = \frac{\sin(A)}{1 + \cos(A)}$$
$$\tan\left(\frac{A}{2}\right) = \frac{1 - \cos(A)}{\sin(A)}.$$

The signs of $\cos\left(\frac{A}{2}\right)$ and $\sin\left(\frac{A}{2}\right)$ depend on the quadrant in which $\frac{A}{2}$ lies.

Cosine Difference and Sum Identities

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B).$$

Sine Difference and Sum Identities

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B).$$

Tangent Difference and Sum Identities

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$
$$\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}.$$

Product-to-Sum Identities

$$\cos(A)\cos(B) = \left(\frac{1}{2}\right)[\cos(A + B) + \cos(A - B)]$$
$$\sin(A)\sin(B) = \left(\frac{1}{2}\right)[\cos(A - B) - \cos(A - B)]$$
$$\sin(A)\cos(B) = \left(\frac{1}{2}\right)[\sin(A + B) + \sin(A - B)].$$

Sum-to-Product Identities

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$
$$\cos(A) - \cos(B) = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$
$$\sin(A) + \sin(B) = 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$
$$\sin(A) - \sin(B) = 2\cos\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right).$$

Exercises for Section 4.5

1. Write each of the following expressions as a sum of trigonometric function values. When possible, determine the exact value of the resulting expression.

- ⋆ **(a)** $\sin(37.5°)\cos(7.5°)$
- **(b)** $\sin(75°)\sin(15°)$
- **(c)** $\cos(44°)\cos(16°)$
- **(d)** $\cos(45°)\cos(15°)$
- ⋆ **(e)** $\cos\left(\dfrac{5\pi}{12}\right)\sin\left(\dfrac{\pi}{12}\right)$
- **(f)** $\sin\left(\dfrac{3\pi}{4}\right)\cos\left(\dfrac{\pi}{12}\right)$

2. Write each of the following expressions as a sum of trigonometric function values. When possible, determine the exact value of the resulting expression.

- ⋆ **(a)** $\sin(50°)+\sin(10°)$
- **(b)** $\sin(195°)-\sin(105°)$
- **(c)** $\cos(195°)-\cos(15°)$
- **(d)** $\cos(76°)+\cos(14°)$
- ⋆ **(e)** $\cos\left(\dfrac{7\pi}{12}\right)+\cos\left(\dfrac{\pi}{12}\right)$
- **(f)** $\sin\left(\dfrac{7\pi}{4}\right)-\sin\left(\dfrac{5\pi}{12}\right)$

3. Find all solutions to the given equation. Use a graphing utility to graph each side of the given equation to check your solutions.

- ⋆ **(a)** $\sin(2x)+\sin(x)=0$
- **(b)** $\sin(x)\cos(x)=\dfrac{1}{4}$
- **(c)** $\cos(2x)+\cos(x)=0$