Part V

Vector Spaces

Section 22

Vector Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a vector space?
- What is a subspace of a vector space?
- What is a linear combination of vectors in a vector space V?
- What is the span of a set of vectors in a vector space V?
- What special structure does the span of a set of vectors in a vector space V have?
- Why is the vector space concept important?

Application: The Hat Puzzle

In a New York Times article (April 10, 2001) "Why Mathematicians Now Care About Their Hat Color", the following puzzle is posed.

> "Three players enter a room and a red or blue hat is placed on each person's head. The color of each hat is determined by a coin toss, with the outcome of one coin toss having no effect on the others. Each person can see the other players' hats but not his own.
>
> No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, the players must simultaneously guess the color of their own hats or pass. The group shares a hypothetical $3 million prize if at least one player guesses correctly and no players guess incorrectly."

The game can be played with more players, and the problem is to find a strategy for the group that maximizes its chance of winning. One strategy is for a designated player to make a random guess and for the others to pass. This gives a 50% chance of winning. However, there are much better strategies that provide a nearly 100% probability of winning as the number of players increases. One such strategy is based on Hamming codes and subspaces of a particular vector space to implement the most effective approach.

Introduction

We have previously seen that $\mathbb{R}^n$ and the set of fixed size matrices have a nice algebraic structure when endowed with the addition and scalar multiplication operations. In fact, as we will see, there are many other sets of elements that have the same kind of structure with natural addition and scalar multiplication operations. Due to this underlying similar structure, these sets are connected in some way and can all be studied jointly. Mathematicians look for these kinds of connections between seemingly dissimilar objects and, from a mathematical standpoint, it is convenient to study all of these similar structures at once by combining them into a larger collection. This motivates the idea of a *vector space* that we will investigate in this chapter.

An example of a set that has a structure similar to vectors is a collection of polynomials. Let $\mathbb{P}_1$ be the collection of all polynomials of degree less than or equal to 1 with real coefficients. That is,

$$\mathbb{P}_1 = \{a_0 + a_1 t : a_0, a_1 \in \mathbb{R}\}.$$

So, for example, the polynomials $2 + t$, $5t$, -7, and $\sqrt{12} - \pi t$ are in $\mathbb{P}_1$, but $\sqrt{t}$ is not in $\mathbb{P}_1$. Two polynomials $a(t) = a_0 + a_1 t$ and $b(t) = b_0 + b_1 t$ in $\mathbb{P}_1$ are equal if $a_0 = b_0$ and $a_1 = b_1$.

We define addition of polynomials in $\mathbb{P}_1$ by adding the coefficients of the like degree terms. So if $a(t) = a_0 + a_1 t$ and $b(t) = b_0 + b_1 t$, then the polynomial sum of $a(t)$ and $b(t)$ is

$$a(t) + b(t) = (a_0 + a_1 t) + (b_0 + b_1 t) = (a_0 + b_0) + (a_1 + b_1)t.$$

So, for example,

$$(2 + 3t) + (-1 + 5t) = (2 + (-1)) + (3 + 5)t = 1 + 8t.$$

We now consider the properties of the addition operation. For example, we can ask if polynomial addition is commutative. That is, if $a(t)$ and $b(t)$ are in $\mathbb{P}_1$, must it be the case that

$$a(t) + b(t) \ = \ b(t) + a(t)?$$

To show that addition is commutative in $\mathbb{P}_1$, we choose arbitrary polynomials $a(t) = a_0 + a_1 t$ and $b(t) = b_0 + b_1 t$ in $\mathbb{P}_1$. Then we have

$$\begin{aligned} a(t) + b(t) &= (a_0 + b_0) + (a_1 + b_1)t \\ &= (b_0 + a_0) + (b_1 + a_1)t \\ &= b(t) + a(t). \end{aligned}$$

Note that in the middle step, we used the definition of equality of polynomials since $a_0 + b_0 = b_0 + a_0$ and $a_1 + b_1 = b_1 + a_1$ due to the fact that addition of real numbers is commutative. So addition of elements in $\mathbb{P}_1$ is a commutative operation.

Preview Activity 22.1.

(1) Now we investigate other properties of addition in $\mathbb{P}_1$.

(a) To show addition is associative in $\mathbb{P}_1$, we need to verify that if $a(t) = a_0 + a_1 t$, $b(t) = b_0 + b_1 t$, and $c(t) = c_0 + c_1 t$ are in $\mathbb{P}_1$, it must be the case that

$$(a(t) + b(t)) + c(t) = a(t) + (b(t) + c(t)).$$

Either verify this property by using the definition of two polynomials being equal, or give a counterexample to show the equality fails in that case.

(b) Find a polynomial $z(t) \in \mathbb{P}_1$ such that

$$a(t) + z(t) = a(t)$$

for all $a(t) \in \mathbb{P}_1$. This polynomial is called the *zero* polynomial or the *additive identity* polynomial in $\mathbb{P}_1$.

(c) If $a(t) = a_0 + a_1 t$ is an element of $\mathbb{P}_1$, is there an element $p(t) \in \mathbb{P}_1$ such that

$$a(t) + p(t) = z(t),$$

where $z(t)$ is the additive identity polynomial you found above? If not, why not? If so, what polynomial is $p(t)$? Explain.

(2) We can also define a multiplication of polynomials by scalars (real numbers).

(a) What element in $\mathbb{P}_1$ could be the scalar multiple $\frac{1}{2}(2 + 3t)$?

(b) In general, if k is a scalar and $a(t) = a_0 + a_1 t$ is in $\mathbb{P}_1$, how do we define the scalar multiple $ka(t)$ in $\mathbb{P}_1$?

(c) If k is a scalar and $a(t) = a_0 + a_1 t$ and $b(t) = b_0 + b_1 t$ are elements in $\mathbb{P}_1$, is it true that

$$k(a(t) + b(t)) = ka(t) + kb(t)?$$

If no, explain why. If yes, verify your answer using the definition of two polynomials being equal.

(d) If k and m are scalars and $a(t) = a_0 + a_1 t$ is an element in $\mathbb{P}_1$, is it true that

$$(k + m)a(t) = ka(t) + ma(t)?$$

If no, explain why. If yes, verify your answer.

(e) If k and m are scalars and $a(t) = a_0 + a_1 t$ is an element in $\mathbb{P}_1$, is it true that

$$(km)a(t) = k(ma(t))?$$

If no, explain why. If yes, verify your answer.

(f) If $a(t) = a_0 + a_1 t$ is an element of $\mathbb{P}_1$, is it true that

$$1a(t) = a(t)?$$

If no, explain why. If yes, verify your answer.

Spaces with Similar Structure to $\mathbb{R}^n$

Mathematicians look for patterns and for similarities between mathematical objects. In doing so, mathematicians often consider larger collections of objects that are sorted according to their similarities and then study these collections rather than just the objects themselves. This perspective can be very powerful – whatever can be shown to be true about an arbitrary element in a collection will then be true for every specific element in the collection. In this section we study the larger collection of sets that share the algebraic structure of vectors in $\mathbb{R}^n$. These sets are called *vector spaces*.

In Preview Activity 22.1, we showed that the set $\mathbb{P}_1$ of polynomials of degree less than or equal to one with real coefficients, with the operations of addition and scalar multiplication defined by

$$(a_0 + a_1t) + (b_0 + b_1t) = (a_0 + b_0) + (a_1 + b_1)t \quad \text{and} \quad k(a_0 + a_1t) = (ka_0) + (ka_1)t,$$

has a structure similar to $\mathbb{R}^2$.

By structure we mean how the elements in the set relate to each other under addition and multiplication by scalars. That is, if $a(t) = a_0 + a_1t$, $b(t)$, and $c(t)$ are elements of $\mathbb{P}_1$ and k and m are scalars, then

(1) $a(t) + b(t)$ is an element of $\mathbb{P}_1$,

(2) $a(t) + b(t) = b(t) + a(t)$,

(3) $(a(t) + b(t)) + c(t) = a(t) + (b(t) + c(t))$,

(4) there is a zero polynomial $z(t)$ (namely, $0 + 0t$) in $\mathbb{P}_1$ so that $a(t) + z(t) = a(t)$,

(5) there is an element $-a(t)$ in $\mathbb{P}_1$ (namely, $(-a_0) + (-a_1)t$) so that $a(t) + (-a(t)) = z(t)$,

(6) $ka(t)$ is an element of $\mathbb{P}_1$,

(7) $(k + m)a(t) = ka(t) + ma(t)$,

(8) $k(a(t) + b(t)) = ka(t) + kb(t)$,

(9) $(km)a(t) = k(ma(t))$,

(10) $1a(t) = a(t)$.

The properties we saw for polynomials in $\mathbb{P}_1$ stated above are the same as the properties for vector addition and multiplication by scalars in $\mathbb{R}^n$, as well as matrix addition and multiplication by scalars identified in Section 8. This indicates that polynomials in $\mathbb{P}_1$, vectors in $\mathbb{R}^n$, and the set of $m \times n$ matrices behave in much the same way as regards their addition and multiplication by scalars. There is an even closer connection between linear polynomials and vectors in $\mathbb{R}^2$. An element $a(t) = a_0 + a_1t$ in $\mathbb{P}_1$ can be naturally associated with the vector $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ in $\mathbb{R}^2$. All the results of polynomial addition and multiplication by scalars then translate to corresponding results of addition and multiplication by scalars of vectors in $\mathbb{R}^2$. So for all intents and purposes, as far as addition and multiplication by scalars is concerned, there is no difference between elements in $\mathbb{P}_1$

and vectors in $\mathbb{R}^2$ – the only difference is how we choose to present the elements (as polynomials or as vectors). This sameness of structure of our sets as it relates to addition and multiplication by scalars is the type of similarity mentioned in the introduction. We can study all of the types of objects that exhibit this same structure at one time by studying vector spaces.

Vector Spaces

We defined vector spaces in the context of subspaces of $\mathbb{R}^n$ in Definition 12.1. In general, any set that has the same kind of additive and multiplicative structure as our sets of vectors, matrices, and linear polynomials is called a vector space. As we will see, the ideas that we introduced about subspaces of $\mathbb{R}^n$ apply to vector spaces in general, so the material in this chapter should have a familiar feel.

Definition 22.1. A set V on which an operation of addition and a multiplication by scalars is defined is a **vector space** if for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars a and b:

(1) $\mathbf{u} + \mathbf{v}$ is an element of V (we say that V is *closed* under the addition in V),

(2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (we say that the addition in V is *commutative*),

(3) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (we say that the addition in V is *associative*),

(4) there is a zero vector $\mathbf{0}$ in V so that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ (we say that V contains an *additive identity* $\mathbf{0}$),

(5) for each $\mathbf{x}$ in V there is an element $\mathbf{y}$ in V so that $\mathbf{x} + \mathbf{y} = \mathbf{0}$ (we say that V contains an *additive inverse* $\mathbf{y}$ for each element $\mathbf{x}$ in V),

(6) $a\mathbf{u}$ is an element of V (we say that V is *closed* under multiplication by scalars),

(7) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ (we say that *multiplication by scalars distributes over scalar addition*),

(8) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ (we say that *multiplication by scalars distributes over addition in* V),

(9) $(ab)\mathbf{u} = a(b\mathbf{u})$,

(10) $1\mathbf{u} = \mathbf{u}$.

Note. Unless otherwise stated, in this book the scalars will refer to real numbers. However, we can define vector spaces where scalars are complex numbers, or rational numbers, or integers modulo p where p is a prime number, or, more generally, elements of a field. A field is an algebraic structure which generalizes the structure of real numbers and rational numbers under the addition and multiplication operations. Since we will focus on the real numbers as scalars, the reader is not required to be familiar with the concept of a field.

Because of the similarity of the way elements in vector spaces behave compared to vectors in $\mathbb{R}^n$, we call the elements in a vector space *vectors*. There are many examples of vectors spaces, which is what makes this idea so powerful.

Example 22.2.

(1) The space $\mathbb{R}^n$ of all vectors with n components is a vector space using the standard vector addition and multiplication by scalars. The zero element is the zero vector $\mathbf{0}$ whose components are all 0.

(2) The set $\mathbb{P}_1$ of all polynomials of degree less than or equal to 1 with addition and scalar multiplication as defined earlier. Recall that $\mathbb{P}_1$ is essentially the same as $\mathbb{R}^2$.

(3) The properties listed in the introduction for $\mathbb{P}_1$ are equally true for the collection of all polynomials of degree less than or equal to some fixed number. We label as $\mathbb{P}_n$ this set of all polynomials of degree less than or equal to n, with the standard addition and scalar multiplication. Note that $\mathbb{P}_n$ is essentially the same as $\mathbb{R}^{n+1}$. More generally, the space $\mathbb{P}$ of all polynomials is also a vector space with standard addition and scalar multiplication.

(4) As a subspace of $\mathbb{R}^n$, the eigenspace of an $n \times n$ matrix corresponding to an eigenvalue λ is a vector space.

(5) As a subspace of $\mathbb{R}^n$, the null space of an $m \times n$ matrix is a vector space.

(6) As a subspace of $\mathbb{R}^m$, the column space of an $m \times n$ matrix is a vector space.

(7) The span of a set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, and is therefore a vector space.

(8) Let V be a vector space and let $\mathbf{0}$ be the additive identity in V. The set $\{\mathbf{0}\}$ is a vector space in which $\mathbf{0}+\mathbf{0} = \mathbf{0}$ and $k\mathbf{0} = \mathbf{0}$ for any scalar k. This space is called the *trivial* vector space.

(9) The space $\mathcal{M}_{m\times n}$ (or $\mathcal{M}_{m\times n}(\mathbb{R})$ when it is important to indicate that the entries of our matrices are real numbers) of all $m \times n$ matrices with real entries with the standard addition and multiplication by scalars we have already defined. In this case, $\mathcal{M}_{m\times n}$ is essentially the same vector space as $\mathbb{R}^{mn}$.

(10) The space $\mathcal{F}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$, where we define the sum of two functions f and g in $\mathcal{F}$ as the function $f + g$ satisfying

$$(f + g)(x) = f(x) + g(x)$$

for all real numbers x, and the scalar multiple cf of the function f by the scalar c to be the function satisfying

$$(cf)(x) = cf(x)$$

for all real numbers x. The verification of the vector space properties for this space is left to the reader.

(11) The space $\mathbb{R}^\infty$ of all infinite real sequences $(x_1, x_2, x_3, \ldots)$. We define addition and scalar multiplication termwise:

$$(x_1, x_2, x_3, \ldots) + (y_1, y_2, y_3, \ldots) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots)\,,$$

$$c(x_1, x_2, x_3, \ldots) = (cx_1, cx_2, cx_3, \ldots)$$

is a vector space. In addition, the set of convergent sequences inside $\mathbb{R}^\infty$ forms a vector space using this addition and multiplication by scalars (as we did in $\mathbb{R}^n$, we will call this set of convergent sequences a subspace of $\mathbb{R}^\infty$).

(12) (For those readers who are familiar with differential equations). The set of solutions to a second order homogeneous differential equation forms a vector space under addition and scalar multiplication defined as in the space $\mathcal{F}$ above.

(13) The set of polynomials of positive degree in $\mathbb{P}_1$ is not a vector space using the standard addition and multiplication by scalars in $\mathbb{P}_1$ is not a vector space. Notice that $t + (-t)$ is not a polynomial of positive degree, and so this set is not closed under addition.

(14) The color space where each color is assigned an RGB (red, green, blue) coordinate between 0 and 255, with addition and scalar multiplication defined component-wise, however, does not define a vector space. The color space is not closed under either operation due to the color coordinates being integers ranging from 0 to 255.

It is important to note that the set of defining properties of a vector space is intended to be a minimum set. Any other properties of a vector space must be verified or proved using the defining properties. For example, in $\mathbb{R}^n$ it is clear that the scalar multiple $0\mathbf{v}$ is the zero vector for any vector $\mathbf{v}$ in $\mathbb{R}^n$. This might be true in any vector space, but it is not a defining property. Therefore, if this property is true, then we must be able to prove it using just the defining properties. To see how this might work, let $\mathbf{v}$ be any vector in a vector space V. We want to show that $0\mathbf{v} = \mathbf{0}$ (the existence of the zero vector is property (4)). Using the fact that $0 + 0 = 0$ and that scalar multiplication distributes over scalar addition, we can see that

$$0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}.$$

Property (5) tells us that V contains an additive inverse for every vector in V, so let $\mathbf{u}$ be an additive inverse of the vector $0\mathbf{v}$ in V. Then $0\mathbf{v} + \mathbf{u} = \mathbf{0}$[1] and so

$$\begin{aligned} 0\mathbf{v} + \mathbf{u} &= (0\mathbf{v} + 0\mathbf{v}) + \mathbf{u} \\ \mathbf{0} &= 0\mathbf{v} + (0\mathbf{v} + \mathbf{u}) \\ \mathbf{0} &= 0\mathbf{v} + \mathbf{0}. \end{aligned}$$

Now $\mathbf{0}$ has the property that $\mathbf{0} + \mathbf{w} = \mathbf{w} + \mathbf{0} = \mathbf{w}$ for any vector $\mathbf{w}$ in V (by properties (4) and (2)), and so we can conclude that

$$\mathbf{0} = 0\mathbf{v}.$$

Activity 22.1. Another property that will be useful is a cancellation property. In the set of real numbers we know that if $a + b = c + b$, then $a = c$, and we verify this by subtracting b from both sides. This is the same as adding the additive inverse of b to both sides, so we ought to be able to make the same argument using additive inverses in a vector space. To see how, let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a vector space and suppose that

$$\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}. \tag{22.1}$$

(a) Why does our space contain an additive inverse $\mathbf{z}$ of $\mathbf{w}$?

[1] It is very important to keep track of the different kinds of zeros here – the boldface zero **0** is the additive identity in the vector space and the non-bold 0 is the scalar zero.

(b) Now add the vector $\mathbf{z}$ to both sides of equation (22.1) to obtain

$$(\mathbf{u}+\mathbf{w})+\mathbf{z}=(\mathbf{v}+\mathbf{w})+\mathbf{z}. \tag{22.2}$$

Which property of a vector space allows us to state the following equality?

$$\mathbf{u}+(\mathbf{w}+\mathbf{z})=\mathbf{v}+(\mathbf{w}+\mathbf{z}). \tag{22.3}$$

(c) Now use the properties of additive inverses and the additive identity to explain why $\mathbf{u}=\mathbf{v}$. Conclude that we have a cancellation law for addition in any vector space.

We should also note that the definition of a vector space only states the existence of a zero vector and an additive inverse for each vector in the space, and does not say that there cannot be more than one zero vector or more than one additive inverse of a vector in the space. The reason why is that the uniqueness of the zero vector and an additive inverse of a vector can be proved from the defining properties of a vector space, and so we don't list this consequence as a defining property. Similarly, the defining properties of a vector space do not state that the additive inverse of a vector $\mathbf{v}$ is the scalar multiple $(-1)\mathbf{v}$. Verification of these properties are left for the exercises. We summarize the results of this section in the following theorem.

Theorem 22.3. *Let V be any vector space with identity $\mathbf{0}$.*

- $0\mathbf{v}=\mathbf{0}$ *for any vector* $\mathbf{v}$ *in* V.
- *The vector* $\mathbf{0}$ *is unique.*
- $c\mathbf{0}=\mathbf{0}$ *for any scalar* c.
- *For any* $\mathbf{v}$ *in* V, *the additive inverse of* $\mathbf{v}$ *is unique.*
- *The additive inverse of a vector* $\mathbf{v}$ *in* V *is the vector* $(-1)\mathbf{v}$.
- *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are in* V *and* $\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}$, *then* $\mathbf{u}=\mathbf{v}$.

Subspaces

In Section 12 we saw that $\mathbb{R}^n$ contained subsets that we called subspaces that had the same algebraic structure as $\mathbb{R}^n$. The same idea applies to vector spaces in general.

Activity 22.2. Let $H=\{at : a\in\mathbb{R}\}$. Notice that H is a subset of $\mathbb{P}_1$.

(a) Is H closed under the addition in $\mathbb{P}_1$? Verify your answer.

(b) Does H contain the zero vector from $\mathbb{P}_1$? Verify your answer.

(c) Is H closed under multiplication by scalars? Verify your answer.

(d) Explain why H satisfies every other property of the definition of a vector space automatically just by being a subset of $\mathbb{P}_1$ and using the same operations as in $\mathbb{P}_1$. Conclude that H is a vector space.

Activity 22.2 illustrates an important point. There is a fundamental difference in the types of properties that define a vector space. Some of the properties that define a vector space are true for any subset of the vector space because they are properties of the operations (such as the commutative and associative properties). The other properties (closure, the inclusion of the zero vector, and the inclusion of additive inverses) are set properties, not properties of the operations. So these three properties have to be specifically checked to see if a subset of a vector space is also a vector space. This leads to the definition of a *subspace*, a subset of a vector space which is a vector space itself.

Definition 22.4. A subset H of a vector space V is a **subspace** of V if

(1) whenever $\mathbf{u}$ and $\mathbf{v}$ are in H it is also true that $\mathbf{u} + \mathbf{v}$ is in H (that is, H is *closed* under addition),

(2) whenever $\mathbf{u}$ is in H and a is a scalar it is also true that $a\mathbf{u}$ is in H (that is, H is *closed* under scalar multiplication),

(3) $\mathbf{0}$ is in H.

Activity 22.3. Is the given subset H a subspace of the indicated vector space V? Verify your answer.

(a) V is any vector space and $H = \{\mathbf{0}\}$

(b) $V = M_{2\times 2}$, the vector space of 2×2 matrices and
$H = \left\{ \begin{bmatrix} 2x & y \\ 0 & x \end{bmatrix} \mid x \text{ and } y \text{ are scalars} \right\}$.

(c) $V = \mathbb{P}_2$, the vector space of all polynomials of degree less than or equal to 2 and $H = \left\{2at^2 + 1 \mid a \text{ is a scalar}\right\}$.

(d) $V = \mathbb{P}_2$ and $H = \{at \mid a \text{ is a scalar}\} \cup \left\{bt^2 \mid b \text{ is a scalar}\right\}$.

(e) $V = \mathcal{F}$ and $H = \mathbb{P}_2$.

There is an interesting subspace relationship between the spaces $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \ldots$ and $\mathbb{P}$. For every i, $\mathbb{P}_i$ is a subspace of $\mathbb{P}$. Furthermore, $\mathbb{P}_1$ is a subspace of $\mathbb{P}_2$, $\mathbb{P}_2$ is a subspace of $\mathbb{P}_3$, and so on. Note however that a similar relationship does NOT hold for $\mathbb{R}^n$, even though $\mathbb{P}_i$ looks like $\mathbb{R}^{i+1}$. For example, $\mathbb{R}^1$ is NOT a subspace of $\mathbb{R}^2$. Similarly, $\mathbb{R}^2$ is NOT a subspace of $\mathbb{R}^3$. Since the vectors in different $\mathbb{R}^n$'s are of different sizes, none of the $\mathbb{R}^i$'s is a subset of another $\mathbb{R}^n$ with $i \neq n$, and hence, $\mathbb{R}^i$ is not a subspace of $\mathbb{R}^n$ when $i < n$.

The Subspace Spanned by a Set of Vectors

In $\mathbb{R}^n$ we showed that the span of any set of vectors forms a subspace of $\mathbb{R}^n$. The same is true in any vector space. Recall that the span of a set of vectors in $\mathbb{R}^n$ is the set of all linear combinations of those vectors. So before we can discuss the span of a set of vectors in a vector space, we need to extend the definition of linear combinations to vector spaces (compare to Definitions 4.4 and 4.6).

Definition 22.5. Let V be a vector space. A **linear combination** of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ in V is a vector of the form

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k,$$

where x_1, x_2, ..., x_k are scalars. The **span** of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ is the collection of all linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$. That is,

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k \mid x_1, x_2, \ldots, x_k \text{ are scalars}\}.$$

The argument that the span of any finite set of vectors in a vector space forms a subspace is the same as we gave for the span of a set of vectors in $\mathbb{R}^n$ (see Theorem 12.5). The proof is left for the exercises.

Theorem 22.6. *Given a vector space V and vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_m$ in V, Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is a subspace of V.*

The subspace Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is called the *subspace of V spanned by* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$.

Activity 22.4.

(a) Let $H = \left\{a_2t^2 - a_1t : a_2 \text{ and } a_1 \text{ are real numbers}\right\}$. Note that H is a subset of $\mathbb{P}_2$. Find two vectors $\mathbf{v}_1, \mathbf{v}_2$ in $\mathbb{P}_2$ so that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and hence conclude that H is a subspace of $\mathbb{P}_2$. (Note that the vectors $\mathbf{v}_1, \mathbf{v}_2$ are not unique.)

(b) Let $p_1(t) = 1 - t^2$ and $p_2(t) = 1 + t^2$, and let $S = \{p_1(t), p_2(t)\}$ in $\mathbb{P}_2$. Is the polynomial $q(t) = 3 - 2t^2$ in Span S? (Hint: Create a matrix equation of the form $A\mathbf{x} = \mathbf{b}$ by setting up an appropriate polynomial equation involving $p_1(t)$, $p_2(t)$ and $q(t)$. Under what conditions on A is the system $A\mathbf{x} = \mathbf{b}$ consistent?)

(c) With S as in part (b), describe as best you can the subspace Span S of $\mathbb{P}_2$.

Given a subspace H, the set S such that $H = \text{Span } S$ is called a *spanning set* of H. In order to determine if a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a spanning set for H, all we need to do is to show that for every $\mathbf{b}$ in H, the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{b}$$

has a solution. We will see important uses of special spanning sets called bases in the rest of this chapter.

Examples

What follows are worked examples that use the concepts from this section.

Example 22.7. Determine if each of the following sets is a vector space.

(a) $V = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ with addition and multiplication by scalars defined by

$$(a, b, c) \oplus (x, y, z) = (a + x, c + z, b + y) \quad \text{and} \quad k(x, y, z) = (kx, kz, ky),$$

where (a, b, c) and (x, y, z) are in V and $k \in \mathbb{R}$

(b) $V = \{x \in \mathbb{R} : x > 0\}$ with addition $\oplus$ and multiplication by scalars defined by

$$x \oplus y = xy \text{ and } kx = x^k,$$

where x and y are in V, $k \in \mathbb{R}$, and xy is the standard product of x and y

(c) The set W of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ where a and b are real numbers using the standard addition and multiplication by scalars on matrices.

(d) The set W of all functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $f(0) \geq 0$ using the standard addition and multiplication by scalars on functions.

Example Solution.

(a) We consider the vector space properties in Definition 22.1. Let (a, b, c), (u, v, w), and (x, y, z) be in V and let $k, m \in \mathbb{R}$. By the definition of addition and multiplication by scalars, both $(a, b, c) + (x, y, z)$ and $k(x, y, z)$ are in V. Note also that

$$\begin{aligned}(a, b, c) \oplus (x, y, z) &= (a + x, c + z, b + y) \\ &= (x + a, z + c, y + b) \\ &= (x, y, z) \oplus (a, b, c),\end{aligned}$$

and so addition is commutative in V.

Since

$$((1, 1, 0) \oplus (0, 1, 1)) \oplus (0, 0, 1) = (1, 1, 2) \oplus (0, 0, 1) = (1, 3, 1)$$

and

$$(1, 1, 0) \oplus ((0, 1, 1) \oplus (0, 0, 1)) = (1, 1, 0) \oplus (0, 2, 1) = (1, 1, 3),$$

we see that addition is not associative and conclude that V is not a vector space. At this point we can stop since we have shown that V is not a vector space.

(b) We consider the vector space properties in Definition 22.1. Let x, y, and z be in V and let $k, m \in \mathbb{R}$. Since x and y are both positive real numbers, we know that xy is a positive real number. Thus, $x \oplus y \in V$ and V is closed under its addition. Also, x^k is a positive real number, so $x^k \in V$ as well.

Now

$$x \oplus y = xy = yx = y \oplus x$$

and addition is commutative in V.

Also,

$$(x \oplus y) \oplus z = (xy) \oplus z = (xy)z = x(yz) = x \oplus (yz) = x \oplus (y \oplus z)$$

and addition is associative in V.

Since

$$1 \oplus x = 1x = x,$$

V contains an additive identity, which is 1. The fact that x is a positive real number implies that $\frac{1}{x}$ is a positive real number. Thus, $\frac{1}{x} \in V$ and

$$x \oplus \frac{1}{x} = x\left(\frac{1}{x}\right) = 1$$

and V contains an additive inverse for each of its elements.

We have that

$$\begin{aligned}
(k+m)x &= x^{k+m} = x^k x^m = x^k \oplus x^m = k(x) \oplus m(x), \\
k(x \oplus +y) &= k(xy) = x^k y^k = x^k \oplus y^k = k(x) \oplus k(y), \\
(km)x &= x^{km} = (x^m)^k = (m(x))^k = k(m(x)) \\
1x &= x^1 = x.
\end{aligned}$$

So V satisfies all of the properties of a vector space.

(c) Recall that $\mathcal{M}_{2\times 2}$ is a vector space using the standard addition and multiplication by scalars on matrices. Any matrix of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ can be written as

$$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

So $W = \text{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$ and W is a subspace of $\mathcal{M}_{2\times 2}$. Thus, W is a vector space.

(d) We will show that W is not a vector space. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1$. Then $f(0) \geq 0$ and $f \in W$. However, if $h = (-1)f$, then $h(0) = (-1)f(0) = -1$ and $h \notin W$. It follows that W is not closed under multiplication by scalars and W is not a vector space.

Example 22.8. Let V be a vector space and $\mathbf{u}$ and $\mathbf{v}$ vectors in V. Also, let a and b be scalars. You may use the result of Exercise 4 that $c\mathbf{0} = \mathbf{0}$ for any scalar c in any vector space.

(a) If $a\mathbf{v} = b\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$, must $a = b$? Use the properties of a vector space or provide a counterexample to justify your answer.

(b) If $a\mathbf{u} = a\mathbf{v}$ and $a \neq 0$, must $\mathbf{u} = \mathbf{v}$? Use the properties of a vector space or provide a counterexample to justify your answer.

(c) If $a\mathbf{u} = b\mathbf{v}$, must $a = b$ and $\mathbf{u} = \mathbf{v}$? Use the properties of a vector space or provide a counterexample to justify your answer.

Example Solution.

(a) We will show that this statement is true. Suppose $a\mathbf{v} = b\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$. Then $\mathbf{0} = a\mathbf{v} - b\mathbf{v} = (a-b)\mathbf{v}$. If $a = b$, then we are done. So suppose $a \neq b$. Then $a - b \neq 0$ and $\frac{1}{a-b}$ is a real number. Then

$$\begin{aligned} \frac{1}{a-b}\mathbf{0} &= \frac{1}{a-b}((a-b)\mathbf{v}) \\ \mathbf{0} &= \left(\frac{1}{a-b}(a-b)\right)\mathbf{v} \\ \mathbf{0} &= \mathbf{v}. \end{aligned}$$

But we assumed that $\mathbf{v} \neq \mathbf{0}$, so we can conclude that $a = b$ as desired.

(b) We will show that this statement is true. Suppose $a\mathbf{u} = a\mathbf{v}$ and $a \neq 0$. Then $\mathbf{0} = a\mathbf{u} - a\mathbf{v} = a(\mathbf{u} - \mathbf{v})$. Since $a \neq 0$, we know that $\frac{1}{a}$ is a real number. Thus,

$$\begin{aligned} \frac{1}{a}\mathbf{0} &= \frac{1}{a}\left(a(\mathbf{u}-\mathbf{v})\right) \\ \mathbf{0} &= \left(\frac{1}{a}a\right)(\mathbf{u}-\mathbf{v}) \\ \mathbf{0} &= \mathbf{u}-\mathbf{v} \\ \mathbf{u} &= \mathbf{v}. \end{aligned}$$

(c) We will demonstrate that this statement is false with a counterexample. Let $a = 1$, $b = 2$, $\mathbf{u} = [2\ 0]^{\mathsf{T}}$ and $\mathbf{v} = [1\ 0]^{\mathsf{T}}$ in $\mathbb{R}^2$. Then

$$a\mathbf{u} = 1[2\ 0]^{\mathsf{T}} = [2\ 0]^{\mathsf{T}} = 2[1\ 0]^{\mathsf{T}} = b\mathbf{v},$$

but $a \neq b$ and $\mathbf{u} \neq \mathbf{v}$.

Summary

- A set V on which an operation of addition and a multiplication by scalars is defined is a vector space if for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars a and b:

  (1) $\mathbf{u} + \mathbf{v}$ is an element of V,

  (2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$,

  (3) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$,

  (4) there is a zero vector $\mathbf{0}$ in V so that $\mathbf{u} + \mathbf{0} = \mathbf{u}$,

  (5) for each $\mathbf{x}$ in V there is an element $\mathbf{y}$ in V so that $\mathbf{x} + \mathbf{y} = \mathbf{0}$,

  (6) $a\mathbf{u}$ is an element of V,

  (7) $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$,

  (8) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$,

  (9) $(ab)\mathbf{u} = a(b\mathbf{u})$,

  (10) $1\mathbf{u} = \mathbf{u}$.

- A subset H of a vector space V is a subspace of V if

(1) whenever $\mathbf{u}$ and $\mathbf{v}$ are in H it is also true that $\mathbf{u}+\mathbf{v}$ is in H,

(2) whenever $\mathbf{u}$ is in H and a is a scalar it is also true that $a\mathbf{u}$ is in H,

(3) $\mathbf{0}$ is in H.

- A linear combination of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ in a vector space V is a vector of the form

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k,$$

where x_1, x_2, ..., x_k are scalars.

- The span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ in a vector space V is the collection of all linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$. That is,

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k : x_1, x_2, \ldots, x_k \text{ are scalars}\}.$$

- The span of any finite set of vectors in a vector space V is always a subspace of V.

- This concept of vector space is important because there are many different types of sets (e.g., $\mathbb{R}^n$, $\mathcal{M}_{m\times n}$, $\mathbb{P}_n$, $\mathcal{F}$) that have similar structure, and we can relate them all as members of this larger collection of vector spaces.

Exercises

(1) The definition of a vector space only states the existence of a zero vector and does not say how many zero vectors the space might have. In this exercise we show that the zero vector in a vector space is unique. To show that the zero vector is unique, we assume that two vectors $\mathbf{0}_1$ and $\mathbf{0}_2$ have the zero vector property.

   (a) Using the fact that $\mathbf{0}_1$ is a zero vector, what vector is $\mathbf{0}_1 + \mathbf{0}_2$?

   (b) Using the fact that $\mathbf{0}_2$ is a zero vector, what vector is $\mathbf{0}_1 + \mathbf{0}_2$?

   (c) How do we conclude that the zero vector is unique?

(2) The definition of a vector space only states the existence of an additive inverse for each vector in the space, but does not say how many additive inverses a vector can have. In this exercise we show that the additive inverse of a vector in a vector space is unique. To show that a vector $\mathbf{v}$ has only one additive inverse, we suppose that $\mathbf{v}$ has two additive inverses, $\mathbf{u}$ and $\mathbf{w}$, and demonstrate that $\mathbf{u} = \mathbf{w}$.

   (a) What equations must $\mathbf{u}$ and $\mathbf{w}$ satisfy if $\mathbf{u}$ and $\mathbf{w}$ are additive inverses of $\mathbf{v}$?

   (b) Use the equations from part (a) to show that $\mathbf{u} = \mathbf{w}$. Clearly identify all vector space properties you use in your argument.

(3) Let V be a vector space and $\mathbf{v}$ a vector in V. In all of the vector spaces we have seen to date, the additive inverse of the vector $\mathbf{v}$ is equal to the scalar multiple $(-1)\mathbf{v}$. This seems reasonable, but it is important to note that this result is not stated in the definition of a vector

space, so this it is something that we need to verify. To show that $(-1)\mathbf{v}$ is an additive inverse of the vector $\mathbf{v}$, we need to demonstrate that

$$\mathbf{v} + (-1)\mathbf{v} = \mathbf{0}.$$

Verify this equation, explicitly stating which properties you use at each step.

(4) It is reasonable to expect that if c is any scalar and $\mathbf{0}$ is the zero vector in a vector space V, then $c\mathbf{0} = \mathbf{0}$. Use the fact that $\mathbf{0} + \mathbf{0} = \mathbf{0}$ to prove this statement.

(5) Let W_1, W_2 be two subspaces of a vector space V. Determine whether $W_1 \cap W_2$ and $W_1 \cup W_2$ are subspaces of V. Justify each answer clearly.

(6) Find three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to express

$$W = \left\{ \begin{bmatrix} a + 2b + c \\ b - 3c \\ a - c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

as $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. How does this justify why W is a subspace of $\mathbb{R}^3$?

(7) Find three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to express

$$W = \left\{ \begin{bmatrix} a + b & a - 2c \\ 3b + c & a + b - c \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

as $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. How does this justify why W is a subspace of $\mathcal{M}_{2\times 2}$?

(8) Let $\mathcal{F}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$, where we define the sum of two functions f and g in $\mathcal{F}$ as the function $f + g$ satisfying

$$(f + g)(x) = f(x) + g(x)$$

for all real numbers x, and the scalar multiple cf of the function f by the scalar c to be the function satisfying

$$(cf)(x) = cf(x)$$

for all real numbers x. Show that $\mathcal{F}$ is a vector space using these operations.

(9) Prove Theorem 22.6. (Hint: Compare to Theorem 12.5).

(10) Determine if each of the following sets of elements is a vector space or not. If appropriate, you can identify a set as a subspace of another vector space, or as a span of a collection of vectors to shorten your solution.

(a) A line through the origin in $\mathbb{R}^n$.

(b) The first quadrant in $\mathbb{R}^2$.

(c) The set of vectors $\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \text{ in } \mathbb{Z} \right\}$.

(d) The set of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.

(e) The set of all functions from $\mathbb{R}$ to $\mathbb{R}$ which are increasing for every x. (Assume that a function f is increasing if $f(a) > f(b)$ whenever $a > b$.)

(f) The set of all functions f from $\mathbb{R}$ to $\mathbb{R}$ for which $f(c) = 0$ for some fixed c in $\mathbb{R}$.

(g) The set of polynomials of the form $a + bt$, where $a + b = 0$.

(h) The set of all upper triangular 4×4 real matrices.

(i) The set of complex numbers $\mathbb{C}$ where scalar multiplication is defined as multiplication by real numbers.

(11) A reasonable way to extend the idea of the vector space $\mathbb{R}^n$ to infinity is to let $\mathbb{R}^\infty$ be the set of all sequences of real numbers. Define addition and multiplication by scalars on $\mathbb{R}^\infty$ by

$$\{x_n\} + \{y_n\} = \{x_n + y_n\} \text{ and } c\{x_n\} = \{cx_n\}$$

where $\{x_n\}$ denotes the sequence $x_1, x_2, x_3, \ldots$, $\{y_n\}$ denotes the sequence $y_1, y_2, y_3, \ldots$ and c is a scalar.

(a) Show that $\mathbb{R}^\infty$ is a vector space using these operations.

(b) Is the set of sequences that have infinitely many zeros a subspace of $\mathbb{R}^\infty$? Verify your answer.

(c) Is the set of sequences which are eventually zero a subspace of $\mathbb{R}^\infty$? Verify your answer. (A sequence $\{x_n\}$ is *eventually zero* if there is an index k_0 such that $x_n = 0$ whenever $n \geq k_0$.)

(d) Is the set of decreasing sequences a subspace of $\mathbb{R}^\infty$? Verify your answer. (A sequence $\{x_n\}$ is *decreasing* if $x_{n+1} \leq x_n$ for each n.)

(e) Is the set of sequences in $\mathbb{R}^\infty$ that have limits at infinity a subspace of $\mathbb{R}^\infty$?

(f) Let ℓ^2 be the set of all square summable sequences in $\mathbb{R}^\infty$, that is sequences $\{x_n\}$ so that $\sum_{k=1}^{\infty} x_k^2$ is finite. So, for example, the sequence $\left\{\frac{1}{n}\right\}$ is in ℓ^2. Show that ℓ^2 is a subspace of $\mathbb{R}^\infty$ (the set ℓ^2 is an example of what is called a *Hilbert space* by defining the inner product $\langle\{x_n\}, \{y_n\}\rangle = \sum_{n=1}^{\infty} x_n y_n$). (Hint: show that $2u^2 + 2v^2 - (u+v)^2 \geq 0$ for any real numbers u and v.)

(12) Given two subspaces H_1, H_2 of a vector space V, define

$$H_1 + H_2 = \{\mathbf{w} \mid \mathbf{w} = \mathbf{u} + \mathbf{v} \text{ where } \mathbf{u} \text{ in } H_1, \mathbf{v} \text{ in } H_2\}.$$

Show that $H_1 + H_2$ is a subspace of V containing both H_1, H_2 as subspaces. The space $H_1 + H_2$ is the sum of the subspaces H_1 and H_2.

(13) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** The intersection of any two subspaces of V is also a subspace.

(b) **True/False** The union of any two subspaces of V is also a subspace.

(c) **True/False** If H is a subspace of a vector space V, then $-H = \{(-1)\mathbf{v} : \mathbf{v} \text{ in } H\}$ is equal to H.

(d) **True/False** If $\mathbf{v}$ is a nonzero vector in H, a subspace of $\mathbb{R}^n$, then H contains the line through the origin and $\mathbf{v}$ in $\mathbb{R}^n$.

(e) **True/False** If $\mathbf{v}_1, \mathbf{v}_2$ are nonzero, non-parallel vectors in H, a subspace of $\mathbb{R}^n$, then H contains the plane through the origin, $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$.

(f) **True/False** The smallest subspace in $\mathbb{R}^n$ containing a vector $\mathbf{v}$ is a line through the origin.

(g) **True/False** The largest subspace of V is V.

(h) **True/False** The space $\mathbb{P}_1$ is a subspace of $\mathbb{P}_n$ for $n \geq 1$.

(i) **True/False** The set of constant functions from $\mathbb{R}$ to $\mathbb{R}$ is a subspace of $\mathcal{F}$.

(j) **True/False** The set of all polynomial functions with rational coefficients is a subspace of $\mathcal{F}$.

Project: Hamming Codes and the Hat Puzzle

Recall the hat problem from the beginning of this section. Three players are assigned either a red or blue hat and can only see the colors of the hats of the other players. The goal is to devise a high probability strategy for one player to correctly guess the color of their hat. The players have a 50% chance of winning if one player guesses randomly and all of the others pass. However, the group can do better than 50% with a reasonably simple strategy. There are 2 possibilities for each hat color for a total of $2^3 = 8$ possible distributions of hat colors. Of these, only red-red-red and blue-blue-blue contain only one hat color, so $6/8$ of $3/4$ of the possible hat distributions have two hats of one color and one of the other color. So if a player sees two hats of the same color, that player guesses the other color and passes otherwise. This gives a 75% chance of winning. This strategy will only work for three players, though. We want to develop an effective strategy that works for larger groups of players.

There is a strategy, based on *Hamming codes* that can be utilized when the number of players is of the form $2^k - 1$ with $k \geq 2$. This strategy will provide a winning probability of

$$1 - 2^{-k}.$$

Note that as $k \to \infty$, this probability has a limit of 1. Note also that if $k = 2$ (so that there are 3 players), then the probability is $\frac{3}{4}$ or 75% – the same strategy we came up with earlier.

To understand this strategy, we need to build a slightly different kind of vector space than we have seen until now, one that is based on a binary choice of red or blue. To do so, we identify the hat colors with numbers – 0 for red and 1 for blue. So let $\mathbb{F} = \{0, 1\}$. Assume there are $n = 2^k - 1$ players for some integer $k \geq 2$. We can then view a distribution of hats among the $n = 2^k - 1$ players as a vector with n components from $\mathbb{F}$. That is,

$$\mathbb{F}^n = \{[\alpha_1\ \alpha_2\ \cdots\ \alpha_n]^{\mathsf{T}} : \alpha_i \in \mathbb{F}\}.$$

We can give some structure to both $\mathbb{F}$ and $\mathbb{F}^n$ by noting that we can define addition and multiplication in $\mathbb{F}$ by

$$\begin{array}{ccc} 0+0=0, & 0+1=1+0=1, & 1+1=0 \\ 0\cdot 0=0, & 0\cdot 1=1\cdot 0=0, & 1\cdot 1=1. \end{array}$$

Project Activity 22.1. Show that $\mathbb{F}$ has the same structure as $\mathbb{R}$. That is, show that for all x, y, and z in $\mathbb{F}$, the following properties are satisfied.

(a) $x+y \in \mathbb{F}$ and $xy \in \mathbb{F}$

(b) $x+y = y+x$ and $xy = yx$

(c) $(x+y)+z = x+(y+z)$ and $(xy)z = x(yz)$

(d) There is an element 0 in $\mathbb{F}$ such that $x+0=x$

(e) There is an element 1 in $\mathbb{F}$ such that $(1)x = x$

(f) There is an element $-x$ in $\mathbb{F}$ such that $x+(-x)=0$

(g) If $x \neq 0$, there is an element $\frac{1}{x}$ in $\mathbb{F}$ such that $x\left(\frac{1}{x}\right)=1$

(h) $x(y+z) = (xy)+(xz)$

Project Activity 22.1 shows that $\mathbb{F}$ has the same properties as $\mathbb{R}$ – that is that $\mathbb{F}$ is a field. Until now, we have worked with vector spaces whose scalars come from the set of real numbers, but that is not necessary. None of the results we have discovered so far about vector spaces require our scalars to come from $\mathbb{R}$. In fact, we can replace $\mathbb{R}$ with any field and all of the same vector space properties hold. It follows that $V = \mathbb{F}^n$ is a vector space over $\mathbb{F}$. As we did in $\mathbb{R}^n$, we define the standard unit vectors $\mathbf{e}_1 = [1\ 0\ 0\ \cdots\ 0]^\mathsf{T}$, $\mathbf{e}_2 = [0\ 1\ 0\ 0\ \ldots\ 0]^\mathsf{T}$, …, $\mathbf{e}_n = [0\ 0\ 0\ \ldots\ 0\ 1]^\mathsf{T}$ in $V = \mathbb{F}^n$.

Now we return to the hat puzzle. We have $n = 2^k - 1$ players. Label the players 1, 2, …, n. We can now represent a random placements of hats on heads as a vector $\mathbf{v} = [\alpha_1\ \alpha_2\ \cdots\ \alpha_n]^\mathsf{T}$ in $V = \mathbb{F}^n$, where $\alpha_i = 0$ in the ith entry represents a red hat and $\alpha_i = 1$ a blue hat on player i. Since player i can see all of the other hats, from player i's perspective the distribution of hats has the form

$$\mathbf{v} = \mathbf{v}_i + \beta_i \mathbf{e}_i,$$

where β_i is the unknown color of hat on player i's head and

$$\mathbf{v}_i = [\alpha_1\ \alpha_2\ \cdots\ \alpha_{i-1}\ 0\ \alpha_{i+1}\ \cdots \alpha_n]^\mathsf{T}.$$

In order to analyze the vectors $\mathbf{v}$ from player i's perspective and to devise an effective strategy, we will partition the set V into an appropriate disjoint union of subsets.

To provide a different way to look at players, we will use a subspace of V. Let W be a subspace of V that has a basis of k vectors. The elements of W are the linear combinations of k basis vectors, and each basis vector in a linear combination has 2 possibilities for its weight (from $\mathbb{F}$). Thus, W contains exactly $2^k = n+1$ vectors. We can then use the $n = 2^k - 1$ nonzero vectors in W to represent our players. Each distribution of hats can be seen as a linear combination of the vectors in W. Let $\mathbf{w}_1$, $\mathbf{w}_2$, ..., $\mathbf{w}_{2^k-1}$ be the nonzero vectors in W. We then define a function $\varphi : V \to W$ as

$$\varphi([\alpha_1\ \alpha_2\ \cdots\ \alpha_n]^{\mathsf{T}}) = \sum_{i=1}^{n} \alpha_i \mathbf{w}_i$$

that identifies a distribution of hats with a vector in W. The subspace that we need to devise our strategy is what is called a Hamming code.

Project Activity 22.2. Let

$$H = \left\{ [\alpha_1\ \alpha_2\ \cdots\ \alpha_n]^{\mathsf{T}} \in V : \sum_{i=1}^{n} \alpha_i \mathbf{w}_i = \mathbf{0} \right\}.$$

Show that H is a subspace of V. (The subspace H is called the $\left(2^k - 1, 2^k - k - 1\right)$ *Hamming code* (where the first component is the number of elements in a basis for V and the second the number of elements in a basis for H). Hamming codes are examples of linear codes – those codes that are subspaces of the larger vector space.)

Now for each i between 0 and n we define $H_i = \mathbf{e}_i + H$ as

$$H_i = \mathbf{e}_i + H = \{\mathbf{e}_i + \mathbf{h} : \mathbf{h} \in H\},$$

where we let $\mathbf{e}_0 = \mathbf{0}$. The sets H_i are called *cosets* of H.

Project Activity 22.3. To complete our strategy for the hat puzzle, we need to know some additional information about the sets H_i.

(a) Show that the sets H_i are disjoint. That is, show that $H_i \cap H_j = \emptyset$ if $i \neq j$. (Hint: If $\mathbf{v} \in H_i$ and $\mathbf{v} \in H_j$, what can we say about $\mathbf{e}_i - \mathbf{e}_j$?)

(b) Since $H_i \subseteq V$ for each i, it follows that $\bigcup_{i=0}^{n} H_i \subseteq V$. Now we show that $V = \bigcup_{i=0}^{n} H_i$ by demonstrating that $\bigcup_{i=0}^{n} H_i$ has exactly the same number of elements as V. We will need one fact for our argument. We will see in a later section that H has a basis of $n - k$ elements, so the number of elements in H is 2^{n-k}.

   i. Since the sets H_i are disjoint, the number of elements in $\bigcup_{i=1}^{n} H_i$ is equal to the sum of the number of elements in each H_i. Show that each H_i has the same number of elements as H.

   ii. Now use the fact that the number of elements in $\bigcup_{i=0}^{n} H_i$ is equal to the sum of the number of elements in each H_i to argue that $V = \bigcup_{i=0}^{n} H_i$.

The useful idea from Project Activity 22.3 is that any hat distribution in V is in exactly one of the sets H_i. Recall that a hat distribution $\mathbf{v} = [\alpha_1\ \alpha_2\ \cdots\ \alpha_n]^{\mathsf{T}}$ in V can be written from player i's perspective as

$$\mathbf{v} = \mathbf{v}_i + \beta_i \mathbf{e}_i,$$

where $\mathbf{v}_i = [\alpha_1\ \alpha_2\ \cdots\ \alpha_{i-1}\ 0\ \alpha_{i+1}\ \cdots\ \cdots\ \alpha_n]^{\mathsf{T}}$. Our strategy for the hat game can now be revealed.

- If $\mathbf{v}_i + \beta_i \mathbf{e}_i$ is not in H for either choice of β_i, then player i should pass.

- If $\mathbf{v}_i + \beta_i \mathbf{e}_i$ is in H, then player i guesses $1 + \beta_i$.

Project Activity 22.4. Let us analyze this strategy.

(a) Explain why every player guesses wrong if $\mathbf{v}$ is in H.

(b) Now we see determine that our strategy is a winning strategy for all hat distributions $\mathbf{v}$ that are not in H. First we need to know that these two options are the only ones. That is, show that it is not possible for $\mathbf{v}_i + \beta_i \mathbf{e}_i$ to be in H for both choices of β_i.

(c) Now we want to demonstrate that this is a winning strategy if $\mathbf{v} \notin H$. That is, at least one player guesses a correct hat color and no one else guesses incorrectly. So assume $\mathbf{v} \notin H$.

i. We know that $\mathbf{v} \in H_i$ for some unique choice of i, so let $\mathbf{v} = \mathbf{e}_i + h$ for some $h \in H$. Explain why player i can correctly choose color $1 + \alpha_i$.

ii. Finally, we need to argue that every player except player i must pass. So consider player j, with $j \neq i$. Recall that

$$\mathbf{v} = \mathbf{v}_j + \alpha_j \mathbf{e}_j.$$

Analyze our strategy and the conditions under which player j does not pass. Show that this leads to a contradiction.

Project Activity 22.4 completes our analysis of this strategy and shows that our strategy results in a win with probability

$$1 - \frac{|H|}{|V|} = 1 - \frac{2^{2^k - k - 1}}{2^{2^k - 1}} = 1 - 2^{-k}.$$

Section 23

Bases for Vector Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What does it mean for a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V to be linearly independent?
- What is another equivalent characterization of a linearly independent set?
- What does is mean for a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V to be linearly dependent?
- Describe another characterization of a linearly dependent set.
- What is a basis for a vector space V?
- What makes a basis for a vector space useful?
- How can we find a basis for a vector space V?

Application: Image Compression

> If you painted a picture with a sky, clouds, trees, and flowers, you would use a different size brush depending on the size of the features. Wavelets are like those brushes.
> – Ingrid Daubechies

The advent of the digital age has presented many new opportunities for the collection, analysis, and dissemination of information. Along with these opportunities come new difficulties as well. All of this digital information must be stored in some way and be retrievable in an efficient manner. One collection of tools that is used to deal with these problems is wavelets. For example, The FBI fingerprint files contain millions of cards, each of which contains 10 rolled fingerprint impressions. Each card produces about 10 megabytes of data. To store all of these cards would require an enormous amount of space, and transmitting one full card over existing data lines is slow and inefficient. Without some sort of image compression, a sortable and searchable electronic fingerprint database would be next to impossible. To deal with this problem, the FBI adopted standards for fingerprint digitization using a wavelet compression standard.

Another problem with electronics is noise. Noise can be a big problem when collecting and transmitting data. Wavelet decomposition filters data by averaging and detailing. The detailing coefficients indicate where the details are in the original data set. If some details are very small in relation to others, eliminating them may not substantially alter the original data set. Similar ideas may be used to restore damaged audio,[1] video, photographs, and medical information.[2]

We will consider wavelets as a tool for image compression. The basic idea behind using wavelets to compress images is that we start with a digital image, made up of pixels. Each pixel can be assigned a number or a vector (depending on the makeup of the image). The image can then be represented as a matrix (or a set of matrices) M, where each entry in M represents a pixel in the image. As a simple example, consider the 16×16 image of a flower as shown at left in Figure 23.1. (We will work with small images like this to make the calculations more manageable, but the ideas work for any size image. We could also extend our methods to consider color images, but for the sake of simplicity we focus on grayscale.) This flower image is a gray-scale image, so each

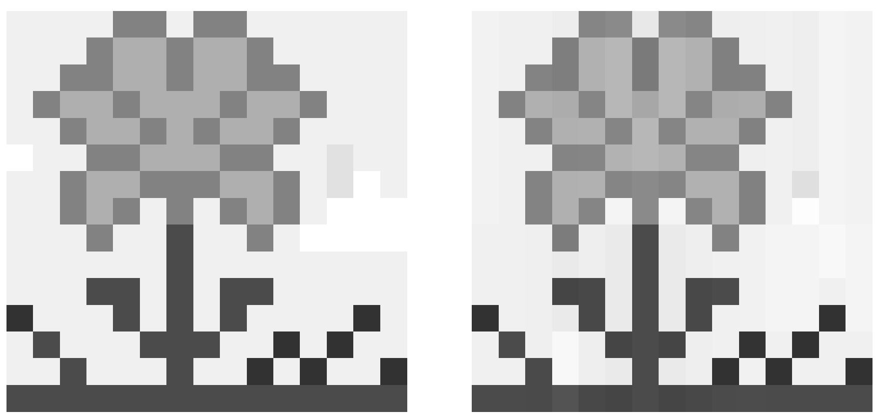

Figure 23.1: Left: A 16 by 16 pixel image. Right: The image compressed.

pixel has a numeric representation between 0 and 255, where 0 is black, 255 is white, and numbers

[1] see https://ccrma.stanford.edu/groups/edison/brahms/brahms.html for a discussion of the denoising of a Brahms recording

[2] A review of wavelets in biomedical applications. M. Unser, A. Aldroubi. *Proceedings of the IEEE*, Volume: 84, Issue: 4 , Apr 1996

between 0 and 255 represent shades of gray. The matrix for this flower image is

$$\begin{bmatrix}
240 & 240 & 240 & 240 & 130 & 130 & 240 & 130 & 130 & 240 & 240 & 240 & 240 & 240 & 240 & 240 \\
240 & 240 & 240 & 130 & 175 & 175 & 130 & 175 & 175 & 130 & 240 & 240 & 240 & 240 & 240 & 240 \\
240 & 240 & 130 & 130 & 175 & 175 & 130 & 175 & 175 & 130 & 130 & 240 & 240 & 240 & 240 & 240 \\
240 & 130 & 175 & 175 & 130 & 175 & 175 & 175 & 130 & 175 & 175 & 130 & 240 & 240 & 240 & 240 \\
240 & 240 & 130 & 175 & 175 & 130 & 175 & 130 & 175 & 175 & 130 & 240 & 240 & 240 & 240 & 240 \\
255 & 240 & 240 & 130 & 130 & 175 & 175 & 175 & 130 & 130 & 240 & 240 & 225 & 240 & 240 & 240 \\
240 & 240 & 130 & 175 & 175 & 130 & 130 & 130 & 175 & 175 & 130 & 240 & 225 & 255 & 240 & 240 \\
240 & 240 & 130 & 175 & 130 & 240 & 130 & 240 & 130 & 175 & 130 & 240 & 255 & 255 & 255 & 240 \\
240 & 240 & 240 & 130 & 240 & 240 & 75 & 240 & 240 & 130 & 240 & 255 & 255 & 255 & 255 & 255 \\
240 & 240 & 240 & 240 & 240 & 240 & 75 & 240 & 240 & 240 & 240 & 240 & 240 & 240 & 240 & 240 \\
240 & 240 & 240 & 75 & 75 & 240 & 75 & 240 & 75 & 75 & 240 & 240 & 240 & 240 & 240 & 240 \\
50 & 240 & 240 & 240 & 75 & 240 & 75 & 240 & 75 & 240 & 240 & 240 & 240 & 50 & 240 & 240 \\
240 & 75 & 240 & 240 & 240 & 75 & 75 & 75 & 240 & 240 & 50 & 240 & 50 & 240 & 240 & 50 \\
240 & 240 & 75 & 240 & 240 & 240 & 75 & 240 & 240 & 50 & 240 & 50 & 240 & 240 & 50 & 240 \\
75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 \\
75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75 & 75
\end{bmatrix}. \tag{23.1}$$

Now we can apply wavelets to the image and compress it. Essentially, wavelets act by averaging and differencing. The averaging creates smaller versions of the image and the differencing keeps track of how far the smaller version is from a previous copy. The differencing often produces many small (close to 0) entries, and so replacing these entries with 0 doesn't have much effect on the image (this is called *thresholding*). By introducing long strings of zeros into our data, we are able to store a (compressed) copy of the image in a smaller amount of space. For example, using a threshold value of 10 produces the flower image shown at right in Figure 23.1.

The averaging and differencing is done with special vectors (wavelets) that form a basis for a suitable function space. More details of this process can be found at the end of this section.

Introduction

In $\mathbb{R}^n$ we defined a basis for a subspace W of $\mathbb{R}^n$ to be a minimal spanning set for W, or a linearly independent spanning set (see Definition 6.6). So to consider the idea of a basis in a vector space, we will need the notion of linear independence in that context.

Since we can add vectors and multiply vectors by scalars in any vector space, and because we have a zero vector in any vector space, we can define linear independence of a finite set of vectors in any vector space as follows (compare to Definition 6.1).

Definition 23.1. A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V is **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

for scalars $x_1, x_2, \ldots, x_k$ has only the trivial solution

$$x_1 = x_2 = x_3 = \cdots = x_k = 0.$$

If a set of vectors is not linearly independent, then the set is **linearly dependent**.

Alternatively, we say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent (or dependent) if the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent (or dependent).

Preview Activity 23.1.

(1) We can use the tools we developed to determine if a set of vectors in $\mathbb{R}^n$ is linearly independent to answer the same questions for sets of vectors in other vector spaces. For example, consider the question of whether the set $\{1+t, 1-t\}$ in $\mathbb{P}_1$ is linearly independent or dependent. To answer this question we need to determine if there is a non-trivial solution to the equation

$$x_1(1+t) + x_2(1-t) = 0. \tag{23.2}$$

Note that equation (23.2) can also be written in the form

$$(x_1+x_2) + (x_1-x_2)t = 0.$$

(a) Recall that two polynomials are equal if all coefficients of like powers are the same. By equating coefficients of like power terms, rewrite equation (23.2) as an equivalent system of two equations in the two unknowns x_1 and x_2, and solve for x_1, x_2.

(b) What does your answer to the previous part tell you about the linear independence or dependence of the set $\{1+t, 1-t\}$ in $\mathbb{P}_1$?

(c) Recall that in $\mathbb{R}^n$, a set of two vectors is linearly dependent if and only if one of the vectors in the set is a scalar multiple of the other and linearly independent if neither vector is a scalar multiple of the other. Verify your answer to part (c) from a similar perspective in $\mathbb{P}_1$.

(2) We can use the same type of method as in problem (1) to address the question of whether the set

$$\left\{ \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -9 \\ 1 & 8 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \right\}$$

is linearly independent or dependent in $\mathcal{M}_{2\times 2}$. To answer this question we need to determine if there is a non-trivial solution to the equation

$$x_1 \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 & -9 \\ 1 & 8 \end{bmatrix} + x_3 \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} = \mathbf{0} \tag{23.3}$$

for some scalars x_1, x_2, and x_3. Note that the linear combination on the left side of equation (23.3) has entries

$$\begin{bmatrix} x_1+x_2+x_3 & 3x_1-9x_2-x_3 \\ x_1+x_2+x_3 & 2x_1+8x_2+4x_3 \end{bmatrix}.$$

(a) Recall that two matrices are equal if all corresponding entries are the same. Equate corresponding entries of the matrices in equation (23.3) to rewrite the equation as an equivalent system of four equations in the three unknowns x_1, x_2, and x_3.

(b) Use appropriate matrix tools and techniques to find all solutions to the system from part (a).

(c) What does the set of solutions to the system from part (a) tell you about the linear independence or dependence of the set

$$\left\{ \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -9 \\ 1 & 8 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \right\}?$$

(d) Recall that in $\mathbb{R}^n$, a set of vectors is linearly dependent if and only if one of the vectors in the set is a linear combination of the others and linearly independent if no vector in the set is a linear combination of the others. Verify your answer to part (c) from a similar perspective in $\mathcal{M}_{2\times 2}$.

(3) We will define a basis for a vector space to be a linearly independent spanning set. Which, if any, of the sets in parts (1) and (2) is a basis for its vector space? Explain.

Linear Independence

The concept of linear independence, which we formally defined in Preview Activity 23.1, provides us with a process to determine if there is redundancy in a spanning set to obtain an efficient spanning set.

The definition tells us that a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V is linearly dependent if there are scalars x_1, x_2, ..., x_n, not all of which are 0 so that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}.$$

As examples, we saw in Preview Activity 23.1 that the set $\{1 + t, 1 - t\}$ is linearly independent in $\mathbb{P}_1$. The set $\{1 + t, -1 + 2t + t^2, 1 - 8t - 3t^2\}$, on the other hand, is linearly dependent in $\mathbb{P}_2$ since $2(1 + t) + 3(-1 + 2t + t^2) + (1 - 8t - 3t^2) = 0$.

In addition to the definition, there are other ways to characterize linearly independent and dependent sets in vector spaces as the next theorems illustrate. These characterizations are the same as those we saw in $\mathbb{R}^n$, and the proofs are essentially the same as well. The proof of Theorem 23.2 is similar to that of Theorem 6.2 and is left for the exercises.

Theorem 23.2. *A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V is linearly dependent if and only if at least one of the vectors in the set can be written as a linear combination of the remaining vectors in the set.*

Theorem 23.2 is equivalent to the following theorem that provides the corresponding result for linearly independent sets.

Theorem 23.3. *A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V is linearly independent if and only if no vector in the set can be written as a linear combination of the remaining vectors in the set.*

One consequence of Theorems 23.2 and 23.3 is that if a spanning set is linearly dependent, then one of the vectors in the set can be written as a linear combination of the others. In other words, at least one of the vectors is redundant. In that case, we can find a smaller spanning set as the next theorem states. The proof of this theorem is similar to that of Theorem 6.5 and is left for the exercises.

Theorem 23.4. *Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space V. If for some i between 1 and k, $\mathbf{v}_i$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_k\}$, then*

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_k\}.$$

Bases

A basis for a vector space is a spanning set that is as small as it can be. We already saw how to define bases formally in $\mathbb{R}^n$. We will now formally define a basis for a vector space and understand why with this definition a basis is a minimal spanning set. Bases are important because any vector in a vector space can be uniquely represented as a linear combination of basis vectors. We will see in later sections that this representation will allow us to identify any vector space with a basis of n vectors with $\mathbb{R}^n$.

To obtain the formal definition of a basis, which is a minimal spanning set, we consider what additional property makes a spanning set a *minimal* spanning set. As a consequence of Theorem 23.4, if S is a spanning set that is linearly dependent, then we can find a proper subset of S that has the same span. Thus, the set S cannot be a minimal spanning set. However, if S is linearly independent, then no vector in S is a linear combination of the others and we need all of the vectors in S to form the span. This leads us to the following formal characterization of a minimal spanning set, called a *basis*.

Definition 23.5. A **basis** for a vector space V is a subset S of V if

(1) Span $S = V$ and

(2) S is a linearly independent set.

In other words, a basis for a vector space V is a linearly independent spanning set for V. To put it another way, a basis for a vector space is a minimal spanning set for the vector space. Similar reasoning will show that a basis is also a maximal linearly independent set.

The key ideas to take from the previous theorems are:

- A basis for a vector space V is a minimal spanning set for V.
- A basis for V is a subset S of V so that
  (1) S spans V and
  (2) S is linearly independent.
- No vector in a basis can be written as a linear combination of the other vectors in the basis.
- If a subset S of a vector space V has the property that one of the vectors in S is a linear combination of the other vectors in S, then S is not a basis for V.

As an example of a basis of a vector space, we saw in Preview Activity 23.1 that the set $S = \{1-t, 1+t\}$ is both linearly independent and spans $\mathbb{P}_1$, and so S is a basis for $\mathbb{P}_1$.

Activity 23.1.

(a) Is $S = \{1+t, t, 1-t\}$ a basis for $\mathbb{P}_1$? Explain.

(b) Explain why the set $S = \{1, t, t^2, \ldots, t^n\}$ is a basis for $\mathbb{P}_n$. This basis is called the *standard basis* for $\mathbb{P}_n$.

(c) Show that the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for $\mathcal{M}_{2\times 2}$.

It should be noted that not every vector space has a finite basis. For example, the space $\mathbb{P}$ of all polynomials with real coefficients (of any degree) is a vector space, but no finite set of vectors will span $\mathbb{P}$. In fact, the infinite set $\{1, t, t^2, \ldots\}$ is both linearly independent and spans $\mathbb{P}$, so $\mathbb{P}$ has an infinite basis.

Finding a Basis for a Vector Space

We already know how to find bases for certain vector spaces, namely Nul A and Col A, where A is any matrix. Finding a basis for a different kind of vector space will require other methods. Since a basis for a vector space is a minimal spanning set, to find a basis for a given vector space we might begin from scratch, starting with a given vector in the space and adding one vector at a time until we have a spanning set.

Activity 23.2. Let $W = \{a + bt + ct^3 \mid a, b, c \text{ are scalars}\}$. We will find a basis of W that contains the polynomial $3 + t - 3t^3$.

(a) Let $\mathcal{S}_1 = \{3 + t - t^3\}$. Find a polynomial $p(t)$ in W that is not in Span $\mathcal{S}_1$. Explain why this means that the set $\mathcal{S}_1$ does not span W.

(b) Let $\mathcal{S}_2 = \{3 + t - t^3, p(t)\}$. Find a polynomial $q(t)$ that is not in Span $\mathcal{S}_2$. What does this mean about $\mathcal{S}_2$ being a possible spanning set of W?

(c) Let $\mathcal{S}_3 = \{3 + t - t^3, p(t), q(t)\}$. Explain why the set $\mathcal{S}_3$ is a basis for W.

Alternatively, we might construct a basis from a known spanning set.

Activity 23.3. Let $W = \left\{ \begin{bmatrix} v+z & w+z \\ x & y \end{bmatrix} \mid v, w, x, y, z \text{ are scalars} \right\}$. Assume that W is a subspace of $\mathcal{M}_{2\times 2}$.

(a) Find a set S of five 2×2 matrices that spans W (since W is a span of a set of vectors in $\mathcal{M}_{2\times 2}$, W is a subspace of $\mathcal{M}_{2\times 2}$). Without doing any computation, can this set S be a basis for W? Why or why not?

(b) Find a subset $\mathcal{B}$ of S that is a basis for W.

Activities 23.2 and 23.3 give us two ways of finding a basis for a subspace W of a vector space V, assuming W has a basis with finitely many vectors. One way (illustrated in Activity 23.2 is to start by choosing any non-zero vector $\mathbf{w}_1$ in W. Let $\mathcal{S}_1 = \{\mathbf{w}_1\}$. If $\mathcal{S}_1$ spans W, then $\mathcal{S}_1$ is a basis for W. If not, there is a vector $\mathbf{w}_2$ in W that is not in Span $\mathcal{S}_1$. Then $\mathcal{S}_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a linearly independent set. If Span $\mathcal{S}_2 = W$, then $\mathcal{S}_2$ is a basis for W and we are done. If not, repeat the process. We will show later that this process must stop as long as we know that W has a basis with finitely many vectors.

Another way (illustrated in Activity 23.3 to find a basis for W is to start with a spanning set $\mathcal{S}_1$ of W. If $\mathcal{S}_1$ is linearly independent, then $\mathcal{S}_1$ is a basis for W. If $\mathcal{S}_1$ is linearly dependent, then one vector in $\mathcal{S}_1$ is a linear combination of the others and we can remove that vector to obtain a new set $\mathcal{S}_2$ that also spans W. If $\mathcal{S}_2$ is linearly independent, then $\mathcal{S}_2$ is a basis for W. If not, we repeat the process as many times as needed until we arrive until at a subset $\mathcal{S}_k$ of $\mathcal{S}_1$ that is linearly independent and spans W. We summarize these results in the following theorem.

Theorem 23.6. *Let W be a subspace of a finite-dimensional vector space V. Then*

(1) any linearly independent subset of W can be extended to a basis of W,

(2) any subset of W that spans W can be reduced to a basis of W.

We conclude this section with the result mentioned in the introduction – that every vector in a vector space with basis $\mathcal{B}$ can be written in one and only one way as a linear combination of basis vectors. The proof is similar to that of Theorem 6.4 and is left to the exercises.

Theorem 23.7. *Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ be vectors in a vector space V that make up a basis $\mathcal{B}$ for V. If $\mathbf{u}$ is a vector in V, then $\mathbf{u}$ can be written in one and only one way as a linear combination of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ in $\mathcal{B}$.*

Examples

What follows are worked examples that use the concepts from this section.

Example 23.8. Let $S = \{1, 1+t, 2-t^2, 1+t+t^2, t-t^2\}$.

(a) Does S span $\mathbb{P}_2$? Explain.

(b) Explain why S is not a basis for $\mathbb{P}_2$.

(c) Find a subset of S that is a basis for $\mathbb{P}_2$. Explain your reasoning.

Example Solution.

(a) Let $p(t) = a_0 + a_1 t + a_2 t^2$ be an arbitrary vector in $\mathbb{P}_2$. If $p(t)$ is in Span S, then there are weights c_1, c_2, c_3, c_4, and c_5 such that

$$a_0 + a_1 t + a_2 t^2 = c_1(1) + c_2(1+t) + c_3(2-t^2) + c_4(1+t+t^2) + c_5(t-t^2).$$

Equating coefficients of like powers gives us the system

$$\begin{aligned} c_1 + c_2 + 2c_3 + c_4 &= a_0 \\ c_2 + c_4 + c_5 &= a_1 \\ -c_3 + c_4 - c_5 &= a_2. \end{aligned}$$

The reduced row echelon form of the coefficient matrix A is

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}.$$

Since there is a pivot in every row of A, the system $A\mathbf{x} = \mathbf{b}$ is always consistent. We conclude that S does span $\mathbb{P}_2$.

(b) The fact that the coefficient matrix A of our system has non-pivot columns means that each vector in $\mathbb{P}_2$ can be written in more than one way as a linear combination of vectors in S. This means that S is not linearly independent and so cannot be a basis for $\mathbb{P}_2$.

(c) That the first three columns of A are pivot columns implies that the polynomials 1, $1+t$, and $2-t^2$ are linearly independent. Since there is a pivot in every row of A, the three polynomials 1, $1+t$, and $2-t^2$ also span $\mathbb{P}_2$. So $\{1, 1+t, 2-t^2\}$ is a subset of S that is a basis for $\mathbb{P}_2$.

Example 23.9. Let U be the set of all matrices of real numbers of the form $\begin{bmatrix} u & -u-x \\ 0 & x \end{bmatrix}$ and W be the set of all real matrices of the form $\begin{bmatrix} v & 0 \\ w & -v \end{bmatrix}$.

(a) Find a basis for U and a basis for W.

(b) Let $U + W = \{A + B : A \text{ is in } U \text{ and } B \text{ is in } W\}$. Show that $U + W$ is a subspace of $\mathcal{M}_{2\times 2}$ and find a basis for $U + W$.

Example Solution.

(a) Every matrix in U has the form

$$\begin{bmatrix} u & -u-x \\ 0 & x \end{bmatrix} = u\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + x\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

Let $S_U = \left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right\}$. Then $U = \text{Span } S_U$ and U is a subspace of $\mathcal{M}_{2\times 2}$. If

$$c_1\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = 0,$$

then $c_1 = c_2 = 0$ and S_U is also linearly independent. This makes S_U a basis for U.

Similarly, every matrix in W has the form

$$\begin{bmatrix} v & 0 \\ w & -v \end{bmatrix} = v\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let $S_W = \left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$. Then $W = \text{Span } S_W$ and W is a subspace of $\mathcal{M}_{2\times 2}$. If

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0,$$

then $c_1 = c_2 = 0$ and S_W is also linearly independent. This makes S_W a basis for W.

(b) Every matrix in $U + W$ has the form

$$\begin{aligned}\begin{bmatrix} u & -u-x \\ 0 & x \end{bmatrix} + \begin{bmatrix} v & 0 \\ w & -v \end{bmatrix} &= \begin{bmatrix} u+v & -u-x \\ w & x-v \end{bmatrix} \\ &= u\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + x\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \\ &\quad + v\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.\end{aligned}$$

Let $S = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$. Then $U + W = \text{Span } S$ and $U + W$ is a subspace of $\mathcal{M}_{2\times 2}$. If

$$c_1\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0,$$

then

$$\begin{aligned} c_1 \quad\quad + c_3 \quad &= 0 \\ -c_1 - c_2 \quad\quad\quad &= 0 \\ c_4 &= 0 \\ c_2 - c_3 \quad &= 0. \end{aligned}$$

The reduced row echelon form of $\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The vectors that correspond to the pivot columns are linearly independent and span $U + W$, so a basis for $U + W$ is

$$\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Summary

The important idea in this section is that of a basis for a vector space. A basis is a minimal spanning set and another equivalent characterization of the "minimal" property is linear independence.

- A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V is linearly independent if the vector equation

  $$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

  for scalars $x_1, x_2, \ldots, x_k$ has only the trivial solution

  $$x_1 = x_2 = x_3 = \cdots = x_k = 0.$$

  If a set of vectors is not linearly independent, then the set is linearly dependent.

- A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V is linearly independent if and only if none of the vectors in the set can be written as a linear combination of the remaining vectors in the set.
- A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V is linearly dependent if and only if at least one of the vectors in the set can be written as a linear combination of the remaining vectors in the set.
- A basis for a vector space V is a subset S of V if
  (1) Span $S = V$ and
  (2) S is a linearly independent set.
- A basis is important in that it provides us with an efficient way to represent any vector in the vector space – any vector can be written in one and only one way as a linear combination of vectors in a basis.
- To find a basis of a vector space, we can start with a spanning set S and toss out any vector in S that can be written as a linear combination of the remaining vectors in S. We repeat the process with the remaining subset of S until we arrive at a linearly independent spanning set. Alternatively, we can find a spanning set for the space and remove any vector that is a linear combination of the others in the spanning set. We can repeat this process until we wind up with a linearly independent spanning set.

Exercises

(1) Determine if the given sets are linearly independent or dependent in the indicated vector space. If dependent, write one of the vectors as a linear combination of the others. If independent, determine if the set is a basis for the vector space.

(a) $\{[1\ 4\ 6]^\mathsf{T}, [2\ -1\ 3]^\mathsf{T}, [0\ 1\ 5]^\mathsf{T}\}$ in $\mathbb{R}^3$

(b) $\{1 - 2t^2 + t^3, 3 - t + 4t^3, 2 - 3t\}$ in $\mathbb{P}_3$

(c) $\{1 + t, -1 - 5t + 4t^2 + t^3, 1 + t^2 + t^3, t + 2t^3\}$ in $\mathbb{P}_3$

(d) $\left\{ \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \right\}$ in $\mathcal{M}_{3\times 2}$.

(2) Let $S = \{1 + t + t^2, t + t^2, 1 + t, 1 + t^2\}$ in $\mathbb{P}_2$.

(a) Show that the set S spans $\mathbb{P}_2$.

(b) Show that the set S is linearly dependent.

(c) Find a subset of S that is a basis for $\mathbb{P}_2$. Be sure to verify that you have a basis.

(3) Find two different bases for $\mathcal{M}_{2\times 2}$. Explain how you know that each set is a basis.

(4) The set $W = \{at + bt^2 \mid a \text{ and } b \text{ are scalars}\}$ is a subspace of $\mathbb{P}_3$.

(a) Find a set of vectors in $\mathbb{P}_3$ that spans W.

(b) Find a basis for W. Be sure to verify that you have a basis.

(5) Suppose that the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for a vector space V. Is the set $\{\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{w}\}$ a basis for V? Verify your result.

(6) Determine all scalars c so that the set $\{c^2 + t^2, c + 2t, 1 + t^2\}$ is a basis for $\mathbb{P}_2$.

(7) A symmetric matrix is a matrix A so that $A^{\mathsf{T}} = A$. Is it possible to find a basis for $\mathcal{M}_{2\times 2}$ consisting entirely of symmetric matrices? If so, exhibit one such basis. If not, explain why not.

(8) Find a basis of the subspace of $\mathcal{M}_{2\times 3}$ consisting of all matrices of the form $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ where $c = a - 2d$ and $f = b + 3e$.

(9) Prove Theorem 23.2. (Hint: Compare to Theorem 6.2.)

(10) Prove Theorem 23.4. (Hint: Compare to Theorem 6.5.)

(11) Prove Theorem 23.7. (Hint: Compare to Theorem 6.4.)

(12) Show that if W_1, W_2 are subspaces of V such that $W_1 \cap W_2 = \{\mathbf{0}\}$, then for any linearly independent vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in W_1 and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\ell$ in W_2, the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\ell\}$ is linearly independent in V.

(13) Label each of the following statements as True or False. Provide justification for your response. Throughout, let V be a vector space.

(a) **True/False** If $\mathbf{v}$ is in V, then the set $\{\mathbf{v}\}$ is linearly independent.

(b) **True/False** If a set of vectors span a subspace, then the set forms a basis of this subspace.

(c) **True/False** If a linearly independent set of vectors spans a subspace, then the set forms a basis of this subspace.

(d) **True/False** If the set S spans V and removing any vector from S makes it not a spanning set anymore, then S is a basis.

(e) **True/False**If S is a linearly independent set in V and for every $\mathbf{u}$ in V, adding $\mathbf{u}$ to S makes it not linearly independent anymore, then S is a basis.

(f) **True/False** If a subset S of V spans V, then S must be linearly independent.

(g) **True/False** If a subset S of V is linearly independent, then S must span V.

(h) **True/False** If S is a linearly dependent set in V, then every vector in S is a linear combination of the other vectors in S.

(i) **True/False** A vector space cannot have more than one basis.

(j) **True/False** If $\mathbf{u}$ is a non-zero vector in V, then there is a basis of V containing $\mathbf{u}$.

(k) **True/False** If $\mathbf{u}, \mathbf{v}$ are two linearly independent vectors in V, then there is a basis of V containing $\mathbf{u}, \mathbf{v}$.

(l) **True/False** If $\mathbf{u}$ is in a basis of V, then $2\mathbf{u}$ cannot be in a basis of V.

Project: Image Compression with Wavelets

We return to the problem of image compression introduced at the beginning of this section. The first step in the wavelet compression process is to digitize an image. There are two important ideas about digitalization to understand here: intensity levels and resolution. In grayscale image processing, it is common to think of 256 different intensity levels, or scales, of gray ranging from 0 (black) to 255 (white). A digital image can be created by taking a small grid of squares (called pixels) and coloring each pixel with some shade of gray. The resolution of this grid is a measure of how many pixels are used per square inch. An example of a 16 by 16 pixel picture of a flower was shown in Figure 23.1.

An image can be thought of in several ways: as a two-dimensional array; as one long vector by stringing the columns together one after another; or as a collection of column vectors. For simplicity, we will use the last approach in this project. We call each column vector in a picture a *signal*. Wavelets are used to process signals. After processing we can apply some technique to compress the processed signals.

To process a signal we select a family of wavelets. There are many different families of wavelets – which family to use depends on the problem to be addressed. The simplest family of wavelets is the Haar family. More complicated families of wavelets are usually used in applications, but the basic ideas in wavelets can be seen through working with the Haar wavelets, and their relative simplicity will make the details easier to follow. Each family of wavelets has a father wavelet (usually denoted φ) and a mother wavelet (ψ).

Wavelets are generated from the mother wavelet by scalings and translations. To further simplify our work we will restrict ourselves to wavelets on [0,1], although this is not necessary. The advantage the wavelets have over other methods of data analysis (Fourier analysis for example) is that with the scalings and translations we are able to analyze both frequency on large intervals and isolate signal discontinuities on very small intervals. The way this is done is by using a large collection (infinite, in fact) of basis functions with which to transform the data. We'll begin by looking at how these basis functions arise.

If we sample data at various points, we can consider our data to represent a piecewise constant function obtained by partitioning [0,1] into n equal sized subintervals, where n represents the number of sample points. For the purposes of this project we will always choose n to be a power of 2. So we can consider all of our data to represent functions. For us, then, it is natural to look at these functions in the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Since our data is piecewise constant, we can really restrict ourselves to a subspace of this larger vector space – subspaces of piecewise constant functions. The most basic piecewise constant function on the interval $[0, 1]$ is the one whose value is 1 on the entire interval. We define φ to be this constant function (called the characteristic function of the unit interval). That is

$$\varphi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

This function φ is the Father Haar wavelet.

This function φ may seem to be a very simple function but it has properties that will be important to us. One property is that φ satisfies a scaling equation. For example, Figure 23.2 shows

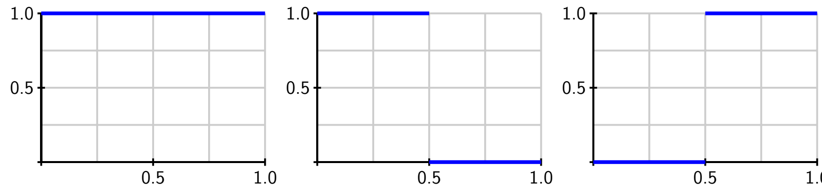

Figure 23.2: Graphs of $\varphi(x)$, $\varphi(2x)$, and $\varphi(2x-1)$ from left to right.

that

$$\varphi(x) = \varphi(2x) + \varphi(2x-1)$$

while Figure 23.3 shows that

$$\varphi(x) = \varphi(2^2x) + \varphi(2^2x-1) + \varphi(2^2x-2) + \varphi(2^2x-3).$$

So φ is a sum of scalings and translations of itself. In general, for each positive integer n and

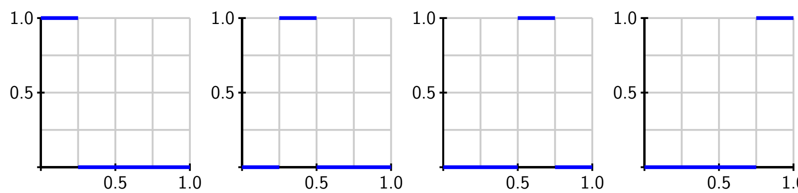

Figure 23.3: Graphs of $\varphi(2^2x)$, $\varphi(2^2x-1)$, $\varphi(2^2x-2)$, and $\varphi(2^2x-3)$, from left to right.

integers k between 0 and 2^n-1 we define

$$\varphi_{n,k}(x) = \varphi\left(2^nx-k\right).$$

Then $\varphi(x) = \sum_{k=0}^{2^n-1} \varphi_{n,k}(x)$ for each n.

These functions $\varphi_{n,k}$ are useful in that they form a basis for the vector space V_n of all piecewise constant functions on $[0,1]$ that have possible breaks at the points $\frac{1}{2^n}$, $\frac{2}{2^n}$, $\frac{3}{2^n}$, $\ldots$, $\frac{2^n-1}{2^n}$. This is exactly the kind of space in which digital signals live, especially if we sample signals at 2^n evenly spaced points on $[0,1]$. Let $\mathcal{B}_n = \{\varphi_{n,k} : 0 \leq k \leq 2^n-1\}$. You may assume without proof that $\mathcal{B}_n$ is a basis of V_n.

Project Activity 23.1.

(a) Draw the linear combination $2\varphi_{2,0} - 3\varphi_{2,1} + 17\varphi_{2,2} + 30\varphi_{2,3}$. What does this linear combination look like? Explain the statement made previously "Notice that these 2^n functions $\varphi_{n,k}$ form a basis for the vector space of all piecewise constant functions on $[0,1]$ that have possible breaks at the points $\frac{1}{2^n}$, $\frac{2}{2^n}$, $\frac{3}{2^n}$, $\ldots$, $\frac{2^n-1}{2^n}$".

(b) Remember that we can consider our data to represent a piecewise constant function obtained by partitioning $[0,1]$ into n subintervals, where n represents the number of sample points. Suppose we collect the following data: 10, 13, 21, 55, 3, 12, 4, 18. Explain how we can use this data to define a piecewise constant function f on $[0,1]$. Express f as a linear combination of suitable functions $\varphi_{n,k}$. Plot this linear combination of $\varphi_{n,k}$ to verify.

Working with functions can be more cumbersome than working with vectors in $\mathbb{R}^n$, but the digital nature of our data makes it possible to view our piecewise constant functions as vectors in $\mathbb{R}^n$ for suitable n. More specifically, if f is an element in V_n, then f is a piecewise constant function on $[0,1]$ with possible breaks at the points $\frac{1}{2^n}$, $\frac{2}{2^n}$, $\frac{3}{2^n}$, $\ldots$, $\frac{2^n-1}{2^n}$. If f has the value of y_i on the interval between $\frac{i-1}{2^n}$ and $\frac{i}{2^n}$, then we can identify f with the vector $[y_1\ y_1\ \ldots\ y_{2^n}]^{\mathsf{T}}$.

Project Activity 23.2.

(a) Determine the vector in $\mathbb{R}^8$ that is identified with φ.

(b) Determine the value of m and the vectors in $\mathbb{R}^m$ that are identified with $\varphi_{2,0}$, $\varphi_{2,1}$, $\varphi_{2,2}$, and $\varphi_{2,3}$.

We can use the functions $\varphi_{n,k}$ to represent digital signals, but to manipulate the data in useful ways we need a different perspective. A different basis for V_n (a *wavelet basis*) will allow us to identify the pieces of the data that are most important. We illustrate in the next activity with the spaces V_1 and V_2.

Project Activity 23.3. The space V_1 consists of all functions that are piecewise constant on $[0,1]$ with a possible break at $x = \frac{1}{2}$. The functions $\varphi = \varphi_{n,k}$ are used to records the values of a signal, and by summing these values we can calculate their average. Wavelets act by averaging and differencing, and so φ does the averaging. We need functions that will perform the differencing.

(a) Define $\{\psi_{0,0}\}$ as

$$\psi_{0,0}(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

A picture of $\psi_{0,0}$ is shown in Figure 23.4. Since $\psi_{0,0}$ assumes values of 1 and -1, we can use $\psi_{0,0}$ to perform differencing. The function $\psi = \psi_{0,0}$ is the Mother Haar wavelet.[3] Show that $\{\varphi, \psi\}$ is a basis for V_1.

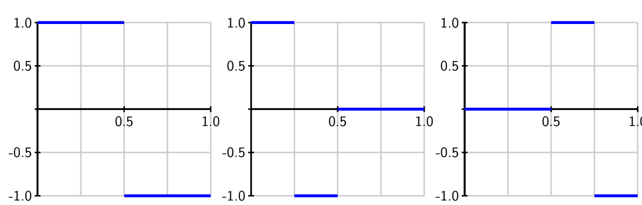

Figure 23.4: The graphs of $\psi_{0,0}$, $\psi_{1,0}$ and $\psi_{1,1}$ from left to right.

(b) We continue in a manner similar to the one in which we constructed bases for V_n. For $k = 0$ and $k = 1$, let $\psi_{1,k} = \psi\left(2^1 x - k\right)$. Graphs of $\psi_{1,0}$ and $\psi_{1,1}$ are shown in Figure 23.4. The functions $\psi_{1,k}$ assume the values of 1 and -1 on smaller intervals, and so can be used to perform differencing on smaller scale than $\psi_{0,0}$. Show that $\{\varphi_{0,0}, \psi_{0,0}, \psi_{1,0}, \psi_{1,1}\}$ is a basis for V_2.

[3] The first mention of wavelets appeared in an appendix to the thesis of A. Haar in 1909.

As Project Activity 23.3 suggests, we can make a basis for V_n from $\varphi_{0,0}$ and functions of the form $\psi_{n,k}$ defined by $\psi_{n,k}(x) = \psi\left(2^n x - k\right)$ for k from 0 to $2^n - 1$. More specifically, if we let $\mathcal{S}_n = \{\psi_{n,k} : 0 \leq k \leq 2^n - 1\}$, then the set

$$\mathcal{W}_n = \{\varphi_{0,0}\} \cup \bigcup_{j=0}^{n-1} \mathcal{S}_j$$

is a basis for $V_n^{\perp}$ (we state this without proof). The functions $\psi_{n,k}$ are the *wavelets*.

Project Activity 23.4. We can now write any function in V_n using the basis $\mathcal{W}_n$. As an example, the string 50, 16, 14, 28 represents a piecewise constant function which can be written as $50\varphi_{2,0} + 16\varphi_{2,1} + 14\varphi_{2,2} + 28\varphi_{2,3}$, an element in V_2.

(a) Specifically identify the functions in $\mathcal{W}_0$, $\mathcal{W}_1$, and $\mathcal{W}_2$, and $\mathcal{W}_3$.

(b) As mentioned earlier, we can identify a signal, and each wavelet function, with a vector in $\mathbb{R}^m$ for an appropriate value of m. We can then use this identification to decompose any signal as a linear combination of wavelets. We illustrate this idea with the signal $[50\ 16\ 14\ 28]^{\mathsf{T}}$ in $\mathbb{R}^4$. Recall that we can represent this signal as the function $f = 50\varphi_{2,0} + 16\varphi_{2,1} + 14\varphi_{2,2} + 28\varphi_{2,3}$.

   i. Find the the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, and $\mathbf{w}_4$ in $\mathbb{R}^m$ that are identified with $\varphi_{0,0}$, $\psi_{0,0}$, $\psi_{1,0}$, and $\psi_{1,1}$, respectively.

   ii. Any linear combination $c_1\varphi_{0,0} + c_2\psi_{0,0} + c_3\psi_{1,0} + c_4\psi_{1,1}$ is then identified with the linear combination $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + c_4\mathbf{w}_4$. Use this idea to find the weights to write the function f as a linear combination of $\varphi_{0,0}$, $\psi_{0,0}$, $\psi_{1,0}$, and $\psi_{1,1}$.

Although is it not necessarily easy to observe, the weights in the decomposition $f = 27\varphi_{0,0} + 6\psi_{0,0} + 17\psi_{1,0} - 7\psi_{1,1}$ are just averages and differences of the original weights in $f = 50\varphi_{2,0} + 16\varphi_{2,1} + 14\varphi_{2,2} + 28\varphi_{2,3}$. To see how, notice that if we take the overall average of the original weights we obtain the value of 27. If we average the original weights in pairs (50 and 16, and 14 and 28) we obtain the values 33 and 21, and if we take average differences of the original weights in pairs (50 and 16, and 14 and 28) we obtain the values 17 and -7. We can treat the signal $[33\ 21]^{\mathsf{T}}$ formed from the average of the pairs of the original weights as a smaller copy of the original signal. The average difference of the entries of this new signal is 6. So the weights in our final decomposition are obtained by differences between successive averages and certain coefficients. The coefficients in our final decomposition $27\varphi_{0,0} + 6\psi_{0,0} + 17\psi_{1,0} - 7\psi_{1,1}$ are called *wavelet coefficients*. This is the idea that makes wavelets so useful for image compression. In many images, pixels that are near to each other often have similar coloring or shading. These pixels are coded with numbers that are close in value. In the differencing process, these numbers are replaced with numbers that are close to 0. If there is little difference in the shading of the adjacent pixels, the image will be changed only a little if the shadings are made the same. This results in replacing these small wavelet coefficients with zeros. If the processed vectors contain long strings of zeros, the vectors can be significantly compressed.

Once we have recognized the pattern in expressing our original function as an overall average and wavelet coefficients we can perform these operations more quickly with matrices.

Project Activity 23.5. The process of averaging and differencing discussed in and following Project Activity 23.4 can be viewed as a matrix-vector problem. As we saw in Project Activity 23.4, we can translate the problem of finding wavelet coefficients to the matrix world.

(a) Consider again the problem of finding the wavelet coefficients contained in the vector $[27\ 6\ 17\ \ -7]^{\mathsf{T}}$ for the signal $[50\ 16\ 14\ 28]^{\mathsf{T}}$. Find the matrix A_4 that has the property that $A_4[50\ 16\ 14\ 28]^{\mathsf{T}} = [27\ 6\ 17\ \ -7]^{\mathsf{T}}$. (You have already done part of this problem in Project Activity 23.4.) Explain how A_4 performs the averaging and differencing discussed earlier.

(b) Repeat the process in part (a) to find the matrix A_8 that converts a signal to its wavelet coefficients.

(c) The matrix A_i is called a *forward wavelet transformation matrix* and A_i^{-1} is the *inverse wavelet transform matrix*. Use A_8 to show that the wavelet coefficients for the data string $[80\ 48\ 4\ 36\ 28\ 64\ 6\ 50]^{\mathsf{T}}$ are contained in the vector $[39.5\ 2.5\ 22\ 9\ 16\ \ -16\ \ -18\ \ -22]^{\mathsf{T}}$.

Now we have all of the necessary background to discuss image compression. Suppose we want to store an image. We partition the image vertically and horizontally and record the color or shade at each grid entry. The grid entries will be our pixels. This gives a matrix, M, of colors, indexed by pixels or horizontal and vertical position. To simplify our examples we will work in gray-scale, where our grid entries are integers between 0 (black) and 255 (white). We can treat each column of our grid as a piecewise constant function. As an example, the image matrix M that produced the picture at left in Figure 23.1 is given in (23.1).

We can then apply a 16 by 16 forward wavelet transformation matrix A_{16} to M to convert the columns to averages and wavelet coefficients that will appear in the matrix $A_{16}M$. These wavelet coefficients allow us to compress the image – that is, create a smaller set of data that contains the essence of the original image.

Recall that the forward wavelet transformation matrix computes weighted differences of consecutive entries in the columns of the image matrix M. If two entries in M are close in values, the weighted difference in $A_{16}M$ will be close to 0. For our example, the matrix $A_{16}M$ is approximately

$$\begin{bmatrix}
208.0 & 202.0 & 178.0 & 165.0 & 155.0 & 172.0 & 118.0 & 172.0 & 155.0 & 153.0 & 176.0 & 202.0 & 208.0 & 210.0 & 209.0 & 208.0 \\
33.4 & 24.1 & -0.625 & 0.938 & -2.50 & -5.94 & 42.8 & -5.94 & -2.50 & 12.8 & 0.938 & 24.7 & 30.6 & 33.4 & 32.5 & 31.6 \\
-1.88 & -13.8 & 19.4 & 2.50 & 0.0 & -2.50 & 8.12 & -2.50 & 0.0 & 2.50 & 19.4 & -13.8 & 1.88 & -3.75 & -1.88 & 0.0 \\
17.5 & 61.9 & 61.9 & 6.88 & 0.0 & 61.9 & 0.0 & 61.9 & 0.0 & 30.6 & 65.0 & 66.9 & 66.9 & 19.4 & 66.9 & 66.9 \\
0.0 & 27.5 & 43.8 & 16.2 & 0.0 & -11.2 & 16.2 & -11.2 & 0.0 & 16.2 & 43.8 & 27.5 & 0.0 & 0.0 & 0.0 & 0.0 \\
3.75 & 0.0 & 27.5 & -11.2 & 0.0 & -16.2 & 22.5 & -16.2 & 0.0 & -11.2 & 27.5 & 0.0 & -3.75 & -7.50 & -3.75 & 0.0 \\
47.5 & 0.0 & 0.0 & 13.8 & 82.5 & 0.0 & 0.0 & 0.0 & 82.5 & 13.8 & 0.0 & 3.75 & 3.75 & 51.2 & 3.75 & 3.75 \\
82.5 & 41.2 & 41.2 & 82.5 & 82.5 & 41.2 & 0.0 & 41.2 & 82.5 & 35.0 & 35.0 & 35.0 & 35.0 & 82.5 & 35.0 & 35.0 \\
0.0 & 0.0 & 0.0 & 55.0 & -22.5 & -22.5 & 55.0 & -22.5 & -22.5 & 55.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 55.0 & -22.5 & -22.5 & 22.5 & 0.0 & -22.5 & 0.0 & 22.5 & -22.5 & -22.5 & 55.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
-7.50 & 0.0 & -55.0 & 22.5 & 22.5 & -22.5 & 0.0 & -22.5 & 22.5 & 22.5 & -55.0 & 0.0 & 7.50 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 22.5 & -55.0 & 0.0 & -55.0 & 22.5 & 0.0 & 0.0 & 0.0 & -15.0 & 0.0 & -7.50 & 0.0 \\
0.0 & 0.0 & 0.0 & -55.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -55.0 & 0.0 & 7.50 & 7.50 & 7.50 & 7.50 & 7.50 \\
95.0 & 0.0 & 0.0 & -82.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -82.5 & 0.0 & 0.0 & 0.0 & 95.0 & 0.0 & 0.0 \\
0.0 & -82.5 & 82.5 & 0.0 & 0.0 & -82.5 & 0.0 & -82.5 & 0.0 & 95.0 & -95.0 & 95.0 & -95.0 & 0.0 & 95.0 & -95.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}.$$

Note that there are many wavelet coefficients that are quite small compared to others – the ones where the weighted averages are close to 0. In a sense, the weighted differences tell us how much "detail" about the whole that each piece of information contains. If a piece of information contains only a small amount of information about the whole, then we shouldn't sacrifice much of the picture if we ignore the small "detail" coefficients. One way to ignore the small "detail" coefficients is to use *thresholding*.

With thresholding (this is *hard thresholding* or *keep or kill*), we decide on how much of the detail we want to remove (this is called the *tolerance*). So we set a tolerance and then replace each entry in our matrix $A_{16}M$ whose absolute value is below the tolerance with 0 to obtain a new matrix M_1. In our example, if you use a threshold value of 10 we obtain the new matrix M_1:

$$\begin{bmatrix}
208.0 & 202.0 & 178.0 & 165.0 & 155.0 & 172.0 & 118.0 & 172.0 & 155.0 & 153.0 & 176.0 & 202.0 & 208.0 & 210.0 & 209.0 & 208.0 \\
33.4 & 24.1 & 0.0 & 0.0 & 0.0 & 0.0 & 42.8 & 0.0 & 0.0 & 12.8 & 0.0 & 24.7 & 30.6 & 33.4 & 32.5 & 31.6 \\
0.0 & -13.8 & 19.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 19.4 & -13.8 & 0.0 & 0.0 & 0.0 & 0.0 \\
17.5 & 61.9 & 61.9 & 0.0 & 0.0 & 61.9 & 0.0 & 61.9 & 0.0 & 30.6 & 65.0 & 66.9 & 66.9 & 19.4 & 66.9 & 66.9 \\
0.0 & 27.5 & 43.8 & 16.2 & 0.0 & -11.2 & 16.2 & -11.2 & 0.0 & 16.2 & 43.8 & 27.5 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 27.5 & -11.2 & 0.0 & -16.2 & 22.5 & -16.2 & 0.0 & -11.2 & 27.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
47.5 & 0.0 & 0.0 & 13.8 & 82.5 & 0.0 & 0.0 & 0.0 & 82.5 & 13.8 & 0.0 & 0.0 & 0.0 & 51.2 & 0.0 & 0.0 \\
82.5 & 41.2 & 41.2 & 82.5 & 82.5 & 41.2 & 0.0 & 41.2 & 82.5 & 35.0 & 35.0 & 35.0 & 35.0 & 82.5 & 35.0 & 35.0 \\
0.0 & 0.0 & 0.0 & 55.0 & -22.5 & -22.5 & 55.0 & -22.5 & -22.5 & 55.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 55.0 & -22.5 & -22.5 & 22.5 & 0.0 & -22.5 & 0.0 & 22.5 & -22.5 & -22.5 & 55.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & -55.0 & 22.5 & 22.5 & -22.5 & 0.0 & -22.5 & 22.5 & 22.5 & -55.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 22.5 & -55.0 & 0.0 & -55.0 & 22.5 & 0.0 & 0.0 & 0.0 & -15.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & -55.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -55.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
95.0 & 0.0 & 0.0 & -82.5 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & -82.5 & 0.0 & 0.0 & 0.0 & 95.0 & 0.0 & 0.0 \\
0.0 & -82.5 & 82.5 & 0.0 & 0.0 & -82.5 & 0.0 & -82.5 & 0.0 & 95.0 & -95.0 & 95.0 & -95.0 & 0.0 & 95.0 & -95.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}.$$

We now have introduced many zeros in our matrix. This is where we compress the image. To store the original image, we need to store every pixel. Once we introduce strings of zeros we can identify a new code (say 256) that indicates we have a string of zeros. We can then follow that code with the number of zeros in the string. So if we had a string of 15 zeros in a signal, we could store that information in 2 bytes rather than 15 and obtain significant savings in storage. This process removes some detail from our picture, but only the small detail. To convert back to an image, we just undo the forward processing by multiplying our thresholded matrix M_1 by A_{16}^{-1}. The ultimate goal is to obtain significant compression but still have $A_{16}^{-1}M_1$ retain all of the essence of the original image.

In our example using M_1, the reconstructed image matrix is $A_{16}^{-1}M_1$ (rounded to the nearest whole number) is

$$\begin{bmatrix}
242 & 240 & 241 & 237 & 132 & 138 & 232 & 138 & 132 & 238 & 239 & 240 & 238 & 244 & 242 & 240 \\
242 & 240 & 241 & 127 & 178 & 183 & 122 & 183 & 178 & 128 & 239 & 240 & 238 & 244 & 242 & 240 \\
242 & 240 & 131 & 127 & 178 & 183 & 122 & 183 & 178 & 128 & 129 & 240 & 238 & 244 & 242 & 240 \\
242 & 130 & 176 & 172 & 132 & 183 & 167 & 183 & 132 & 172 & 174 & 130 & 238 & 244 & 242 & 240 \\
242 & 240 & 131 & 177 & 178 & 133 & 183 & 133 & 178 & 178 & 129 & 240 & 238 & 244 & 242 & 240 \\
242 & 240 & 241 & 132 & 132 & 178 & 183 & 178 & 132 & 132 & 239 & 240 & 238 & 244 & 242 & 240 \\
242 & 240 & 131 & 177 & 178 & 133 & 138 & 133 & 178 & 178 & 129 & 240 & 223 & 244 & 242 & 240 \\
242 & 240 & 131 & 177 & 132 & 243 & 138 & 243 & 132 & 178 & 129 & 240 & 253 & 244 & 242 & 240 \\
240 & 240 & 239 & 124 & 238 & 234 & 75 & 234 & 238 & 130 & 241 & 244 & 244 & 248 & 244 & 244 \\
240 & 240 & 239 & 234 & 238 & 234 & 75 & 234 & 238 & 240 & 241 & 244 & 244 & 248 & 244 & 244 \\
240 & 240 & 239 & 69 & 73 & 234 & 75 & 234 & 73 & 75 & 241 & 244 & 244 & 240 & 244 & 244 \\
50 & 240 & 239 & 234 & 73 & 234 & 75 & 234 & 73 & 240 & 241 & 244 & 244 & 50 & 244 & 244 \\
240 & 75 & 239 & 248 & 238 & 69 & 75 & 69 & 238 & 240 & 51 & 240 & 50 & 240 & 240 & 50 \\
240 & 240 & 74 & 248 & 238 & 234 & 75 & 234 & 238 & 50 & 241 & 50 & 240 & 240 & 50 & 240 \\
75 & 75 & 74 & 83 & 73 & 69 & 75 & 69 & 73 & 75 & 76 & 75 & 75 & 75 & 75 & 75 \\
75 & 75 & 74 & 83 & 73 & 69 & 75 & 69 & 73 & 75 & 76 & 75 & 75 & 75 & 75 & 75
\end{bmatrix}.$$

We convert this into a gray-scale image and obtain the image at right in Figure 23.1. Compare this image to the original at right in Figure 23.1. It is difficult to tell the difference.

There is a Sage file you can use at

http://faculty.gvsu.edu/schlicks/Wavelets_Sage.html

that allows you to create your own 16 by 16 image and process, process your image with the Haar wavelets in $\mathbb{R}^{16}$, apply thresholding, and reconstruct the compressed image. matrix. You can create your own image, experiment with several different threshold levels, and choose the one that you feel gives the best combination of strings of 0s while reproducing a reasonable copy of the original image.

Section 24

The Dimension of a Vector Space

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a finite dimensional vector space?
- What is the dimension of a finite dimensional vector space? What important result about bases of finite dimensional vector spaces makes dimension well-defined?
- What must be true about any linearly independent subset of n vectors in a vector space with dimension n? Why?
- What must be true about any subset of n vectors in a vector space with dimension n that spans the vector space? Why?

Application: Principal Component Analysis

The discipline of statistics is based on the idea of analyzing data. In large data sets it is usually the case that one wants to understand the relationships between the different data in the set. This can be difficult to do when the data set is large and it is impossible to visually examine the data for patterns. Principal Component Analysis (PCA) is a tool for identifying and representing underlying patterns in large data sets, and PCA has been called one of the most important and valuable linear algebra tools for statistical analysis. PCA is used to transform a collection of variables into a (usually smaller) number of uncorrelated variables called principal components. The principal components form the most meaningful basis from which to view data by removing extraneous information and revealing underlying relationships in the data. This presents a framework for how to reduce a complex data set to a lower dimension while retaining the important attributes of the data set. The output helps the experimenter determine which dynamics in the data are important and which can be ignored.

Introduction

In Section 15 we learned that any two bases for a subspace of $\mathbb{R}^n$ contain the same number of vectors. This allowed us to define the dimension of a subspace of $\mathbb{R}^n$. In this section we extend the arguments we made in Section 15 to arbitrary vector spaces and define the dimension of a vector space.

Preview Activity 24.1. The main tool we used to prove that any two bases for a subspace of $\mathbb{R}^n$ must contain the same number of elements was Theorem 15.1. In this preview activity we will show that the same argument can be used for vector spaces. More specifically, we will prove a special case of the following theorem generalizing Theorem 15.1.

Theorem 24.1. *If V is a vector space with a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of k vectors, then any subset of V containing more than k vectors is linearly dependent.*

Suppose V is a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Consider the set $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of vectors in V. We will show that U is linearly dependent using a similar approach to the Preview Activity 15.1.

(1) What vector equation involving $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ do we need to solve to determine linear independence/dependence of these vectors? Use x_1, x_2, x_3 for coefficients.

(2) Since $\mathcal{B}$ is a basis of V, it spans V. Using this information, rewrite the vectors $\mathbf{u}_i$ in terms of $\mathbf{v}_j$ and substitute into the above equation to obtain another equation in terms of $\mathbf{v}_j$.

(3) Since $\mathcal{B}$ is a basis of V, the vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent. Using the equation in the previous part, determine what this means about the coefficients x_1, x_2, x_3.

(4) Express the conditions on x_1, x_2, x_3 in the form of a matrix-vector equation. Explain why there are infinitely many solutions for x_i's and why this means the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent.

Finite Dimensional Vector Spaces

Theorem 24.1 shows that if sets $\mathcal{B}_1$ and $\mathcal{B}_2$ are finite bases for a vector space V, which are linearly independent by definition, then each cannot contain more elements than the other, so the number of elements in each basis must be equal.

Theorem 24.2. *If a non-trivial vector space V has a basis of n vectors, then every basis of V contains exactly n vectors.*

Theorem 24.2 states that if a vector space V has a basis with a finite number of vectors, then the number of vectors in a basis for that vector space is a well-defined number. In other words, the number of vectors in a basis is an *invariant* of the vector space. This important number is given a name.

Definition 24.3. A **finite-dimensional** vector space is a vector space that can be spanned by a finite number of vectors. The **dimension** of a non-trivial finite-dimensional vector space is the number of vectors in a basis for V. The dimension of the trivial vector space is defined to be 0.

We denote the dimension of a finite dimensional vector space V by $\dim(V)$.

Not every vector space is finite dimensional. We have seen, for example, that the vector space $\mathbb{P}$ of all polynomials, regardless of degree, is not a finite-dimensional vector space. In fact, the polynomials

$$1, t, t^2, \ldots, t^n, \ldots$$

are linearly independent, so $\mathbb{P}$ has an infinite linearly independent set and therefore has no finite basis. A vector space that has an infinite basis is called an *infinite dimensional* vector space.

Activity 24.1. Since columns of the $n \times n$ identity matrix span $\mathbb{R}^n$ and are linearly independent, the columns of I_n form a basis for $\mathbb{R}^n$ (the standard basis). Consequently, we have that $\dim(\mathbb{R}^n) = n$. In this activity we determine the dimensions of other familiar vector spaces. Find the dimensions of each of the indicated vector spaces. Verify your answers.

(a) $\mathbb{P}_1$ (b) $\mathbb{P}_2$ (c) $\mathbb{P}_n$

(d) $\mathcal{M}_{2\times 3}$ (e) $\mathcal{M}_{3\times 4}$ (f) $\mathcal{M}_{k\times n}$

Finding the dimension of a finite-dimensional vector space amounts to finding a basis for the space.

Activity 24.2. Let $W = \{(a+b) + (a-b)t + (2a+3b)t^2 \mid a, b \text{ are scalars}\}$.

(a) Find a finite set of polynomials in W that span W.

(b) Determine if the spanning set from part (a) is linearly independent or dependent. Clearly explain your process.

(c) What is $\dim(W)$? Explain.

The Dimension of a Subspace

Every subspace of a finite-dimensional vector space is a vector space, and since a subspace is contained in a vector space it is natural to think that the dimension of a subspace should be less than or equal to the dimension of the larger vector space. We verify that fact in this section.

Activity 24.3. Let V be a finite dimensional vector space of dimension n and let W be a subspace of V. Explain why W cannot have dimension larger than $\dim(V)$, and if $W \neq V$ then $\dim(W) < \dim(V)$. (Hint: Use Theorem 24.1.)

Conditions for a Basis of a Vector Space

There are two items we need to confirm before we can state that a subset $\mathcal{B}$ of a subspace W of a vector space is a basis for W: the set $\mathcal{B}$ must be linearly independent and span W. We can reduce the amount of work it takes to show that a set is a basis if we know the dimension of the vector space in advance.

Activity 24.4. Let W be a subspace of a vector space V with $\dim(W) = k$. We know that every basis of W contains exactly k vectors.

(a) Suppose that S is a subset of W that contains k vectors and is linearly independent. In this part of the activity we will show that S must span W.

  i. Suppose that S does not span W. Explain why this implies that W contains a set of $k+1$ linearly independent vectors.

  ii. Explain why the result of i tells us that S is a basis for W.

(b) Now suppose that S is a subset of W with k vectors that spans W. In this part of the activity we will show that S must be linearly independent.

  i. Suppose that S is not linearly independent. Explain why we can then find a proper subset of S that is linearly independent but has the same span as S.

  ii. Explain why the result of i tells us that S is a basis for W.

The result of Activity 24.4 is summarized in the following theorem (compare to Theorem 15.4).

Theorem 24.4. *Let W be a subspace of dimension k of a vector space V and let S be a subset of W containing exactly k vectors.*

(1) If S is linearly independent, then S is a basis for W.

(2) If S spans W, then S is a basis for W.

Examples

What follows are worked examples that use the concepts from this section.

Example 24.5. Find a basis and dimension for each of the indicated subspaces of the given vector spaces.

(a) $\{a + b(t + t^2) : a, b \in \mathbb{R}\}$ in $\mathbb{P}_2$

(b) Span $\left\{1, \frac{1}{1+x^2}, \frac{2+x^2}{1+x^2}, \arctan(x)\right\}$ in $\mathcal{F}$

(c) $\{p(t) \in \mathbb{P}_n : p(-t) = p(t)\}$ in $\mathbb{P}_4$ (The polynomials with the property that $p(-t) = p(t)$ are called *even* polynomials.)

Example Solution.

(a) Let $W = \{a + b(t + t^2) : a, b \in \mathbb{R}\}$. Every element in W has the form

$$a + b(t + t^2) = a(1) + b(t + t^2).$$

So $W = \text{Span}\{1, t + t^2\}$. Since neither 1 nor $t + t^2$ is a scalar multiple of the other, the set $\{1, t + t^2\}$ is linearly independent. Thus, $\{1, t + t^2\}$ is a basis for W and $\dim(W) = 2$.

(b) Let $W = \text{Span}\left\{1, \frac{1}{1+x^2}, \frac{2+x^2}{1+x^2}, \arctan(x)\right\}$. To find a basis for W, we find a linearly independent subset of $\left\{1, \frac{1}{1+x^2}, \frac{2+x^2}{1+x^2}, \arctan(x)\right\}$. Consider the equation

$$c_1(1) + c_2\left(\frac{1}{1+x^2}\right) + c_3\left(\frac{2+x^2}{1+x^2}\right) + c_4\arctan(x) = 0.$$

To find the weights c_i for which this equality of functions holds, we use the fact that the we must have equality for every x. So we pick four different values for x to obtain a linear system that we can solve for the weights. Evaluating both sides of the equation at $x = 0$, $x = 1$, $x = -1$, and $x = 2$ yields the equations

$$\begin{aligned} c_1 + \quad c_2 + 2c_3 \qquad\qquad\qquad &= 0 \\ c_1 + \frac{1}{2}c_2 + \frac{3}{2}c_3 + \qquad\quad \frac{\pi}{4}c_4 &= 0 \\ c_1 + \frac{1}{2}c_2 + \frac{3}{2}c_3 - \qquad\quad \frac{\pi}{4}c_4 &= 0 \\ c_1 + \frac{1}{5}c_2 + \frac{6}{5}c_3 + \arctan(2)c_4 &= 0. \end{aligned}$$

The reduced row echelon form of the coefficient matrix

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & \frac{1}{2} & \frac{3}{2} & \frac{\pi}{4} \\ 1 & \frac{1}{2} & \frac{3}{2} & -\frac{\pi}{4} \\ 1 & \frac{1}{5} & \frac{6}{5} & \arctan(2) \end{bmatrix}$$

is $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution to this linear system is $c_1 = c_2 = -c_3$ and $c_4 = 0$. Notice that

$$(-1)(1) + (-1)\left(\frac{1}{1+x^2}\right) + \frac{2+x^2}{1+x^2} = 0$$

or

$$\frac{2+x^2}{1+x^2} = 1 + \left(\frac{1}{1+x^2}\right),$$

so $\frac{2+x^2}{1+x^2}$ is a linear combination of the other vectors. The vectors corresponding to the pivot columns are linearly independent, so it follows that 1, $\frac{1}{1+x^2}$, and $\arctan(x)$ are linearly independent. We conclude that $\left\{1, \frac{1}{1+x^2}, \arctan(x)\right\}$ is a basis for W and $\dim(W) = 3$.

(c) Let $W = \{p(t) \in \mathbb{P}_n : p(-t) = p(t)\}$ in $\mathbb{P}_4$. Let $p(t) \in W$ and suppose that $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4$. Since $p(-t) = p(t)$, we must have $p(1) = p(-1)$ and $p(3) = p(-3)$. Since $p(1) = p(-1)$, it follows that

$$a_0 + a_1 + a_2 + a_3 + a_4 = a_0 - a_1 + a_2 - a_3 + a_4$$

or

$$a_1 + a_3 = 0.$$

Similarly, the fact that $p(3) = p(-3)$ yields the equation

$$a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4 = a_0 - 3a_1 + 9a_2 - 27a_3 + 81a_4$$

or

$$a_1 + 9a_3 = 0.$$

The reduced row echelon form of the coefficient matrix of system $a_1 + a_3 = 0$ and $a_1 + 9a_3 = 0$ is I_2, so it follows that $a_1 = a_3 = 0$. Thus, $p(t) = a_0 + a_2t^2 + a_4t^4$ and so $W = \text{Span}\{1, t^2, t^4\}$. Equating like terms in the equation

$$c_1(1) + c_2(t^2) + c_3(t^4) = 0$$

yields $c_1 = c_2 = c_3 = 0$. We conclude that $\{1, t^2, t^4\}$ is linearly independent and is therefore a basis for W. Thus, $\dim(W) = 3$.

As an alternative solution, notice that $p(t) = t$ is not in W. So $W \neq V$ and we know that $\dim(W) < \dim(V)$. Since 1, t^2, and t^4 are in W, we can show as above that 1, t^2, and t^4 are linearly independent. We can conclude that $\dim(W) = 3$ since it cannot be 4.

Example 24.6. Let $U_{n\times n}$ be the set of all $n \times n$ upper triangular matrices. Recall that a matrix $A = [a_{ij}]$ is upper triangular if $a_{ij} = 0$ whenever $i > j$. That is, a matrix is upper triangular if all entries below the diagonal are 0.

(a) Show that $U_{n\times n}$ is a subspace of $\mathcal{M}_{n\times n}$.

(b) Find the dimensions of $U_{2\times 2}$ and $U_{3\times 3}$. Explain. Make a conjecture as to what $\dim(U_{n\times n})$ is in terms of n.

Example Solution.

(a) Since the $n \times n$ zero matrix $0_{n\times n}$ has all entries equal to 0, it follows that $0_{n\times n}$ is in $U_{n\times n}$. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be in $U_{n\times n}$, and let $C = [c_{ij}] = A + B$. Then $c_{ij} = a_{ij} + b_{ij} = 0 + 0$ when $i > j$. So C is an upper triangular matrix and $U_{n\times n}$ is closed under addition. Let c be a scalar. The ijth entry of cA is $ca_{ij} = c(0) = 0$ whenever $i > j$. So cA is an upper triangular matrix and $U_{n\times n}$ is closed under multiplication by scalars. We conclude that $U_{n\times n}$ is a subspace of $\mathcal{M}_{n\times n}$.

(b) Let $M_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $M_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We will show that $S = \{M_{11}, M_{12}, M_{22}\}$ is a basis for $U_{2\times 2}$. Consider the equation

$$x_1M_{11} + x_2M_{12} + x_3M_{22} = 0.$$

Equating like entries shows that $x_1 = x_2 = x_3 = 0$, and so S is linearly independent. If $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is in $U_{2\times 2}$, then

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = aM_{11} + bM_{12} + cM_{22}$$

and so S spans $U_{2\times 2}$. Thus, S is a basis for $U_{2\times 2}$ and so $\dim(U_{2\times 2}) = 3$.

Similarly, for the 3×3 case let M_{ij} for $i \leq j$ be the 3×3 matrix with a 1 in the ij position and 0 in every other position. Let

$$S = \{M_{11}, M_{12}, M_{13}, M_{22}, M_{23}, M_{33}\}.$$

Equating corresponding entries shows that if

$$x_1M_{11} + x_2M_{12} + x_3M_{13} + x_4M_{22} + x_5M_{23} + x_6M_{33} = 0,$$

then $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$. So S is a linearly independent set. If $A = [a_{ij}]$ is in $U_{3\times 3}$, then $A = \sum_{i\geq j} a_{ij}M_{ij}$ and S spans $U_{3\times 3}$. We conclude that S is a basis for $U_{3\times 3}$ and $\dim(U_{3\times 3}) = 6$.

In general, for an $n \times n$ matrix, the set of matrices M_{ij}, one for each entry on and above the diagonal, is a basis for $U_{n\times n}$. There are n such matrices for the entries on the diagonal. The number of entries above the diagonal is equal to half the total number of entries (n^2) minus half the number of entries on the diagonal (n). So there is a total of $\frac{n^2-n}{2}$ such matrices for the entries above the diagonal. Therefore,

$$\dim(U_{n\times n}) = n + \frac{n^2 - n}{2} = \frac{n^2 + n}{2}.$$

Summary

- A finite dimensional vector space is a vector space that can be spanned by a finite set of vectors.
- We showed that any two bases for a finite dimensional vector space *must* contain the same number of vectors. Therefore, we can define the *dimension* of a finite dimensional vector space V to be the number of vectors in any basis for V.
- If V is a vector space with dimension n and S is any linearly independent subset of V with n vectors, then S is a basis for V. Otherwise, we could add vectors to S to make a basis for V and then V would have a basis of more than n vectors.
- If V is a vector space with dimension n and S is any subset of V with n vectors that spans V, then S is a basis for V. Otherwise, we could remove vectors from S to obtain a basis for V and then V would have a basis of fewer than n vectors.
- For any finite dimensional space V and a subspace W of V, $\dim(W) \leq \dim(V)$.

Exercises

(1) Let $W = \text{Span}\{1 + t^2, 2 + t + 2t^2 + t^3, 1 + t + t^3, t - t^2 + t^3\}$ in $\mathbb{P}_3$. Find a basis for W. What is the dimension of W?

(2) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$.

(a) Are A and B linearly independent or dependent? Verify your result.

(b) Extend the set $S = \{A, B\}$ to a basis for $\mathcal{M}_{2\times 2}$. That is, find a basis for $\mathcal{M}_{2\times 2}$ that contains both A and B.

(3) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 4 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 1 & -3 \\ 0 & -12 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 4 & -2 \\ 5 & -8 & 6 \end{bmatrix}$, and $E = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ in $\mathcal{M}_{2\times 3}$ and let $S = \{A, B, C, D, E\}$.

(a) Is S a basis for $\mathcal{M}_{2\times 3}$? Explain.

(b) Determine if S is a linearly independent or dependent set. Verify your result.

(c) Find a basis $\mathcal{B}$ for Span S that is a subset of S and write all of the vectors in S as linear combinations of the vectors in $\mathcal{B}$.

(d) Extend your basis $\mathcal{B}$ from part (c) to a basis $\mathcal{M}_{2\times 3}$. Explain your method.

(4) Determine the dimension of each of the following vector spaces.

(a) Span$\{2, 1+t, t^2\}$ in $\mathbb{P}_2$

(b) The space of all polynomials in $\mathbb{P}_3$ whose constant terms is 0.

(c) Nul $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(d) Span $\left\{[1\ 2\ 0\ 1\ \ -1]^{\mathsf{T}}, [0\ 1\ 1\ 1\ 0]^{\mathsf{T}}, [0\ 3\ 2\ 3\ 0]^{\mathsf{T}}, [-1\ 0\ 1\ 1\ 1]^{\mathsf{T}}\right\}$ in $\mathbb{R}^5$

(5) Let W be the set of matrices in $\mathcal{M}_{2\times 2}$ whose diagonal entries sum to 0. Show that W is a subspace of $\mathcal{M}_{2\times 2}$, find a basis for W, and then find $\dim(W)$.

(6) Show that if W is a subspace of a finite dimensional vector space V, then any basis of W can be extended to a basis of V.

(7) Let W be the set of all polynomials $a + bt + ct^2$ in $\mathbb{P}_2$ such that $a + b + c = 0$. Show that W is a subspace of $\mathbb{P}_2$, find a basis for W, and then find $\dim(W)$.

(8) Suppose W_1, W_2 are subspaces in a finite-dimensional space V .

(a) Show that it is not true in general that $\dim(W_1) + \dim(W_2) \leq \dim(V)$.

(b) Are there any conditions on W_1 and W_2 that will ensure that $\dim(W_1)+\dim(W_2) \leq \dim(V)$? (Hint: See problem 12 in the previous section.)

(9) Suppose $W_1 \subseteq W_2$ are two subspaces of a finite-dimensional space. Show that if $\dim(W_1) = \dim(W_2)$, then $W_1 = W_2$.

(10) Suppose W_1, W_2 are both three-dimensional subspaces inside $\mathbb{R}^4$. In this exercise we will show that $W_1 \cap W_2$ contains a plane. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for W_1 and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for W_2.

(a) If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are all in W_1, explain why $W_1 \cap W_2$ must contain a plane.

(b) Now we consider the case where not all of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are in W_1. Since the arguments will be the same, let us assume that $\mathbf{v}_1$ is not in W_1.

i. Explain why the set $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}_1\}$ is a basis for $\mathbb{R}^4$.

ii. Explain why $\mathbf{v}_2$ and $\mathbf{v}_3$ can be written as linear combinations of the vectors in $\mathcal{S}$. Use these linear combinations to find two vectors that are in $W_1 \cap W_2$. Then show that these vectors span a plane in $W_1 \cap W_2$.

(11) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** The dimension of a finite dimensional vector space is the minimum number of vectors needed to span that space.

(b) **True/False** The dimension of a finite dimensional vector space is the maximum number of linearly independent vectors that can exist in that space.

(c) **True/False** If n vectors span an n-dimensional vector space V, then these vectors form a basis of V.

(d) **True/False** Any set of n vectors form a basis in an n-dimensional vector space.

(e) **True/False** Every vector in a vector space V spans a one-dimensional subspace of V.

(f) **True/False** Any set of n linearly independent vectors in a vector space V of dimensional n is a basis for V.

(g) **True/False** If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent in V, then $\dim(V) \geq k$.

(h) **True/False** If a set of k vectors span V, then any set of more than k vectors in V is linearly dependent.

(i) **True/False** If an infinite set of vectors span V, then V is infinite-dimensional.

(j) **True/False** If W_1, W_2 are both two-dimensional subspaces of $\mathbb{R}^3$, then $W_1 \cap W_2 \neq \{\mathbf{0}\}$.

(k) **True/False** If $\dim(V) = n$ and W is a subspace of V with dimension n, then $W = V$.

Project: Understanding Principal Component Analysis

Suppose we were to film an experiment involving a ball that is bouncing up and down. Naively, we set up several cameras to follow the process of the experiment from different perspectives and collect the data. All of this data tells us something about the bouncing ball, but there may be no perspective that tells us the most important piece of information – that axis along which the ball bounces. The question, then, is how we can extract from the data this most important piece of information. Principal Component Analysis (PCA) is a tool for just this type of analysis.

We will use an example to illustrate important concepts we will need. To realistically apply PCA we will have much more data than this, but for now we will restrict ourselves to only two variables so that we can visualize our results. Table 24.1 presents information from ten states on two attributes related to the SAT – Evidence-Based Reading and Writing (EBRW) score and Math score. The SAT is made up of three sections: Reading, Writing and Language (also just called

State	1	2	3	4	5	6	7	8	9	10
EBRW	595	540	522	508	565	512	643	574	534	539
Math	571	536	493	493	554	501	655	566	534	539

Table 24.1: SAT data.

Writing), and Math. The The EBRW score is calculated by combining the Reading and Writing section scores – both the Math and EBRW are scored on a scale of 200-800.

Each attribute (Math, EBRW score) creates a vector whose entries are the student responses for that attribute. The data provides the average scores from participating students in each state. In this example we have two attribute vectors:

$$\mathbf{x}_1 = [595\ 540\ 522\ 508\ 565\ 512\ 643\ 574\ 534\ 539]^{\mathsf{T}} \text{ and}$$
$$\mathbf{x}_2 = [571\ 536\ 493\ 493\ 554\ 501\ 655\ 566\ 534\ 539]^{\mathsf{T}}.$$

These vectors form the rows of a 2×10 matrix

$$X_0 = \begin{bmatrix} \mathbf{x}_1^{\mathsf{T}} \\ \mathbf{x}_2^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} 595 & 540 & 522 & 508 & 565 & 512 & 643 & 574 & 534 & 539 \\ 571 & 536 & 493 & 493 & 554 & 501 & 655 & 566 & 534 & 539 \end{bmatrix}$$

that makes up our data set. A plot of the data is shown at left in Figure 24.1, where the EBRW score is along the horizontal axis and the math score is along the vertical axis. The question we want to

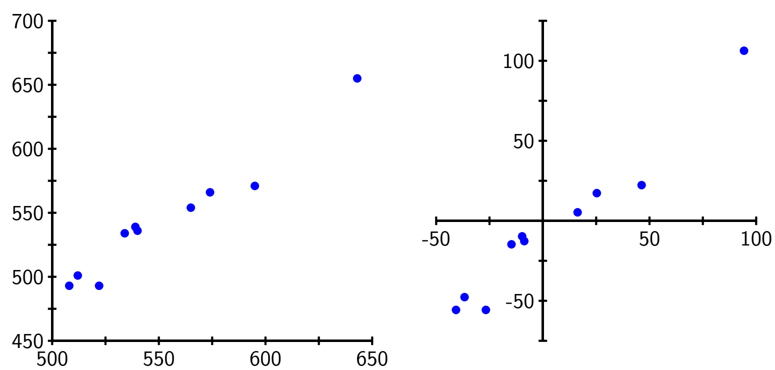

Figure 24.1: Two views of the data set (EBRW horizontal, math vertical).

answer is, how do we represent our data set so that the most important features in the data set are revealed?

Project Activity 24.1. Before analyzing a data set there is often some initial preparation that needs to be made. Issues that have to be dealt with include the problem that attributes might have different units and scales. For example, in a data set with attributes about people, height could be measured in inches while weight is measured in pounds. It is difficult to compare variables when they are on different scales. Another issue to consider is that some attributes are independent of the others (height, for example does not depend on hair color), while some are interrelated (body mass index depends on height and weight). To simplify our work, we will not address these type of problems. The only preparation we will do with our data is to center it.

(a) An important piece of information about a one-dimensional data set $\mathbf{x} = [x_1\ x_2\ x_3\ \cdots\ x_n]^{\mathsf{T}}$ is the sample average or mean

$$\overline{\mathbf{x}} = \sum_{i=1}^{n} x_i.$$

Calculate the means $\overline{\mathbf{x}_1}$ and $\overline{\mathbf{x}_2}$ for our SAT data from the matrix X_0.

(b) We can use these sample means to center our data at the origin by translating the data so that each column of our data matrix has mean 0. We do this by subtracting the mean for that row vector from each component of the vector. Determine the matrix X that contains the centered data for our SAT data set from matrix X_0.

A plot of the centered data for our SAT data is shown at right in Figure 24.1. Later we will see why centering the data is useful – it will more easily allow us to project onto subspaces. The goal of PCA is to find a matrix P so that $PX = Y$, and Y is suitably transformed to identify the important aspects of the data. We will discuss what the important aspects are shortly. Before we do so, we need to discuss a way to compare the one dimensional data vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Project Activity 24.2. To compare the two one dimensional data vectors, we need to consider variance and covariance.

(a) With data it is useful to know how spread out the data is, something the average doesn't tell us. For example, the data sets $[1\ 2\ 3]^{\mathsf{T}}$ and $[-2\ 0\ 8]^{\mathsf{T}}$ both have averages of 2, but the data in $[-2\ 0\ 8]^{\mathsf{T}}$ is more spread out. Variance provides one measure of how spread out a one-dimensional data set $\mathbf{x} = [x_1\ x_2\ x_3\ \cdots\ x_n]^{\mathsf{T}}$ is. Variance is defined as

$$\operatorname{var}(\mathbf{x}) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{\mathbf{x}})^2.$$

The variance provides a measure of how far from the average the data is spread.[1]

Determine the variances of the two data vectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Which is more spread out?

(b) In general, we will have more than one-dimensional data, as in our SAT data set. It will be helpful to have a way to compare one-dimensional data sets to try to capture the idea of variance for different data sets – how much the data in two different data sets varies from the mean with respect to each other. One such measure is covariance – essentially the average of all corresponding products of deviations from the means. We define the covariance of two data vectors $\mathbf{x} = [x_1\ x_2\ \cdots\ x_n]^{\mathsf{T}}$ and $\mathbf{y} = [y_1\ y_2\ \cdots\ y_n]^{\mathsf{T}}$ as

$$\operatorname{cov}(\mathbf{x},\mathbf{y}) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{\mathbf{x}})(y_i - \overline{\mathbf{y}}).$$

Determine all covariances

$$\operatorname{cov}(\mathbf{x}_1,\mathbf{x}_1),\ \operatorname{cov}(\mathbf{x}_1,\mathbf{x}_2),\ \operatorname{cov}(\mathbf{x}_2,\mathbf{x}_1),\ \text{ and } \operatorname{cov}(\mathbf{x}_2,\mathbf{x}_2).$$

[1] It might seem that we should divide by n instead of $n-1$ in the variance, but it is generally accepted to do this for reasons we won't get into. Suffice it to say that if we are using a sample of the entire population, then dividing by $n-1$ provides a variance whose square root is closer to the standard deviation than we would get if we divide by n. If we are calculating the variance of an entire population, then we would divide by n.

How are $\text{cov}(\mathbf{x}_1, \mathbf{x}_2)$ and $\text{cov}(\mathbf{x}_2, \mathbf{x}_1)$ related? How are $\text{cov}(\mathbf{x}_1, \mathbf{x}_1)$ and $\text{cov}(\mathbf{x}_2, \mathbf{x}_2)$ related to variances?

(c) What is most important about covariance is its sign. Suppose $\mathbf{y} = [y_1 \ y_2 \ \ldots \ y_n]^{\mathsf{T}}$, $\mathbf{z} = [z_1 \ z_2 \ \ldots \ z_n]^{\mathsf{T}}$ and $\text{cov}(\mathbf{y}, \mathbf{z}) > 0$. Then if y_i is larger than y_j it is likely that z_i is also larger than z_j. For example, if $\mathbf{y}$ is a vector that records a persons height from age 2 to 10 and $\mathbf{z}$ records tha same person's weight in the same years, we might expect that when y_i increases so does z_i. Similarly, if $\text{cov}(\mathbf{y}, \mathbf{z}) < 0$, then as one data set increases, the other decreases. As an example, if $\mathbf{y}$ records the number of hours a student spends playing video games each semester ad $\mathbf{z}$ gives the students GPA for each semester, then we might expect that z_i decreases as y_i increases. When $\text{cov}(\mathbf{y}, \mathbf{z}) = 0$, then $\mathbf{y}$ and $\mathbf{z}$ are said to be uncorrelated or independent of each other.

For our example $\mathbf{x}_1$ and $\mathbf{x}_2$, what does $\text{cov}(\mathbf{x}_1, \mathbf{x}_2)$ tell us about the relationship between $\mathbf{x}_1$ and $\mathbf{x}_2$? Why should we expect this from the context?

(d) The covariance gives us information about the relationships between the attributes. So instead of working with the original data, we work with the covariance data. If we have m data vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots$, $\mathbf{x}_m$ in $\mathbb{R}^n$, the *covariance matrix* $C = [c_{ij}]$ satisfies $c_{ij} = \text{cov}(\mathbf{x}_i, \mathbf{x}_j)$. Calculate the covariance matrix for our SAT data. Then explain why $C = \frac{1}{n-1}XX^{\mathsf{T}}$.

Recall that the goal of PCA is to find a matrix P such that $PX = Y$ where P transforms the data set to a coordinate system in which the important aspects of the data are revealed. We are now in a position to discuss what that means.

An ideal view of our data would be one in which we can see the direction of greatest variance and one that minimizes redundancy. With redundant data the variables are not independent – that is, covariance is nonzero. So we would like the covariances to all be zero (or as close to zero as possible) to remove redundancy in our data. In other words, we want covariance for Y to be diagonal.

Project Activity 24.3. Consider the covariance matrix $C = \begin{bmatrix} 1760.18 & 1967.62 \\ 1967.62 & 2319.29 \end{bmatrix}$. Find a matrix P whose columns are unit vectors that diagonalizes C. Use technology as appropriate.

The matrix P from Project Activity 24.3 has some especially important properties: the columns of P are unit vectors and, moreover, $P^{-1} = P^{\mathsf{T}}$ (you should check this - such a matrix is called an *orthogonal* matrix). For our purposes, we want to diagonalize XX^{T} with $PXX^{\mathsf{T}}P^{-1}$, so the matrix P that serve our purposes is the one whose *rows* are the eigenvectors of XX^{T}. To understand why this matrix is the one we want, recall that we want to have $PX = Y$, and we want to diagonalize XX^{T} to a diagonal covariance matrix YY^{T}. In this situation we will have (recalling that $P^{-1} = P^{\mathsf{T}}$)

$$\frac{1}{n-1}YY^{\mathsf{T}} = \frac{1}{n-1}(PX)(PX)^{\mathsf{T}} = \frac{1}{n-1}P\left(XX^{\mathsf{T}}\right)P^{\mathsf{T}} = P\left(XX^{\mathsf{T}}\right)P^{-1}.$$

So the matrix P that we want is exactly the one that diagonalizes XX^{T}.

Project Activity 24.4. There are two useful ways we can interpret the results of our work so far. The eigenvector of XX^{T} that corresponds to the largest (also called the *dominant*) eigenvalue λ_1

is $[-0.66 \ \ -0.76]^{\mathsf{T}}$. A plot of the centered data along with the eigenspace E_{λ_1} of XX^{T} spanned by $\mathbf{v}_1 = [-0.66 \ \ -0.76]^{\mathsf{T}}$ is shown at left in Figure 24.2. The eigenvector $\mathbf{v}_1$ is called the *first principal component* of X. Notice that this line E_{λ_1} indicates the direction of greatest variation in the data. In other words, when we project the data points onto E_{λ_1}, as shown at right in Figure 24.2, the variation of the resulting points is larger than it is for any other line. In other words, the data is most spread out in this direction.

(a) There is another way we can interpret this result. If we drop a perpendicular from one of our data points to the space E_{λ_1} it creates a right triangle with sides of length a, b, and c as illustrated in the middle of Figure 24.2. Use this idea to explain why maximizing the variation also minimizes the sum of the squares of the distances from the data points to this line. As a result, we have projected our two-dimensional data onto the one-dimensional space that maximizes the variance of the data.

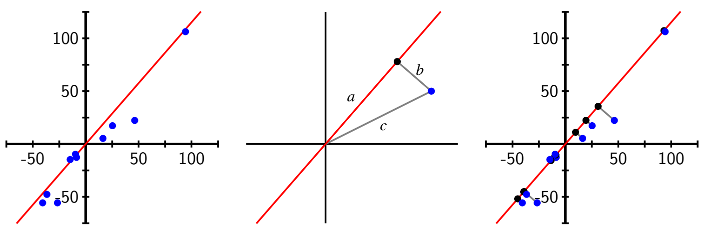

Figure 24.2: The principal component.

(b) Recall that the matrix (to two decimal places) $P = \begin{bmatrix} -0.66 & -0.76 \\ -0.76 & 0.66 \end{bmatrix}$ transforms the data set X to a new data set Y whose covariance matrix is diagonal. Notice that P looks like a rotation matrix, but is not exactly a rotation matrix. To understand what P does to X, a plot of the original data from X is shown with solid blue circles and a plot of the transformed data from Y in solid magenta diamonds is at left in Figure 24.3. The form of the matrix P looks similar to a rotation matrix, but is not exactly a rotation matrix. Show that P is a combination of a rotation matrix (and find the rotation angle) and a reflection (and specifically identify the reflection). Explain how this relates to Figure 24.3. Also explain how the x-axis is related to the transformed data set Y.

The result of Project Activity 24.4 is that we have reduced the problem from considering the data in a two-dimensional space to a one-dimensional space E_{λ_1} where the most important aspect of the data is revealed. Of course, we eliminate some of the characteristics of the data, but the most important aspect is still included and highlighted.

The second eigenvector of XX^{T} also has meaning. A picture of the eigenspace E_{λ_2} corresponding to the smaller eigenvector λ_2 of XX^{T} is shown in Figure 24.4. The second eigenvector of XX^{T} is perpendicular to the first, and the direction of the second eigenvector tells us the direction of the second most amount of variance as can be seen in Figure 24.4.

To summarize, the unit eigenvector for the largest eigenvalue of XX^{T} indicates the direction in which the data has the greatest variance. The direction of the unit eigenvector for the smaller eigenvalue shows the direction in which the data has the second largest variance. This direction is

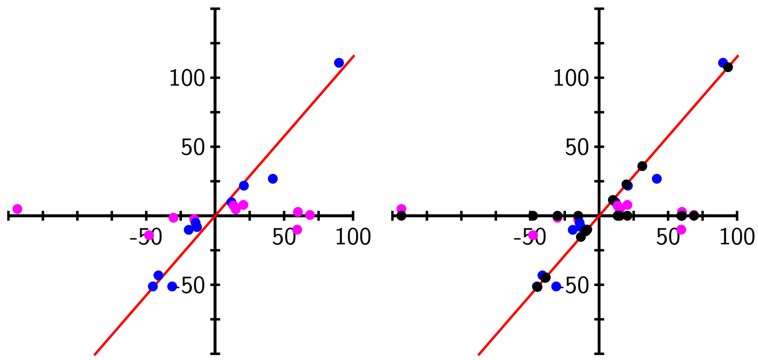

Figure 24.3: Applying P.

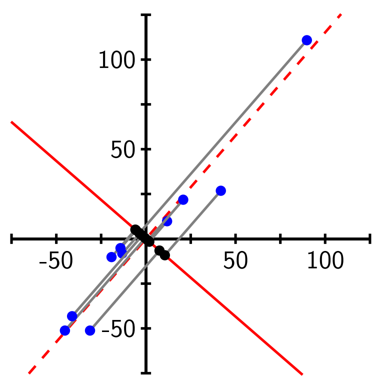

Figure 24.4: The second principal component.

also perpendicular to the first (indicating 0 covariance). The directions of the eigenvectors are called the *principal components* of X. The eigenvector with the highest eigenvalue is the first principal component of the data set and the other eigenvectors are ordered by the eigenvalues, highest to lowest. The principal components provide a new coordinate system from which to view our data – one in which we can see the maximum variance and in which there is zero covariance.

Project Activity 24.5. We can use the eigenvalues of XX^{T} to quantify the amount of variance that is accounted for by our projections. Notice that the points along the x-axis at right in Figure 24.3 are exactly the numbers in the first row of Y. These numbers provide the projections of the data in Y onto the x-axis – the axis along which the data has its greatest variance.

(a) Calculate the variance of the data given by the first row of Y. This is the variance of the data i the direction of the eigenspace E_{λ_1}. How does the result compare to entries of the covariance matrix for Y.

(b) Repeat part (a) for the data along the second row of Y.

(c) The total variance of the data set is the sum of the variances. Calculate the sum of the variances for X and Y. What do you notice? Explain why the percentage of variance in the data that is accounted for in the direction of E_{λ_1} is

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Then calculate this amount for the SAT data.

In general, PCA is most useful for larger data sets. The process is the same.

- Start with a set of data that forms the rows of an $m \times n$ matrix. We center the data by subtracting the mean of each row from the entries of that row to create a centered data set in a matrix X.
- The principal components of X are the eigenvectors of XX^{T}, ordered so that they correspond to the eigenvalues of XX^{T} in decreasing order.
- Let P be the matrix whose rows are the principal components of X, ordered from highest to lowest. Then $Y = PX$ is suitably transformed to identify the important aspects of the data.
- If λ_1, λ_2, . . ., λ_n are the eigenvalues of XX^{T} in decreasing order, then the amount of variance in the data accounted for by the first r principal components is given by

$$\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_r}{\lambda_1 + \lambda_2 + \cdots + \lambda_n}.$$

- The first r rows of $Y = PX$ provide the projection of the data set X onto an r-dimensional space spanned by the first r principal components of X.

Project Activity 24.6. Let us now consider a problem with more than two variables. We continue to keep the data set small so that we can conveniently operate with it. Table 24.2 presents additional information from ten states on four attributes related to the SAT – Participation rates, Evidence-Based Reading and Writing (EBRW) score, Math score, and average SAT score. Use technology as appropriate for this activity.

State	1	2	3	4	5	6	7	8	9	10
Rate	6	60	97	100	64	99	4	23	79	70
EBRW	595	540	522	508	565	512	643	574	534	539
Math	571	536	493	493	554	501	655	566	534	539
SAT	1166	1076	1014	1001	1120	1013	1296	1140	1068	1086

Table 24.2: SAT data.

(a) Determine the centered data matrix X for this data set.

(b) Find the covariance matrix for this data set. Round to four decimal places.

(c) Find the principal components of X. Include at least four decimal places accuracy.

(d) How much variation is accounted for in the data by the first principal component? In other words, if we reduce this data to one dimension, how much of the variation do we retain? Explain.

(e) How much variation is accounted for in the data by the first two principal components? In other words, if we reduce this data to two dimensions, how much of the variation do we retain? Explain.

Section 25

Coordinate Vectors and Coordinate Transformations

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- How do we find the coordinate vector of a vector $\mathbf{x}$ with respect to a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$?
- How can we visualize coordinate systems?
- How do we identify a vector space of dimension n with $\mathbb{R}^n$?
- What properties does the coordinate transformation $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ have?
- How are the coordinates of a vector with respect to a basis related to its coordinates with respect to the standard basis?

Application: Calculating Sums

Consider the problem of calculating sums of powers of integers. For example,

$$\sum_{t=0}^{n-1} t = \frac{1}{2}\left(n^2 - n\right)$$
$$\sum_{t=0}^{n-1} t^2 = \frac{1}{6}\left(2n^3 - 3n^2 + 1\right)$$
$$\sum_{t=0}^{n-1} t^3 = \frac{1}{4}\left(n^4 - 2n^3 + n^2\right)$$

and so on. It is possible to prove these formulas if we have an idea of what the formula is, but how can one come up with these formulas in the first place? As we will see later in this section, we can make use of coordinate vectors to find and verify such formulas.

Introduction

In this section we will investigate how a basis of a vector space provides a coordinate system in which each vector in the vector space has a unique set of coordinates. Such a coordinate system will make any n-dimensional vector space look like $\mathbb{R}^n$. If the vector space is already $\mathbb{R}^n$, then each basis will provide us with a new perspective to visualize $\mathbb{R}^n$. We begin our analysis of coordinate systems by looking at how a basis in $\mathbb{R}^2$ gives us a different view of $\mathbb{R}^2$.

Preview Activity 25.1. Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are shown in Figure 25.1.

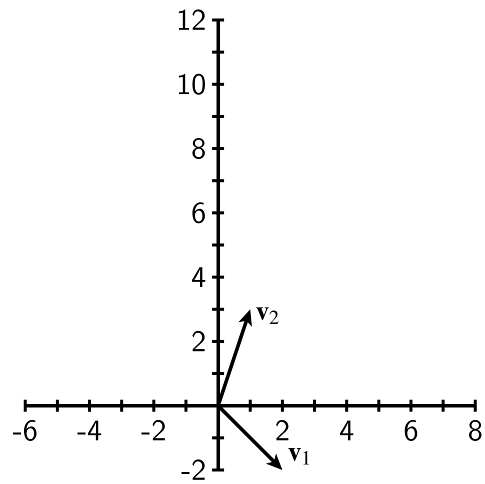

Figure 25.1: Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(1) Explain why $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$.

(2) Draw the vector $\mathbf{b} = -3\mathbf{v}_1 + 2\mathbf{v}_2$ in Figure 25.1. Explain your process.

(3) In part (1) we were given the weights (-3 and 2) of the linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ that produced $\mathbf{b}$. We call the vector $\left[\begin{array}{r} -3 \\ 2 \end{array}\right]$ the *coordinate vector* of $\mathbf{b}$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Explain why any vector $\mathbf{b}$ in $\mathbb{R}^2$ can be written as a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2$. This shows that each vector in $\mathbb{R}^2$ has a coordinate vector in the coordinate system defined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(4) Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$, any vector $\mathbf{b}$ in $\mathbb{R}^2$ has a coordinate vector in the coordinate system defined by $\mathbf{v}_1$ and $\mathbf{v}_2$. But we also need to make sure that each vector has a unique coordinate vector. Explain why there is no vector in $\mathbb{R}^2$ which has two different coordinate vectors.

(5) We can think of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as defining a coordinate system, with Span$\{\mathbf{v}_1\}$ as the "x"-axis and Span$\{\mathbf{v}_2\}$ as the "y"-axis. Any vector $\mathbf{b}$ in $\mathbb{R}^2$ can be written uniquely in the form

$$\mathbf{b} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$$

and the weights serve as the coordinates of $\mathbf{b}$ in the $\mathbf{v}_1$, $\mathbf{v}_2$ coordinate system. In this case the coordinate vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of $\mathbf{b}$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is written as $[\mathbf{b}]_{\mathcal{B}}$.

Let $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$.

(a) Show that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

(b) Find $[\mathbf{b}]_{\mathcal{B}}$ if $\mathbf{b} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$. Draw a picture to illustrate how $\mathbf{b}$ is expressed as a linear combination of the vectors in $\mathcal{B}$.

Bases as Coordinate Systems

Bases are useful for many reasons. A basis provides us with a unique representation of the elements in the space as linear combinations of the coordinate system defined by the basis vectors. The characterization of a vector space in terms of such a coordinate system will ultimately provide us with a powerful way to identify any vector space of dimension n geometrically and algebraically with $\mathbb{R}^n$ through coordinate assignment.

As we saw in Preview Activity 25.1, we can think of a basis $\mathcal{B}$ of $\mathbb{R}^2$ as determining a coordinate system of $\mathbb{R}^2$. For example, let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The vector $\mathbf{b} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}$ can be written as $\mathbf{b} = -3\mathbf{v}_1 + 2\mathbf{v}_2$. Figure 25.2 shows that if we plot the point that is -3 units in the $\mathbf{v}_1$ direction (where a "unit" is a copy of $\mathbf{v}_1$) and 2 units in the $\mathbf{v}_2$ direction, then the result is the vector from the origin to point defined by $\mathbf{b}$.

As discussed in Preview Activity 25.1, we can think of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as defining a coordinate system, with Span$\{\mathbf{v}_1\}$ as the "x"-axis and Span$\{\mathbf{v}_2\}$ as the "y"-axis. Since $\mathcal{B}$ is a basis, any vector $\mathbf{b}$ in $\mathbb{R}^2$ can be written uniquely in the form

$$\mathbf{b} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$$

and the weights serve as the coordinates of $\mathbf{b}$ in the $\mathbf{v}_1$, $\mathbf{v}_2$ coordinate system. We call the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ the *coordinate vector* of $\mathbf{b}$ with respect to the basis $\mathcal{B}$ and write this vector as $[\mathbf{b}]_{\mathcal{B}}$.

This is actually a familiar idea, one we have used for years. The standard coordinates of a vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ in $\mathbb{R}^2$ are just the coordinates of $\mathbf{a}$ with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$.

While we can draw pictures in $\mathbb{R}^2$, there is no reason to restrict this idea to $\mathbb{R}^2$.

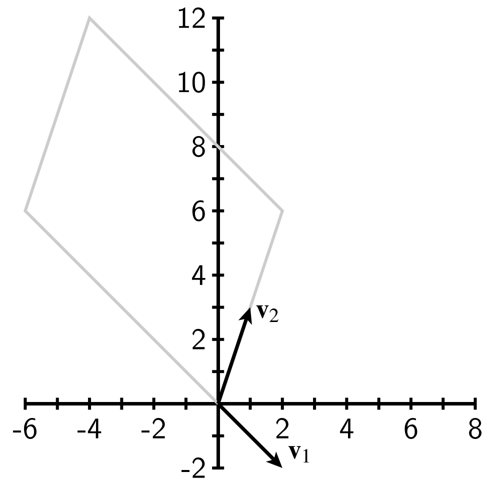

Figure 25.2: A Coordinate System Defined by a Basis.

Definition 25.1. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. For any vector $\mathbf{x}$ in V, the **coordinate vector** of $\mathbf{x}$ with respect to $\mathcal{B}$ is the vector

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where

$$\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n.$$

The scalars x_1, x_2, ..., x_n are the **coordinates of the vector $\mathbf{x}$ with respect to the basis**.

Recall that there is exactly one way to write a vector as a linear combination of basis vectors, so there is only one coordinate vector of a given vector with respect to a basis. Therefore, the coordinate vector of any vector is well-defined.

Activity 25.1. Let $\mathcal{S} = \{1, t\}$ and $\mathcal{B} = \{3 + 2t, 1 - t\}$. Assume that $\mathcal{S}$ and $\mathcal{B}$ are bases for $\mathbb{P}_1$. Find $[3 + 7t]_{\mathcal{S}}$ and $[3 + 7t]_{\mathcal{B}}$. Note that the coordinate vector depends on the basis that is used.

IMPORTANT NOTE: We have defined the coordinate vector of a vector $\mathbf{x}$ in a vector space V with respect to a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ as the vector $[x_1 \ x_2 \ \ldots \ x_n]^{\mathsf{T}}$ if

$$\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n.$$

Until now we have listed a basis as a set without regard to the order in which the basis elements are written. That is, the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the same as the set $\{\mathbf{v}_2, \mathbf{v}_1\}$. Notice, however, that if we change the order of the vectors in our basis, say from $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ to $\{\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$, then the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$ will be different. To avoid this problem, when discussing coordinate vectors we will consider our bases to be *ordered bases*, so that the order in which we write the elements in our basis is fixed. So, for example, the ordered basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is different than the ordered basis $\{\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$.

The Coordinate Transformation

We have seen that both $\mathbb{R}^2$ and $\mathbb{P}_1$ are vector spaces of dimension 2. Moreover, we can pair the polynomial $a_0 + a_1 t$ in $\mathbb{P}_1$ with the vector $[a_0\ a_1]$ in $\mathbb{R}^2$ as the coordinate vector of $a_0 + a_1 t$ with respect to the standard basis for $\mathbb{P}_1$. In this way we can see that the vector space $\mathbb{P}_1$ "looks" like the vector space $\mathbb{R}^2$. In fact, using this same idea, we can show that any vector space of dimension n "looks" like $\mathbb{R}^n$. The key idea to make this work is the coordinate transformation.

As we saw in Activity 25.1, the coordinate vector of a vector with respect to a basis depends on the basis we choose. In this activity we found that $[3+7t]_{\mathcal{S}} = [3\ 7]^{\mathsf{T}}$ and $[3+7t]_{\mathcal{B}} = [2\ {-3}]^{\mathsf{T}}$, where $\mathcal{S} = \{1, t\}$ and $\mathcal{B} = \{3 + 2t, 1 - t\}$. These coordinate vectors allow us to identify polynomials in $\mathbb{P}_1$ with vectors in $\mathbb{R}^2$. In this way, the coordinate vectors provide an identification between the vector space $\mathbb{P}_1$ and the vector space $\mathbb{R}^2$ via a *coordinate transformation*.

Definition 25.2. Let V be a vector space of dimension n with basis $\mathcal{B}$. The **coordinate transformation** T from V to $\mathbb{R}^n$ with respect to the basis $\mathcal{B}$ is the mapping defined by

$$T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$$

for any vector $\mathbf{x}$ in V.

Coordinate transformations have several useful properties that allow us to use them to compare the structure of vector spaces.

Activity 25.2. Let $a(t) = 8 + 2t$ and $b(t) = -5 + t$ in $\mathbb{P}_1$. Suppose we know $\mathcal{C} = \{1 + t, 2 - t\}$ is a basis of $\mathbb{P}_1$. Let $T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x}$ in $\mathbb{P}_1$.

(a) What are $T(a(t))$ and $T(b(t))$?

(b) Find $T(a(t)) + T(b(t))$.

(c) What is $T(a(t) + b(t)) = [a(t) + b(t)]_{\mathcal{C}}$?

(d) What is the relationship between $T(a(t)) + T(b(t))$ and $T(a(t) + b(t))$?

(e) Show that if c is any scalar, then $T(ca(t)) = cT(a(t))$.

(f) Where have we seen functions with these properties before?

Activity 25.2 shows that coordinate transformations behave in ways similar to matrix transformations. Recall that if A is an $m \times n$ matrix, then

$$A(\mathbf{u} + \mathbf{w}) = A\mathbf{u} + A\mathbf{w} \ \text{ and } \ A(c\mathbf{u}) = cA\mathbf{u}$$

for any vectors $\mathbf{u}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and any scalar c. Because of these properties, the transformation A preserves linear combinations. That is, if $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_n$ are any vectors in $\mathbb{R}^n$ and c_1, c_2, ..., c_n are any scalars, then

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + \cdots + c_n A\mathbf{v}_n.$$

Activity 25.2 contains the essential ideas to show that if T is a coordinate transformation from a vector space V with basis $\mathcal{B}$ to $\mathbb{R}^n$, then

$$T(\mathbf{u}+\mathbf{w}) = T(\mathbf{u}) + T(\mathbf{w}) \text{ and } T(c\mathbf{u}) = cT(\mathbf{u})$$

for any vectors $\mathbf{u}$ and $\mathbf{w}$ in V and any scalar c. The fact that any coordinate transformation satisfies these properties is contained in the following theorem.

Theorem 25.3. *If a vector space V has a basis $\mathcal{B}$ of n vectors, then the coordinate mapping $T : V \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ satisfies*

(1) $T(\mathbf{u}+\mathbf{w}) = T(\mathbf{u}) + T(\mathbf{w})$ *and*

(2) $T(c\mathbf{u}) = cT(\mathbf{u})$

for any vectors $\mathbf{u}$ and $\mathbf{w}$ in V and any scalar c.

Proof. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, and $\mathbf{v}_n$ be vectors that form a basis $\mathcal{B}$ for a vector space V. To verify the first property, let $\mathbf{u}$ and $\mathbf{w}$ be two arbitrary vectors from V. We will show that $T(\mathbf{u})+T(\mathbf{w}) = T(\mathbf{u}+\mathbf{w})$ for these two vectors. Consider $T(\mathbf{u})$. If $T(\mathbf{u}) = [u_1 \ u_2 \ \ldots \ u_n]^{\mathsf{T}}$, then the fact that $T(\mathbf{u}) = [\mathbf{u}]_{\mathcal{B}}$ implies that

$$\mathbf{u} = u_1\mathbf{v}_1 + u_2\mathbf{v}_2 + \cdots + u_n\mathbf{v}_n .$$

Similarly, if $T(\mathbf{w}) = [w_1 \ w_2 \ \ldots \ w_n]^{\mathsf{T}}$, then $T(\mathbf{w}) = [\mathbf{w}]_{\mathcal{B}}$ implies that

$$\mathbf{w} = w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \cdots + w_n\mathbf{v}_n .$$

We then obtain

$$\begin{aligned} \mathbf{u}+\mathbf{w} &= (u_1\mathbf{v}_1 + u_2\mathbf{v}_2 + \cdots + u_n\mathbf{v}_n) + (w_1\mathbf{v}_1 + w_2\mathbf{v}_2 + \cdots + w_n\mathbf{v}_n) \\ &= (u_1+w_1)\mathbf{v}_1 + (u_2+w_2)\mathbf{v}_2 + \cdots + (u_n+w_n)\mathbf{v}_n. \end{aligned}$$

Thus, by definition of T again,

$$T(\mathbf{u}+\mathbf{w}) = [\mathbf{u}+\mathbf{w}]_{\mathcal{B}} = [(u_1+w_1) \ (u_2+w_2) \ \ldots \ (u_n+w_n)]^{\mathsf{T}}.$$

To show that $T(\mathbf{u}+\mathbf{w}) = T(\mathbf{u}) + T(\mathbf{w})$, note that

$$\begin{aligned} T(\mathbf{u}) + T(\mathbf{w}) &= [u_1 \ u_2 \ \ldots \ u_n]^{\mathsf{T}} + [w_1 \ w_2 \ \ldots \ w_n]^{\mathsf{T}} \\ &= [(u_1+w_1) \ (u_2+w_2) \ \ldots \ (u_n+w_n)]^{\mathsf{T}} \\ &= [\mathbf{u}+\mathbf{w}]_{\mathcal{B}} \\ &= T(\mathbf{u}+\mathbf{w}). \end{aligned}$$

The proof of the second property is left for the exercises.

■

The first result of Theorem 25.3 can be extended to any finite number of vectors by mathematical induction. That is, if $T : V \to \mathbb{R}^n$ is the coordinate mapping defined by $T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ for some basis $\mathcal{B}$, and if $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_m$ are vectors in V, then $T(\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_m) = T(\mathbf{u}_1) + T(\mathbf{u}_2) + \cdots + T(\mathbf{u}_m)$. In other words,

$$[\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_m]_{\mathcal{B}} = [\mathbf{u}_1]_{\mathcal{B}} + [\mathbf{u}_2]_{\mathcal{B}} + \cdots + [\mathbf{u}_m]_{\mathcal{B}}$$

for any vectors $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_m$ in a vector space V with basis $\mathcal{B}$.

In essence, a coordinate transformation makes an identification of any vector space of dimension n with $\mathbb{R}^n$. One of the driving forces behind this identification is the fact that, just as in $\mathbb{R}^n$, if a vector space V has a basis of n elements, then any basis for V will have exactly n elements. The fact that any coordinate transformation from a vector space V to $\mathbb{R}^n$ is linear means that elements in V behave the same way with respect to addition and multiplication by scalars in V as do their images in $\mathbb{R}^n$ under the coordinate transformation. To make this identification complete, there are still two questions to address. First, given a coordinate transformation T from a vector space V to $\mathbb{R}^n$, is is possible for two vectors in V to have the same image in $\mathbb{R}^n$ (in other words, is T one-to-one), and is every vector in $\mathbb{R}^n$ the image of some vector in V under T (in other words, does T map V onto $\mathbb{R}^n$)? If the coordinate transformations are one-to-one and onto, then, in essence, any vector space of dimension n is just a copy of $\mathbb{R}^n$. As a result, to understand any vector space with a basis of n vectors, it is enough to understand $\mathbb{R}^n$.

Activity 25.3. Let V be a vector space with an ordered basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Then T maps V into $\mathbb{R}^n$. We want to show that T maps V onto $\mathbb{R}^n$. Recall that a function f from a set X to a set Y is onto if for any element y in Y, there is an element x in X such that $f(x) = y$. Let $\mathbf{b} = [b_1\ b_2\ \ldots\ b_n]^{\mathsf{T}}$ be a vector in $\mathbb{R}^n$. Must there be a vector $\mathbf{v}$ in V so that $T(\mathbf{v}) = \mathbf{b}$? If so, find such a vector. If not, explain why not.

Activity 25.3 shows that a coordinate transformation maps an n-dimensional vector space V onto $\mathbb{R}^n$. What's more, any coordinate transformation is also one-to-one (the proof that T is one-to-one is left for the exercises). We summarize these results in the following theorem.

Theorem 25.4. *If a vector space V has a basis $\mathcal{B}$ of n vectors, then the coordinate mapping $T : V \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ is both one-to-one and onto.*

Theorems 25.3 and 25.4 show that a coordinate transformation from an n-dimensional vector space V with basis $\mathcal{B}$ to $\mathbb{R}^n$ provides an identification of V with $\mathbb{R}^n$, where a vector $\mathbf{v}$ in V is identified with the vector $[\mathbf{v}]_{\mathcal{B}}$. This identification allows us to transfer questions (linear dependence, independence, span) in V to $\mathbb{R}^n$ where we can apply our knowledge of matrices to answer the questions.

Activity 25.4. Let $V = \mathbb{P}_3$ and let $\mathcal{B} = \{1, t, t^2, t^3\}$. Let $S = \{1 + t + t^2 + t^3, t - t^3, 1 + 2t^2, 1 + 5t - t^3\}$.

(a) Find each of $[1 + t + t^2 + t^3]_{\mathcal{B}}$, $[t - t^3]_{\mathcal{B}}$, $[1 + 2t^2]_{\mathcal{B}}$, and $[1 + 5t - t^3]_{\mathcal{B}}$.

(b) Are the vectors $[1+t+t^2+t^3]_{\mathcal{B}}$, $[t-t^3]_{\mathcal{B}}$, $[1+2t^2]_{\mathcal{B}}$, and $[1+5t-t^3]_{\mathcal{B}}$ linearly independent or dependent? Explain. If the vectors are linearly independent, write one of the vectors as a linear combination of the others.

(c) The coordinate transformation identifies the vectors in $S = \{1 + t + t^2 + t^3, t - t^3, 1 + 2t^2, 1 + 5t - t^3\}$ with their coordinate vectors in $\mathbb{R}^4$. Use that information to determine if S is a linearly independent or dependent set. If dependent, write one of the vectors in S as a linear combination of the others.

Examples

What follows are worked examples that use the concepts from this section.

Example 25.5.

(a) Find the coordinate vector of $\mathbf{v}$ with respect to the ordered basis $\mathcal{B}$ in the indicated vector space.

i. $\mathcal{B} = \{1 + t, 2 - t\}$ in $\mathbb{P}_1$ with $\mathbf{v} = 4 + t$

ii. $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ in $\mathcal{M}_{2\times 2}$ with $\mathbf{v} = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

(b) Find the vector $\mathbf{v}$ in the indicated vector space V given the basis $\mathcal{B}$ of V and $[\mathbf{v}]_{\mathcal{B}}$.

i. $V = \mathbb{P}_2$, $\mathcal{B} = \{1 + t^2, 1 + t, t + t^2\}$, $[\mathbf{v}]_{\mathcal{B}} = [2\ 1\ 3]^{\mathsf{T}}$

ii. $\mathcal{B} = \left\{\cos(x), \frac{1}{1+x^2}\right\}$, $V = \text{Span } \mathcal{B}$ as a subspace of $\mathcal{F}$, $[\mathbf{x}]_{\mathcal{B}} = [2\ \ -1]^{\mathsf{T}}$

Example Solution.

(a) Find the coordinate vector of $\mathbf{v}$ with respect to the ordered basis $\mathcal{B}$ in the indicated vector space.

i. We need to write $\mathbf{v} = 4 + t$ as a linear combination of $1 + t$ and $2 - t$. If $4 + t = c_1(1+t)+c_2(2-t)$, then equating coefficients of like power terms yields the equations $4 = c_1 + 2c_2$ and $1 = c_1 - c_2$. The solution to this system is $c_1 = 2$ and $c_2 = 1$, so $[4 + t]_{\mathcal{B}} = [2\ 1]^{\mathsf{T}}$.

ii. We need to write $\mathbf{v}$ as a linear combination of the vectors in $\mathcal{B}$. If

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

equating corresponding components produces the system

$$\begin{aligned} c_1 + c_2 + c_3 \quad &= 2 \\ c_3 \quad &= 3 \\ -c_2 \quad &= 1 \\ c_1 + c_2 + c_3 + c_4 &= 0. \end{aligned}$$

The solution to this system is $c_1 = 0$, $c_2 = -1$, $c_3 = 3$, and $c_4 = -2$, so $[\mathbf{v}]_{\mathcal{B}} = [0\ -1\ 3\ -2]^{\mathsf{T}}$.

(b) Find the vector $\mathbf{v}$ in the indicated vector space V given the basis $\mathcal{B}$ of V and $[\mathbf{v}]_{\mathcal{B}}$.

i. Since $[\mathbf{v}]_{\mathcal{B}} = [2\ 1\ 3]^{\mathsf{T}}$, it follows that

$$\mathbf{v} = 2(1+t^2) + 1(1+t) + 3(t+t^2) = 3 + 4t + 5t^2.$$

ii. Since $[\mathbf{x}]_{\mathcal{B}} = [2\ -1]^{\mathsf{T}}$, it follows that

$$\mathbf{v} = 2\cos(x) - \frac{1}{1+x^2}.$$

Example 25.6. Let $p_1(t) = 1$, $p_2(t) = 2 - t$, $p_3(t) = 3 + t - t^2$, $p_4(t) = t + t^3$, and $p_5(t) = 2t + t^2 + t^4$ be in $\mathbb{P}_4$. Also, let $\mathcal{B} = \{1, t, t^2, t^3, t^4\}$ be the standard basis for $\mathbb{P}_4$.

(a) Find $[p_1(t)]_{\mathcal{B}}$, $[p_2(t)]_{\mathcal{B}}$, $[p_3(t)]_{\mathcal{B}}$, $[p_4(t)]_{\mathcal{B}}$, and $[p_5(t)]_{\mathcal{B}}$.

(b) Use the result of part (a) to explain why $\{p_1(t), p_2(t), p_3(t), p_4(t), p_5(t)\}$ is a basis for $\mathbb{P}_4$.

(c) Let $p(t) = 2 - t + t^2 - 3t^3 + 4t^4$. Find $[p(t)]_{\mathcal{B}}$.

(d) Use the coordinate vectors in parts (a) and (c) to write $p(t)$ as a linear combination of $p_1(t)$, $p_2(t)$, $p_3(t)$, $p_4(t)$, and $p_5(t)$

Example Solution.

(a) The coordinate vectors of a polynomial with respect to the standard basis in $\mathbb{P}_4$ is found by just reading off the coefficients of the polynomial. So

$$\begin{array}{ll} [p_1(t)]_{\mathcal{B}} = [1\ 0\ 0\ 0\ 0]^{\mathsf{T}} & [p_2(t)]_{\mathcal{B}} = [2\ -1\ 0\ 0\ 0]^{\mathsf{T}} \\ [p_3(t)]_{\mathcal{B}} = [3\ 1\ -1\ 0\ 1]^{\mathsf{T}} & [p_4(t)]_{\mathcal{B}} = [0\ 1\ 0\ 1\ 0]^{\mathsf{T}} \\ [p_5(t)]_{\mathcal{B}} = [0\ 2\ 1\ 0\ 1]^{\mathsf{T}}. & \end{array}$$

(b) Let

$$\begin{aligned} \mathcal{C} &= \{p_1(t), p_2(t), p_3(t), p_4(t), p_5(t)\} \text{ and} \\ \mathcal{S} &= \{[p_1(t)]_{\mathcal{B}}, [p_2(t)]_{\mathcal{B}}, [p_3(t)]_{\mathcal{B}}, [p_4(t)]_{\mathcal{B}}, [p_5(t)]_{\mathcal{B}}\}. \end{aligned}$$

Since the coordinate transformation is one-to-one and onto, the two sets $\mathcal{C}$ and $\mathcal{S}$ are either both linearly dependent or linearly independent. Technology shows that the reduced row echelon form of

$$[[p_1(t)]_{\mathcal{B}}\ [p_2(t)]_{\mathcal{B}}\ [p_3(t)]_{\mathcal{B}}\ [p_4(t)]_{\mathcal{B}}\ [p_5(t)]_{\mathcal{B}}]$$

is I_5, so the sets $\mathcal{C}$ and $\mathcal{S}$ are both linearly linearly independent. Since $\dim(\mathbb{P}_4) = 5$, it follows that any linearly independent set of five vectors in $\mathbb{P}_4$ is a basis for $\mathbb{P}_4$. Therefore, the set $\mathcal{C}$ is a basis for $\mathbb{P}_4$.

(c) Similar to part (a), we have $[p(t)]_{\mathcal{B}} = [2\ -1\ 1\ -3\ 4]^{\mathsf{T}}$.

(d) Technology shows that the reduced row echelon form of the augmented matrix

$$[[p_1(t)]_{\mathcal{B}}\ [p_2(t)]_{\mathcal{B}}\ [p_3(t)]_{\mathcal{B}}\ [p_4(t)]_{\mathcal{B}}\ [p_5(t)]_{\mathcal{B}} \mid [p(t)]_{\mathcal{B}}]$$

is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -25 \\ 0 & 1 & 0 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right].$$

So

$$[p(t)]_{\mathcal{B}} = -25[p_1(t)]_{\mathcal{B}} + 9[p_2(t)]_{\mathcal{B}} + 3[p_3(t)]_{\mathcal{B}} - 3[p_4(t)]_{\mathcal{B}} + 4[p_5(t)]_{\mathcal{B}}$$

and

$$p(t) = -25p_1(t) + 9p_2(t) + 3p_3(t) - 3p_4(t) + 4p_5(t).$$

Summary

The key idea in this handout is the coordinate vector with respect to a basis.

- If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then the coordinate vector of $\mathbf{x}$ in V with respect to $\mathcal{B}$ is the vector

$$[\mathbf{x}]_{\mathcal{B}} = [x_1, x_2, \ldots, x_n]^{\mathsf{T}},$$

  where

$$\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n.$$

- The coordinate transformation $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one and onto transformation from an n-dimensional vector space V to $\mathbb{R}^n$ which preserves linear combinations.
- The coordinate transformation $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ allows us to translate problems in arbitrary vector spaces to $\mathbb{R}^n$ where we have already developed tools to solve the problems.

Exercises

(1) Let $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ be a basis of the subspace defined by the equation $y - 4x + z = 0$. Find the coordinates of the vector $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ with respect to the basis $\mathcal{B}$.

(2) Given basis $\mathcal{B} = \{1 + t, 2 + t^2, t + t^2\}$ of $\mathbb{P}_2$,

(a) For which $p(t)$ in $\mathbb{P}_2$ is $[p(t)]_{\mathcal{B}} = [1\ -1\ 3]^{\mathsf{T}}$?

(b) Determine coordinates of $q(t) = -1 + t + 2t^2$ in $\mathbb{P}_2$ with respect to basis $\mathcal{B}$.

(3) Find two different bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathbb{R}^2$ so that $[\mathbf{b}]_{\mathcal{B}_1} = [\mathbf{b}]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, where $\mathbf{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

(4) If $[\mathbf{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $[\mathbf{b}_2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with respect to some basis $\mathcal{B}$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, what are the coordinates of $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$?

(5) If $[\mathbf{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $[\mathbf{b}_2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with respect to some basis $\mathcal{B}$, where $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$, what are the vectors in $\mathcal{B}$?

(6) Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Describe how the coordinates of a vector with respect to $\mathcal{B}$ will change if $\mathbf{v}_1$ is replaced with $\frac{1}{2}\mathbf{v}_1$.

(7) Let $\mathcal{B} = \{1, t, 1 + t^2\}$ in $\mathbb{P}_2$.

(a) Show that $\mathcal{B}$ is a basis for $\mathbb{P}_2$.

(b) Let $p_1(t) = 1 + 2t^2$, $p_2(t) = 1 + t + 2t^2$, and $p_3(t) = 2 - t + t^2$ in $\mathbb{P}_2$.

i. Find $[p_1(t)]_{\mathcal{B}}$, $[p_2(t)]_{\mathcal{B}}$, and $[p_3(t)]_{\mathcal{B}}$.

ii. Use the coordinate vectors in part i. to determine if the set $\{p_1(t), p_2(t), p_3(t)\}$ is linearly independent or dependent.

(8) Let $W = \text{Span}\{2+4t+6t^3, 3-t^2, 3-t^2+9t^3\}$ in $\mathbb{P}_3$. Let $\mathcal{B} = \{1, t, t^2, t^3\}$ be the standard basis for $\mathbb{P}_3$.

(a) Calculate $[2 + 4t + 6t^3]_{\mathcal{B}}$, $[3 - t^2]_{\mathcal{B}}$ and $[3 - t^2 + 9t^3]_{\mathcal{B}}$.

(b) Use the coordinate vectors from part (a) to determine if the polynomials $2+4t+6t^3$, $3 - t^2$, and $3 - t^2 + 9t^3$ are linearly independent or dependent.

(c) Let $p(t) = 4 + 2t - t^2 + 9t^3$. Find $[p(t)]_{\mathcal{B}}$.

(d) Use the calculations from parts (a) and (c) to determine if $p(t)$ is in W. If so, write $p(t)$ as a linear combination of the polynomials $2+4t+6t^3$, $3-t^2$, and $3-t^2+9t^3$. If not, explain why not.

(9) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right\}$ in $M_{2\times 2}$.

(a) Show that $\mathcal{B}$ is a basis of $M_{2\times 2}$.

(b) Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Find $[A]_{\mathcal{B}}, [B]_{\mathcal{B}}, [C]_{\mathcal{B}}, [D]_{\mathcal{B}}$.

(c) Determine if the set $\{A, B, C, D\}$ is linearly dependent or independent.

(10) Let V be a vector space of dimension n and let $\mathcal{B}$ be a basis for V. Show that the coordinate transformation T from V to $\mathbb{R}^n$ defined by $T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ satisfies $T(\mathbf{0}_V) = \mathbf{0}$, where $\mathbf{0}_V$ is the additive identity in V.

(11) Prove the second property of Theorem 25.3. That is, if a vector space V has a basis of n vectors, then the coordinate mapping $T : V \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ satisfies

$$T(c\mathbf{u}) = cT(\mathbf{u})$$

for any vector $\mathbf{u}$ in V and any scalar c.

(12) Prove Theorem 25.4 by demonstrating that if V is a vector space with ordered basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then the coordinate mapping $T : V \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$ is one-to-one.

(13) The coordinate transformation T is one-to-one, so it has an inverse T^{-1}. Let V be an n-dimensional vector space that has a basis $\mathcal{B}$, and let $T : V \to \mathbb{R}^n$ be the coordinate transformation defined by $T(\mathbf{x}) = [\mathbf{x}]_{\mathcal{B}}$. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of V and let $R = \{[\mathbf{u}_1]_{\mathcal{B}}, [\mathbf{u}_2]_{\mathcal{B}}, \ldots, [\mathbf{u}_k]_{\mathcal{B}}\}$ in $\mathbb{R}^n$.

(a) Suppose $\mathbf{x}$ is in V with $\mathbf{x} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_k\mathbf{u}_k$. Write the vector $[\mathbf{x}]_{\mathcal{B}}$ as a linear combination of the vectors in R. Explain your reasoning and explain how your result shows that T preserves linear combinations.

(b) Now suppose $\mathbf{w}$ is in V so that $[\mathbf{w}]_{\mathcal{B}} = w_1[\mathbf{u}_1]_{\mathcal{B}} + w_2[\mathbf{u}_2]_{\mathcal{B}} + \cdots + w_k[\mathbf{u}_k]_{\mathcal{B}}$. Write the vector $\mathbf{w}$ as a linear combination of the vectors in S. Explain your reasoning and explain how your result shows that T^{-1} preserves linear combinations.

(14) Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of an n-dimensional vector space V with basis $\mathcal{B}$. Let $R = \{[\mathbf{u}_1]_{\mathcal{B}}, [\mathbf{u}_2]_{\mathcal{B}}, \ldots, [\mathbf{u}_k]_{\mathcal{B}}\}$ in $\mathbb{R}^n$.

(a) Show that if S is linearly independent in V, then R is linearly independent in $\mathbb{R}^n$.

(b) Is the converse of part (a) true? That is, if R is linearly independent in $\mathbb{R}^n$, must S be linearly independent in V? Justify your answer.

(c) Repeat parts (a) and (b), replacing 'linearly independent" with "linearly dependent".

(15) Let V be an n-dimensional vector space with basis $\mathcal{B}$, and let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be a subset of V that spans V. Prove that $\{[\mathbf{w}_1]_{\mathcal{B}}, [\mathbf{w}_2]_{\mathcal{B}}, \ldots, [\mathbf{w}_k]_{\mathcal{B}}\}$ spans $\mathbb{R}^n$.

(16) Suppose $\mathcal{B}_1$ is a basis of a vector space V with n elements and $\mathcal{B}_2$ is a basis of W with n elements. Show that the map T_{VW} which sends every $\mathbf{x}$ in V to the vector $\mathbf{y}$ in W such that $[\mathbf{x}]_{\mathcal{B}_1} = [\mathbf{y}]_{\mathcal{B}_2}$ is one-to-one and onto.

(17) Label each of the following statements as True or False. Provide justification for your response.

(a) **True/False** The coordinates of a non-zero vector cannot be the same in the coordinate systems defined by two different bases.

(b) **True/False** The coordinate vector of the zero vector with respect to any basis is always the zero vector.

(c) **True/False** If W is a k dimensional subset of an n dimensional vector space V, and $\mathcal{B}$ is a basis of W, then $[\mathbf{w}]_{\mathcal{B}}$ is a vector in $\mathbb{R}^n$ for any $\mathbf{w}$ in W.

(d) **True/False** The order of vectors in a basis do not affect the coordinates of vectors with respect to this basis.

(e) **True/False** If T is a coordinate transformation from a vector space V with basis $\mathcal{B}$ to $\mathbb{R}^n$, then the vector $[\mathbf{x}]_{\mathcal{B}}$ is unique to the vector $\mathbf{x}$ in V.

(f) **True/False** If T is a coordinate transformation from a vector space V with basis $\mathcal{B}$ to $\mathbb{R}^n$, then there is always a vector $\mathbf{x}$ in V that maps to any vector $\mathbf{b}$ in $\mathbb{R}^n$.

(g) **True/False** A coordinate transformation from a vector space V with basis $\mathcal{B}$ to $\mathbb{R}^n$ always maps the additive inverse of a vector $\mathbf{x}$ in V to the additive inverse of the vector $[\mathbf{x}]_{\mathcal{B}}$ in $\mathbb{R}^n$.

(h) **True/False** A coordinate transformation provides a unique identification of vectors in an n-dimensional vector space with vectors in $\mathbb{R}^n$ in a way that preserves the algebraic structure of the spaces.

(i) **True/False** If the coordinate vector of $\mathbf{x}$ in a vector space V is $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, then the coordinate vector of $2\mathbf{x}$ is $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$.

Project: Finding Formulas for Sums of Powers

One way to derive formulas for sums of powers of whole numbers is to use different bases and coordinate vectors. One basis that will be useful is a basis of polynomials created by the binomial coefficients. Recall that the binomial coefficient $\binom{n}{k}$ is equal to

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

with $\binom{n}{k} = 0$ if $n < k$.

The binomial coefficient can be rewritten in a way to make it applicable to polynomials as

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \Pi_{i=1}^{n}\frac{1}{i}(n-i+1).$$

With this representation of $\binom{n}{k}$, we can define a new polynomial in t of degree k as

$$p_k(t) = \frac{1}{k!}(t)(t-1)(t-2)\cdots(t-k+1)$$

with $p_0(t) = 1$. For example,

$$\begin{aligned} p_1(t) &= \frac{1}{1!}t = t \\ p_2(t) &= \frac{1}{2}(t)(t-1) = \frac{1}{2}\left(t^2 - t\right) \\ p_3(t) &= \frac{1}{6}(t)(t-1)(t-2) = \frac{1}{6}\left(t^3 - 3t^2 + 2t\right). \end{aligned}$$

These "generalized binomial coefficients" appear in Newton's generalized binomial theorem.

Two facts will make these polynomials useful for our sums.

Project Activity 25.1. Our polynomials $p_k(t)$ are defined in terms of binomial coefficients. A useful identity will relate sums of binomial coefficients to other binomial coefficients. This identity, called the *hockey-stick identity* after the way it can be visualized on Pascal's triangle, is as follows:

$$\sum_{k=0}^{n-1} \binom{k}{r} = \binom{n}{r+1}$$

for positive integers r. Use the definition of the binomial coefficients and some algebra to verify the hockey-stick identity.

The second useful fact about our polynomials $p_k(t)$ is that they form a basis for $\mathbb{P}_n$.

Project Activity 25.2.

(a) Let k be a positive integer. Explain why $p_k(0) = p_k(1) = p_k(2) = \cdots = p_k(k-1) = 0$ and $p_k(k) = 1$.

(b) Let $\mathcal{P}_n = \{p_0(t), p_1(t), p_2(t), \ldots, p_n(t)\}$. Show that $\mathcal{P}_n$ is a basis for $\mathbb{P}_n$. (Hint: Let c_1, c_2, ..., c_n be scalars and consider the equation

$$c_0p_0(t) + c_1p_1(t) + c_2p_2(t) + \cdots + c_np_n(t) = 0.$$

Evaluate this equation at $t = 0$, $t = 1$, ..., $t = n$ and use the result of part (a).)

Now we have the tools we need to derive our formulas. To simplify computations, we will change coordinates to the standard basis $\mathcal{S}_n = \{1, t, t^2, \ldots, t^n\}$ for $\mathbb{P}_n$.

We will illustrate the process of deriving formulas for our sums with the sum $\sum_{t=0}^{n-1} t$. We want to write t as a linear combination of the vectors in $\mathcal{P}_1$ so that we can utilize the hockey-stick identity. In this case, by definition we have $t = p_1(t)$. It follows by the hockey-stick identity and the fact that $p_1(t) = \binom{t}{1}$ that

$$\begin{aligned} \sum_{t=0}^{n-1} t &= \sum_{t=0}^{n-1} p_1(t) \\ &= \sum_{t=0}^{n-1} \binom{t}{1} \\ &= \binom{n}{2} \\ &= \frac{1}{2}\left(n^2 - n\right). \end{aligned}$$

This is exactly the formula we saw at the beginning of this section. The cases for sums of higher powers work the same way, but we will need to do a little more work to write t^n in terms of the basis vectors in $\mathcal{P}_n$.

Project Activity 25.3. Consider the sum $\sum_{t=0}^{n-1} t^2$. We want to write t^2 as a linear combination of $p_0(t)$, $p_1(t)$ and $p_2(t)$. To do so, we will use the coordinate vectors with respect to $\mathcal{S}_2$ and do our work in $\mathbb{R}^3$.

(a) Find $[p_0(t)]_{\mathcal{S}_2}$, $[p_1(t)]_{\mathcal{S}_2}$, $[p_2(t)]_{\mathcal{S}_2}$, and $[t^2]_{\mathcal{S}_2}$.

(b) Use the coordinate vectors from part (a) to write t^2 as a linear combination of the vectors in $\mathcal{P}_2$.

(c) Use the result of pat (b) and, the hockey-stick identity, and the fact that $p_k(t) = \binom{t}{k}$ to find a formula for $\sum_{t=0}^{n-1} t^2$.

Deriving formulas for higher powers involves the same process, just with more algebra.

Project Activity 25.4. Use the process outlined in Project Activity 25.3 to derive formulas for the following sums.

(a) $\sum_{t=0}^{n-1} t^3$

(b) $\sum_{t=0}^{n-1} t^4$

Section 26

Change of Basis

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a change of basis matrix?
- Why is a change of basis useful?
- How can we use coordinate transformations to find a change of basis matrix?
- What are three important properties of change of basis matrices?

Application: Describing Orbits of Planets

Consider a planet orbiting the sun (or an object like a satellite orbiting the Earth). According to Kepler's Laws, we assume an elliptical orbit. There are many different ways to describe this orbit, and which description we use depends on our perspective and the application. One important perspective is to make the description of the orbit as simple as possible for earth-based observations. Two problems arise. One is that the earth's orbit and the orbit of the planet do not lie in the same plane. A second problem is that it is complicated to describe the orbit of a planet using the perspective of the plane of the earth's orbit. A reasonable approach, then, is to establish two different coordinate systems, one for the earth's orbit and one for the planet's orbit. We can then use a change of basis to move back and forth from these two perspectives.

Introduction

In calculus we changed coordinates, from rectangular to polar, for example, to make certain calculations easier. In order for us to be able to work effectively in different coordinate systems, and to easily change back and forth as needed, we will want to have a way to effectively transition from one coordinate system to another. In other words, if we have two different bases for a vector space V, we want a straightforward way to translate between the coordinate vectors of any given vector in V with respect to the two bases.

Preview Activity 26.1.

(1) Let $b_1(t) = 4 + 2t$, $b_2(t) = -6 + 8t$, $c_1(t) = 1 + t$, $c_2(t) = 1 - t$, and let $\mathcal{B} = \{b_1(t), b_2(t)\}$ and $\mathcal{C} = \{c_1(t), c_2(t)\}$.

(a) Show that $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{P}_1$.

(b) Let $p(t) = 3b_1(t) + 2b_2(t)$. What is $[p(t)]_{\mathcal{B}}$?

(c) Since $\mathcal{C}$ is also a basis for $\mathbb{P}_1$, there is also a coordinate vector for $p(t)$ with respect to $\mathcal{C}$, and it is reasonable to ask how $[p(t)]_{\mathcal{C}}$ is related to $[p(t)]_{\mathcal{B}}$. Recall that a coordinate transformation respects linear combinations – that is

$$[r\mathbf{x} + s\mathbf{y}]_{\mathcal{S}} = r[\mathbf{x}]_{\mathcal{S}} + s[\mathbf{y}]_{\mathcal{S}}$$

for any vectors $\mathbf{x}$ and $\mathbf{y}$ in a vector space with basis $\mathcal{S}$, and any scalars r and s. Use the fact that $p(t) = 3b_1(t) + 2b_2(t)$ and the linearity of the coordinate transformation with respect to the basis $\mathcal{C}$ to express $[p(t)]_{\mathcal{C}}$ in terms of $[b_1(t)]_{\mathcal{C}}$ and $[b_2(t)]_{\mathcal{C}}$ (don't actually calculate $[b_1(t)]_{\mathcal{C}}$ and $[b_2(t)]_{\mathcal{C}}$ yet, just leave your result in terms of the symbols $[b_1(t)]_{\mathcal{C}}$ and $[b_2(t)]_{\mathcal{C}}$.)

(d) The result of part (c) can be expressed as a matrix-vector product of the form

$$[p(t)]_{\mathcal{C}} = P[p(t)]_{\mathcal{B}}.$$

Describe how the columns of the matrix P are related to $[b_1(t)]_{\mathcal{C}}$ and $[b_2(t)]_{\mathcal{C}}$.

(e) Now calculate $[b_1(t)]_{\mathcal{C}}$, $[b_2(t)]_{\mathcal{C}}]$, and $[p(t)]_{\mathcal{C}}$. Determine the entries of the matrix P and verify in this example that $[p(t)]_{\mathcal{C}} = P[p(t)]_{\mathcal{B}}$.

(2) The matrix P that we constructed in problem (1) allows us to quickly and easily switch from coordinates with respect to a basis $\mathcal{B}$ to coordinates with respect to another basis $\mathcal{C}$, providing a way to effectively transition from one coordinate system to another as described in the introduction. This matrix P is called a *change of basis matrix*. In problem (1) we explained why the change of basis matrix exists, and in this problem we will see another perspective from which to view this matrix. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be two bases for a vector space V. The change of basis matrix P from $\mathcal{B}$ to $\mathcal{C}$ has the property that $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ for every vector $\mathbf{x}$ in V. We can determine the entries of P by applying this formula to specific vectors in V.

(a) What are $[\mathbf{b}_1]_{\mathcal{B}}$ and $[\mathbf{b}_2]_{\mathcal{B}}$? Why?

(b) If A is an $n \times n$ matrix and $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$ are the standard unit vectors in $\mathbb{R}^n$ (that is, $\mathbf{e}_i$ is the ith column of the $n \times n$ identity matrix), then what does the product $A\mathbf{e}_i$ tell us about the matrix A?

(c) Combine the results of parts (a) and (b) and the equation $P[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$ to explain why $P = [[\mathbf{b}_1]_{\mathcal{C}}\ [\mathbf{b}_2]_{\mathcal{C}}]$.

The Change of Basis Matrix

Suppose we have two different finite bases $\mathcal{B}$ and $\mathcal{C}$ for a vector space V. In Preview Activity 26.1 we learned how to translate between the two bases in the 2-dimensional case – if $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, then the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is the matrix $[[\mathbf{b}_1]_{\mathcal{C}} \ [\mathbf{b}_2]_{\mathcal{C}}]$. This result in the 2-dimensional case generalizes to the n-dimensional case, and we can determine a straightforward method for calculating a change of basis matrix. The essential idea was introduced in Preview Activity 26.1.

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ be two bases for a vector space V. If $\mathbf{x}$ is in V, we have defined the coordinate vectors $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x}$ with respect to $\mathcal{B}$ and $\mathcal{C}$, respectively. Recall that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ if

$$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n.$$

To see how to convert from the coordinates of $\mathbf{x}$ with respect to $\mathcal{B}$ to coordinates of $\mathbf{x}$ with respect to $\mathcal{C}$, note that

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n]_{\mathcal{C}} \\ &= x_1[\mathbf{b}_1]_{\mathcal{C}} + x_2[\mathbf{b}_2]_{\mathcal{C}} + \cdots + x_n[\mathbf{b}_n]_{\mathcal{C}} \\ &= [[\mathbf{b}_1]_{\mathcal{C}} \ [\mathbf{b}_2]_{\mathcal{C}} \ \cdots \ [\mathbf{b}_n]_{\mathcal{C}}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= [[\mathbf{b}_1]_{\mathcal{C}} \ [\mathbf{b}_2]_{\mathcal{C}} \ \cdots \ [\mathbf{b}_n]_{\mathcal{C}}][\mathbf{x}]_{\mathcal{B}}. \end{aligned}$$

So we can convert from coordinates with respect to the basis $\mathcal{B}$ to coordinates with respect to the basis $\mathcal{C}$ by multiplying $[\mathbf{x}]_{\mathcal{B}}$ on the left by the matrix

$$[[\mathbf{b}_1]_{\mathcal{C}} \ [\mathbf{b}_2]_{\mathcal{C}} \ \cdots \ [\mathbf{b}_n]_{\mathcal{C}}].$$

This matrix is called the *change of basis matrix* from $\mathcal{B}$ to $\mathcal{C}$ and is denoted $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$.

Definition 26.1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ be two bases for a vector space V. The **change of basis matrix** from $\mathcal{B}$ to $\mathcal{C}$ is the matrix

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = [[\mathbf{b}_1]_{\mathcal{C}} \ [\mathbf{b}_2]_{\mathcal{C}} \ \cdots \ [\mathbf{b}_n]_{\mathcal{C}}].$$

The change of basis matrix allows us to convert from coordinates with respect to one basis to coordinates with respect to another. The result is summarized in the following theorem.

Theorem 26.2. *Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ be two bases for a vector space V. Then*

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$$

for any vector $\mathbf{x}$ in V.

Finding a Change of Basis Matrix

We can use matrix techniques we have developed to quickly find a change of basis matrix in a finite dimensional vector space. The coordinate transformation will allow us to transfer our calculations into $\mathbb{R}^n$, and especially useful are the coordinate vectors with respect to a standard basis. These coordinate vectors are easy to find – for example, if $p(t) = 1 + t - 2t^2$ in $\mathbb{P}_2$ and $\mathcal{S} = \{1, t, t^2\}$ is the standard basis for $\mathbb{P}_2$, then we can see by inspection that $[p(t)]_{\mathcal{S}} = [1\ 1\ \ -2]^{\mathsf{T}}$. We will also utilize the same process we developed for calculating the inverse of an invertible $n \times n$ matrix A. Recall that this process involved solving the n matrix equations $A\mathbf{x} = \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith column of the $n \times n$ identity matrix I_n. We were able to solve these n systems all at one time by augmenting A with all of the vectors $\mathbf{e}_i$ at once, which amounted to augmenting A with I_n and row reducing. Keep these ideas in mind for the following activity.

Activity 26.1. Let $b_1(t) = 4 + t$, $b_2(t) = 2 + 5t$, $c_1(t) = -1 + 2t$, and $c_2(t) = -1 - t$. The sets $\mathcal{B} = \{b_1(t), b_2(t)\}$ and $\mathcal{C} = \{c_1(t), c_2(t)\}$ are bases for $\mathbb{P}_1$. Our goal is to find the change of basis matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$. We will use the coordinate transformation with respect to the standard basis $\mathcal{S} = \{1, t\}$ for $\mathbb{P}_1$ to transfer our work into $\mathbb{R}^2$.

(a) What are $[b_1(t)]_{\mathcal{S}}$, $[b_2(t)]_{\mathcal{S}}$, $[c_1(t)]_{\mathcal{S}}$, and $[c_2(t)]_{\mathcal{S}}$?

(b) What matrix equation must we solve to write $[b_1(t)]_{\mathcal{S}}$ as a linear combination of the vectors $[c_1(t)]_{\mathcal{S}}$ and $[c_2(t)]_{\mathcal{S}}$? How do we solve this equation? (Don't solve the equation yet.)

(c) What matrix equation must we solve to write $[b_2(t)]_{\mathcal{S}}$ as a linear combination of the vectors $[c_1(t)]_{\mathcal{S}}$ and $[c_2(t)]_{\mathcal{S}}$? How do we solve this equation? (Don't solve the equation yet.)

(d) Let A be the coefficient matrix of the systems you wrote in parts (b) and (c). Find the reduced row echelon form of the augmented matrix $[A \mid [b_1(t)]_{\mathcal{S}}\ [b_2(t)]_{\mathcal{S}}]$.

(e) Use the result of part (d) to write $b_1(t)$ and $b_2(t)$ as linear combinations of $c_1(t)$ and $c_2(t)$.

(f) Based on your responses to (b) and (c), if $[I_2 \mid P]$ is the reduced row echelon form of the matrix $[A \mid [b_1(t)]_{\mathcal{S}}\ [b_2(t)]_{\mathcal{S}}]$, what property will the matrix P have? Explain.

Activity 26.1 demonstrates how we can find a change of basis matrix in an n-dimensional vector space V. If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ are two bases for V, then the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is given by

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = [[\mathbf{b}_1]_{\mathcal{C}}\ [\mathbf{b}_2]_{\mathcal{C}}\ \cdots\ [\mathbf{b}_n]_{\mathcal{C}}].$$

To find $[\mathbf{b}_i]_{\mathcal{C}}$, we need to write $\mathbf{b}_i$ as a linear combination of the vectors in $\mathcal{C}$. That is, we need to find weights $x_{1.i}$, $x_{2,i}$, …, $x_{n,i}$ so that

$$\mathbf{b}_i = x_{1,i}\mathbf{c}_1 + x_{2,i}\mathbf{c}_2 + \cdots + x_{n,i}\mathbf{c}_n. \tag{26.1}$$

The weights in equation (26.1) are also the weights that satisfy the equation

$$[\mathbf{b}_i]_{\mathcal{S}} = x_{1,i}[\mathbf{c}_1]_{\mathcal{S}} + x_{2,i}[\mathbf{c}_2]_{\mathcal{S}} + \cdots + x_{n,i}[\mathbf{c}_n]_{\mathcal{S}}$$

where $\mathcal{S}$ is any basis for V. So to find these weights, we choose a convenient basis $\mathcal{S}$ (often the standard basis, if one exists, is a good choice) and then row reduce the matrix

$$[[\mathbf{c}_1]_{\mathcal{S}}\ [\mathbf{c}_2]_{\mathcal{S}}\ \cdots\ [\mathbf{c}_n]_{\mathcal{S}} \mid [\mathbf{b}_i]_{\mathcal{S}}].$$

The row operations we will apply to row reduce the coefficient matrix

$$[[\mathbf{c}_1]_{\mathcal{S}}\ [\mathbf{c}_2]_{\mathcal{S}}\ \cdots\ [\mathbf{c}_n]_{\mathcal{S}}]$$

will be the same regardless of the augmented column, so we can solve all of the systems at one time by row reducing the matrix

$$[[\mathbf{c}_1]_{\mathcal{S}}\ [\mathbf{c}_2]_{\mathcal{S}}\ \cdots\ [\mathbf{c}_n]_{\mathcal{S}} \mid [\mathbf{b}_1]_{\mathcal{S}}\ [\mathbf{b}_2]_{\mathcal{S}}\ \cdots\ [\mathbf{b}_n]_{\mathcal{S}}].$$

The result of the row reduction will be the matrix

$$\left[I_n \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right].$$

Let us return to our example in Activity 26.1. For ease of computation we choose $\mathcal{S} = \{1, t\}$ to be the standard basis for $\mathbb{P}_1$. Then $[b_1(t)]_{\mathcal{S}} = [4\ 1]^{\mathsf{T}}$, $[b_2(t)]_{\mathcal{S}} = [2\ 5]^{\mathsf{T}}$, $[c_1(t)]_{\mathcal{S}} = [-1\ 2]^{\mathsf{T}}$, and $[c_2(t)]_{\mathcal{S}} = [-1\ \ -1]^{\mathsf{T}}$. Row reducing

$$[[c_1(t)]_{\mathcal{S}}\ [c_2(t)]_{\mathcal{S}} \mid [b_1(t)]_{\mathcal{S}}\ [b_2(t)]_{\mathcal{S}}] = \left[\begin{array}{rr|rr} -1 & -1 & 4 & 2 \\ 2 & -1 & 1 & 5 \end{array}\right]$$

gives us

$$\left[\begin{array}{rr|rr} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & -3 \end{array}\right].$$

So the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[\begin{array}{rr} -1 & 1 \\ -3 & -3 \end{array}\right].$$

Note that columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ tell us how to write our basis vectors in $\mathcal{B}$ as linear combinations of the basis vectors in $\mathcal{C}$. So we can check our work by noting that $b_1(t) = -c_1(t) - 3c_2(t)$ and $b_2(t) = c_1(t) - 3c_2(t)$.

The coordinate transformations, along with the change of basis matrix, allow us to visualize finite dimensional vector spaces in $\mathbb{R}^n$. In our example from Activity 26.1 we can view the vectors in the bases $\mathcal{B}$ and $\mathcal{C}$ as their coordinate vectors $[b_1(t)]_{\mathcal{S}}$, $[b_2(t)]_{\mathcal{S}}$, $[c_1(t)]_{\mathcal{S}}$, and $[c_2(t)]_{\mathcal{S}}$ as illustrated at left in Figure 26.1. The change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ tells us how to transform a vector whose coordinates are given in the coordinate system defined by the basis $\mathcal{B}$ to the coordinate system defined by the basis $\mathcal{C}$. For example, let $p(t) = 6 - 3t$. Then $[p(t)]_{\mathcal{S}} = [6\ -3]$. We can find $[p(t)]_{\mathcal{B}}$ by row reducing the matrix $[[b_1(t)]_{\mathcal{S}}\ [b_2(t)]_{\mathcal{S}} \mid [p(t)]_{\mathcal{S}}]$ to its reduced row echelon form $\left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$. So $[p(t)]_{\mathcal{S}} = 2[b_1(t)]_{\mathcal{S}} - [b_2(t)]_{\mathcal{S}}$ (and $p(t) = 2b_1(t) - b_2(t)$). Therefore, $[p(t)]_{\mathcal{B}} = [2\ \ -1]^{\mathsf{T}}$ as shown at right in Figure 26.1. Using the change of basis matrix we can see that

$$[p(t)]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[p(t)]_{\mathcal{B}} = \left[\begin{array}{rr} -1 & 1 \\ -3 & -3 \end{array}\right]\left[\begin{array}{r} 2 \\ -1 \end{array}\right] = \left[\begin{array}{r} -3 \\ -3 \end{array}\right].$$

So $[p(t)]_{\mathcal{C}} = (-3)[c_1(t)]_{\mathcal{S}} - 3[c_2(t)]_{\mathcal{S}}$ (and $p(t) = (-3)c_1(t) - 3c_2(t)$). Therefore, $[p(t)]_{\mathcal{C}} = [-3\ \ -3]^{\mathsf{T}}$ as shown at right in Figure 26.1.

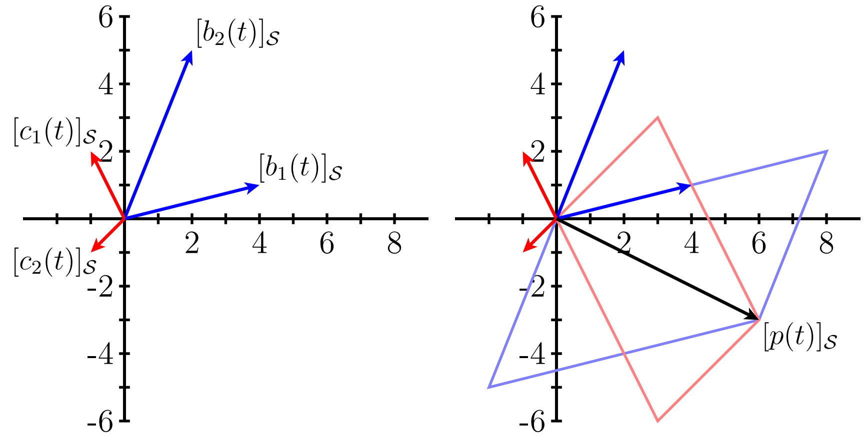

Figure 26.1: Two coordinate systems in $\mathbb{R}^2$ determined by the bases $\mathcal{B}$ and $\mathcal{C}$.

Properties of the Change of Basis Matrix

A vector space can have more than one basis, so it is natural to ask how change of bases matrices might be related to one another.

Activity 26.2. The sets $\mathcal{B} = \{3, 4-t\}$ and $\mathcal{C} = \{1+2t, -1+t\}$ are bases for $\mathbb{P}_1$.

(a) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from the basis $\mathcal{B}$ to the basis $\mathcal{C}$.

(b) Let $p(t) = 2 + 4t$. Find $[p(t)]_{\mathcal{B}}$ and $[p(t)]_{\mathcal{C}}$.

(c) Verify by matrix multiplication that $[p(t)]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[p(t)]_{\mathcal{B}}$.

(d) Find the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ from the basis $\mathcal{C}$ to the basis $\mathcal{B}$.

(e) Verify by matrix multiplication that $[p(t)]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[p(t)]_{\mathcal{C}}$.

(f) How, specifically, are the matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ related? (Hint: If you don't see a relationship right away, what is the product of these two matrices?)

Activity 26.2 seems to indicate that the inverse of a change of basis matrix is also a change of basis matrix, which assumes that a change of basis matrix is always invertible. The following theorem provides some properties about change of basis matrices. The proofs are left for the exercises.

Theorem 26.3. *Let V be a finite dimensional vector space, and let $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{S}$ be bases for V. Then*

(1) the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is invertible,

(2) $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$,

(3) $\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{S}\leftarrow\mathcal{B}}{P}$.

Examples

What follows are worked examples that use the concepts from this section.

Example 26.4. Let $\mathcal{B} = \{1+t^2, 2-t, 3+t\}$ and $\mathcal{C} = \{t+2t^2, 3+t^2, t\}$. Let $p(t) = 6-4t+8t^2$ and $q(t) = 9-t+7t^2$.

(a) Show that $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{P}_2$.

(b) Find $[p(t)]_{\mathcal{B}}$ and $[q(t)]_{\mathcal{B}}$.

(c) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$.

(d) Use $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ to calculate $[p(t)]_{\mathcal{C}}$ and $[q(t)]_{\mathcal{C}}$. Verify that these are the correct coordinate vectors.

Example Solution.

(a) Let $\mathcal{S} = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$. If $\{[1+t^2]_{\mathcal{S}}, [2-t]_{\mathcal{S}}, [3+t]_{\mathcal{S}}\}$ and $\{[t+2t^2]_{\mathcal{S}}, [3+t^2]_{\mathcal{S}}, [t]_{\mathcal{S}}\}$ are bases for $\mathbb{R}^3$, then by the coordinate transformation, both $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{P}_2$. Now $[1+t^2]_{\mathcal{S}} = [1\ 0\ 1]^{\mathsf{T}}$, $[2-t]_{\mathcal{S}} = [2\ -1\ 0]^{\mathsf{T}}$, and $[3+t]_{\mathcal{S}} = [3\ 1\ 0]^{\mathsf{T}}$. Technology shows that the reduced row echelon form of $[[1+t]_{\mathcal{S}}\ [2-t]_{\mathcal{S}}\ [3+t]_{\mathcal{S}}]$ is I_3, so $\mathcal{B}$ is a basis for $\mathbb{P}_2$. Similarly, $[t+2t^2]_{\mathcal{S}} = [0\ 1\ 2]^{\mathsf{T}}$, $[3+t^2]_{\mathcal{S}} = [0\ 0\ 1]^{\mathsf{T}}$, and $[t]_{\mathcal{S}} = [0\ 1\ 0]^{\mathsf{T}}$. Technology shows that the reduced row echelon form of $[[t+2t^2]_{\mathcal{S}}\ [3+t^2]_{\mathcal{S}}\ [t]_{\mathcal{S}}]$ is I_3, so $\mathcal{C}$ is also a basis for $\mathbb{P}_2$.

(b) To find $[p(t)]_{\mathcal{B}}$ and $[q(t)]_{\mathcal{B}}$, we need to write $p(t)$ and $q(t)$ as linear combinations of the basis elements in $\mathcal{B}$. This is equivalent to writing $[p(t)]_{\mathcal{S}}$ and $[q(t)]_{\mathcal{S}}$ in terms of $[1+t^2]_{\mathcal{S}}$, $[2-t]_{\mathcal{S}}$, $[3+t]_{\mathcal{S}}$. Technology shows that the reduced row echelon form of

$$\left[\begin{array}{rrr|rr} 1 & 2 & 3 & 6 & 9 \\ 0 & -1 & 1 & -4 & -1 \\ 1 & 0 & 0 & 8 & 7 \end{array}\right]$$

is

$$\left[\begin{array}{rrr|rr} 1 & 0 & 0 & 8 & 7 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right].$$

It follows that

$$[p(t)]_{\mathcal{B}} = [8\ 2\ -2]^{\mathsf{T}} \text{ and } [q(t)]_{\mathcal{B}} = [7\ 1\ 0]^{\mathsf{T}}.$$

(c) We know that the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$ can be found by row reducing

$$[[t+2t^2]_{\mathcal{S}}\ [3+t^2]_{\mathcal{S}}\ [t]_{\mathcal{S}} \mid [1+t]_{\mathcal{S}}\ [2-t]_{\mathcal{S}}\ [3+t]_{\mathcal{S}}].$$

Technology shows that the reduced row echelon form of

$$\left[\begin{array}{rrr|rrr} 0 & 3 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{3}{2} \end{array}\right].$$

So

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \frac{1}{6}\left[\begin{array}{rrr} 2 & -2 & -3 \\ 2 & 4 & 6 \\ -2 & -4 & 9 \end{array}\right].$$

(d) We know that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[r(t)]_{\mathcal{B}} = [r(t)]_{\mathcal{C}}$ for any polynomial $r(t)$ in $\mathbb{P}_2$. Technology shows that

$$[p(t)]_{\mathcal{C}} = \frac{1}{6}\left[\begin{array}{rrr} 2 & -2 & -3 \\ 2 & 4 & 6 \\ -2 & -4 & 9 \end{array}\right]\left[\begin{array}{r} 8 \\ 2 \\ -2 \end{array}\right] = \left[\begin{array}{r} 3 \\ 2 \\ -7 \end{array}\right]$$

$$[q(t)]_{\mathcal{C}} = \frac{1}{6}\left[\begin{array}{rrr} 2 & -2 & -3 \\ 2 & 4 & 6 \\ -2 & -4 & 9 \end{array}\right]\left[\begin{array}{r} 7 \\ 1 \\ 0 \end{array}\right] = \left[\begin{array}{r} 2 \\ 3 \\ -3 \end{array}\right].$$

Since $3(t+2t^2) + 2(3+t^2) - 7(t) = p(t)$ and $2(t+2t^2) + 3(3+t^2) - 3(t) = q(t)$, we see that we have the correct coordinate vectors.

Example 26.5. Let $D_{2\times 2}$ be the set of diagonal matrices in $\mathcal{M}_{2\times 2}$.

(a) Show that the set $\mathcal{B} = \{B_1, B_2\}$, where $B_1 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]$ and $B_2 = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]$ is a basis for D_2.

(b) If $M = \left[\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right]$ in $D_{2\times 2}$, find $[M]_{\mathcal{B}}$.

(c) Now show that the set $\mathcal{C} = \{C_1, C_2\}$, where $C_1 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array}\right]$ and $C_2 = \left[\begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array}\right]$ is a basis for $D_{2\times 2}$. If $M = \left[\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right]$ in D, find $[M]_{\mathcal{C}}$.

(d) Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. Use the change of basis matrix to verify your result from part (b).

Example Solution.

(a) If $x_1B_1 + x_2B_2 = 0$, then equating the $1,1$ entries shows that $x_1 = 0$ and equating the $2,2$ entries gives us $x_2 = 0$. So B_1 and B_2 are linearly independent. If $A = \left[\begin{array}{cc} x_1 & 0 \\ 0 & x_2 \end{array}\right]$ is in D, then $A = x_1B_1 + x_2B_2$ and $\mathcal{B}$ spans $D_{2\times 2}$. Therefore, $\mathcal{B}$ is a basis for $D_{2\times 2}$.

(b) Since $M = aB_1 + bB_2$, we have $[M]_{\mathcal{B}} = \left[\begin{array}{c} a \\ b \end{array}\right]$.

(c) Consider the equation $x_1C_1 + x_2C_2 = C$, where C is any matrix in $D_{2\times 2}$. The coefficient matrix of this system is $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Since $\det(A) = -3 \neq 0$, it follows that A is invertible and so the Invertible Matrix Theorem tells us that the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution and that the system $A\mathbf{x} = \mathbf{b}$ is always consistent. Thus, $\mathcal{C}$ is linearly independent and spans $D_{2\times 2}$. So $\mathcal{C}$ is a basis for $D_{2\times 2}$.

Let $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then $x_1C_1 + x_2C_2 = M$ if $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. So

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\begin{bmatrix} a \\ b \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} a-2b \\ b-2a \end{bmatrix}.$$

So $[M]_{\mathcal{C}} = -\frac{1}{3}\begin{bmatrix} a-2b \\ b-2a \end{bmatrix}$.

(d) We use the basis $\mathcal{B}$ and row reduce

$$[[C_1]_{\mathcal{B}}\ [C_2]_{\mathcal{B}} \mid [B_1]_{\mathcal{B}}\ [B_2]_{\mathcal{B}}] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right]$$

to

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right].$$

So $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$. Let $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then

$$[M]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[M]_{\mathcal{B}} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}a + \frac{2}{3}b \\ \frac{2}{3}a - \frac{1}{3}b \end{bmatrix}$$

as calculated in part (b).

Summary

- If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ are two bases for a vector space V, then the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is the matrix

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [[\mathbf{b}_1]_{\mathcal{C}}\ [\mathbf{b}_2]_{\mathcal{C}}\ \cdots\ [\mathbf{b}_n]_{\mathcal{C}}]$$

  that satisfies

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$$

  for any vector $\mathbf{x}$ in V.

- Change of basis matrices allow us to effectively and efficiently transition from one coordinate system to another.

- To find a change of basis matrix we can use the coordinate transformation to transfer all of our calculations to $\mathbb{R}^n$. In particular, if $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ are two bases for a vector space V, then for any basis $\mathcal{S}$ of V the reduced row echelon form of the augmented matrix

$$[[\mathbf{c}_1]_{\mathcal{S}}\ [\mathbf{c}_2]_{\mathcal{S}}\ \cdots\ [\mathbf{c}_n]_{\mathcal{S}} \mid [\mathbf{b}_1]_{\mathcal{S}}\ [\mathbf{b}_2]_{\mathcal{S}}\ \cdots\ [\mathbf{b}_n]_{\mathcal{S}}]$$

is

$$\left[I_n \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right].$$

- A change of basis matrix is always invertible. What's more, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$, and $\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}\ \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{S}\leftarrow\mathcal{B}}{P}$ for any bases $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{S}$ of a vector space V.

Exercises

(1) Calculate the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in each of the following cases.

  (a) $\mathcal{B} = \{[1\ 2\ -1]^{\mathsf{T}}, [-1\ 1\ 0]^{\mathsf{T}}, [0\ 0\ 1]^{\mathsf{T}}\}$ and $\mathcal{C} = \{[0\ 1\ 0]^{\mathsf{T}}, [1\ -1\ 1]^{\mathsf{T}}, [0\ 1\ 1]^{\mathsf{T}}\}$ in $\mathbb{R}^3$.

  (b) $\mathcal{B} = \{1 + t^3, t - t^2, t, t^3\}$ and $\mathcal{C} = \{1, t, 1 + t^2, t + t^3\}$ in $\mathbb{P}_3$.

  (c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ in $\mathcal{M}_{2\times 2}$.

(2) A *permutation matrix* is a change of basis matrix that is obtained when the order of the basis vectors is switched. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ and $\mathcal{C} = \{\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_1, \mathbf{b}_4\}$ be two ordered bases for a vector space V. Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

(3)

  (a) Suppose $V = \mathbb{P}_1$, $\mathcal{B} = \{1 + 2t, 2 - 3t\}$ and $\mathcal{C} = \{t, 2 - t\}$. You may assume that $\mathcal{B}$ and $\mathcal{C}$ are bases for V. Calculate $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$. How are these matrices related?

  (b) Now let V be an arbitrary n-dimensional vector space and $\mathcal{B}$ and $\mathcal{C}$ arbitrary bases for V. In the remainder of this exercise we demonstrate that the result of part (a) is true in general.

    i. Show that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is an invertible matrix.

    ii. Explain why $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}$ is the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

(4)

  (a) Let $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$, $A_5 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$, $A_6 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Suppose $V = M_{2\times 2}$, and that $\mathcal{B} = \{A_1, A_2, A_3, A_4\}$, $\mathcal{C} = \{A_1, A_3, A_4, A_5\}$, and $\mathcal{S} = \{A_2, A_4, A_5, A_6\}$. Calculate each of $\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}$, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, and $\underset{\mathcal{S}\leftarrow\mathcal{B}}{P}$. How is the product $\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}\ \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ related to $\underset{\mathcal{S}\leftarrow\mathcal{B}}{P}$?

(b) Now let V be an arbitrary n-dimensional vector space and $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{S}$ arbitrary bases for V. Prove that the result of part (a) is true in general. That is, show that $\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{S}\leftarrow\mathcal{B}}{P}$.

(5) We can view the matrix transformation that performs a counterclockwise rotation by an angle θ around the origin in $\mathbb{R}^2$ as a change of basis matrix. Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$, and let $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = [\cos(\theta)\ \sin(\theta)]^{\mathsf{T}}$ and $\mathbf{v}_2 = [\cos(\theta+\pi/2)\ \sin(\theta+\pi/2)]^{\mathsf{T}}$. Note that $\mathbf{v}_1$ is a vector rotated counterclockwise from the positive x-axis by the angle θ, and $\mathbf{v}_2$ is a vector rotated counterclockwise from the positive y-axis by the angle θ.

(a) Use necessary trigonometric identities to show that the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$ is

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Then find the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

(b) Let $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^2$. Find $[\mathbf{x}]_{\mathcal{B}}$. Then find $[\mathbf{x}]_{\mathcal{C}}$, where $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ with $\theta = 30^\circ$. Draw a picture to illustrate how the components of $[\mathbf{x}]_{\mathcal{C}}$ determine coordinates of $(2, 1)$ in the coordinate system with axes $\mathbf{v}_1$ and $\mathbf{v}_2$.

(c) Let $\mathbf{y}$ be the vector such that $[\mathbf{y}]_{\mathcal{C}} = [2\ 3]^{\mathsf{T}}$. Find $[\mathbf{y}]_{\mathcal{B}}$. Draw a picture to illustrate how the components of $[\mathbf{y}]_{\mathcal{B}}$ determine coordinates of $\mathbf{y}$ in the coordinate system with axes $\mathbf{e}_1$ and $\mathbf{e}_2$.

(6) Let $\mathcal{B} = \{t, 1+t, t^2\}$ be a basis for $\mathbb{P}_2$. Suppose $\mathcal{C}$ is another basis for $\mathbb{P}_2$ and

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 2 & 0 & 2 \end{bmatrix}.$$

Find the polynomials in the basis $\mathcal{C}$.

(7) Activity 26.1 showed how we can find a change of basis matrix from a basis $\mathcal{B}$ to a basis $\mathcal{C}$. We can approach this problem in other ways as well. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for a vector space V, and assume that there is some convenient standard basis $\mathcal{S}$ for V.

(a) Explain why the matrices $\underset{\mathcal{S}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}$ will be easy to find.

(b) How can we combine the matrices $\underset{\mathcal{S}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}$ to calculate the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$? (Hint: There may be inverses involved.)

(c) Explain why row reducing the augmented matrix $\left[\underset{\mathcal{S}\leftarrow\mathcal{C}}{P} \mid \underset{\mathcal{S}\leftarrow\mathcal{B}}{P}\right]$ will result in the matrix $\left[I_n \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right]$.

(8) Label each of the following statements as True or False. Provide justification for your response. Throughout, the sets $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ are bases for an n-dimensional vector space V.

(a) **True/False** The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ span V.

(b) **True/False** The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ span $\mathbb{R}^n$.

(c) **True/False** The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are linearly independent.

(d) **True/False** The matrix $[\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_n \mid \mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n]$ row reduces to $[I_n \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}]$.

Project: Planetary Orbits and Change of Basis

We are interested in determining the orbit of planet that orbits the sun. Finding the equation of such an orbit is not difficult, but just having an equation is not enough. For many purposes, it is important to know where the planet is fro the perspective or earth observation. This is a more complicated question, one we can address through change of bases matrices.[1]

Project Activity 26.1. Since planetary orbits are elliptical, not circular, we need to understand ellipses. An ellipse is a shape like a flattened circle. More specifically, while a circle is the set of points equidistant from a fixed point, and ellipse is a set of points so that the sum of the distances from a point on the ellipse to two fixed points (called foci) is a constant. We can use this definition to derive an equation for an ellipse. We will simplify our we by rotating and translating an ellipse so that its foci are at points $(-c, 0)$ and $(c, 0)$, and the constant sum is $2a$. Let (x, y) be a point on the ellipse as illustrated in Figure 26.2. Use the fact that the sum of the distances from (x, y) to the foci is $2a$ to show that (x, y) satisfies the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{26.2}$$

where the points $(0, b)$ and $(0, -b)$ are the y intercepts of the ellipse.

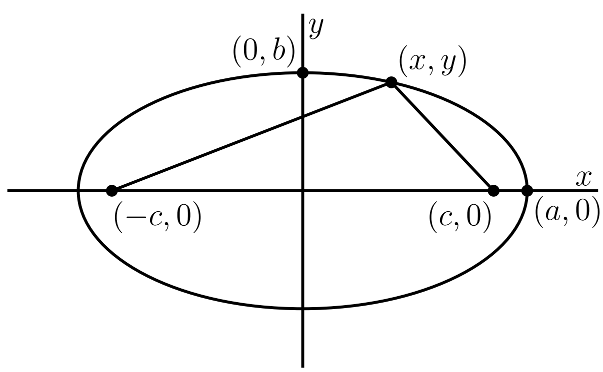

Figure 26.2: An ellipse.

The longer axis of an ellipse is called the major axis and the axis perpendicular to the major axis through the origin is the minor axis. Half of these axes (from the origin) are the semi-major

[1] This project is based on the paper "Planetary Orbits: Change of Basis in R[3]", Donald Teets, *Teaching Mathematics and its Applications: An International Journal of the IMA*, Volume 17, Issue 2, 1 June 1998, Pages 66-68.

axis and the semi-minor axis. So the parameter a in (26.2) is the length of the semi-major axis and the parameter b is the length of the semi-minor axis. Note that the points $(0, b)$ and $(0, -b)$ are the y intercepts and the points $(a, 0)$ and $(-a, 0)$ are the x intercepts of this ellipse. Note that if a and b are equal, then the ellipse is a circle. How far the ellipse deviates from a circle is called the *eccentricity* (usually denoted as e) of the ellipse. In other words, the eccentricity is a measure of how flattened en ellipse is, and this is determined by how close c is to a, or how close the ratio $\frac{c}{a}$ is to 1. Thus, we define the eccentricity of an ellipse by

$$e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}.$$

Now we assume we have a planet (different from the earth) orbiting the sun and we establish how to convert back and forth from the coordinate system of earth's orbit to the coordinate system of the planet's orbit. To do so we need to establish some coordinate systems. We assume the orbit of earth is in the standard xy plane, with the sun (one of the foci) at the origin. The elliptical orbit of the planet is in some other plane with coordinate axes x' and y'. The two orbital planes intersect in a line. Let this line be the x' axis and let α be the angle the positive x' axis makes with the positive x axis. We can represent the elliptical orbit of the planet in the $x'y'$ plane, but the x' and y' axes are not likely to be the best axes for this orbit. So we define a third coordinate system $x''y''$ in the $x'y'$ plane so that the origin (the position of the sun) is at one focus of the planet's orbit and the x'' axis is the major axis of the orbit and the y'' axis is the minor axis of the orbit of the planet. The unit vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ in the positive x, y, and z directions define a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ for $\mathbb{R}^3$, the unit vectors $\mathbf{b}'_1$, $\mathbf{b}'_2$, $\mathbf{b}'_3$ in the positive x', y', and z' directions define a basis $\mathcal{B}' = \{\mathbf{b}'_1, \mathbf{b}'_2, \mathbf{b}'_3\}$ for $\mathbb{R}^3$, and the unit vectors $\mathbf{b}''_1$, $\mathbf{b}''_2$, $\mathbf{b}''_3$ in the positive x'', y'', and z'' directions define a basis $\mathcal{B}'' = \{\mathbf{b}''_1, \mathbf{b}''_2, \mathbf{b}''_3\}$ for $\mathbb{R}^3$. See Figure 26.3 for illustrations. Finally, let γ be the angle between the

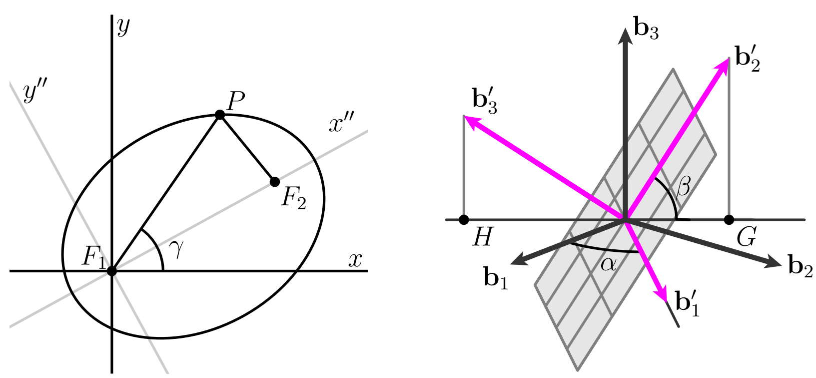

Figure 26.3: Left: The planet's orbit in the $x''y''$ system. Right: The planes of the planet and earth orbits.

positive x' axis and the positive x'' axis as shown at left in Figure 26.3. Our first step is to find the change of basis matrix from $\mathcal{B}''$ to $\mathcal{B}'$.

Project Activity 26.2. Explain why the change of basis matrix $\underset{\mathcal{B}' \leftarrow \mathcal{B}''}{P}$ is given by

$$\underset{\mathcal{B}' \leftarrow \mathcal{B}''}{P} = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

More complicated is the change of basis matrix from $\mathcal{B}'$ to $\mathcal{B}$.

Project Activity 26.3. Now we look for $\underset{\mathcal{B}\leftarrow\mathcal{B}'}{P} = [[\mathbf{b}_1']_{\mathcal{B}}\ [\mathbf{b}_2']_{\mathcal{B}}\ [\mathbf{b}_3']_{\mathcal{B}}]$. Assume that the plane p in which the planet's orbit lies has equation $z = ax + by$.

(a) Explain why $\mathbf{b}_1' = \cos(\alpha)\mathbf{b}_1 + \sin(\alpha)\mathbf{b}_2$.

(b) The x' axis is the intersection of the plane $z = ax+by$ with the plane $z = 0$, so the equation of x' axis in terms of x and y is $ax + by = 0$. Now we determine the coordinates of $\mathbf{b}_2'$ in terms of the basis $\mathcal{B}$.

i. Explain why the vector $[b\ \ -a\ 0]^{\mathsf{T}}$ lies on the x' axis. We take this vector to point in the positive x' direction. This gives us another representation of $\mathbf{b}_1'$ – namely that $\mathbf{b}_1' = \frac{1}{\sqrt{a^2+b^2}}[b\ \ -a\ 0]^{\mathsf{T}}$.

ii. Explain why a vector in the plane $z = ax + by$ orthogonal to $\mathbf{b}_1'$ is $\left[a\ b\ a^2+b^2\right]^{\mathsf{T}}$.

iii. From the previous part we have

$$\mathbf{b}_2' = \frac{1}{\sqrt{a^2+b^2+(a^2+b^2)^2}}\left[b\ a\ a^2+b^2\right]^{\mathsf{T}}.$$

Let $G = \left(\frac{b}{\sqrt{a^2+b^2+(a^2+b^2)^2}}, \frac{a}{\sqrt{a^2+b^2+(a^2+b^2)^2}}, 0\right)$ be the terminal point of the projection of $\mathbf{b}_2'$ onto the xy plane. Show that $\overrightarrow{OG}$ is orthogonal to $\mathbf{b}_1'$.

iv. Let β be the angle between the plane p and the xy plane as illustrated at right in Figure 26.3. Explain why $||\overrightarrow{OG}|| = \cos(\beta)$. Then explain why

$$\mathbf{b}_2' = [-\cos(\beta)\sin(\alpha)\ \ \cos(\beta)\cos(\alpha)\ \ \sin(\beta)]^{\mathsf{T}}.$$

(Hint: Use the trigonometric identities $\cos\left(A+\frac{\pi}{2}\right) = -\sin(A)$ and $\sin\left(A+\frac{\pi}{2}\right) = \cos(A)$.)

(c) Finally, we find $[\mathbf{b}_3']_{\mathcal{B}}$. The cross product of $\mathbf{b}_1'$ and $\mathbf{b}_2'$ is a vector orthogonal to $\mathbf{b}_1'$ and $\mathbf{b}_2'$, so

$$\mathbf{b}_3' = \frac{1}{\sqrt{\left(a^2+b^2\right)^2+4a^2b^2}}\left[-a(a^2+b^2)\ \ -b(a^2+b^2)\ 2ab\right]^{\mathsf{T}}.$$

Let H be the terminal point of the projection of $\mathbf{b}_3'$ onto the xy plane as illustrated at right in Figure 26.3.

i. Explain why the angle between $\mathbf{b}_1'$ and $\overrightarrow{OH}$ is $\frac{\pi}{2}$.

ii. Explain why $||\overrightarrow{OH}|| = \sin(\beta)$. (Hint: Use the trigonometric identity $\cos\left(\frac{\pi}{2}-A\right) = \sin(A)$.)

iii. Since the angle from $\mathbf{b}_1$ to $\overrightarrow{OH}$ is negative, this angle is $\alpha - \frac{\pi}{2}$. Use this angle and the previous information to find the coordinates of the point H and, consequently, explain why

$$\mathbf{b}_3' = [\sin(\beta)\sin(\alpha)\ \ -\sin(\beta)\cos(\alpha)\ \ \cos(\beta)]^{\mathsf{T}}.$$

(Hint: Use the trigonometric identities $\cos\left(A-\frac{\pi}{2}\right) = \sin(A)$ and $\sin\left(A-\frac{\pi}{2}\right) = -\cos(A)$.)

(d) Explain why the change of basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{B}'}{P}$ from $\mathcal{B}'$ to $\mathcal{B}$ is

$$\underset{\mathcal{B}\leftarrow\mathcal{B}'}{P} = \begin{bmatrix} \cos(\alpha) & -\cos(\beta)\sin(\alpha) & \sin(\beta)\sin(\alpha) \\ \sin(\alpha) & \cos(\beta)\cos(\alpha) & -\sin(\beta)\cos(\alpha) \\ 0 & \sin(\beta) & \cos(\beta) \end{bmatrix}.$$

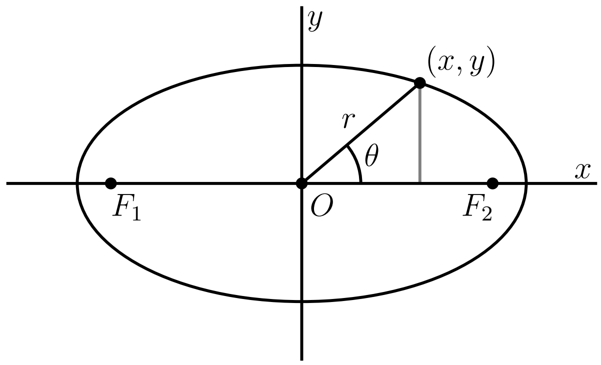

Figure 26.4: Points on the ellipse in terms of angles.

With the change of basis matrices we can convert from any one coordinate system to the other. Note that all of the change of basis matrices are written in terms of angles, so it will be convenient to have a way to express points on our ellipses using angles as well. Given any point on an ellipse (or any point in the plane), we can represent the coordinates of that point in terms of the angle θ the vector through the origin and the point makes with the positive x-axis and the distance r from the origin to the point as shown in Figure 26.4. In this representation we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

So we can start in the $x''y''$ coordinate system with the coordinate vector of a point $\left[\overrightarrow{OP}\right]_{\mathcal{B}''} = [r\cos(\theta)\ r\sin(\theta)\ 0]^{\mathsf{T}}$. Then to view this point in the xy system, we apply the change of basis matrices

$$\left[\overrightarrow{OP}\right]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{B}'}{P}\underset{\mathcal{B}'\leftarrow\mathcal{B}''}{P}[r\cos(\theta)\ r\sin(\theta)\ 0]^{\mathsf{T}}.$$

Of course we can also covert from $\mathcal{B}$ coordinates to $\mathcal{B}''$ coordinates by applying the inverses of our change of basis matrices.