Chapter 4

Mathematical Induction

4.1 The Principle of Mathematical Induction

Beginning Activity 1 (Exploring Statements of the Form $(\forall n \in \mathbb{N})\,(P(n))$)
One of the most fundamental sets in mathematics is the set of natural numbers $\mathbb{N}$. In this section, we will learn a new proof technique, called mathematical induction, that is often used to prove statements of the form $(\forall n \in \mathbb{N})\,(P(n))$. In Section 4.2, we will learn how to extend this method to statements of the form $(\forall n \in T)\,(P(n))$, where T is a certain type of subset of the integers $\mathbb{Z}$.

For each natural number n, let $P(n)$ be the following open sentence:

$$4 \text{ divides } \left(5^n - 1\right).$$

1. Does this open sentence become a true statement when $n = 1$? That is, is 1 in the truth set of $P(n)$?

2. Does this open sentence become a true statement when $n = 2$? That is, is 2 in the truth set of $P(n)$?

3. Choose at least four more natural numbers and determine whether the open sentence is true or false for each of your choices.

All of the examples that were used should provide evidence that the following proposition is true:

For each natural number n, 4 divides $(5^n - 1)$.

We should keep in mind that no matter how many examples we try, we cannot prove this proposition with a list of examples because we can never check if 4 divides $(5^n - 1)$ for every natural number n. Mathematical induction will provide a method for proving this proposition.

For another example, for each natural number n, we now let $Q(n)$ be the following open sentence:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{1}$$

The expression on the left side of the previous equation is the sum of the squares of the first n natural numbers. So when $n = 1$, the left side of equation (1) is 1^2. When $n = 2$, the left side of equation (1) is $1^2 + 2^2$.

4. Does $Q(n)$ become a true statement when

- $n = 1$? (Is 1 in the truth set of $Q(n)$?)
- $n = 2$? (Is 2 in the truth set of $Q(n)$?)
- $n = 3$? (Is 3 in the truth set of $Q(n)$?)

5. Choose at least four more natural numbers and determine whether the open sentence is true or false for each of your choices. A table with the columns n, $1^2 + 2^2 + \cdots + n^2$, and $\dfrac{n(n+1)(2n+1)}{6}$ may help you organize your work.

All of the examples we have explored, should indicate the following proposition is true:

For each natural number n, $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

In this section, we will learn how to use mathematical induction to prove this statement.

Beginning Activity 2 (A Property of the Natural Numbers)
Intuitively, the natural numbers begin with the number 1, and then there is 2, then 3, then 4, and so on. Does this process of "starting with 1" and "adding 1 repeatedly" result in all the natural numbers? We will use the concept of an inductive set to explore this idea in this activity.

Definition. A set T that is a subset of $\mathbb{Z}$ is an **inductive set** provided that for each integer k, if $k \in T$, then $k + 1 \in T$.

1. Carefully explain what it means to say that a subset T of the integers $\mathbb{Z}$ is not an inductive set. This description should use an existential quantifier.

2. Use the definition of an inductive set to determine which of the following sets are inductive sets and which are not. Do not worry about formal proofs, but if a set is not inductive, be sure to provide a specific counterexample that proves it is not inductive.

   (**a**) $A = \{1, 2, 3, \ldots, 20\}$
   (**b**) The set of natural numbers, $\mathbb{N}$
   (**c**) $B = \{n \in \mathbb{N} \mid n \geq 5\}$
   (**d**) $S = \{n \in \mathbb{Z} \mid n \geq -3\}$
   (**e**) $R = \{n \in \mathbb{Z} \mid n \leq 100\}$
   (**f**) The set of integers, $\mathbb{Z}$
   (**g**) The set of odd natural numbers.

3. This part will explore one of the underlying mathematical ideas for a proof by induction. Assume that $T \subseteq \mathbb{N}$ and assume that $1 \in T$ and that T is an inductive set. Use the definition of an inductive set to answer each of the following:

   (**a**) Is $2 \in T$? Explain.
   (**b**) Is $3 \in T$? Explain.
   (**c**) Is $4 \in T$? Explain.
   (**d**) Is $100 \in T$? Explain.
   (**e**) Do you think that $T = \mathbb{N}$? Explain.

Inductive Sets

The two open sentences in Beginning Activity 1 appeared to be true for all values of n in the set of natural numbers, $\mathbb{N}$. That is, the examples in this beginning activity provided evidence that the following two statements are true.

- For each natural number n, 4 divides $(5^n - 1)$.
- For each natural number n, $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

One way of proving statements of this form uses the concept of an inductive set introduced in Beginning Activity 2. The idea is to prove that if one natural number makes the open sentence true, then the next one also makes the open sentence true. This is how we handle the phrase "and so on" when dealing with the natural numbers. In Beginning Activity 2, we saw that the number systems $\mathbb{N}$ and $\mathbb{Z}$ and other sets are inductive. What we are trying to do is somehow distinguish $\mathbb{N}$ from the other inductive sets. The way to do this was suggested in Part (3) of Beginning Activity 2. Although we will not prove it, the following statement should seem true.

Statement 1: For each subset T of $\mathbb{N}$, if $1 \in T$ and T is inductive, then $T = \mathbb{N}$.

Notice that the integers, $\mathbb{Z}$, and the set $S = \{n \in \mathbb{Z} \mid n \geq -3\}$ both contain 1 and both are inductive, but they both contain numbers other than natural numbers. For example, the following statement is false:

Statement 2: For each subset T of $\mathbb{Z}$, if $1 \in T$ and T is inductive, then $T = \mathbb{Z}$.

The set $S = \{n \in \mathbb{Z} \mid n \geq -3\} = \{-3, -2, -1, 0, 1, 2, 3, \ldots\}$ is a counterexample that shows that this statement is false.

Progress Check 4.1 (Inductive Sets)
Suppose that T is an inductive subset of the integers. Which of the following statements are true, which are false, and for which ones is it not possible to tell?

1. $1 \in T$ and $5 \in T$.
2. If $1 \in T$, then $5 \in T$.
3. If $5 \notin T$, then $2 \notin T$.
4. For each integer k, if $k \in T$, then $k + 7 \in T$.
5. For each integer k, $k \notin T$ or $k + 1 \in T$.
6. There exists an integer k such that $k \in T$ and $k + 1 \notin T$.
7. For each integer k, if $k + 1 \in T$, then $k \in T$.
8. For each integer k, if $k + 1 \notin T$, then $k \notin T$.

The Principle of Mathematical Induction

Although we proved that Statement (2) is false, in this text, we will not prove that Statement (1) is true. One reason for this is that we really do not have a formal definition of the natural numbers. However, we should be convinced that

Statement (1) is true. We resolve this by making Statement (1) an axiom for the natural numbers so that this becomes one of the defining characteristics of the natural numbers.

The Principle of Mathematical Induction
If T is a subset of $\mathbb{N}$ such that

1. $1 \in T$, and
2. For every $k \in \mathbb{N}$, if $k \in T$, then $(k+1) \in T$,

then $T = \mathbb{N}$.

Using the Principle of Mathematical Induction

The primary use of the Principle of Mathematical Induction is to prove statements of the form

$$(\forall n \in \mathbb{N})\,(P\,(n))\,,$$

where $P(n)$ is some open sentence. Recall that a universally quantified statement like the preceding one is true if and only if the truth set T of the open sentence $P(n)$ is the set $\mathbb{N}$. So our goal is to prove that $T = \mathbb{N}$, which is the conclusion of the Principle of Mathematical Induction. To verify the hypothesis of the Principle of Mathematical Induction, we must

1. Prove that $1 \in T$. That is, prove that $P(1)$ is true.
2. Prove that if $k \in T$, then $(k+1) \in T$. That is, prove that if $P(k)$ is true, then $P(k+1)$ is true.

The first step is called the **basis step** or the **initial step**, and the second step is called the **inductive step**. This means that a proof by mathematical induction will have the following form:

Procedure for a Proof by Mathematical Induction
To prove: $(\forall n \in \mathbb{N})\,(P\,(n))$

Basis step: Prove $P(1)$.

Inductive step: Prove that for each $k \in \mathbb{N}$,
if $P(k)$ is true, then $P(k+1)$ is true.

We can then conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Note that in the inductive step, we want to prove that the conditional statement "for each $k \in \mathbb{N}$, if $P(k)$ then $P(k+1)$" is true. So we will start the inductive step by assuming that $P(k)$ is true. This assumption is called the **inductive assumption** or the **inductive hypothesis.**

The key to constructing a proof by induction is to discover how $P(k+1)$ is related to $P(k)$ for an arbitrary natural number k. For example, in Beginning Activity 1, one of the open sentences $P(n)$ was

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Sometimes it helps to look at some specific examples such as $P(2)$ and $P(3)$. The idea is not just to do the computations, but to see how the statements are related. This can sometimes be done by writing the details instead of immediately doing computations.

$$\begin{array}{lll} P(2) & \text{is} & 1^2 + 2^2 = \dfrac{2 \cdot 3 \cdot 5}{6} \\ P(3) & \text{is} & 1^2 + 2^2 + 3^2 = \dfrac{3 \cdot 4 \cdot 7}{6} \end{array}$$

In this case, the key is the left side of each equation. The left side of $P(3)$ is obtained from the left side of $P(2)$ by adding one term, which is 3^2. This suggests that we might be able to obtain the equation for $P(3)$ by adding 3^2 to both sides of the equation in $P(2)$. Now for the general case, if $k \in \mathbb{N}$, we look at $P(k+1)$ and compare it to $P(k)$.

$$\begin{array}{lll} P(k) & \text{is} & 1^2 + 2^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6} \\ P(k+1) & \text{is} & 1^2 + 2^2 + \cdots + (k+1)^2 = \dfrac{(k+1)\left[(k+1)+1\right]\left[2\,(k+1)+1\right]}{6} \end{array}$$

The key is to look at the left side of the equation for $P(k+1)$ and realize what this notation means. It means that we are adding the squares of the first $(k+1)$ natural numbers. This means that we can write

$$1^2 + 2^2 + \cdots + (k+1)^2 = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2.$$

This shows us that the left side of the equation for $P(k+1)$ can be obtained from the left side of the equation for $P(k)$ by adding $(k+1)^2$. This is the motivation for proving the inductive step in the following proof.

Proposition 4.2. *For each natural number n,*

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

***Proof*.** We will use a proof by mathematical induction. For each natural number n, we let $P(n)$ be

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

We first prove that $P(1)$ is true. Notice that $\dfrac{1\,(1+1)\,(2\cdot 1+1)}{6} = 1$. This shows that

$$1^2 = \frac{1\,(1+1)\,(2\cdot 1+1)}{6},$$

which proves that $P(1)$ is true.

For the inductive step, we prove that for each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true. So let k be a natural number and assume that $P(k)$ is true. That is, assume that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \tag{1}$$

The goal now is to prove that $P(k+1)$ is true. That is, it must be proved that

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{(k+1)\,[(k+1)+1]\,[2(k+1)+1]}{6} \\ &= \frac{(k+1)\,(k+2)\,(2k+3)}{6}. \end{aligned} \tag{2}$$

To do this, we add $(k+1)^2$ to both sides of equation (1) and algebraically rewrite the right side of the resulting equation. This gives

$$\begin{aligned}
1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6} \\
&= \frac{(k+1)\left(2k^2 + 7k + 6\right)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

Comparing this result to equation (2), we see that if $P(k)$ is true, then $P(k+1)$ is true. Hence, the inductive step has been established, and by the Principle of Mathematical Induction, we have proved that for each natural number n, $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$. ■

Writing Guideline

The proof of Proposition 4.2 shows a standard way to write an induction proof. When writing a proof by mathematical induction, we should follow the guideline that we always keep the reader informed. This means that at the beginning of the proof, we should state that a proof by induction will be used. We should then clearly define the open sentence $P(n)$ that will be used in the proof.

Summation Notation

The result in Proposition 4.2 could be written using summation notation as follows:

$$\text{For each natural number } n, \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

In this case, we use j for the index for the summation, and the notation $\sum_{j=1}^{n} j^2$ tells us to add all the values of j^2 for j from 1 to n, inclusive. That is,

$$\sum_{j=1}^{n} j^2 = 1^2 + 2^2 + \cdots + n^2.$$

So in the proof of Proposition 4.2, we would let $P(n)$ be $\sum_{j=1}^{n} j^2 = \dfrac{n(n+1)(2n+1)}{6}$, and we would use the fact that for each natural number k,

$$\sum_{j=1}^{k+1} j^2 = \left(\sum_{j=1}^{k} j^2\right) + (k+1)^2.$$

Progress Check 4.3 (An Example of a Proof by Induction)

1. Calculate $1 + 2 + 3 + \cdots + n$ and $\dfrac{n(n+1)}{2}$ for several natural numbers n. What do you observe?

2. Use mathematical induction to prove that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

   To do this, let $P(n)$ be the open sentence, "$1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$."

   For the basis step, notice that the equation $1 = \dfrac{1(1+1)}{2}$ shows that $P(1)$ is true. Now let k be a natural number and assume that $P(k)$ is true. That is, assume that

   $$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2},$$

   and complete the proof.

Some Comments about Mathematical Induction

1. The basis step is an essential part of a proof by induction. See Exercise (19) for an example that shows that the basis step is needed in a proof by induction.

2. Exercise (20) provides an example that shows the inductive step is also an essential part of a proof by mathematical induction.

3. It is important to remember that the inductive step in an induction proof is a proof of a conditional statement. Although we did not explicitly use the forward-backward process in the inductive step for Proposition 4.2, it was implicitly used in the discussion prior to Proposition 4.2. The key question was, "How does knowing the sum of the first k squares help us find the sum of the first $(k+1)$ squares?"

4. When proving the inductive step in a proof by induction, the key question is,

How does knowing $P(k)$ help us prove $P(k+1)$?

In Proposition 4.2, we were able to see that the way to answer this question was to add a certain expression to both sides of the equation given in $P(k)$. Sometimes the relationship between $P(k)$ and $P(k+1)$ is not as easy to see. For example, in Beginning Activity 1, we explored the following proposition:

For each natural number n, 4 divides $(5^n - 1)$.

This means that the open sentence, $P(n)$, is "4 divides $(5^n - 1)$." So in the inductive step, we assume $k \in \mathbb{N}$ and that 4 divides $\left(5^k - 1\right)$. This means that there exists an integer m such that

$$5^k - 1 = 4m. \tag{1}$$

In the backward process, the goal is to prove that 4 divides $\left(5^{k+1} - 1\right)$. This can be accomplished if we can prove that there exists an integer s such that

$$5^{k+1} - 1 = 4s. \tag{2}$$

We now need to see if there is anything in equation (1) that can be used in equation (2). The key is to find something in the equation $5^k - 1 = 4m$ that is related to something similar in the equation $5^{k+1} - 1 = 4s$. In this case, we notice that

$$5^{k+1} = 5 \cdot 5^k.$$

So if we can solve $5^k - 1 = 4m$ for 5^k, we could make a substitution for 5^k. This is done in the proof of the following proposition.

Proposition 4.4. *For every natural number n, 4 divides* $(5^n - 1)$.

Proof. (Proof by Mathematical Induction) For each natural number n, let $P(n)$ be "4 divides $(5^n - 1)$." We first prove that $P(1)$ is true. Notice that when $n = 1$, $(5^n - 1) = 4$. Since 4 divides 4, $P(1)$ is true.

For the inductive step, we prove that for all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true. So let k be a natural number and assume that $P(k)$ is true. That is, assume that

$$4 \text{ divides } \left(5^k - 1\right).$$

This means that there exists an integer m such that

$$5^k - 1 = 4m.$$

Thus,

$$5^k = 4m + 1. \tag{1}$$

In order to prove that $P(k+1)$ is true, we must show that 4 divides $\left(5^{k+1} - 1\right)$. Since $5^{k+1} = 5 \cdot 5^k$, we can write

$$5^{k+1} - 1 = 5 \cdot 5^k - 1. \tag{2}$$

We now substitute the expression for 5^k from equation (1) into equation (2). This gives

$$\begin{aligned} 5^{k+1} - 1 &= 5 \cdot 5^k - 1 \\ &= 5(4m+1) - 1 \\ &= (20m+5) - 1 \\ &= 20m + 4 \\ &= 4(5m+1) \end{aligned} \tag{3}$$

Since $(5m + 1)$ is an integer, equation (3) shows that 4 divides $\left(5^{k+1} - 1\right)$. Therefore, if $P(k)$ is true, then $P(k+1)$ is true and the inductive step has been established. Thus, by the Principle of Mathematical Induction, for every natural number n, 4 divides $(5^n - 1)$. ■

Proposition 4.4 was stated in terms of "divides." We can use congruence to state a proposition that is equivalent to Proposition 4.4. The idea is that the sentence, 4 divides $(5^n - 1)$ means that $5^n \equiv 1 \pmod 4$. So the following proposition is equivalent to Proposition 4.4.

Proposition 4.5. *For every natural number n, $5^n \equiv 1 \pmod 4$.*

Since we have proved Proposition 4.4, we have in effect proved Proposition 4.5. However, we could have proved Proposition 4.5 first by using the results in Theorem 3.28 on page 147. This will be done in the next progress check.

Progress Check 4.6 (Proof of Proposition 4.5)
To prove Proposition 4.5, we let $P(n)$ be $5^n \equiv 1 \pmod 4$ and notice that $P(1)$ is true since $5 \equiv 1 \pmod 4$. For the inductive step, let k be a natural number and assume that $P(k)$ is true. That is, assume that $5^k \equiv 1 \pmod 4$.

1. What must be proved in order to prove that $P(k+1)$ is true?
2. Since $5^{k+1} = 5 \cdot 5^k$, multiply both sides of the congruence $5^k \equiv 1 \pmod 4$ by 5. The results in Theorem 3.28 on page 147 justify this step.
3. Now complete the proof that for each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true and complete the induction proof of Proposition 4.5.

It might be nice to compare the proofs of Propositions 4.4 and 4.5 and decide which one is easier to understand.

Exercises for Section 4.1

⋆ **1.** Which of the following sets are inductive sets? Explain.

(**a**) $\mathbb{Z}$
(**b**) $\{x \in \mathbb{N} \mid x \geq 4\}$
(**c**) $\{x \in \mathbb{Z} \mid x \leq 10\}$
(**d**) $\{1, 2, 3, \ldots, 500\}$

⋆ **2.** (**a**) Can a finite, nonempty set be inductive? Explain.
(**b**) Is the empty set inductive? Explain.

3. Use mathematical induction to prove each of the following:

⋆ (**a**) For each natural number n, $2 + 5 + 8 + \cdots + (3n - 1) = \dfrac{n(3n+1)}{2}$.
(**b**) For each natural number n, $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$.
(**c**) For each natural number n, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$.

4. Based on the results in Progress Check 4.3 and Exercise (3c), if $n \in \mathbb{N}$, is there any conclusion that can be made about the relationship between the sum $\left(1^3 + 2^3 + 3^3 + \cdots + n^3\right)$ and the sum $(1 + 2 + 3 + \cdots + n)$?

5. Instead of using induction, we can sometimes use previously proven results about a summation to obtain results about a different summation.

(**a**) Use the result in Progress Check 4.3 to prove the following proposition:

$$\text{For each natural number } n,\ 3 + 6 + 9 + \cdots + 3n = \frac{3n\,(n+1)}{2}.$$

(**b**) Subtract n from each side of the equation in Part (a). On the left side of this equation, explain why this can be done by subtracting 1 from each term in the summation.

(**c**) Algebraically simplify the right side of the equation in Part (b) to obtain a formula for the sum $2 + 5 + 8 + \cdots + (3n - 1)$. Compare this to Exercise (3a).

* **6.** (**a**) Calculate $1 + 3 + 5 + \cdots + (2n - 1)$ for several natural numbers n.

(**b**) Based on your work in Exercise (6a), if $n \in \mathbb{N}$, make a conjecture about the value of the sum $1 + 3 + 5 + \cdots + (2n - 1) = \sum_{j=1}^{n} (2j - 1)$.

(**c**) Use mathematical induction to prove your conjecture in Exercise (6b).

7. In Section 3.1, we defined congruence modulo n for a natural number n, and in Section 3.5, we used the Division Algorithm to prove that each integer is congruent, modulo n, to precisely one of the integers $0, 1, 2, \ldots, n - 1$ (Corollary 3.32).

(**a**) Find the value of r so that $4 \equiv r \pmod 3$ and $r \in \{0, 1, 2\}$.

(**b**) Find the value of r so that $4^2 \equiv r \pmod 3$ and $r \in \{0, 1, 2\}$.

(**c**) Find the value of r so that $4^3 \equiv r \pmod 3$ and $r \in \{0, 1, 2\}$.

(**d**) For two other values of n, find the value of r so that $4^n \equiv r \pmod 3$ and $r \in \{0, 1, 2\}$.

* (**e**) If $n \in \mathbb{N}$, make a conjecture concerning the value of r where $4^n \equiv r \pmod 3$ and $r \in \{0, 1, 2\}$. This conjecture should be written as a self-contained proposition including an appropriate quantifier.

* (**f**) Use mathematical induction to prove your conjecture.

8. Use mathematical induction to prove each of the following:

* (**a**) For each natural number n, 3 divides $(4^n - 1)$.

(**b**) For each natural number n, 6 divides $\left(n^3 - n\right)$.

9. In Exercise (7), we proved that for each natural number n, $4^n \equiv 1 \pmod 3$. Explain how this result is related to the proposition in Exercise (8a).

10. Use mathematical induction to prove that for each natural number n, 3 divides $n^3 + 23n$. Compare this proof to the proof from Exercise (19) in Section 3.5.

11. (**a**) Calculate the value of $5^n - 2^n$ for $n = 1$, $n = 2$, $n = 3$, $n = 4$, $n = 5$, and $n = 6$.

(**b**) Based on your work in Part (a), make a conjecture about the values of $5^n - 2^n$ for each natural number n.

(**c**) Use mathematical induction to prove your conjecture in Part (b).

12. Let x and y be integers. Prove that for each natural number n, $(x - y)$ divides $(x^n - y^n)$. Explain why your conjecture in Exercise (11) is a special case of this result.

$\star$ **13.** Prove Part (3) of Theorem 3.28 from Section 3.4. Let $n \in \mathbb{N}$ and let a and b be integers. For each $m \in \mathbb{N}$, if $a \equiv b \pmod n$, then $a^m \equiv b^m \pmod n$.

$\star$ **14.** Use mathematical induction to prove that the sum of the cubes of any three consecutive natural numbers is a multiple of 9.

15. Let a be a real number. We will explore the derivatives of the function $f(x) = e^{ax}$. By using the chain rule, we see

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}.$$

Recall that the second derivative of a function is the derivative of the derivative function. Similarly, the third derivative is the derivative of the second derivative.

(**a**) What is $\dfrac{d^2}{dx^2}\left(e^{ax}\right)$, the second derivative of e^{ax}?

(**b**) What is $\dfrac{d^3}{dx^3}\left(e^{ax}\right)$, the third derivative of e^{ax}?

(**c**) Let n be a natural number. Make a conjecture about the n^{th} derivative of the function $f(x) = e^{ax}$. That is, what is $\dfrac{d^n}{dx^n}\left(e^{ax}\right)$? This conjecture should be written as a self-contained proposition including an appropriate quantifier.

(**d**) Use mathematical induction to prove your conjecture.

16. In calculus, it can be shown that

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{2}\sin x\cos x + c \quad \text{and}$$
$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{2}\sin x\cos x + c.$$

Using integration by parts, it is also possible to prove that for each natural number n,

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \quad \text{and}$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

(**a**) Determine the values of

$$\int_0^{\pi/2} \sin^2 x\,dx \qquad \text{and} \qquad \int_0^{\pi/2} \sin^4 x\,dx.$$

(**b**) Use mathematical induction to prove that for each natural number n,

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,\frac{\pi}{2} \quad \text{and}$$
$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)}.$$

These are known as the ***Wallis sine formulas***.

(**c**) Use mathematical induction to prove that

$$\int_0^{\pi/2} \cos^{2n} x\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,\frac{\pi}{2} \quad \text{and}$$
$$\int_0^{\pi/2} \cos^{2n+1} x\,dx = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)}.$$

These are known as the ***Wallis cosine formulas***.

17. (**a**) Why is it not possible to use mathematical induction to prove a proposition of the form

$$(\forall x \in \mathbb{Q})\,(P(x)),$$

where $P(x)$ is some predicate?

(**b**) Why is it not possible to use mathematical induction to prove a proposition of the form

For each real number x with $x \geq 1$, $P(x)$,

where $P(x)$ is some predicate?

18. Evaluation of proofs
See the instructions for Exercise (19) on page 100 from Section 3.1.

(**a**) For each natural number n, $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n-1)}{2}$.

Proof. We will prove this proposition using mathematical induction. So we let $P(n)$ be the open sentence

$$1 + 4 + 7 + \cdots + (3n - 2).$$

Using $n = 1$, we see that $3n - 2 = 1$ and hence, $P\ (1)$ is true.
We now assume that $P(k)$ is true. That is,

$$1 + 4 + 7 + \cdots + (3k - 2) = \frac{k(3k-1)}{2}.$$

We then see that

$$\begin{aligned}
1 + 4 + 7 + \cdots + (3k - 2) + (3(k + 1) - 2) &= \frac{(k+1)(3k+2)}{2} \\
\frac{k(3k-1)}{2} + (3k + 1) &= \frac{(k+1)(3k+2)}{2} \\
\frac{\left(3k^2 - k\right) + (6k + 2)}{2} &= \frac{3k^2 + 5k + 2}{2} \\
\frac{3k^2 + 5k + 2}{2} &= \frac{3k^2 + 5k + 2}{2}.
\end{aligned}$$

We have thus proved that $P(k + 1)$ is true, and hence, we have proved the proposition. ■

(**b**) For each natural number n, $1 + 4 + 7 + \cdots + (3n - 2) = \dfrac{n(3n-1)}{2}$.

Proof. We will prove this proposition using mathematical induction. So we let

$$P(n) = 1 + 4 + 7 + \cdots + (3n - 2).$$

Using $n = 1$, we see that $P(1) = 1$ and hence, $P(1)$ is true.
We now assume that $P(k)$ is true. That is,

$$1 + 4 + 7 + \cdots + (3k - 2) = \frac{k\,(3k - 1)}{2}.$$

We then see that

$$\begin{aligned} P(k+1) &= 1 + 4 + 7 + \cdots + (3k - 2) + (3(k + 1) - 2) \\ &= \frac{k\,(3k-1)}{2} + 3\,(k+1) - 2 \\ &= \frac{3k^2 - k + 6k + 6 - 4}{2} \\ &= \frac{3k^2 + 5k + 2}{2} \\ &= \frac{(k+1)\,(3k+2)}{2}. \end{aligned}$$

We have thus proved that $P(k + 1)$ is true, and hence, we have proved the proposition. ■

(c) All dogs are the same breed.

Proof. We will prove this proposition using mathematical induction. For each natural number n, we let $P(n)$ be

Any set of n dogs consists entirely of dogs of the same breed.

We will prove that for each natural number n, $P(n)$ is true, which will prove that all dogs are the same breed. A set with only one dog consists entirely of dogs of the same breed and, hence, $P(1)$ is true.
So we let k be a natural number and assume that $P(k)$ is true, that is, that every set of k dogs consists of dogs of the same breed. Now consider a set D of $k + 1$ dogs, where

$$D = \{d_1, d_2, \ldots, d_k, d_{k+1}\}.$$

If we remove the dog d_1 from the set D, we then have a set D_1 of k dogs, and using the assumption that $P(k)$ is true, these dogs must all be of the same breed. Similarly, if we remove d_{k+1} from the set D, we again have a set D_2 of k dogs, and these dogs must all be of the same breed. Since $D = D_1 \cup D_2$, we have proved that all of the dogs in D must be of the same breed.

This proves that if $P(k)$ is true, then $P(k+1)$ is true and, hence, by mathematical induction, we have proved that for each natural number n, any set of n dogs consists entirely of dogs of the same breed. ■

Explorations and Activities

19. The Importance of the Basis Step. Most of the work done in constructing a proof by induction is usually in proving the inductive step. This was certainly the case in Proposition 4.2. However, the basis step is an essential part of the proof. Without it, the proof is incomplete. To see this, let $P(n)$ be

$$1 + 2 + \cdots + n = \frac{n^2 + n + 1}{2}.$$

(**a**) Let $k \in \mathbb{N}$. Complete the following proof that if $P(k)$ is true, then $P(k+1)$ is true.

Let $k \in \mathbb{N}$. Assume that $P(k)$ is true. That is, assume that

$$1 + 2 + \cdots + k = \frac{k^2 + k + 1}{2}. \qquad (1)$$

The goal is to prove that $P(k+1)$ is true. That is, we need to prove that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)^2 + (k+1) + 1}{2}. \qquad (2)$$

To do this, we add $(k+1)$ to both sides of equation (1). This gives

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k^2 + k + 1}{2} + (k+1) \\ &= \cdots. \end{aligned}$$

(**b**) Is $P(1)$ true? Is $P(2)$ true? What about $P(3)$ and $P(4)$? Explain how this shows that the basis step is an essential part of a proof by induction.

20. Regions of a Circle. Place n equally spaced points on a circle and connect each pair of points with the chord of the circle determined by that pair of points. See Figure 4.1.

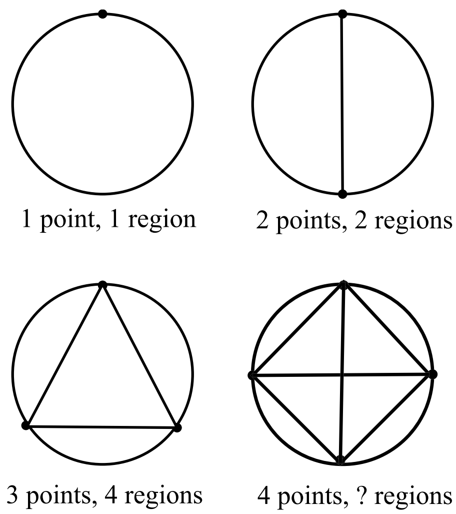

Figure 4.1: Regions of Circles

Count the number of distinct regions within each circle. For example, with three points on the circle, there are four distinct regions. Organize your data in a table with two columns: "Number of Points on the Circle" and "Number of Distinct Regions in the Circle."

(**a**) How many regions are there when there are four equally spaced points on the circle?

(**b**) Based on the work so far, make a conjecture about how many distinct regions would you get with five equally spaced points.

(**c**) Based on the work so far, make a conjecture about how many distinct regions would you get with six equally spaced points.

(**d**) Figure 4.2 shows the figures associated with Parts (b) and (c). Count the number of regions in each case. Are your conjectures correct or incorrect?

(**e**) Explain why this activity shows that the inductive step is an essential part of a proof by mathematical induction.

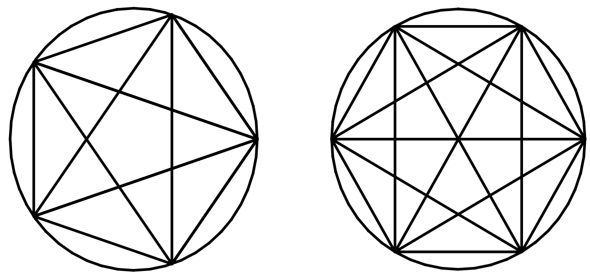

Figure 4.2: Regions of Circles

4.2 Other Forms of Mathematical Induction

Beginning Activity 1 (Exploring a Proposition about Factorials)

Definition. If n is a natural number, we define ***n* factorial**, denoted by $n!$, to be the product of the first n natural numbers. In addition, we define 0! to be equal to 1.

Using this definition, we see that

$$\begin{aligned} 0! &= 1 & 3! &= 1 \cdot 2 \cdot 3 = 6 \\ 1! &= 1 & 4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24 \\ 2! &= 1 \cdot 2 = 2 & 5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120. \end{aligned}$$

In general, we write $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$ or $n! = n \cdot (n-1) \cdots 2 \cdot 1$. Notice that for any natural number n, $n! = n \cdot (n-1)!$.

1. Compute the values of 2^n and $n!$ for each natural number n with $1 \leq n \leq 7$.

Now let $P(n)$ be the open sentence, "$n! > 2^n$."

2. Which of the statements $P(1)$ through $P(7)$ are true?
3. Based on the evidence so far, does the following proposition appear to be true or false? For each natural number n with $n \geq 4$, $n! > 2^n$.

Let k be a natural number with $k \geq 4$. Suppose that we want to prove that if $P(k)$ is true, then $P(k+1)$ is true. (This could be the inductive step in an induction proof.) To do this, we would be assuming that $k! > 2^k$ and would need to prove that $(k+1)! > 2^{k+1}$. Notice that if we multiply both sides of the inequality $k! > 2^k$ by $(k+1)$, we obtain

$$(k+1) \cdot k! > (k+1)2^k. \tag{1}$$

4. In the inequality in (1), explain why $(k+1) \cdot k! = (k+1)!$.

5. Now look at the right side of the inequality in (1). Since we are assuming that $k \geq 4$, we can conclude that $(k+1) > 2$. Use this to help explain why $(k+1)2^k > 2^{k+1}$.

6. Now use the inequality in (1) and the work in steps (4) and (5) to explain why $(k+1)! > 2^{k+1}$.

Beginning Activity 2 (**Prime Factors of a Natural Number**)
Recall that a natural number p is a **prime number** provided that it is greater than 1 and the only natural numbers that divide p are 1 and p. A natural number other than 1 that is not a prime number is a **composite number**. The number 1 is neither prime nor composite.

1. Give examples of four natural numbers that are prime and four natural numbers that are composite.

2. Write each of the natural numbers 20, 40, 50, and 150 as a product of prime numbers.

3. Do you think that any composite number can be written as a product of prime numbers?

4. Write a useful description of what it means to say that a natural number is a composite number (other than saying that it is not prime).

5. Based on your work in Part (2), do you think it would be possible to use induction to prove that any composite number can be written as a product of prime numbers?

The Domino Theory

Mathematical induction is frequently used to prove statements of the form

$$(\forall n \in \mathbb{N})\,(P(n)), \tag{1}$$

where $P(n)$ is an open sentence. This means that we are proving that every statement in the following infinite list is true.

$$P(1), P(2), P(3), \ldots \tag{2}$$

The inductive step in a proof by induction is to prove that if one statement in this infinite list of statements is true, then the next statement in the list must be true. Now imagine that each statement in (2) is a domino in a chain of dominoes. When we prove the inductive step, we are proving that if one domino is knocked over, then it will knock over the next one in the chain. Even if the dominoes are set up so that when one falls, the next one will fall, no dominoes will fall unless we start by knocking one over. This is why we need the basis step in an induction proof. The basis step guarantees that we knock over the first domino. The inductive step, then, guarantees that all dominoes after the first one will also fall.

Now think about what would happen if instead of knocking over the first domino, we knock over the sixth domino. If we also prove the inductive step, then we would know that every domino after the sixth domino would also fall. This is the idea of the *Extended Principle of Mathematical Induction*. It is not necessary for the basis step to be the proof that $P(1)$ is true. We can make the basis step be the proof that $P(M)$ is true, where M is some natural number. The Extended Principle of Mathematical Induction can be generalized somewhat by allowing M to be any integer. We are still only concerned with those integers that are greater than or equal to M.

The Extended Principle of Mathematical Induction
Let M be an integer. If T is a subset of $\mathbb{Z}$ such that

1. $M \in T$, and
2. For every $k \in \mathbb{Z}$ with $k \geq M$, if $k \in T$, then $(k+1) \in T$,

then T contains all integers greater than or equal to M. That is, $\{n \in \mathbb{Z} \mid n \geq M\} \subseteq T$.

Using the Extended Principle of Mathematical Induction

The primary use of the Principle of Mathematical Induction is to prove statements of the form

$$(\forall n \in \mathbb{Z}, \text{ with } n \geq M)\,(P(n)),$$

where M is an integer and $P(n)$ is some open sentence. (In most induction proofs, we will use a value of M that is greater than or equal to zero.) So our goal is to prove that the truth set T of the predicate $P(n)$ contains all integers greater than or equal to M. So to verify the hypothesis of the Extended Principle of Mathematical Induction, we must

1. Prove that $M \in T$. That is, prove that $P(M)$ is true.
2. Prove that for every $k \in \mathbb{Z}$ with $k \geq M$, if $k \in T$, then $(k+1) \in T$. That is, prove that if $P(k)$ is true, then $P(k+1)$ is true.

As before, the first step is called the **basis step** or the **initial step**, and the second step is called the **inductive step**. This means that a proof using the Extended Principle of Mathematical Induction will have the following form:

Using the Extended Principle of Mathematical Induction
Let M be an integer. To prove: $(\forall n \in \mathbb{Z} \text{ with } n \geq M)\,(P(n))$

Basis step:	Prove $P(M)$.
Inductive step:	Prove that for every $k \in \mathbb{Z}$ with $k \geq M$, if $P(k)$ is true, then $P(k+1)$ is true.

We can then conclude that $P(n)$ is true for all $n \in \mathbb{Z}$ with $n \geq M$.

This is basically the same procedure as the one for using the Principle of Mathematical Induction. The only difference is that the basis step uses an integer M other than 1. For this reason, when we write a proof that uses the Extended Principle of Mathematical Induction, we often simply say we are going to use a proof by mathematical induction. We will use the work from Beginning Activity 1 to illustrate such a proof.

Proposition 4.7. *For each natural number n with $n \geq 4$, $n! > 2^n$.*

Proof. We will use a proof by mathematical induction. For this proof, we let

$$P(n) \text{ be "} n! > 2^n. \text{"}$$

We first prove that $P(4)$ is true. Using $n = 4$, we see that $4! = 24$ and $2^4 = 16$. This means that $4! > 2^4$ and, hence, $P(4)$ is true.

For the inductive step, we prove that for all $k \in \mathbb{N}$ with $k \geq 4$, if $P(k)$ is true, then $P(k+1)$ is true. So let k be a natural number greater than or equal to 4, and assume that $P(k)$ is true. That is, assume that

$$k! > 2^k. \tag{1}$$

The goal is to prove that $P(k+1)$ is true or that $(k+1)! > 2^{k+1}$. Multiplying both sides of inequality (1) by $k+1$ gives

$$\begin{aligned}(k+1)\cdot k! &> (k+1)\cdot 2^k, \text{ or}\\ (k+1)! &> (k+1)\cdot 2^k. \qquad (2)\end{aligned}$$

Now, $k \geq 4$. Thus, $k+1 > 2$, and hence $(k+1)\cdot 2^k > 2\cdot 2^k$. This means that

$$(k+1)\cdot 2^k > 2^{k+1}. \qquad (3)$$

Inequalities (2) and (3) show that

$$(k+1)! > 2^{k+1},$$

and this proves that if $P(k)$ is true, then $P(k+1)$ is true. Thus, the inductive step has been established, and so by the Extended Principle of Mathematical Induction, $n! > 2^n$ for each natural number n with $n \geq 4$. ■

Progress Check 4.8 (Formulating Conjectures)
Formulate a conjecture (with an appropriate quantifier) that can be used as an answer to each of the following questions.

1. For which natural numbers n is 3^n greater than $1 + 2^n$?
2. For which natural numbers n is 2^n greater than $(n+1)^2$?
3. For which natural numbers n is $\left(1 + \dfrac{1}{n}\right)^n$ greater than 2.5?

The Second Principle of Mathematical Induction

Let $P(n)$ be

n is a prime number or n is a product of prime numbers.

(This is related to the work in Beginning Activity 2.)

Suppose we would like to use induction to prove that $P(n)$ is true for all natural numbers greater than 1. We have seen that the idea of the inductive step in a proof by induction is to prove that if one statement in an infinite list of statements is true, then the next statement must also be true. The problem here is that when we factor a composite number, we do not get to the previous case. For example, if assume that $P(39)$ is true and we want to prove that $P(40)$ is true, we could factor 40 as $40 = 2 \cdot 20$. However, the assumption that $P(39)$ is true does not help us prove that $P(40)$ is true.

This work is intended to show the need for another principle of induction. In the inductive step of a proof by induction, we assume one statement is true and prove the next one is true. The idea of this new principle is to assume that *all* of the previous statements are true and use this assumption to prove the next statement is true. This is stated formally in terms of subsets of natural numbers in the *Second Principle of Mathematical Induction*. Rather than stating this principle in two versions, we will state the extended version of the Second Principle. In many cases, we will use $M = 1$ or $M = 0$.

The Second Principle of Mathematical Induction
Let M be an integer. If T is a subset of $\mathbb{Z}$ such that

1. $M \in T$, and
2. For every $k \in \mathbb{Z}$ with $k \geq M$, if $\{M, M+1, \ldots, k\} \subseteq T$, then $(k+1) \in T$,

then T contains all integers greater than or equal to M. That is, $\{n \in \mathbb{Z} \mid n \geq M\} \subseteq T$.

Using the Second Principle of Mathematical Induction

The primary use of mathematical induction is to prove statements of the form

$$(\forall n \in \mathbb{Z}, \text{with } n \geq M)\,(P(n)),$$

where M is an integer and $P(n)$ is some predicate. So our goal is to prove that the truth set T of the predicate $P(n)$ contains all integers greater than or equal to M. To use the Second Principle of Mathematical Induction, we must

1. Prove that $M \in T$. That is, prove that $P(M)$ is true.
2. Prove that for every $k \in \mathbb{N}$, if $k \geq M$ and $\{M, M+1, \ldots, k\} \subseteq T$, then $(k+1) \in T$. That is, prove that if $P(M), P(M+1), \ldots, P(k)$ are true, then $P(k+1)$ is true.

As before, the first step is called the **basis step** or the **initial step**, and the second step is called the **inductive step**. This means that a proof using the Second Principle of Mathematical Induction will have the following form:

Using the Second Principle of Mathematical Induction
Let M be an integer. To prove: $(\forall n \in \mathbb{Z} \text{ with } n \geq M)\,(P(n))$

Basis step:	Prove $P(M)$.
Inductive step:	Let $k \in \mathbb{Z}$ with $k \geq M$. Prove that if $P(M), P(M+1), \ldots, P(k)$ are true, then $P(k+1)$ is true.

We can then conclude that $P(n)$ is true for all $n \in \mathbb{Z}$ with $n \geq M$.

We will use this procedure to prove the proposition suggested in Beginning Activity 2.

Theorem 4.9. *Each natural number greater than 1 either is a prime number or is a product of prime numbers.*

***Proof*.** We will use the Second Principle of Mathematical Induction. We let $P(n)$ be

$$n \text{ is a prime number or } n \text{ is a product of prime numbers.}$$

For the basis step, $P(2)$ is true since 2 is a prime number.

To prove the inductive step, we let k be a natural number with $k \geq 2$. We assume that $P(2), P(3), \ldots, P(k)$ are true. That is, we assume that each of the natural numbers $2, 3, \ldots, k$ is a prime number or a product of prime numbers. The goal is to prove that $P(k+1)$ is true or that $(k+1)$ is a prime number or a product of prime numbers.

Case 1: If $(k+1)$ is a prime number, then $P(k+1)$ is true.

Case 2: If $(k+1)$ is not a prime number, then $(k+1)$ can be factored into a product of natural numbers with each one being less than $(k+1)$. That is, there exist natural numbers a and b with

$$k + 1 = a \cdot b, \quad \text{and} \quad 1 < a \leq k \text{ and } 1 < b \leq k.$$

Using the inductive assumption, this means that $P(a)$ and $P(b)$ are both true. Consequently, a and b are prime numbers or are products of prime numbers. Since $k + 1 = a \cdot b$, we conclude that $(k + 1)$ is a product of prime numbers. That is, we conclude that $P(k + 1)$ is true. This proves the inductive step.

Hence, by the Second Principle of Mathematical Induction, we conclude that $P(n)$ is true for all $n \in \mathbb{N}$ with $n \geq 2$, and this means that each natural number greater than 1 is either a prime number or is a product of prime numbers. ■

We will conclude this section with a progress check that is really more of an activity. We do this rather than including the activity at the end of the exercises since this activity illustrates a use of the Second Principle of Mathematical Induction in which it is convenient to have the basis step consist of the proof of more than one statement.

Progress Check 4.10 (Using the Second Principle of Induction)
Consider the following question:

> For which natural numbers n do there exist nonnegative integers x and y such that $n = 3x + 5y$?

To help answer this question, we will let $\mathbb{Z}^* = \{x \in \mathbb{Z} \mid x \geq 0\}$, and let $P(n)$ be

$$\text{There exist } x, y \in \mathbb{Z}^* \text{ such that } n = 3x + 5y.$$

Notice that $P(1)$ is false since if both x and y are zero, then $3x + 5y = 0$ and if either $x > 0$ or $y > 0$, then $3x + 5y \geq 3$. Also notice that $P(6)$ is true since $6 = 3 \cdot 2 + 5 \cdot 0$ and $P(8)$ is true since $8 = 3 \cdot 1 + 5 \cdot 1$.

1. Explain why $P(2)$, $P(4)$, and $P(7)$ are false and why $P(3)$ and $P(5)$ are true.

2. Explain why $P(9)$, $P(10)$, $P(11)$, and $P(12)$ are true.

We could continue trying to determine other values of n for which $P(n)$ is true. However, let us see if we can use the work in part (2) to determine if $P(13)$ is true. Notice that $13 = 3 + 10$ and we know that $P(10)$ is true. We should be able to use this to prove that $P(13)$ is true. This is formalized in the next part.

3. Let $k \in \mathbb{N}$ with $k \geq 10$. Prove that if $P(8)$, $P(9)$, …, $P(k)$ are true, then $P(k + 1)$ is true. **Hint**: $k + 1 = 3 + (k - 2)$.

4. Prove the following proposition using mathematical induction. Use the Second Principle of Induction and have the basis step be a proof that $P(8)$, $P(9)$, and $P(10)$ are true. (The inductive step is part (3).)

Proposition 4.11. *For each $n \in \mathbb{N}$ with $n \geq 8$, there exist nonnegative integers x and y such that $n = 3x + 5y$.*

Exercises for Section 4.2

1. Use mathematical induction to prove each of the following:

   ⋆ **(a)** For each natural number n with $n \geq 2$, $3^n > 1 + 2^n$.

   (b) For each natural number n with $n \geq 6$, $2^n > (n+1)^2$.

   (c) For each natural number n with $n \geq 3$, $\left(1 + \dfrac{1}{n}\right)^n < n$.

⋆ 2. For which natural numbers n is $n^2 < 2^n$? Justify your conclusion.

3. For which natural numbers n is $n! > 3^n$? Justify your conclusion.

4. **(a)** Verify that $\left(1 - \dfrac{1}{4}\right) = \dfrac{3}{4}$ and that $\left(1 - \dfrac{1}{4}\right)\left(1 - \dfrac{1}{9}\right) = \dfrac{4}{6}$.

   (b) Verify that $\left(1 - \dfrac{1}{4}\right)\left(1 - \dfrac{1}{9}\right)\left(1 - \dfrac{1}{16}\right) = \dfrac{5}{8}$ and that $\left(1 - \dfrac{1}{4}\right)\left(1 - \dfrac{1}{9}\right)\left(1 - \dfrac{1}{16}\right)\left(1 - \dfrac{1}{25}\right) = \dfrac{6}{10}$.

   (c) For $n \in \mathbb{N}$ with $n \geq 2$, make a conjecture about a formula for the product $\left(1 - \dfrac{1}{4}\right)\left(1 - \dfrac{1}{9}\right)\left(1 - \dfrac{1}{16}\right)\cdots\left(1 - \dfrac{1}{n^2}\right)$.

   (d) Based on your work in Parts (4a) and (4b), state a proposition and then use the Extended Principle of Mathematical Induction to prove your proposition.

⋆ 5. Is the following proposition true or false? Justify your conclusion.

   For each nonnegative integer n, $8^n \mid (4n)!$.

6. Let $y = \ln x$.

(**a**) Determine $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, $\dfrac{d^3y}{dx^3}$, and $\dfrac{d^4y}{dx^4}$.

(**b**) Let n be a natural number. Formulate a conjecture for a formula for $\dfrac{d^ny}{dx^n}$. Then use mathematical induction to prove your conjecture.

7. For which natural numbers n do there exist nonnegative integers x and y such that $n = 4x + 5y$? Justify your conclusion.

⋆ **8.** Can each natural number greater than or equal to 4 be written as the sum of at least two natural numbers, each of which is a 2 or a 3? Justify your conclusion. For example, $7 = 2+2+3$, and $17 = 2+2+2+2+3+3+3$.

9. Can each natural number greater than or equal to 6 be written as the sum of at least two natural numbers, each of which is a 2 or a 5? Justify your conclusion. For example, $6 = 2+2+2$, $9 = 2+2+5$, and $17 = 2+5+5+5$.

10. Use mathematical induction to prove the following proposition:

> Let x be a real number with $x > 0$. Then for each natural number n with $n \geq 2$, $(1 + x)^n > 1 + nx$.

Explain where the assumption that $x > 0$ was used in the proof.

11. Prove that for each odd natural number n with $n \geq 3$,

$$\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{4}\right)\cdots\left(1 + \frac{(-1)^n}{n}\right) = 1.$$

⋆ **12.** Prove that for each natural number n,

$$\text{any set with } n \text{ elements has } \frac{n\,(n-1)}{2} \text{ two-element subsets.}$$

13. Prove or disprove each of the following propositions:

(**a**) For each $n \in \mathbb{N}$, $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

(**b**) For each natural number n with $n \geq 3$,

$$\frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{n(n+1)} = \frac{n-2}{3n+3}.$$

(**c**) For each $n \in \mathbb{N}$, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

14. Is the following proposition true or false? Justify your conclusion.

For each natural number n, $\left(\dfrac{n^3}{3} + \dfrac{n^2}{2} + \dfrac{7n}{6}\right)$ is a natural number.

15. Is the following proposition true or false? Justify your conclusion.

For each natural number n, $\left(\dfrac{n^5}{5} + \dfrac{n^4}{2} + \dfrac{n^3}{3} - \dfrac{n}{30}\right)$ is an integer.

* **16.** (**a**) Prove that if $n \in \mathbb{N}$, then there exists an odd natural number m and a nonnegative integer k such that $n = 2^k m$.

(**b**) For each $n \in \mathbb{N}$, prove that there is only one way to write n in the form described in Part (a). To do this, assume that $n = 2^k m$ and $n = 2^q p$ where m and p are odd natural numbers and k and q are nonnegative integers. Then prove that $k = q$ and $m = p$.

17. Evaluation of proofs
See the instructions for Exercise (19) on page 100 from Section 3.1.

(**a**) For each natural number n with $n \geq 2$, $2^n > 1 + n$.

Proof. We let k be a natural number and assume that $2^k > 1 + k$. Multiplying both sides of this inequality by 2, we see that $2^{k+1} > 2 + 2k$. However, $2 + 2k > 2 + k$ and, hence,

$$2^{k+1} > 1 + (k + 1).$$

By mathematical induction, we conclude that $2^n > 1 + n$. ■

(**b**) Each natural number greater than or equal to 6 can be written as the sum of natural numbers, each of which is a 2 or a 5.

Proof. We will use a proof by induction. For each natural number n, we let $P(n)$ be, "There exist nonnegative integers x and y such that $n = 2x + 5y$." Since

$$6 = 3 \cdot 2 + 0 \cdot 5 \qquad 7 = 2 + 5$$
$$8 = 4 \cdot 2 + 0 \cdot 5 \qquad 9 = 2 \cdot 2 + 1 \cdot 5$$

we see that $P(6)$, $P(7)$, $P(8)$, and $P(9)$ are true.
We now suppose that for some natural number k with $k \geq 10$ that $P(6), P(7), \ldots, P(k)$ are true. Now

$$k + 1 = (k - 4) + 5.$$

Since $k \geq 10$, we see that $k - 4 \geq 6$ and, hence, $P(k-4)$ is true. So $k - 4 = 2x + 5y$ and, hence,

$$\begin{aligned} k + 1 &= (2x + 5y) + 5 \\ &= 2x + 5(y + 1). \end{aligned}$$

This proves that $P(k+1)$ is true, and hence, by the Second Principle of Mathematical Induction, we have proved that for each natural number n with $n \geq 6$, there exist nonnegative integers x and y such that $n = 2x + 5y$. ■

Explorations and Activities

18. **The Sum of the Angles of a Convex Quadrilateral**. There is a famous theorem in Euclidean geometry that states that the sum of the interior angles of a triangle is 180°.

(**a**) Use the theorem about triangles to determine the sum of the angles of a convex quadrilateral. **Hint**: Draw a convex quadrilateral and draw a diagonal.

(**b**) Use the result in Part (a) to determine the sum of the angles of a convex pentagon.

(**c**) Use the result in Part (b) to determine the sum of the angles of a convex hexagon.

(**d**) Let n be a natural number with $n \geq 3$. Make a conjecture about the sum of the angles of a convex polygon with n sides and use mathematical induction to prove your conjecture.

19. **De Moivre's Theorem**. One of the most interesting results in trigonometry is De Moivre's Theorem, which relates the complex number i to the trigonometric functions. Recall that the number i is a complex number whose square is -1, that is, $i^2 = -1$. One version of the theorem can be stated as follows:

If x is a real number, then for each nonnegative integer n,

$$[\cos x + i(\sin x)]^n = \cos(nx) + i(\sin(nx)).$$

This theorem is named after Abraham de Moivre (1667 – 1754), a French mathematician.

(**a**) The proof of De Moivre's Theorem requires the use of the trigonometric identities for the sine and cosine of the sum of two angles. Use the Internet or a book to find identities for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.

(**b**) To get a sense of how things work, expand $[\cos x + i(\sin x)]^2$ and write the result in the form $a + bi$. Then use the identities from part (1) to prove that $[\cos x + i(\sin x)]^2 = \cos(2x) + i(\sin(2x))$.

(**c**) Use mathematical induction to prove De Moivre's Theorem.

4.3 Induction and Recursion

Beginning Activity 1 (**Recursively Defined Sequences**)
In a proof by mathematical induction, we "start with a first step" and then prove that we can always go from one step to the next step. We can use this same idea to define a sequence as well. We can think of a **sequence** as an infinite list of numbers that are indexed by the natural numbers (or some infinite subset of $\mathbb{N} \cup \{0\}$). We often write a sequence in the following form:

$$a_1, a_2, \ldots, a_n, \ldots.$$

The number a_n is called the n^{th} term of the sequence. One way to define a sequence is to give a specific formula for the n^{th} term of the sequence such as $a_n = \dfrac{1}{n}$.

Another way to define a sequence is to give a specific definition of the first term (or the first few terms) and then state, in general terms, how to determine a_{n+1} in terms of n and the first n terms $a_1, a_2, \ldots, a_n$. This process is known as **definition by recursion** and is also called a **recursive definition**. The specific definition of the first term is called the **initial condition**, and the general definition of a_{n+1} in terms of n and the first n terms $a_1, a_2, \ldots, a_n$ is called the **recurrence relation**. (When more than one term is defined explicitly, we say that these are the initial conditions.) For example, we can define a sequence recursively as follows:

$$b_1 = 16, \text{ and for each } n \in \mathbb{N},\ b_{n+1} = \frac{1}{2}b_n.$$

Using $n = 1$ and then $n = 2$, we then see that

$$b_2 = \frac{1}{2}b_1 \qquad\qquad b_3 = \frac{1}{2}b_2$$
$$= \frac{1}{2} \cdot 16 \qquad\qquad = \frac{1}{2} \cdot 8$$
$$= 8 \qquad\qquad = 4$$

1. Calculate b_4 through b_{10}. What seems to be happening to the values of b_n as n gets larger?

2. Define a sequence recursively as follows:

   $T_1 = 16$, and for each $n \in \mathbb{N}$, $T_{n+1} = 16 + \frac{1}{2}T_n$.

   Then $T_2 = 16 + \frac{1}{2}T_1 = 16 + 8 = 24$. Calculate T_3 through T_{10}. What seems to be happening to the values of T_n as n gets larger?

The sequences in Parts (1) and (2) can be generalized as follows: Let a and r be real numbers. Define two sequences recursively as follows:

$a_1 = a$, and for each $n \in \mathbb{N}$, $a_{n+1} = r \cdot a_n$.

$S_1 = a$, and for each $n \in \mathbb{N}$, $S_{n+1} = a + r \cdot S_n$.

3. Determine formulas (in terms of a and r) for a_2 through a_6. What do you think a_n is equal to (in terms of a, r, and n)?

4. Determine formulas (in terms of a and r) for S_2 through S_6. What do you think S_n is equal to (in terms of a, r, and n)?

In Beginning Activity 1 in Section 4.2, for each natural number n, we defined $n!$, read n **factorial**, as the product of the first n natural numbers. We also defined $0!$ to be equal to 1. Now recursively define a sequence of numbers $a_0, a_1, a_2, \ldots$ as follows:

$a_0 = 1$, and

for each nonnegative integer n, $a_{n+1} = (n + 1) \cdot a_n$.

Using $n = 0$, we see that this implies that $a_1 = 1 \cdot a_0 = 1 \cdot 1 = 1$. Then using $n = 1$, we see that

$$a_2 = 2a_1 = 2 \cdot 1 = 2.$$

5. Calculate a_3, a_4, a_5, and a_6.

6. Do you think that it is possible to calculate a_{20} and a_{100}? Explain.

7. Do you think it is possible to calculate a_n for any natural number n? Explain.

8. Compare the values of $a_0, a_1, a_2, a_3, a_4, a_5$, and a_6 with those of 0!, 1!, 2!, 3!, 4!, 5!, and 6!. What do you observe? We will use mathematical induction to prove a result about this sequence in Exercise (1).

Beginning Activity 2 (**The Fibonacci Numbers**)
The **Fibonacci numbers** are a sequence of natural numbers $f_1, f_2, f_3, \ldots, f_n, \ldots$ defined recursively as follows:

- $f_1 = 1$ and $f_2 = 1$, and
- For each natural number n, $f_{n+2} = f_{n+1} + f_n$.

In words, the recursion formula states that for any natural number n with $n \geq 3$, the n^{th} Fibonacci number is the sum of the two previous Fibonacci numbers. So we see that

$$\begin{aligned} f_3 &= f_2 + f_1 = 1 + 1 = 2, \\ f_4 &= f_3 + f_2 = 2 + 1 = 3, \text{ and} \\ f_5 &= f_4 + f_3 = 3 + 2 = 5. \end{aligned}$$

1. Calculate f_6 through f_{20}.

2. Which of the Fibonacci numbers f_1 through f_{20} are even? Which are multiples of 3?

3. For $n = 2$, $n = 3$, $n = 4$, and $n = 5$, how is the sum of the first $(n-1)$ Fibonacci numbers related to the $(n+1)^{st}$ Fibonacci number?

4. Record any other observations about the values of the Fibonacci numbers or any patterns that you observe in the sequence of Fibonacci numbers. If necessary, compute more Fibonacci numbers.

The Fibonacci Numbers

The Fibonacci numbers form a famous sequence in mathematics that was investigated by Leonardo of Pisa (1170 – 1250), who is better known as Fibonacci. Fibonacci introduced this sequence to the Western world as a solution of the following problem:

> Suppose that a pair of adult rabbits (one male, one female) produces a pair of rabbits (one male, one female) each month. Also, suppose that newborn rabbits become adults in two months and produce another pair of rabbits. Starting with one adult pair of rabbits, how many pairs of rabbits will be produced each month for one year?

Since we start with one adult pair, there will be one pair produced the first month, and since there is still only one adult pair, one pair will also be produced in the second month (since the new pair produced in the first month is not yet mature). In the third month, two pairs will be produced, one by the original pair and one by the pair which was produced in the first month. In the fourth month, three pairs will be produced, and in the fifth month, five pairs will be produced.

The basic rule is that in a given month after the first two months, the number of adult pairs is the number of adult pairs one month ago plus the number of pairs born two months ago. This is summarized in Table 4.1, where the number of pairs produced is equal to the number of adult pairs, and the number of adult pairs follows the Fibonacci sequence of numbers that we developed in Beginning Activity 2.

Months	1	2	3	4	5	6	7	8	9	10
Adult Pairs	1	1	2	3	5	8	13	21	34	55
Newborn Pairs	1	1	2	3	5	8	13	21	34	55
Month-Old Pairs	0	1	1	2	3	5	8	13	21	34

Table 4.1: Fibonacci Numbers

Historically, it is interesting to note that Indian mathematicians were studying these types of numerical sequences well before Fibonacci. In particular, about fifty years before Fibonacci introduced his sequence, Acharya Hemachandra (sometimes spelled Hemchandra) (1089 – 1173) considered the following problem, which is from the biography of Hemachandra in the *MacTutor History of Mathematics Archive* at https://mathshistory.st-andrews.ac.uk/Biographies/Hemchandra/.

> Suppose we assume that lines are composed of syllables which are either short or long. Suppose also that each long syllable takes twice as long to articulate as a short syllable. A line of length n contains n units where each short syllable is one unit and each long syllable is two units. Clearly a line of length n units takes the same time to articulate regardless of how it is composed. Hemchandra asks: How many different combinations of short and long syllables are possible in a line of length n?

This is an important problem in the Sanskrit language since Sanskrit meters are based on duration rather than on accent as in the English Language. The answer to this question generates a sequence similar to the Fibonacci sequence. Suppose that h_n is the number of patterns of syllables of length n. We then see that $h_1 = 1$ and $h_2 = 2$. Now let n be a natural number and consider pattern of length $n + 2$. This pattern either ends in a short syllable or a long syllable. If it ends in a short syllable and this syllable is removed, then there is a pattern of length $n + 1$, and there are h_{n+1} such patterns. Similarly, if it ends in a long syllable and this syllable is removed, then there is a pattern of length n, and there are h_n such patterns. From this, we conclude that

$$h_{n+2} = h_{n+1} + h_n.$$

This actually generates the sequence $1, 2, 3, 5, 8, 13, 21, \ldots$. For more information about Hemachandra, see the article *Math for Poets and Drummers* by Rachel Wells Hall in the February 2008 issue of *Math Horizons*.

We will continue to use the Fibonacci sequence in this book. This sequence may not seem all that important or interesting. However, it turns out that this sequence occurs in nature frequently and has applications in computer science. There is even a scholarly journal, *The Fibonacci Quarterly*, devoted to the Fibonacci numbers.

The sequence of Fibonacci numbers is one of the most studied sequences in mathematics, due mainly to the many beautiful patterns it contains. Perhaps one observation you made in Beginning Activity 2 is that every third Fibonacci number is even. This can be written as a proposition as follows:

For each natural number n, f_{3n} is an even natural number.

As with many propositions associated with definitions by recursion, we can prove this using mathematical induction. The first step is to define the appropriate open sentence. For this, we can let $P(n)$ be, "f_{3n} is an even natural number."

Notice that $P(1)$ is true since $f_3 = 2$. We now need to prove the inductive step. To do this, we need to prove that for each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

That is, we need to prove that for each $k \in \mathbb{N}$, if f_{3k} is even, then $f_{3(k+1)}$ is even.

So let's analyze this conditional statement using a know-show table.

Step	Know	Reason
P	f_{3k} is even.	Inductive hypothesis
$P1$	$(\exists m \in \mathbb{N})\,(f_{3k} = 2m)$	Definition of "even integer"
$\vdots$	$\vdots$	$\vdots$
$Q1$	$(\exists q \in \mathbb{N})\,\left(f_{3(k+1)} = 2q\right)$	
Q	$f_{3(k+1)}$ is even.	Definition of "even integer"
Step	**Show**	**Reason**

The key question now is, "Is there any relation between $f_{3(k+1)}$ and f_{3k}?" We can use the recursion formula that defines the Fibonacci sequence to find such a relation.

The recurrence relation for the Fibonacci sequence states that a Fibonacci number (except for the first two) is equal to the sum of the two previous Fibonacci numbers. If we write $3(k+1) = 3k+3$, then we get $f_{3(k+1)} = f_{3k+3}$. For f_{3k+3}, the two previous Fibonacci numbers are f_{3k+2} and f_{3k+1}. This means that

$$f_{3k+3} = f_{3k+2} + f_{3k+1}.$$

Using this and continuing to use the Fibonacci relation, we obtain the following:

$$\begin{aligned} f_{3(k+1)} &= f_{3k+3} \\ &= f_{3k+2} + f_{3k+1} \\ &= (f_{3k+1} + f_{3k}) + f_{3k+1}. \end{aligned}$$

The preceding equation states that $f_{3(k+1)} = 2f_{3k+1} + f_{3k}$. This equation can be used to complete the proof of the induction step.

Progress Check 4.12 (Every Third Fibonacci Number Is Even)
Complete the proof of Proposition 4.13.

Proposition 4.13. *For each natural number n, the Fibonacci number f_{3n} is an even natural number.*

Hint: We have already defined the predicate $P(n)$ to be used in an induction proof and have proved the basis step. Use the information in and after the preceding know-show table to help prove that if f_{3k} is even, then $f_{3(k+1)}$ is even.

Geometric Sequences and Geometric Series

Let $a, r \in \mathbb{R}$. The following sequence was introduced in Beginning Activity 1.

Initial condition: $a_1 = a$.
Recurrence relation: For each $n \in \mathbb{N}$, $a_{n+1} = r \cdot a_n$.

This is a recursive definition for a **geometric sequence** with **initial term** a and (common) **ratio** r. The basic idea is that the next term in the sequence is obtained by multiplying the previous term by the ratio r. The work in Beginning Activity 1 suggests that the following proposition is true.

Theorem 4.14. *Let $a, r \in \mathbb{R}$. If a geometric sequence is defined by $a_1 = a$ and for each $n \in \mathbb{N}$, $a_{n+1} = r \cdot a_n$, then for each $n \in \mathbb{N}$, $a_n = a \cdot r^{n-1}$.*

The proof of this proposition is Exercise (6).

Another sequence that was introduced in Beginning Activity 1 is related to geometric series and is defined as follows:

Initial condition: $S_1 = a$.
Recurrence relation: For each $n \in \mathbb{N}$, $S_{n+1} = a + r \cdot S_n$.

For each $n \in \mathbb{N}$, the term S_n is a (finite) **geometric series** with **initial term** a and (common) **ratio** r. The work in Beginning Activity 1 suggests that the following proposition is true.

Theorem 4.15. *Let $a, r \in \mathbb{R}$. If the sequence $S_1, S_2, \ldots, S_n, \ldots$ is defined by $S_1 = a$ and for each $n \in \mathbb{N}$, $S_{n+1} = a + r \cdot S_n$, then for each $n \in \mathbb{N}$, $S_n = a + a \cdot r + a \cdot r^2 + \cdots + a \cdot r^{n-1}$. That is, the geometric series S_n is the sum of the first n terms of the corresponding geometric sequence.*

The proof of Proposition 4.15 is Exercise (7). The recursive definition of a geometric series and Proposition 4.15 give two different ways to look at geometric series. Proposition 4.15 represents a geometric series as the sum of the first n terms of the corresponding geometric sequence. Another way to determine the sum of a geometric series is given in Theorem 4.16, which gives a formula for the sum of a geometric series that does not use a summation.

Theorem 4.16. *Let $a, r \in \mathbb{R}$ and $r \neq 1$. If the sequence $S_1, S_2, \ldots, S_n, \ldots$ is defined by $S_1 = a$ and for each $n \in \mathbb{N}$, $S_{n+1} = a + r \cdot S_n$, then for each $n \in \mathbb{N}$, $S_n = a\left(\dfrac{1 - r^n}{1 - r}\right)$.*

The proof of Proposition 4.16 is Exercise (8).

Exercises for Section 4.3

⋆ **1.** For the sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$, assume that $a_0 = 1$ and that for each $n \in \mathbb{N} \cup \{0\}$, $a_{n+1} = (n+1)\, a_n$. Use mathematical induction to prove that for each $n \in \mathbb{N} \cup \{0\}$, $a_n = n!$.

2. Assume that $f_1, f_2, \ldots, f_n, \ldots$ are the Fibonacci numbers. Prove each of the following:

⋆ **(a)** For each $n \in \mathbb{N}$, f_{4n} is a multiple of 3.

(b) For each $n \in \mathbb{N}$, f_{5n} is a multiple of 5.

⋆ **(c)** For each $n \in \mathbb{N}$ with $n \geq 2$, $f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - 1$.

(d) For each $n \in \mathbb{N}$, $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$.

(e) For each $n \in \mathbb{N}$, $f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$.

⋆ **(f)** For each $n \in \mathbb{N}$, $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$.

(g) For each $n \in \mathbb{N}$ such that $n \not\equiv 0 \pmod 3$, f_n is an odd integer.

3. Use the result in Part (f) of Exercise (2) to prove that

$$\frac{f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2}{f_1^2 + f_2^2 + \cdots + f_n^2} = 1 + \frac{f_{n+1}}{f_n}.$$

4. The quadratic formula can be used to show that $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ are the two real number solutions of the quadratic equation $x^2 - x - 1 = 0$. Notice that this implies that

$$\alpha^2 = \alpha + 1, \text{ and}$$
$$\beta^2 = \beta + 1.$$

It may be surprising to find out that these two irrational numbers are closely related to the Fibonacci numbers.

(a) Verify that $f_1 = \dfrac{\alpha^1 - \beta^1}{\alpha - \beta}$ and that $f_2 = \dfrac{\alpha^2 - \beta^2}{\alpha - \beta}$.

(b) (This part is optional, but it may help with the induction proof in part (c).) Work with the relation $f_3 = f_2 + f_1$ and substitute the expressions for f_1 and f_2 from part (a). Rewrite the expression as a single fraction and then in the numerator use $\alpha^2 + \alpha = \alpha(\alpha + 1)$ and a similar equation involving β. Now prove that $f_3 = \dfrac{\alpha^3 - \beta^3}{\alpha - \beta}$.

(**c**) Use induction to prove that for each natural number n, if $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$, then $f_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$. **Note**: This formula for the n^{th} Fibonacci number is known as Binet's formula, named after the French mathematician Jacques Binet (1786 – 1856).

5. Is the following conjecture true or false? Justify your conclusion.

Conjecture. Let $f_1, f_2, \ldots, f_m, \ldots$ be the sequence of the Fibonacci numbers. For each natural number n, the numbers $f_n f_{n+3}$, $2f_{n+1} f_{n+2}$, and $\left(f_{n+1}^2 + f_{n+2}^2\right)$ form a Pythagorean triple.

★ **6.** Prove Proposition 4.14. Let $a, r \in \mathbb{R}$. If a geometric sequence is defined by $a_1 = a$ and for each $n \in \mathbb{N}$, $a_{n+1} = r \cdot a_n$, then for each $n \in \mathbb{N}$, $a_n = a \cdot r^{n-1}$.

7. Prove Proposition 4.15. Let $a, r \in \mathbb{R}$. If the sequence $S_1, S_2, \ldots, S_n, \ldots$ is defined by $S_1 = a$ and for each $n \in \mathbb{N}$, $S_{n+1} = a + r \cdot S_n$, then for each $n \in \mathbb{N}$, $S_n = a + a \cdot r + a \cdot r^2 + \cdots + a \cdot r^{n-1}$. That is, the geometric series S_n is the sum of the first n terms of the corresponding geometric sequence.

★ **8.** Prove Proposition 4.16. Let $a, r \in \mathbb{R}$ and $r \neq 1$. If the sequence $S_1, S_2, \ldots, S_n, \ldots$ is defined by $S_1 = a$ and for each $n \in \mathbb{N}$, $S_{n+1} = a + r \cdot S_n$, then for each $n \in \mathbb{N}$, $S_n = a\left(\dfrac{1-r^n}{1-r}\right)$.

9. For the sequence $a_1, a_2, \ldots, a_n, \ldots$, assume that $a_1 = 2$ and that for each $n \in \mathbb{N}$, $a_{n+1} = a_n + 5$.

★ **(a)** Calculate a_2 through a_6.

★ **(b)** Make a conjecture for a formula for a_n for each $n \in \mathbb{N}$.

(c) Prove that your conjecture in Exercise (9b) is correct.

10. The sequence in Exercise (9) is an example of an **arithmetic sequence**. An arithmetic sequence is defined recursively as follows:

Let c and d be real numbers. Define the sequence $a_1, a_2, \ldots, a_n, \ldots$ by $a_1 = c$ and for each $n \in \mathbb{N}$, $a_{n+1} = a_n + d$.

(**a**) Determine formulas for a_3 through a_8.

(**b**) Make a conjecture for a formula for a_n for each $n \in \mathbb{N}$.

(**c**) Prove that your conjecture in Exercise (10b) is correct.

11. For the sequence $a_1, a_2, \ldots, a_n, \ldots$, assume that $a_1 = 1$, $a_2 = 5$, and that for each $n \in \mathbb{N}$, $a_{n+1} = a_n + 2a_{n-1}$. Prove that for each natural number n, $a_n = 2^n + (-1)^n$.

⋆ **12.** For the sequence $a_1, a_2, \ldots, a_n, \ldots$, assume that $a_1 = 1$ and that for each $n \in \mathbb{N}$, $a_{n+1} = \sqrt{5 + a_n}$.

(**a**) Calculate, or approximate, a_2 through a_6.

(**b**) Prove that for each $n \in \mathbb{N}$, $a_n < 3$.

13. For the sequence $a_1, a_2, \ldots, a_n, \ldots$, assume that $a_1 = 1$, $a_2 = 3$, and that for each $n \in \mathbb{N}$, $a_{n+2} = 3a_{n+1} - 2a_n$.

⋆ **(a)** Calculate a_3 through a_6.

⋆ **(b)** Make a conjecture for a formula for a_n for each $n \in \mathbb{N}$.

(c) Prove that your conjecture in Exercise (13b) is correct.

14. For the sequence $a_1, a_2, \ldots, a_n, \ldots$, assume that $a_1 = 1$, $a_2 = 1$, and that for each $n \in \mathbb{N}$, $a_{n+2} = \dfrac{1}{2}\left(a_{n+1} + \dfrac{2}{a_n}\right)$.

⋆ **(a)** Calculate a_3 through a_6.

(b) Prove that for each $n \in \mathbb{N}$, $1 \le a_n \le 2$.

15. For the sequence $a_1, a_2, \ldots, a_n, \ldots$, assume that $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, and for that each natural number n,

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n.$$

(**a**) Compute a_4, a_5, a_6, and a_7.

(**b**) Prove that for each natural number n with $n > 1$, $a_n \le 2^{n-2}$.

16. For the sequence $a_1, a_2, \ldots, a_n, \ldots$, assume that $a_1 = 1$, and that for each natural number n,

$$a_{n+1} = a_n + n \cdot n!.$$

(**a**) Compute $n!$ for the first 10 natural numbers.

⋆ **(b)** Compute a_n for the first 10 natural numbers.

(c) Make a conjecture about a formula for a_n in terms of n that does not involve a summation or a recursion.

(d) Prove your conjecture in Part (c).

17. For the sequence $a_1, a_2, \ldots, a_n, \ldots$, assume that $a_1 = 1$, $a_2 = 1$, and for each $n \in \mathbb{N}$, $a_{n+2} = a_{n+1} + 3a_n$. Determine which terms in this sequence are divisible by 4 and prove that your answer is correct.

18. The **Lucas numbers** are a sequence of natural numbers $L_1, L_2, L_3, \ldots, L_n, \ldots$, which are defined recursively as follows:

- $L_1 = 1$ and $L_2 = 3$, and
- For each natural number n, $L_{n+2} = L_{n+1} + L_n$.

List the first 10 Lucas numbers and the first ten Fibonacci numbers and then prove each of the following propositions. The Second Principle of Mathematical Induction may be needed to prove some of these propositions.

$\star$ **(a)** For each natural number n, $L_n = 2f_{n+1} - f_n$.

(b) For each $n \in \mathbb{N}$ with $n \geq 2$, $5f_n = L_{n-1} + L_{n+1}$.

(c) For each $n \in \mathbb{N}$ with $n \geq 3$, $L_n = f_{n+2} - f_{n-2}$.

19. There is a formula for the Lucas numbers similar to the formula for the Fibonacci numbers in Exercise (4). Let $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. Prove that for each $n \in \mathbb{N}$, $L_n = \alpha^n + \beta^n$.

20. Use the result in Exercise (19), previously proven results from Exercise (18), or mathematical induction to prove each of the following results about Lucas numbers and Fibonacci numbers.

(a) For each $n \in \mathbb{N}$, $L_n = \dfrac{f_{2n}}{f_n}$.

(b) For each $n \in \mathbb{N}$, $f_{n+1} = \dfrac{f_n + L_n}{2}$.

(c) For each $n \in \mathbb{N}$, $L_{n+1} = \dfrac{L_n + 5f_n}{2}$.

(d) For each $n \in \mathbb{N}$ with $n \geq 2$, $L_n = f_{n+1} + f_{n-1}$.

21. Evaluation of proofs
See the instructions for Exercise (19) on page 100 from Section 3.1.

(**a**) Let f_n be the n^{th} Fibonacci number, and let α be the positive solution of the equation $x^2 = x + 1$. So $\alpha = \dfrac{1+\sqrt{5}}{2}$. For each natural number n, $f_n \le \alpha^{n-1}$.

Proof. We will use a proof by mathematical induction. For each natural number n, we let $P(n)$ be, "$f_n \le \alpha^{n-1}$."

We first note that $P(1)$ is true since $f_1 = 1$ and $\alpha^0 = 1$. We also notice that $P(2)$ is true since $f_2 = 1$ and, hence, $f_2 \le \alpha^1$.

We now let k be a natural number with $k \ge 2$ and assume that $P(1)$, $P(2)$, …, $P(k)$ are all true. We now need to prove that $P\,(k+1)$ is true or that $f_{k+1} \le \alpha^k$.

Since $P(k-1)$ and $P(k)$ are true, we know that $f_{k-1} \le \alpha^{k-2}$ and $f_k \le \alpha^{k-1}$. Therefore,

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ f_{k+1} &\le \alpha^{k-1} + \alpha^{k-2} \\ f_{k+1} &\le \alpha^{k-2}\,(\alpha + 1). \end{aligned}$$

We now use the fact that $\alpha + 1 = \alpha^2$ and the preceding inequality to obtain

$$\begin{aligned} f_{k+1} &\le \alpha^{k-2}\alpha^2 \\ f_{k+1} &\le \alpha^k. \end{aligned}$$

This proves that if $P(1)$, $P(2)$, …, $P(k)$ are true, then $P(k+1)$ is true. Hence, by the Second Principle of Mathematical Induction, we conclude that or each natural number n, $f_n \le \alpha^{n-1}$. ■

Explorations and Activities

22. Compound Interest. Assume that R dollars is deposited in an account that has an interest rate of i for each compounding period. A compounding period is some specified time period such as a month or a year.

For each integer n with $n \ge 0$, let V_n be the amount of money in an account at the end of the nth compounding period. Then

$$\begin{aligned} V_1 &= R + i \cdot R & \qquad V_2 &= V_1 + i \cdot V_1 \\ &= R\,(1+i) & &= (1+i)\,V_1 \\ & & &= (1+i)^2\,R. \end{aligned}$$

(**a**) Explain why $V_3 = V_2 + i \cdot V_2$. Then use the formula for V_2 to determine a formula for V_3 in terms of i and R.

(**b**) Determine a recurrence relation for V_{n+1} in terms of i and V_n.

(**c**) Write the recurrence relation in Part (22b) so that it is in the form of a recurrence relation for a geometric sequence. What is the initial term of the geometric sequence and what is the common ratio?

(**d**) Use Proposition 4.14 to determine a formula for V_n in terms of I, R, and n.

23. **The Future Value of an Ordinary Annuity**. For an **ordinary annuity**, R dollars is deposited in an account at the end of each compounding period. It is assumed that the interest rate, i, per compounding period for the account remains constant.

Let S_t represent the amount in the account at the end of the tth compounding period. S_t is frequently called the **future value** of the ordinary annuity.

So $S_1 = R$. To determine the amount after two months, we first note that the amount after one month will gain interest and grow to $(1+i)S_1$. In addition, a new deposit of R dollars will be made at the end of the second month. So

$$S_2 = R + (1+i)S_1.$$

(**a**) For each $n \in \mathbb{N}$, use a similar argument to determine a recurrence relation for S_{n+1} in terms of R, i, and S_n.

(**b**) By recognizing this as a recursion formula for a geometric series, use Proposition 4.16 to determine a formula for S_n in terms of R, i, and n that does not use a summation. Then show that this formula can be written as

$$S_n = R\left(\frac{(1+i)^n - 1}{i}\right).$$

(**c**) What is the future value of an ordinary annuity in 20 years if \$200 dollars is deposited in an account at the end of each month where the interest rate for the account is 6% per year compounded monthly? What is the amount of interest that has accumulated in this account during the 20 years?

4.4 Chapter 4 Summary

Important Definitions

- Inductive set, page 171
- Factorial, page 201
- Recursive definition, page 200
- Fibonacci numbers, page 202
- Geometric sequence, page 206
- Geometric series, page 206

The Various Forms of Mathematical Induction

1. **The Principle of Mathematical Induction**
If T is a subset of $\mathbb{N}$ such that

   (**a**) $1 \in T$, and
   (**b**) For every $k \in \mathbb{N}$, if $k \in T$, then $(k+1) \in T$,

   then $T = \mathbb{N}$.

   Procedure for a Proof by Mathematical Induction
   To prove $(\forall n \in \mathbb{N})\,(P(n))$

   Basis step: Prove $P(1)$.

   Inductive step: Prove that for each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

2. **The Extended Principle of Mathematical Induction**
Let M be an integer. If T is a subset of $\mathbb{Z}$ such that

   (**a**) $M \in T$, and
   (**b**) For every $k \in \mathbb{Z}$ with $k \geq M$, if $k \in T$, then $(k+1) \in T$,

   then T contains all integers greater than or equal to M.

   Using the Extended Principle of Mathematical Induction
   Let M be an integer. To prove $(\forall n \in \mathbb{Z} \text{ with } n \geq M)\,(P(n))$

   Basis step: Prove $P(M)$.

   Inductive step: Prove that for every $k \in \mathbb{Z}$ with $k \geq M$, if $P(k)$ is true, then $P(k+1)$ is true.

   We can then conclude that $P(n)$ is true for all $n \in \mathbb{Z}$ with $n \geq M$.

3. **The Second Principle of Mathematical Induction**
Let M be an integer. If T is a subset of $\mathbb{Z}$ such that

   (**a**) $M \in T$, and

   (**b**) For every $k \in \mathbb{Z}$ with $k \geq M$, if $\{M, M+1, \ldots, k\} \subseteq T$, then $(k+1) \in T$,

   then T contains all integers greater than or equal to M.

   Using the Second Principle of Mathematical Induction
   Let M be an integer. To prove $(\forall n \in \mathbb{Z} \text{ with } n \geq M)\,(P(n))$

   Basis step: Prove $P(M)$.

   Inductive step: Let $k \in \mathbb{Z}$ with $k \geq M$. Prove that if $P(M), P(M+1), \ldots, P(k)$ are true, then $P(k+1)$ is true.

   We can then conclude that $P(n)$ is true for all $n \in \mathbb{Z}$ with $n \geq M$.

Important Results

- **Theorem 4.9**. *Each natural number greater than 1 either is a prime number or is a product of prime numbers.*

- **Theorem 4.14**. *Let $a, r \in \mathbb{R}$. If a geometric sequence is defined by $a_1 = a$ and for each $n \in \mathbb{N}$, $a_{n+1} = r \cdot a_n$, then for each $n \in \mathbb{N}$, $a_n = a \cdot r^{n-1}$.*

- **Theorem 4.15**. *Let $a, r \in \mathbb{R}$. If the sequence $S_1, S_2, \ldots, S_n, \ldots$ is defined by $S_1 = a$ and for each $n \in \mathbb{N}$, $S_{n+1} = a + r \cdot S_n$, then for each $n \in \mathbb{N}$, $S_n = a + a \cdot r + a \cdot r^2 + \cdots + a \cdot r^{n-1}$. That is, the geometric series S_n is the sum of the first n terms of the corresponding geometric sequence.*

- **Theorem 4.16**. *Let $a, r \in \mathbb{R}$ and $r \neq 1$. If the sequence $S_1, S_2, \ldots, S_n, \ldots$ is defined by $S_1 = a$ and for each $n \in \mathbb{N}$, $S_{n+1} = a + r \cdot S_n$, then for each $n \in \mathbb{N}$, $S_n = a\left(\dfrac{1-r^n}{1-r}\right)$.*