Section 5

The Greatest Lower Bound

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a lower bound and a greatest lower bound of a subset of $\mathbb{R}$?
- What is an upper bound and a least upper bound of a subset of $\mathbb{R}$?
- How is a greatest lower bound used to define the distance from a point to a set? Why is it necessary to use a greatest lower bound?

Introduction

The real numbers have a special property that allows us to, among other things, define the distance between a point and a set in a metric space. It also allows us to define distances between subsets of certain types of metric spaces, which creates a whole new metric space whose elements are the subsets of the metric space. We will examine that property of the real numbers in this activity.

We begin by considering the problem of defining the distance between a real number and an interval in $\mathbb{R}$ with the Euclidean metric d_E defined by

$$d_E(x, y) = |x - y|.$$

Let $x = 1$ and let A be the closed interval $[-1, 0]$. It is natural to suggest that the distance between the point x and the set A, denoted $d_E(x, A)$, should be the distance from the point x to the point in A closest to x. So in this case we would say

$$d(x, A) = d(x, [-1, 0]) = d_E(x, 0) = 1.$$

This might lead us to suggest that the distance from a point x to a set A, denoted by $d(x, A)$ is the minimum distance from the point to any point in the set, or $d(x, A) = \min\{d_E(x, a) \mid a \in A\}$.

What if we changed the set A to be the open interval $(-1, 0)$? What then should $d(x, A)$ be, or should this distance even exist? If we think of the distance between a point and a set as measuring how far we have to travel from the point until we reach the set, then in the case of $x = 1$ and $A = (-1, 0)$, as soon as we travel a distance more than 1 from x in the direction of A, we reach the set A. So we might intuitively say that $d_E(x, (-1, 0)) = 1$ as well. But we cannot define this distance as a distance from x to a point in A since $0 \notin A$. We need a different way to formulate the notion of a distance from a point to a set.

In a case like this, with $x = 1$ and $A = (-1, 0)$, we can examine the set $T = \{d_E(x, a) \mid a \in A\}$ and notice some facts about this set. For example, the set T is a subset of the nonnegative real numbers. Also, in this example there are no numbers in T that are smaller than 1. Because of this property, we will call the number 1 a *lower bound* for T. More generally,

Definition 5.1. Let S be a nonempty subset of $\mathbb{R}$. A **lower bound** for S is a real number m such that $m \leq s$ for all $s \in S$.

If a subset S of $\mathbb{R}$ has a lower bound, we say that S is *bounded below*. So the set $T = \{d_E(1, a) \mid a \in (-1, 0)\}$ is bounded below by 1. The set T is also bounded below by 0.5 and 0. In fact, any number less than 1 is a lower bound for T. The critical idea, though, is that no number larger than 1 is a lower bound for T. Because of this we call 1 a *greatest lower bound* of T. More generally,

Definition 5.2. Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. A **greatest lower bound** for S is a real number m such that

(1) m is a lower bound for S and

(2) if k is a lower bound for S, then $m \geq k$.

A greatest lower bound is also called an *infimum*. We might now use this idea of a greatest lower bound to define the distance between 1 and $A = (-1, 0)$ as the greatest lower bound of the set $\{d_E(1, a) \mid a \in (-1, 0)\}$. However, there are questions we need to address before we can do so. One question is whether or not every nonempty subset of $\mathbb{R}$ that is bounded below has an infimum. The answer to this question is yes, and we will take this result as an axiom of the real number system (often called the *completeness axiom*).

Preview Activity 5.1.

(1) Does every subset of $\mathbb{R}$ have a lower bound? Explain. (When a subset of $\mathbb{R}$ has a lower bound we say that the set is *bounded below*.)

(2) Which of the following subsets S of $\mathbb{R}$ are bounded below? If the set is bounded below, what is its infimum?

  i. $S = \{x \mid 3x^2 - 12x + 3 < 0\}$
  ii. $S = \{3x^3 - 1 \mid x \in \mathbb{R}\}$
  iii. $S = \{2^r + 3^s \mid r, s \in \mathbb{Z}^+\}$

(3) How would you define a least upper bound of a subset S of $\mathbb{R}$?

The Distance from a Point to a Set

Metrics are used to establish separation between objects. Topological spaces can be placed into different categories based on how well certain types of sets can be separated. We have defined metrics as measuring distances between points in a metric space, and in this activity we extend that idea to measure the distance between a point and a subset in a metric space. However, there are two questions we need to address before we can do so. The first we mentioned in our preview activity. We will assume the *completeness axiom* of the reals, that is that any subset of $\mathbb{R}$ that is bounded below always has a greatest lower bound. The second question is whether or not a greatest lower bound is unique.

Activity 5.1. Let S be a subset of $\mathbb{R}$ that is bounded below, and assume that S has a greatest lower bound. In this activity we will show that the infimum of S is unique.

(a) What method can we use to prove that there is only one greatest lower bound for S?

(b) Suppose m and m' are both greatest lower bounds for S. Why are m and m' both lower bounds for S?

(c) What two things does the second property of a greatest lower bound tell us about the relationship between m and m'?

(d) Why must the greatest lower bound of S be unique?

With the existence and uniqueness of greatest lower bounds considered, we can now say that any nonempty subset S of $\mathbb{R}$ that is bounded below has a unique greatest lower bound. We use the notation glb(S) (or $\inf(S)$ for *infimum* of S) for the greatest lower bound of S. There is also a *least upper bound* (lub(S), or $\sup(S)$ for *supremum*) of a subset S of $\mathbb{R}$ that is bounded above.

Now we can formally define the distance between a point and a subset in a metric space.

Definition 5.3. Let (X, d) be a metric space, let $x \in X$, and let A be a nonempty subset of X. The **distance from x to A** is

$$\inf\{d(x, a) \mid a \in A\}.$$

We denote the distance from x to A by $d(x, A)$. When calculating these distances, it must be understood what the underlying metric is.

Activity 5.2. There are a couple of facts about the distance between a point and a set that we examine in this activity. Let (X, d) be a metric space, let $x \in X$, and let A be a nonempty subset of X

(a) Why must $d(x, A)$ exist?

(b) If $d(x, A) = 0$, must $x \in A$?

Summary

Important ideas that we discussed in this section include the following.

- A lower bound or a nonempty subset S of $\mathbb{R}$ that is bounded below is a real number m such that $m \leq s$ for all $s \in S$. A greatest lower bound (or infimum) for a nonempty subset S of $\mathbb{R}$ that is bounded below is a real number m such that

  i. m is a lower bound for S and

  ii. if k is a lower bound for S, then $m \geq k$.

- An upper bound for a nonempty subset S of $\mathbb{R}$ that is bounded above is a real number M such that $M \geq s$ for all $s \in S$. A least upper bound (or supremum) for a nonempty subset S of $\mathbb{R}$ that is bounded above is a real number M such that

  i. M is an upper bound for S and

  ii. if k is an upper bound for s, then $M \leq k$.

- The distance from a point x to a set A in a metric space (X, d) is $d(x, A) = \inf\{d(x, a) \mid a \in A\}$. There may be no point $a \in A$ such that $d(x, A) = d(x, a)$, so it is necessary to use an infimum to define this distance.

Exercises

(1) Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Let $a \in \mathbb{R}$, and define $a + S$ to be $a + S = \{a + s \mid s \in S\}$.

  (a) Explain why $a + \inf(S)$ is a lower bound for $a + S$. Explain why $a + S$ has an infimum.

  (b) Let b be a lower bound for $a + S$. Show that $a + \inf(S) \geq b$. Then explain why $a + \inf(S) = \inf(a + S)$.

(2) Let S be a nonempty subset of $\mathbb{R}$.

  (a) Assume that S is bounded above, and let $t = \sup(S)$. Show that for every $r < t$, there is a number $s \in S$ such that $r < s \leq t$.

  (b) Assume that S is bounded below, and let $t = \inf(S)$. Show that for every $r > t$, there is a number $s \in S$ such that $t \leq s < r$.

(3) Let A and B be nonempty subsets of $\mathbb{R}$ that are bounded above and below. Let $A + B = \{a + b \mid a \in A, b \in B\}$.

  (a) Follow the steps below to show that $\sup(A + B) = \sup(A) + \sup(B)$.

    i. Let $x = \sup(A)$ and $y = \sup(B)$. Show that $x+y$ is an upper bound for $A+B$.

    ii. The previous part shows that $A + B$ is bounded above and so has a supremum. Let $z = \sup(A + B)$. Explain why $z \leq x + y$.

    iii. To show that $z = x+y$ we have to prove that z cannot be strictly less than $x+y$. Suppose to the contrary that $z < x + y$. Let $\epsilon = x + y - z$. Use the result of Exercise 2 to arrive at a contradiction.

  (b) Prove that $\inf(A + B) = \inf(A) + \inf(B)$.

(c) Prove or disprove: $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$

(d) Prove or disprove: $\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$

(4) Let $X = C[a, b]$, the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ on an interval $[a, b]$. Define $d : X \times X \to \mathbb{R}$ by

$$d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [a, b]\}.$$

(a) What is $d(x^2, 1 - 2x)$ on $[0, 1]$?

(b) Prove that d is a metric on X. Describe in geometric terms how this metric measures the distance between functions f and g. (This metric is called the *supremum metric* or the *uniform metric* on X.

(5) In this exercise we prove the *Archimedean property* of the natural numbers. Note that the set of natural numbers, denoted $\mathbb{N}$ or $\mathbb{Z}^+$, is the set of all positive integers.

Theorem 5.4 (The Archimedean Property.)**.** *Given any real number x, there exists a natural number N such that $N > x$.*

Let x be a real number.

(a) Suppose that there is no positive integer N such that $N > x$. Explain how we can conclude that $\mathbb{Z}^+$ is bounded above.

(b) Assuming that $\mathbb{Z}^+$ is bounded above, explain why $\mathbb{Z}^+$ must have a least upper bound M.

(c) Explain why M cannot be a least upper bound for $\mathbb{Z}^+$. Explain why this proves the Archimedean property.

(6) In this exercise we prove two statements that are equivalent to the Archimedean property (see Exercise (5)). One of the statements is the following theorem:

Theorem 5.5. *Given real numbers x and y with $x > 0$, there exists a natural number N such that $Nx > y$.*

(a) Let x and y be real numbers with $x > 0$.

i. Show that if the Archimedean property is true, then so is Theorem 5.5.

ii. Show that if Theorem 5.5 is true, then so is the Archimedean property. Conclude that Theorem 5.5 is equivalent to the Archimedean property.

(b) A second statement that is equivalent to the Archimedean property is the following.

Theorem 5.6. *If x is a positive real number, then there exists a positive integer N such that $\frac{1}{N} < x$.*

Prove that Theorem 5.6 is equivalent to the Archimedean property.

(7) We can use greatest lower bounds to prove the following theorem.

Theorem 5.7. *Given any two distinct real numbers x and y, there is a rational number that lies between them.*

This theorem tells us an important fact – that the rational numbers are what is called *dense* in the set of real numbers. We prove this theorem in this exercise. Let x and y be real numbers and assume $x < y$. By the Archimedean property of the natural numbers (see Exercises 5 and 6), there is a positive integer n such that $n(y - x) > 1$. Let $S = \{k \in \mathbb{Z} \mid k > nx\}$.

(a) Show that S is bounded below in $\mathbb{R}$.

(b) Explain why S contains an integer m such that if $q \in \mathbb{Z}$ with $q < m$, then $q \leq nx$. It may be helpful to use the Well-Ordering Principle that states

Every subset of the integers that is bounded below contains its infimum.

(The Well-Ordering Principle is one of many axioms that are equivalent to the Principle of Mathematical Induction. These principles are taken as axioms and are assumed to be true.)

(c) Explain why $m > nx$ and $m - 1 \leq nx$. Use these inequalities, along with $n(y-x) > 1$, to show that $nx < m < ny$. Then find a rational number that is strictly between x and y.

(8) Show that every open ball in $(\mathbb{R}^2, d_E)$ contains a point $x = (x_1, x_2)$ with both x_1 and x_2 rational.

(9) We are familiar with solving the quadratic equation $x^2 - 2 = 0$ to obtain the solutions $\pm\sqrt{2}$. But do we really know that the number $\sqrt{2}$ exists? We address that question in this exercise and demonstrate the existence of the number $\sqrt{2}$ using the greatest lower bound.

(a) To begin, let $S = \{x \in \mathbb{R}^+ \mid x^2 > 2\}$. Explain why S must have a greatest lower bound m.

(b) In what follows we demonstrate that $m^2 = 2$, which makes $m = \sqrt{2}$. We consider the cases $m^2 < 2$ and $m^2 > 2$.

i. Suppose $m^2 < 2$. Show that there is a positive integer n such that

$$\left(m + \frac{1}{n}\right)^2 < 2.$$

Explain why this also cannot happen.

ii. Suppose $m^2 > 2$. Show that there is a positive integer n such that

$$\left(m - \frac{1}{n}\right)^2 > 2.$$

Explain why this also cannot happen.

(c) Explain how we have demonstrated the existence of $\sqrt{2}$.

(10) Similar to Exercise 7 we can prove the following theorem.

Theorem 5.8. *Given any two distinct real numbers x and y, there is an irrational number that lies between them.*

(a) The first step is to demonstrate the existence of an irrational number. We will do that by proving that $\sqrt{2}$ is irrational. Proceed by contradiction and assume that $\sqrt{2}$ is a rational number. That is, $\sqrt{2} = \frac{r}{s}$ for some positive integers r and s such that r and s have no positive common factors other than 1.

i. Explain why $r^2 = 2s^2$. Since 2 is prime, it follows that 2 divides r.

ii. Show that 2 divides s. Explain how this proves that $\sqrt{2}$ is an irrational number.

(b) Let x and y be distinct real numbers. Show that there exists an integer q and a positive integer N such that $z = \frac{q\sqrt{2}}{2^N}$ is an irrational number between x and y. (Hint: Consider the approach in Exercise 7 .)

(11) Let (X, d) be a metric space and A a nonempty subset of X. For $x, y \in X$, prove that $d(x, A) \leq d(x, y) + d(y, A)$.

(12) Prove that if (X, d) is a metric space and B and C are nonempty subsets of X, then

$$d(a, B \cup C) = \min\{d(a, B), d(a, C)\}$$

for every $a \in X$.

(13) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why. Throughout, let S and T be bounded subsets of $\mathbb{R}$ (a subset of $\mathbb{R}$ is bounded if it is both bounded above and bounded below).

(a) Any nonempty subset of S is bounded.

(b) If $S + T = \{s + t \mid s \in S, t \in T\}$, then $\sup(S + T) = \max\{\sup(S), \sup(T)\}$.

(c) Let $S + T = \{s + t \mid s \in S, t \in T\}$, then $\inf(S + T) = \min\{\inf(S), \inf(T)\}$.

(d) If U is a nonempty subset of S, then $\sup(U) \leq \sup(S)$.

(e) If U is a nonempty subset of S, then $\inf(S) \leq \inf(U)$.

(f) If A is a subset of $\mathbb{R}$ and $x \in \mathbb{R}$ with $d(x, A) = 0$, then $x \in A$.