Appendix A

Functions

Focus Questions

By the end of this investigation, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the investigation.

- What is a function?
- What does it mean to say that a function is an injection? How can we prove that a function is (or is not) an injection?
- What does it mean to say that a function is a surjection? How can we prove that a function is (or is not) a surjection?
- What is a bijection?
- What is the composition of two functions, and what is a composite function? What are some important theorems about composite functions?
- What is the inverse of a function? Under what conditions is the inverse of a function $f : A \to B$ a function from B to A?
- What are some important theorems about functions and their inverses?

Functions are frequently used in mathematics to define and describe certain relationships between sets and other mathematical objects. In this appendix, we will first study special types of functions known as injections and surjections. Before defining these types of functions, we will review the definition of a function and explore certain functions with finite domains.

Definition A.1. A function f from a set A to a set B is a collection of ordered pairs

$$\{(a, b) : a \in A \text{ and } b \in B\}$$

such that for each element a in A, there is one and only one element in B such that (a, b) is in f.

There is a special notation, called *functional notation*, that is commonly used to describe functions and the way they act on sets. In particular, if (a, b) is in the function f, we write $f(a) = b$ (read as "f of a equals b"). It is important to note the dual use of the symbol f here; we use f to represent a collection of ordered pairs and also to describe an action (pairing a with b in $f(a) = b$). In general practice, we use functional notation and think of a function as assigning to an element a a unique element b. In this context, we think of the elements in A as the input of the assignment and the elements in B as the output. In this way, we can consider f as a *mapping* from A to B and write

$f : A \to B$ to indicate this mapping action. How we read this notation depends on the context in which it appears. For instance, the statement

Consider the function $f : A \to B$

would be read as, "Consider the function f from A to B." On the other hand, if we write

Let $f : A \to B$,

then this statement would be read as "Let f be a function from A to B" or "Let f map from A to B."

There is some familiar terminology and notation associated with functions. Let f be a function from a set A to a set B.

- The set A is called the **domain** of f, and we write $\text{dom}(f) = A$.
- The set B is called the **codomain** of f, and we write $\text{codom}(f) = B$.
- The subset $\{f(a) : a \in A\}$ of B is called the **range** of f, which we denote by range(f). Note that the range of f could equivalently be defined as follows:

$$\text{range}(f) = \{y \in B \mid y = f(x) \text{ for some } x \in A\}.$$

- If $a \in A$, then $f(a)$ is the **image** of a under f.
- If $b \in B$ and $b = f(a)$ for some $a \in A$, then a is called a **pre-image** of b.

Notice that, according to these definitions, $\text{range}(f) \subseteq \text{codom}(f)$, but it is not necessarily the case that $\text{range}(f) = \text{codom}(f)$. Whether we have this set equality or not depends on the function f, as we will see in the next section.

Special Types of Functions: Injections and Surjections

Preview Activity A.2. Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, and $C = \{s, t\}$. Define

$f : A \to B$ by	$g : A \to B$ by	$h : A \to C$ by
$f(1) = a$	$g(1) = a$	$h(1) = s$
$f(2) = b$	$g(2) = b$	$h(2) = t$
$f(3) = c$	$g(3) = a$	$h(3) = s$

(a) Consider the following property, defined for an arbitrary function F:

For all $x, y \in \text{dom}(F)$, if $x \neq y$, then $F(x) \neq F(y)$.

Which of the functions defined above satisfy this property?

(b) Which of the functions defined above satisfy the following property (defined in terms of an arbitrary function F)?

For all $x, y \in \text{dom}(F)$, if $F(x) = F(y)$, then $x = y$.

(c) Determine the range of each of the functions f, g, and h.

(d) Which of these functions have their range equal to their codomain?

(e) Which of the these functions satisfy the following property (again, defined in terms of an arbitrary function F)?

For all y in the codomain of F, there exists an $x \in \text{dom}(F)$ such that $F(x) = y$.

(f) Let F be a function from a set S to a set T.

(i) Is it possible to have two elements x_1 and x_2 in S with $x_1 \neq x_2$ and $F(x_1) = F(x_2)$? If no, explain why not. If yes, give an example and explain why this does not violate the definition of a function.

(ii) Are the range and codomain of a function the same or different? If they are the same, explain why. If different, give an example to illustrate the difference and explain any relationships that must exist between the two sets.

Preview Activity A.3. Let A and B be nonempty sets, and let $f : A \to B$. In Preview Activity A.2, we determined whether or not certain functions satisfied some specified properties. These properties were written in the form of statements, and we will now examine these statements in more detail.

(a) Consider the following statement:

For all $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

(i) Write the contrapositive of this conditional statement.

(ii) Write the negation of this conditional statement.

(b) Now consider the following statement:

For all $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

Write the negation of this statement.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 5x + 3$, for all $x \in \mathbb{R}$. Complete the proofs of the following propositions about the function g.

Proposition 1. For all $a, b \in \mathbb{R}$, if $g(a) = g(b)$, then $a = b$.

Proof. Let $a, b \in \mathbb{R}$, and assume that $g(a) = g(b)$. We will prove that $a = b$. Since $g(a) = g(b)$, we know that

$$5a + 3 = 5b + 3.$$

(Now prove that in this situation, $a = b$.)

Proposition 2. For all $b \in \mathbb{R}$, there exists an $a \in \mathbb{R}$ such that $g(a) = b$.

Proof. Let $b \in \mathbb{R}$. We will prove that there exists an $a \in \mathbb{R}$ such that $g(a) = b$ by constructing such an a in $\mathbb{R}$. In order for this to happen, we need $g(a) = 5a + 3 = b$.

(Now solve the equation for a, and then show that for this real number a, $g(a) = b$.)

Injections

We have now seen examples of functions for which there exist different inputs that produce the same output. Using more formal notation, this means that there are functions $f : A \to B$ for which there exist $x_1, x_2 \in A$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. The work in the preview activities was intended to motivate the following definition.

Definition A.4. Let $f : A \to B$ be a function from the set A to the set B. The function f is called an **injection** provided that

$$\text{for all } x_1, x_2 \in A, \text{ if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2).$$

When f is an injection, we also say that f is a **one-to-one function**, or that f is an **injective function**.

Notice that the condition that specifies that a function f is an injection is given in the form of a conditional statement. As we will see, in proofs it is usually easier to use the contrapositive of this conditional statement. Although we did not define the term then, we have already written the contrapositive for the conditional statement in the definition of an injection in part (a) of Preview Activity A.3. In that preview activity, we also wrote the negation of the definition of an injection. The box below summarizes this work by giving the conditions that are equivalent to f being an injection or not being an injection.

Let $f : A \to B$.

"The function f is an injection" means that

- for all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$; or
- for all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

"The function f is not an injection" means that

- there exist $x_1, x_2 \in A$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

Activity A.5. Now that we have defined what it means for a function to be an injection, we can see that in part (c) of Preview Activity A.3, we proved that the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = 5x+3$ for all $x \in \mathbb{R}$, is an injection. Use the definition (or its negation) to determine whether or not the following functions are injections.

(a) $k : A \to B$, where $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, and $k(a) = 4$, $k(b) = 1$, and $k(c) = 3$

(b) $f : A \to C$, where $A = \{a, b, c\}$, $C = \{1, 2, 3\}$, and $f(a) = 2$,$f(b) = 3$, and $f(c) = 2$

(c) $F : \mathbb{Z} \to \mathbb{Z}$ defined by $F(m) = 3m + 2$ for all $m \in \mathbb{Z}$

(d) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2 - 3x$ for all $x \in \mathbb{R}$

(e) $s : \mathbb{Z}_5 \to \mathbb{Z}_5$ defined by $s(x) = x^3$ for all $x \in \mathbb{Z}_5$

Surjections

In previous mathematics courses and in Preview Activity A.2, we have seen that there exist functions $f : A \to B$ for which the codomain and range of f are equal—that is, range$(f) = B$. This means that every element of B is an output of the function f for some input from the set A. Using quantifiers, this means that for every $y \in B$, there exists an $x \in A$ such that $f(x) = y$. One of the objectives of the preview activities was to motivate the following definition:

Definition A.6. Let $f : A \to B$ be a function from the set A to the set B. The function f is called a **surjection** provided that the range of f equals the codomain of f. This means that

for every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

When f is a surjection, we also say that f is an **onto function**, that f maps A **onto** B, or that f is a **surjective function**.

Note that the main condition defining what it means for a function f to be a surjection is given in the form of a universally quantified statement. Although we did not define the term then, we have already written the negation of this statement defining a surjection in part (b) of Preview Activity A.3. The box below summarizes the conditions for f being a surjection or not being a surjection.

Let $f : A \to B$.

"The function f is a surjection" means that

- $\text{range}(f) = \text{codom}(f) = B$; or
- for every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

"The function f is not a surjection" means that

- $\text{range}(f) \neq \text{codom}(f)$; or
- there exists a $y \in B$ such that for all $x \in A$, $f(x) \neq y$.

Activity A.7. Now that we have defined what it means for a function to be a surjection, we can see that in part (c) of Preview Activity A.3, we proved that the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = 5x+3$ for all $x \in \mathbb{R}$, is a surjection. Determine whether or not the following functions are surjections. Are any of these functions injections?

(a) $k : A \to B$, where $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, and $k(a) = 4$, $k(b) = 1$, and $k(c) = 3$.

(b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$ for all $x \in \mathbb{R}$.

(c) $F : \mathbb{Z} \to \mathbb{Z}$ defined by $F(m) = 3m + 2$ for all $m \in \mathbb{Z}$.

(d) $s : \mathbb{Z}_5 \to \mathbb{Z}_5$ defined by $s(x) = x^3$ for all $x \in \mathbb{Z}_5$.

Another important class of functions are those that are both injective and surjective. Any such function is called a *bijection*.

Definition A.8. A **bijection** is a function that is both an injection and a surjection. If the function f is a bijection, we also say that f is **one-to-one and onto** and that f is a **bijective function**.

Activity A.9. Which of the functions in Activity A.7 are bijections?

The Importance of the Domain and Codomain

The functions in the next activity will illustrate why the domain and the codomain are just as important as the rule defining the outputs when we are trying to determine if a given function is injective and/or surjective.

Activity A.10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$. Notice that

$$f(2) = 5 \text{ and } f(-2) = 5.$$

This observation is enough to prove that the function f is not an injection since we can see that there exist two different inputs that produce the same output.

Since $f(x) = x^2 + 1$, we know that $f(x) \geq 1$ for all $x \in \mathbb{R}$. This implies that the function f is not a surjection. For example, -2 is in the codomain of f and $f(x) \neq -2$ for all x in the domain of f.

(a) Now let $T = \{y \in \mathbb{R} \mid y \geq 1\}$, and define $F : \mathbb{R} \to T$ by $F(x) = x^2 + 1$. Notice that the function F uses the same formula as the function f and has the same domain as f, but has a different codomain than f.

   (i) Explain why F is not an injection.

   (ii) Is F a surjection? Justify your conclusion.

(b) Let $\mathbb{Z}^* = \{x \in \mathbb{Z} \mid x \geq 0\} = \mathbb{N} \cup \{0\}$. Define $g : \mathbb{Z}^* \to \mathbb{N}$ by $g(x) = x^2 + 1$. (Notice that this is the same formula used in part (a).)

   (i) Calculate $g(0), g(1), g(2), g(3), g(4)$, and $g(5)$. Based on this information, does the function g appear to be an injection? Does the function g appear to be a surjection?

   (ii) Is the function g an injection? Justify your conclusion with a proof or a counterexample.

   (iii) Is the function g a surjection? Justify your conclusion with a proof of a counterexample.

In Activity A.10, the same mathematical formula was used to determine the outputs for the functions. However:

- One of the functions was neither an injection nor a surjection.
- Another one of the functions was not an injection but was a surjection.
- The third function was an injection but was not a surjection.

This illustrates the important fact that whether a function is injective or surjective not only depends on the formula that defines the output of the function but also on the domain and codomain of the function.

Composition of Functions

The basic idea of function composition is that, when possible, the output of a function f is used as the input of a function g. The resulting function can be referred to as "f followed by g" and is called the composition of f with g. For example, if

$$f(x) = 3x^2 + 2 \quad \text{and} \quad g(x) = \sin(x),$$

then we can compute $g(f(x))$ as follows:

$$\begin{aligned} g(f(x)) &= g\big(3x^2 + 2\big) \\ &= \sin\big(3x^2 + 2\big). \end{aligned}$$

In this case, $f(x)$, the output of the function f, was used as the input for the function g. This idea motivates the formal definition of the composition of two functions.

Definition A.11. Let A, B, and C be nonempty sets, and let $f : A \to B$ and $g : B \to C$ be functions. The **composition of f and g** is the function $g \circ f : A \to C$ defined by

$$(g \circ f)(x) = g\,(f(x))$$

for all $x \in A$. We often refer to the function $g \circ f$ as a **composite function**.

Activity A.12. Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, and $C = \{s, t\}$. Define $f : A \to B$ by

$$f(1) = a, f(2) = b, f(3) = c,$$

$g : A \to B$ by

$$g(1) = c, g(2) = d, g(3) = c,$$

and $h : B \to C$ by

$$h(a) = s, h(b) = s, h(c) = t, h(d) = s.$$

(a) Find the images of the elements in A under the function $h \circ f$.

(b) Find the images of the elements in A under the function $h \circ g$.

(c) Is $h \circ f$ an injection? Is $h \circ f$ a surjection? Explain.

(d) Is $h \circ g$ an injection? Is $h \circ g$ a surjection? Explain.

In Activity A.12, we asked questions about whether certain composite functions were injections and/or surjections. In mathematics, it is typical to explore whether certain properties of an object transfer to related objects. In particular, we might want to know whether or not the composite of two injective functions is also an injection. (Of course, we could ask a similar question for surjections.) These types of questions are explored in the next activity.

Activity A.13. Let the sets A, B, C, and D be as follows:

$$A = \{a, b, c\}, \quad B = \{p, q, r\}, \quad C = \{u, v, w, x\}, \quad \text{and} \quad D = \{u, v\}.$$

(a) Construct a function $f : A \to B$ that is an injection and a function $g : B \to C$ that is an injection. In this case, is the composite function $g \circ f : A \to C$ an injection? Explain.

(b) Construct a function $f : A \to B$ that is a surjection and a function $g : B \to D$ that is a surjection. In this case, is the composite function $g \circ f : A \to D$ a surjection? Explain.

(c) Construct a function $f : A \to B$ that is a bijection and a function $g : B \to A$ that is a bijection. In this case, is the composite function $g \circ f : A \to A$ a bijection? Explain.

In Activity A.13, we explored some properties of composite functions related to injections, surjections, and bijections. The following theorem summarizes the results that these explorations were intended to illustrate.

Theorem A.14. *Let A, B, and C be nonempty sets, and assume that $f : A \to B$ and $g : B \to C$.*

(i) *If f and g are both injections, then $(g \circ f) : A \to C$ is an injection.*

(ii) *If f and g are both surjections, then $(g \circ f) : A \to C$ is a surjection.*

(iii) *If f and g are both bijections, then $(g \circ f) : A \to C$ is a bijection.*

The proof of part (i) is Exercise 4, and part (iii) is a direct consequence of the first two parts. Therefore, we will focus here on constructing a proof of part (ii). Our goal is to prove that $g \circ f$ is a surjection. Since $g \circ f : A \to C$, this is equivalent to proving that

for all $c \in C$, there exists an $a \in A$ such that $(g \circ f)(a) = c$.

Thus, we need to find an $a \in A$ such that $(g \circ f)(a) = c$.

Now we can look at the hypotheses. In particular, we are assuming that both $f : A \to B$ and $g : B \to C$ are surjections. Since we have chosen $c \in C$, and $g : B \to C$ is a surjection, we know that there exists a $b \in B$ such that $g(b) = c$. Now, $b \in B$ and $f : A \to B$ is a surjection. Therefore, there exists an $a \in A$ such that $f(a) = b$. If we now compute $(g \circ f)(a)$, we will see that

$$(g \circ f)(a) = g\left(f(a)\right) = g(b) = c.$$

We can now write the complete proof as follows:

Proof of Theorem A.14, part (ii). Let A, B, and C be nonempty sets, and assume that $f : A \to B$ and $g : B \to C$ are both surjections. We will prove that $g \circ f : A \to C$ is a surjection.

Let c be an arbitrary element of C. We will prove there exists an $a \in A$ such that $(g \circ f)(a) = c$. Since $g : B \to C$ is a surjection, it follows that there exists a $b \in B$ such that $g(b) = c$. Now $b \in B$ and $f : A \to B$ is a surjection. Hence, there exists an $a \in A$ such that $f(a) = b$. We now see that

$$\begin{aligned}(g \circ f)(a) &= g\left(f(a)\right) \\ &= g(b) \\ &= c.\end{aligned}$$

We have therefore shown that for every $c \in C$, there exists an $a \in A$ such that $(g \circ f)(a) = c$. This proves that $g \circ f$ is a surjection. ■

Inverse Functions

Now that we have studied composite functions, we will move on to consider another important idea: the inverse of a function. In order to study inverse functions, we will need to use the concept of the

Cartesian product of two sets A and B, denoted by $A \times B$, which is the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$. That is,

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

In previous mathematics courses, you probably learned that the exponential function (with base e) and the natural logarithm functions are inverses of each other. You may have seen this relationship expressed as follows:

For each $x \in \mathbb{R}$ with $x > 0$ and for each $y \in \mathbb{R}$,
$y = \ln(x)$ if and only if $x = e^y$.

Notice that x is the input and y is the output for the natural logarithm function if and only if y is the input and x is the output for the exponential function. In essence, the inverse function (in this case, the exponential function) reverses the action of the original function (in this case, the natural logarithm function). In terms of ordered pairs (input-output pairs), this means that if (x, y) is an ordered pair for a function, then (y, x) is an ordered pair for its inverse. The idea of reversing the roles of the first and second coordinates is the basis for our definition of the inverse of a function.

Definition A.15. Let $f : A \to B$ be a function. The **inverse** of f, denoted by f^{-1}, is the set of ordered pairs $\{(b, a) \in B \times A \mid f(a) = b\}$. That is,

$$f^{-1} = \{(b, a) \in B \times A : f(a) = b\}.$$

If we use the ordered pair representation for f, we could also write

$$f^{-1} = \{(b, a) \in B \times A : (a, b) \in f\}.$$

Notice that this definition does not state that f^{-1} is a function. Rather, f^{-1} is simply a subset of $B \times A$. In Activity A.16, we will explore the conditions under which the inverse of a function $f : A \to B$ is itself a function from B to A.

Activity A.16. Let $A = \{a, b, c\}$, $B = \{a, b, c, d\}$, and $C = \{p, q, r\}$. Define

$f : A \to C$ by	$g : A \to C$ by	$h : B \to C$ by
$f(a) = r$	$g(a) = p$	$h(a) = p$
$f(b) = p$	$g(b) = q$	$h(b) = q$
$f(c) = q$	$g(c) = p$	$h(c) = r$
		$h(d) = q$

(a) Determine the inverse of each function as a set of ordered pairs.

(b) (i) Is f^{-1} a function from C to A? Explain.

(ii) Is g^{-1} a function from C to A? Explain.

(iii) Is h^{-1} a function from C to B? Explain.

(c) Make a conjecture about what conditions on a function $F : S \to T$ will ensure that its inverse is a function from T to S.

We will now consider a general argument suggested by the explorations in Activity A.16. By definition, if $f : A \to B$ is a function, then f^{-1} is a subset of $B \times A$. However, f^{-1} may or may

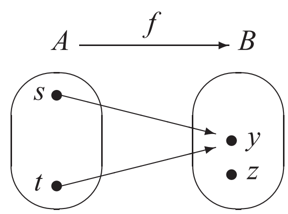

Figure A.1
The inverse is not a function.

not be a function from B to A. For example, suppose that $s, t \in A$ with $s \neq t$ and $f(s) = f(t)$ (as illustrated in Figure A.1).

In this case, if we try to reverse the arrows, we will not get a function from B to A. This is because $(y, s) \in f^{-1}$ and $(y, t) \in f^{-1}$ with $s \neq t$. Consequently, f^{-1} is not a function. This observation suggests that if f is not an injection, then f^{-1} is not a function.

Also, if f is not a surjection, then there exists a $z \in B$ such that $f(a) \neq z$ for all $a \in A$, as in the diagram in Figure A.1. In other words, there is no ordered pair in f with z as the second coordinate. This means that there would be no ordered pair in f^{-1} with z as a first coordinate. Consequently, f^{-1} cannot be a function from B to A.

Theorem A.17 formalizes these observations. In the proof of the theorem, we will use both the input-output representation and the ordered pair representation of a function. The idea is that if $G : S \to T$ is a function, then for $s \in S$ and $t \in T$,

$$G(s) = t \text{ if and only if } (s, t) \in G.$$

When we use the ordered pair representation of a function, we will also use the ordered pair representation of its inverse. In this case, we know that

$$(s, t) \in G \text{ if and only if } (t, s) \in G^{-1}.$$

Theorem A.17. *Let A and B be nonempty sets, and let $f : A \to B$. The inverse of f is a function from B to A if and only if f is a bijection.*

Proof. Let A and B be nonempty sets, and let $f : A \to B$. We will first assume that f is a bijection and prove that f^{-1} is a function from B to A. To do this, we will show that f^{-1} satisfies the conditions of Definition A.1.

Let $b \in B$. Since the function f is a surjection, there exists an $a \in A$ such that $f(a) = b$. This implies that $(a, b) \in f$ and hence that $(b, a) \in f^{-1}$. Thus, each element of B is the first coordinate of an ordered pair in f^{-1}. We must now prove that each element of B is the first coordinate of exactly one ordered pair in f^{-1}. So let $b \in B$, $a_1, a_2 \in A$ and assume that

$$(b, a_1) \in f^{-1} \text{ and } (b, a_2) \in f^{-1}.$$

This means that $(a_1, b) \in f$ and $(a_2, b) \in f$. We can then conclude that

$$f(a_1) = b \text{ and } f(a_2) = b.$$

But this means that $f(a_1) = f(a_2)$. Since f is a bijection, f is by definition an injection, and we can conclude that $a_1 = a_2$. This proves that b is the first element of only one ordered pair in f^{-1}. Consequently, we have proved that f^{-1} satisfies the conditions of Definition A.1 and hence f^{-1} is a function from B to A.

We will now assume that f^{-1} is a function from B to A and prove that f is a bijection. First, to prove that f is an injection, we will assume that $a_1, a_2 \in A$ and that $f(a_1) = f(a_2)$. We wish to show that $a_1 = a_2$. If we let $b = f(a_1) = f(a_2)$, we can conclude that

$$(a_1, b) \in f \text{ and } (a_2, b) \in f.$$

But this means that

$$(b, a_1) \in f^{-1} \text{ and } (b, a_2) \in f^{-1}.$$

Since we have assumed that f^{-1} is a function, we can conclude that $a_1 = a_2$. Hence, f is an injection.

Now to prove that f is a surjection, we will choose an arbitrary $b \in B$ and show that there exists an $a \in A$ such that $f(a) = b$. Since f^{-1} is a function, b must be the first coordinate of some ordered pair in f^{-1}. Consequently, there exists an $a \in A$ such that

$$(b, a) \in f^{-1}.$$

Now this implies that $(a, b) \in f$, and so $f(a) = b$. This proves that f is a surjection. Since we have also proved that f is an injection, we can conclude that f is a bijection, as desired. ■

Theorems about Inverse Functions

In the situation where $f : A \to B$ is a bijection and f^{-1} is a function from B to A, we can write $f^{-1} : B \to A$. In this case, we frequently say that f is an **invertible function**, and we usually do not use the ordered pair representation for either f or f^{-1}. Instead of writing $(a, b) \in f$, we write $f(a) = b$, and instead of writing $(b, a) \in f^{-1}$, we write $f^{-1}(b) = a$. Using the fact that $(a, b) \in f$ if and only if $(b, a) \in f^{-1}$, we can now write $f(a) = b$ if and only if $f^{-1}(b) = a$. Theorem A.18 formalizes this observation.

Theorem A.18. *Let A and B be nonempty sets, and let $f : A \to B$ be a bijection. Then $f^{-1} : B \to A$ is a function, and for every $a \in A$ and $b \in B$,*

$$f(a) = b \textit{ if and only if } f^{-1}(b) = a.$$

The next two results are two important theorems about inverse functions. The first can be considered to be a corollary of Theorem A.18.

Corollary A.19. *Let A and B be nonempty sets, and let $f : A \to B$ be a bijection. Then*

(i) *For every x in A, $\left(f^{-1} \circ f\right)(x) = x$.*

(ii) *For every y in B, $\left(f \circ f^{-1}\right)(y) = y$.*

Activity A.20. Prove Corollary A.19. For the first part, let $x \in A$, write $f(x) = y$, and then use the result in Theorem A.18.

We will now consider the case where $f : A \to B$ and $g : B \to C$ are both bijections. In this case, $f^{-1} : B \to A$ and $g^{-1} : C \to B$. Figure A.2 illustrates this situation.

By Theorem A.14, $g \circ f : A \to C$ is also a bijection. Hence, by Theorem A.17, $(g \circ f)^{-1}$ is a function and, in fact, $(g \circ f)^{-1} : C \to A$. Notice that we can also form the composition of g^{-1} followed by f^{-1} to get $f^{-1} \circ g^{-1} : C \to A$. Figure A.2 helps illustrate the result of the next theorem.

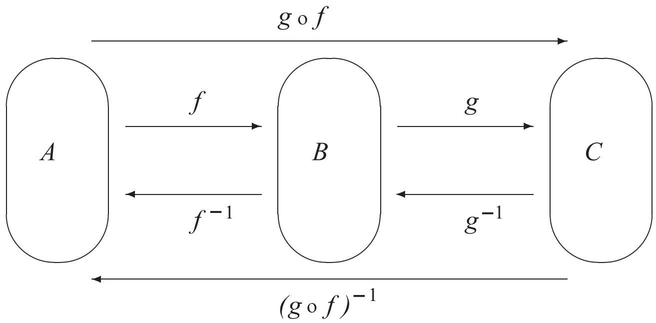

Figure A.2
Composition of two bijections.

Theorem A.21. *Let $f : A \to B$ and $g : B \to C$ be bijections. Then $g \circ f$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Proof. Let $f : A \to B$ and $g : B \to C$ be bijections. Then $f^{-1} : B \to A$ and $g^{-1} : C \to B$. Hence, $f^{-1} \circ g^{-1} : C \to A$. Also, by Theorem A.14, $g \circ f : A \to C$ is a bijection, and hence $(g \circ f)^{-1} : C \to A$. We will now prove that for each $z \in C$, $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$.

Let $z \in C$. Since the function g is a surjection, there exists a $y \in B$ such that

$$g(y) = z. \tag{A.1}$$

Also, since f is a surjection, there exists an $x \in A$ such that

$$f(x) = y. \tag{A.2}$$

Now equations (A.1) and (A.2) can be written in terms of the respective inverse functions as

$$g^{-1}(z) = y \text{ and} \tag{A.3}$$

$$f^{-1}(y) = x. \tag{A.4}$$

Using equations (A.3) and (A.4), we see that

$$\begin{aligned} \left(f^{-1} \circ g^{-1}\right)(z) &= f^{-1}\left(g^{-1}(z)\right) \\ &= f^{-1}(y) \\ &= x. \end{aligned} \tag{A.5}$$

Using equations (A.1) and (A.2) again, we see that $(g \circ f)(x) = z$. However, in terms of the inverse function, this means that

$$(g \circ f)^{-1}(z) = x. \tag{A.6}$$

Comparing equations (A.5) and (A.6), we have shown that for all $z \in C$, $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$. This proves that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. ∎

Concluding Activities

Activity A.22. Prove the following:

If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection.

Exercises

(1) For each of the following functions, determine if the function is an injection, a surjection, a bijection, or none of these. Justify all of your conclusions.

(a) $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 5x + 3$, for all $x \in \mathbb{R}$.

(b) $G : \mathbb{Z} \to \mathbb{Z}$ defined by $G(x) = 5x + 3$, for all $x \in \mathbb{Z}$.

(c) $f : (\mathbb{R} - \{4\}) \to \mathbb{R}$ defined by $f(x) = \dfrac{3x}{x-4}$, for all $x \in (\mathbb{R} - \{4\})$.

(d) $g : (\mathbb{R} - \{4\}) \to (\mathbb{R} - \{3\})$ defined by $g(x) = \dfrac{3x}{x-4}$, for all $x \in (\mathbb{R} - \{4\})$.

(2) Define $f : \mathbb{N} \to \mathbb{Z}$ as follows: For each $n \in \mathbb{N}$,

$$f(n) = \frac{1 + (-1)^n(2n-1)}{4}.$$

Is the function f an injection? Is the function f a surjection? Justify your conclusions.

Suggestions: Start by calculating several outputs for the function before you attempt to write a proof. In exploring whether or not the function is an injection, it might be a good idea to use cases based on whether the inputs are even or odd. In exploring whether f is a surjection, consider using cases based on whether the output is positive or less than or equal to zero.

(3) An operation $*$ on a set S is a function from $S \times S$ to S that assigns to the pair $(x, y) \in S \times S$ the element $x * y$ in S. For example, addition of integers can be defined as a function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ that maps the pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ to the integer $f(a, b) = a + b$.

(a) Is the function f an injection? Justify your conclusion.

(b) Is the function f a surjection? Justify your conclusion.

(4) Prove Part (i) of Theorem A.14:

Let A, B, and C be nonempty sets, and let $f : A \to B$ and $g : B \to C$. If f and g are both injections, then $g \circ f : A \to C$ is an injection.

(5) Suppose $f : A \to B$ and $g : B \to C$ are functions.

(a) Is it true that if $g \circ f$ is an injection, then both f and g are injections? If the answer is no, are there any conditions that f or g must satisfy to make $g \circ f$ an injection? Prove your answers.

(b) Is it true that if $g \circ f$ is a surjection, then both f and g are surjections? If the answer is no, are there any conditions that f or g must satisfy to make $g \circ f$ a surjection? Prove your answers.

(6) Is composition of functions a commutative operation? Prove your answer.

(7) Is composition of functions an associative operation? Prove your answer.

(8) (a) Define $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $f([x]) = \left[x^2 + 4\right]$ for all $[x] \in \mathbb{Z}_5$. Write the inverse of f as a set of ordered pairs, and explain why f^{-1} is not a function.

(b) Define $g : \mathbb{Z}_5 \to \mathbb{Z}_5$ by $f([x]) = \left[x^3 + 4\right]$ for all $[x] \in \mathbb{Z}_5$. Write the inverse of g as a set of ordered pairs, and explain why g^{-1} is a function.

(c) Is it possible to write a formula for $g^{-1}([y])$, where $[y] \in \mathbb{Z}_5$? The answer to this question depends on whether or not it is possible to define a cube root of elements of $\mathbb{Z}_5$. Recall that for a real number x, we define the cube root of x to be the real number y such that $y^3 = x$. That is,

$$y = \sqrt[3]{x} \text{ if and only if } y^3 = x.$$

Using this idea, is it possible to define the cube root of each element of $\mathbb{Z}_5$? If so, what is $\sqrt[3]{[0]}$, $\sqrt[3]{[1]}$, $\sqrt[3]{[2]}$, $\sqrt[3]{[3]}$, and $\sqrt[3]{[4]}$.

(d) Now answer the question posed at the beginning of part (c). If possible, determine a formula for $g^{-1}([y])$ where $g^{-1} : \mathbb{Z}_5 \to \mathbb{Z}_5$.